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A Model of a Systemic Bank Run

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Abstract

This paper provides a model of the view that the 2008 financial crisis is reminiscent of a bank run, focussing on six stylized key features. In particular, core financial institutions have invested their funds in asset-backed securities rather than committed to long-term projects: in distress, these can potentially be sold to a large pool of outside investors at steep discounts. I consider two different motives for outside investors and their interaction with banks trading asset-backed

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securities: uncertainty aversion versus adverse selection. I shall argue that the version with uncertainty averse investors is more consistent with the stylized facts than the adverse selection perspective: in the former, the crisis deepens, the larger the market share of distressed core banks, while a run becomes less likely instead as a result in the adverse selection version. Therefore, the outright purchase of troubled assets by the government at prices above current market prices may both alleviate the financial crises as well as provide tax payers with returns above those for safe securities.

Keywords: systemic bank run, bank run, systemic risk, financial crisis, firesale pricing, adverse selection, uncertainty aversion

JEL codes: G21, G12, G14, G01

1 Introduction

Bryant (1980) and Diamond and Dybvig (1983) have provided us with the classic benchmark model for a bank run. The financial crisis of 2007 and 2008 is reminiscent of a bank run, but not quite, see Brunnermeier (2008) and Gorton (2009). The following six features summarize the prevalent view by many observers:

1. The withdrawal of funds was done by financial institutions (in particular, money market funds and other banks) at some core financial institutions (I shall call them "core banks" for the purpose of this paper), rather than depositors at their local bank.

2. The troubled financial institutions held their portfolio in asset-backed securities (most notably tranches of mortgage-backed securities and credit default swaps) rather than being invested directly in long-term projects.

3. These securities are traded on markets.

4. There is a large pool of investors willing to purchase securities. For example, in the 2008 financial crisis, newly issued US government bonds were purchased at moderate discounts and the volume on stock markets was not low.

5. Nonetheless, investors were willing to buy the asset-backed securities during the crisis only at prices that are low compared to standard discounting of the entire pool of these securities.

6. The larger the market share of troubled financial institutions, the steeper the required discounts.

This perspective has possibly been crucial for a number of policy interventions, despite the inapplicability of the original Diamond-Dybvig framework.

This creates a gap in our understanding. A new or at least a modified theory is needed.

This paper seeks to contribute to filling that gap, and provide a model (in two variants) of a systemic bank run. A systemic bank run is a situation, in which early liquidity withdrawals by long-term depositors at some bank are larger and a bank run more likely, if other banks are affected by liquidity withdrawals too, i.e. the market interaction of the distressed banks is crucial. This is different from a system-wide run, which may occur if all depositors view their banks as not viable, regardless of whether the depositors at other banks do to.

The goal is for the model to produce the stylized view, i.e. the six items listed above. That stylized view may be entirely incorrect as a description of the 2008 financial crisis. It is possible that the appropriate perspective is one of insolvency rather than illiquidity, and future research will hopefully sort this out. Absent that clarification, it is worthwhile to analyze the situation from a variety of perspectives, including the one above.

It will turn out, that items 1 to 3 are straightforward to incorporate, merely requiring some additional notation. Item 4 is easy to incorporate in principle, but hard once one demands item 5 and 6 as well. Item 6 will turn out to be particulary thorny to achieve, and decisive in selecting one of two variants for modeling outside investors.

Here is the key argument. Common to both variants, suppose that there are some unforeseen early withdrawals. Therefore, core financial institutions need to sell part of their long-term securities, thereby incurring opportunity costs in terms of giving up returns at some later date. Suppose that the remaining depositors (or depositing institutions) are the more inclined to withdraw early as well, the larger these opportunity costs are. If a larger market share of distressed banks and therefore larger additional liquidity needs drive these opportunity costs up, then a wide spread run on the core banks is more likely: this creates a systemic bank run. Whether this happens depends on the market for the long-term securities, the outside investors, and the reasons for steep discounts of these securities, and it is here where the two variants differ.

In the first variant, I hypothesize that expert investors have finite resources, while the remaining vast majority of investors is highly uncertainty averse: they fear getting “stuck” with the worst asset among a diverse portfolio, and are therefore not willing to bid more than the lowest price, see section 3. For the second variant, I assume that risk-neutral investors together with adverse selection create an Akerlof-style lemons problem: liquid core banks have an incentive to sell their worst assets at a given market price, leading to a low equilibrium price, see section 5. Both models generate a downward sloping demand curve or, more accurately, an upward sloping period-2 opportunity cost for providing period-1 resources per selling long-term securities from the perspective of the individual core bank, holding aggregate liquidity demands unchanged.

However, the two variants have sharply different implications regarding the last feature in the list above. With uncertainty aversion, a larger market share of troubled institutions dilutes the set of expert investors faster, leading more quickly to steep period-2 opportunity costs for providing period-1 liquidity, and thereby setting the stage for a systemic bank run. By contrast and with adverse selection, a larger pool of distressed institutions leads to less free-riding of unaffected core banks, thereby lowering the opportunity costs for providing liquidity, see section 6. I shall therefore argue to rather analyze the 2008 financial crisis and draw policy conclusions, using the tools of uncertainty aversion. For example, with uncertainty aversion, a government purchase of assets above market price may be a good deal for the tax payers under uncertainty aversion, but not under adverse selection.

There obviously is a large literature expanding the Diamond-Dybvig bank run paradigm, including investigations of systemic risk and the occurrence of fire sales. A thorough discussion is beyond the scope here, but is available elsewhere. Allen and Gale (2007) have succinctly summarized much of the bank run literature and in particular their own contributions. Rochet (2008) has collected a number of his contributions with his co-authors on banking crises and bank regulation. Several papers on the recent financial crises and on resolution proposals have been collected in Acharya and Richardson (2009).

Nonetheless, it may be good to provide to explore at least some relationships. While the Diamond-Dybvig model is originally about multiple equilibria ("bank run" vs "no bank run"), Allen and Gale (2007) have emphasized fundamental equilibria, in which it is individually rational for a depositor to "run", even if nobody else does. Here, I also employ this fundamental view. Heterogeneous beliefs by depositors will create partial bank runs here. Allen and Gale (1994, 2004b) have investigated the scope and consequences of cash-in-the-market pricing to generate fire sale pricing and bank runs. Technical and legal details as well as institutional frictions and barriers surely play a key role in preventing outside investors to enter this market quickly, see Duffie (2009). It still remains surprising that outside investors remain reluctant to enter, if there truly is underpricing. Thinking about this reluctance and its implications is a goal of this paper.

Diamond and Rajan (2009) have argued that banks have become reluctant to sell their securities at present, if they foresee the possibility of insolvency due to firesale prices in the future, as they will gain on the upside. Additional reasons are needed to generate the firesale price in the first place: the latter is the focus of this paper.

While the popular press views financial crises and bank runs as undesirable disasters, e.g. Allen and Gale (1998, 2004a) have shown that they may serve a socially useful rule by partially substituting for a missing market, and policy should not necessarily seek to avoid them. On a more subtle level, Ennis and Keister (2008) have shown that ex-post efficient policy responses to a bank run of allowing urgent depositors to withdraw may actually increase the incentives to participate in a bank run and the conditions for a self-fulfilling bank run in the first place. In this paper, the main focus is on the positive rather than the normative analysis.

There is a large literature on systemic risk and contagion. For example, Cifuentes et al (2005) have studied the interplay between uncontingent capital adequacy requirements and the endogenous collapse of prices and balance sheets, as banks need to unload assets in order to meet these requirements. They assume that demand for these assets is downward sloping: this paper seeks to investigate why. Allen and Gale (2000) have studied the possibility for contagion in a sparse network of banks interlinked by mutual demand deposits, where a collapse of one bank can lead to a domino effect per withdrawals on their neighbor. Here, I assume a hierarchy, where local banks hold deposit contracts on core banks, who in turn use the market to obtain liquidity. Diamond and Rajan (2005) have investigated the contagious nature of bank failures, when returns on long-term projects can only be obtained by banks. Here instead, I assume that a asset-backed securities will pay its return, irrespective of ownership.

The role of adverse selection in firesale pricing has been analyzed before in Eisfeldt (2004). In contrast to that paper, I compare different versions of "bad times" here to obtain my key insight. There is a growing literature on the role of uncertainty aversion (or Knightian uncertainty) for asset markets and macroeconomics, see e.g. Hansen and Sargent (2008) and the references therein. Its importance as a key ingredient in the current crisis has been stressed also by e.g. Caballero and Krishnamurthy (2008), Easley and O'Hara (2008) and Backus et al (2009).

The structure of the paper is as follows. After a literature review, I describe the model in section 2 and provide a brief description of the equilibrium under "normal" conditions, see subsection 4. I then analyze the uncertainty averse case in section 3 and the adverse selection case in section 5. Section 6 offers some policy discussion, followed by a short conclusion. The appendix offers additional details, in particular on the equilibrium under normal conditions, see appendix B.

2 The model

The model can be summarized as follows. I consider an environment inspired by Smith (1991), in which depositors interact with a local bank, which in turn refinances itself via an (uncontingent) demand deposit account with one of a few core banks, who in turn invest in long-term ("mortgage-backed") securities. Clearly, the observable world of securities is considerably richer (and harder to describe), but this framework may capture the essence of the interactions. I assume that there are two aggregate states, a "boom" state and a (rare) "bust" state. In the "boom" state, everything follows from the well-known analysis in the benchmark bank run literature, see section B: essentially, things are fine. More serious problems arise in the bust state. I assume that the long-term securities become heterogeneous in terms of their long-term returns, and that local banks (together with their local depositors) hold heterogeneous beliefs regarding the portfolio of their core bank. Therefore, some local banks may withdraw early, even if local consumption demands are "late". Long term securities will then be sold to outside investors, who may be heterogeneous in their information and their beliefs regarding these assets.

I shall now describe the details. There are three periods, $t = 0, 1, 2$. There are two fundamental aggregate states: "boom" and "bust". The aggregate state will be learned by all participants in period 1. There are four types of agents or agencies:

1. Depositors in locations $s \in [0, 1]$.

2. Local banks in locations $s \in [0, 1]$.

3. Core banks, $n = 1, \ldots, N$.

4. Outside investors $i \in [0, \infty)$.

There are two types of assets

1. A heterogeneous pool of long-term securities.

2. A short term security, providing a safe return of 1.

Figure 1 provides a graphical representation of the model: there, I have drawn the unit interval as a unit circle.

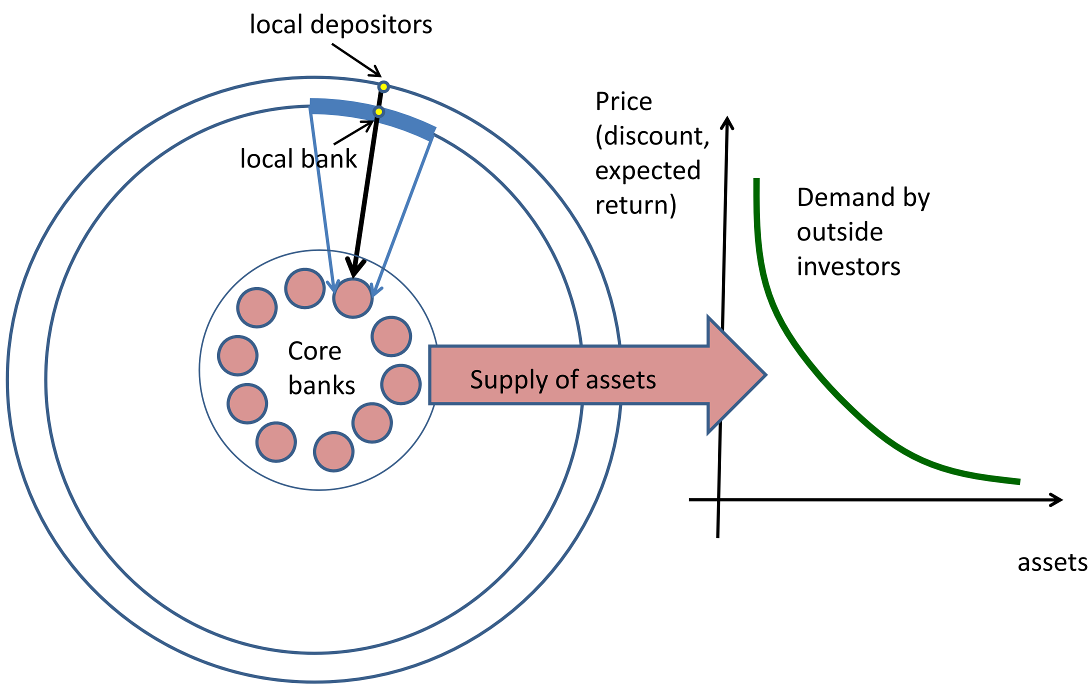

Figure 1: *A graphical representation of the model.*

2.1 Depositors and local banks

As in Diamond and Dybvig (1983), I assume that depositors have one unit of resources in period 0, but that they care about consumption either in period 1 ("early consumer") or in period 2 ("late consumer"). As in Smith (1983), I assume that all depositors at one location are of the same type. They learn their type in period 1. I assume that a fraction $0 < \varphi < 1$ of locations has early consumers and a fraction $1 - \varphi$ has late consumers. I assume that the realization of the early/late resolution is iid across locations and that depositors are evenly distributed across locations. I assume that depositors learn of their type in period 1. Ex-ante utility is therefore given by

$$U = \varphi E[u(c_1)] + (1 - \varphi)E[u(c_2)] \tag{1}$$

where c_1 and c_2 denotes consumption at date 1, if the consumer is of the early type and c_2 denotes consumption at date 2, if the consumer is of the late type and where $u(\cdot)$ satisfies standard properties. This heterogeneity in consumption preference induces a role for liquidity provision and maturity transformation, as in Diamond and Dybvig (1983) and the related literature. I assume that depositors only bank with the local bank in the same location, e.g. due to some unspecified cost to diversification.

Local banks open uncontingent demand deposit accounts with the core banks, depositing resources in period 0 and taking withdrawals in period 1 and/or period 2. Again, for some unspecified cost reasons, I assume that local banks operate a deposit account only with one of the core banks.

2.2 Core banks and securities

Core banks, local banks and depositors all can invest in short term securities in period 0 or in period 1, returning 1 unit next period. Additionally, core banks can invest in a set of long-term securities in period 0 and trade them on asset markets in period 1. One may wish to view these as "mortgage backed securities": appendix A details, how the model can be extended to describe that feature.

In the aggregate "boom" state, the long-term securities are all assumed to return R_{boom} per unit invested. I assume that

$$\varphi u(0) + (1 - \varphi)u(R_{\text{boom}}) \;\; < \;\; u(1) \tag{2}$$

$$1 < R_{\text{boom}} < \frac{u'(1)}{u'(R_{\text{boom}})} \tag{3}$$

If $u(c)$ is CRRA with an intertemporal elasticity of substitution below unity and if $R_{\text{boom}} > 1$, both equations are satisfied. Further, (2) is generally satisfied, if $(1 - \varphi)R_{\text{boom}} < 1$.

In the "bust" state, each long term securities offers a safe[1] return R per unit invested in period 0, but these returns are heterogeneous and distributed according to $R \sim F$, drawn from some distribution F on some interval $[\underline{R}, \bar{R}]$, where $0 < \underline{R} \leq \bar{R} < \infty$, with unconditional expectation R_{bust}, satisfying

$$R_{\text{bust}} \leq R_{\text{boom}} \tag{4}$$

Once the aggregate state is revealed to be a "bust" in period 1, I assume that core banks all know the type and therefore the period-2 payoff of the securities in their portfolio, and by implication the return distribution of their securities. The entire portfolio of the long-term securities has the safe return R_{bust}. Particular long-term securities within that portfolio have different returns, however. An outside investor who buys one particular long-term security, and e.g. draws a random security from the entire pool therefore exposes himself to that return risk.

But even for the entire portfolio at a core bank, the composition is assumed to be unknown to depositors and local banks. Instead, they form heterogeneous beliefs about that. I assume that local banks at location s and its depositors believe their core bank to hold a portfolio with return distribution $F(\cdot; s)$, where $F(\cdot; \cdot)$ is measurable and $F(\cdot; s)$ is a distribution function. This may arise e.g. due to unmodeled heterogeneous signals arriving at each location: with that interpretation, one needs to insist on local banks not updating their beliefs in light of the actions of other local banks.

One may wish to impose that

$$F(R) = \int F(R; s) ds \tag{5}$$

[1]It is not hard to generalize this to risky returns, but the additional insights may be small. From the perspective of outside investors, who do not know the specific R, the returns will be uncertain, and this is what matters.

so that aggregate beliefs accurately reflect the aggregate distribution (or a similar assumption at each core bank), but none of the results appear to depend on this.

For simplicity, I shall assume that core banks actually all hold exactly the same portfolio, i.e. there is a mismatch between the beliefs of the local banks and the portfolio of their core bank. Note that the portfolio of any core bank has the safe return R_{bust}. Core banks are not allowed to contract or condition on the bust-state beliefs $F(\cdot; s)$, e.g. due to unmodelled informational reasons.

2.3 Outside investors

Finally, there is a large pool of outside investors $i \in [0, \infty)$. These investors can invest in the long-term securities or the short-term securities in period 1, though not in period 0. Each investor is endowed with one unit of resources. I do not allow them to engage in short-selling. They are assumed to be risk neutral, discounting the future at some rate β, with

$$\beta R_{\text{boom}} < 1 \tag{6}$$

It remains to specify the information and beliefs of these investors. I shall investigate three variants.

1. [**Benchmark:**] As a benchmark, I assume that outside investors are risk-neutral, discounting resources between period 1 and period 2 at rate β. Furthermore, I assume that core banks sell their portfolio as proportional bundles (or, equivalently, sell randomly selected long-term securities, without adverse selection).

2. [**Uncertainty Aversion:**] I assume the investors to be uncertainty averse, following Schmeidler (1989) or Epstein (1999) or to follow robust control rules against downside risks, following Hansen and Sargent (2008). There may also be an interpretation as extreme loss aversion,

see Tversky and Kahnemann (1991) and Barberis, Huang and Santos (2001). In either case, I presume the following starkly simplified structure: given a security drawn from a pool of securities with some interval as the support of its payoffs, these investors are willing to pay β times the lower bound of this interval as the price per unit invested, i.e. the investor is risk neutral, but minimizes over all probability distributions with support on that interval. Put differently, if offered to trade a security from the described set, they will fear that they will always be offered the security with the lowest of these returns, even though this cannot happen to all investors in equilibrium. Another potential justification may be that these are traders working on behalf of institutional investors and face lopsided incentives: due to the complexity of these securities, they cannot afford to risk loosing money ex post during the bust, as their managers may not be able to tell whether this was bad luck or poor research.

Let $\omega \geq 0$. I assume the group $i \in [0, \omega]$ of these investors to have the expertise of discerning the quality of the long-term securities, i.e. they know the return of a given long-term security, the support interval is a single number, and they are therefore willing to buy them when the return exceeds $1/\beta$. I call them the expert investors. All other investors $i > \omega$ only know the distribution F and the equilibrium, but not the specific return of some offered long-term security. They use the support interval $[\underline{R}, \bar{R}]$ and are therefore willing to pay

$$\beta \underline{R}$$

per unit invested.

3. [**Adverse Selection:**] I assume that outside investors are risk-neutral, discounting period-2 payoffs at rate β, but cannot distinguish between the qualities of the long-term securities sold to them. I assume that core banks can "adversely select" the long-term security they wish to sell. I assume in this scenario, that all investors know that all core

banks hold a portfolio of long-term securities with return distribution $R \sim F(R)$.

2.4 Timing and equilibrium

The timing is as follows. In period 0, core banks offer deposit contracts to local banks, offering state-uncontingent withdrawals of r in period 1 per unit deposited. Likewise, local banks offer state-uncontingent withdrawals of $\tilde{r}$ in period 1 per unit deposited. In period 1 and depending on the aggregate state, local banks may withdraw r from their core bank. The core banks match these withdrawal demands from payoffs of their portfolio of short-term securities as well as sales of long-term securities. If they cannot meet all withdrawal demands, they declare bankruptcy. In that case, all local banks, who have decided to withdraw, obtain an equal pro-rata payment from the remaining resources.

I assume that Bertrand competition in these contracts makes local banks pay out everything to their depositors[2] and likewise makes core banks pay out everything to local banks. Therefore, $\tilde{r} = r$ and all resources left in period 2 will be paid in proportion to the remaining deposits. Furthermore, local banks will be indifferent which particular core bank to choose. Let

$$\nu : [0, 1] \to \{1, \ldots, N\}$$

be the core bank selection function, i.e. let $\nu(s)$ be the core bank selected by the local bank s. I assume $\nu(\cdot)$ to be measurable[3] To analyze what happens when the number of distressed banks increases, I consider in particular the case, where a fraction μ of the core banks (in terms of their market share) face the same heterogeneous beliefs of their local banks, whereas a fraction $1-\mu$ of core banks has local banks, who all (accurately) believe the portfolio of their core bank to be given by securities with $R \sim F(R)$.

[2] For that, one may want to assume that there are at least two local banks in each location, though that assumption is immaterial for the rest of the analysis

[3] An alternative is to assume $\nu(\cdot)$ to be random and use Pettis integration, see Uhlig (1996).

Finally and for simplicity, I assume that the "bust" state is sufficiently unlikely a priori, so that $r = \tilde{r}$ are determined entirely from the "boom" state calculus. With this, deposit contracts will be written as is standard in the banking literature. Appendix B contains the details. Given these contracts, I analyze the unfolding of the equilibrium in the bust state, with either uncertainty averse investors or with adverse selection.

3 Variant 1: uncertainty aversion.

The analysis will proceed as follows. Given a "conjectured" fraction θ of withdrawals by late-consumer local banks at some core bank and given aggregate liquidations L, I shall calculate the actual fraction of withdrawals by local banks, assuming that only local banks believing in otherwise receiving less consumption in period 2 will withdraw early. This then provides me with a mapping from conjectured withdrawal fractions (and therefore aggregate liquidations) to actual withdrawal fractions (and therefore aggregate liquidations): the lowest fix point of that is a fundamental, partial bank run. It is systemic, if L exceeds ω and if this matters for the withdrawal decisions.

Consider first the dependence of the market price for any security on the aggregate liquidation L of long-term securities. If $L < \omega$, there is an "excess supply" of expert investors. They will bid more than non-expert uncertainty-averse investors for the securities sold: therefore, the market price will be the final payoff, discounted at β. If $L > \omega$ (and, by assumption, if $L = \omega$), however, the "marginal" investor is an uncertainty-averse investor, willing only to pay $\beta \underline{R}$, regardless of the asset. This then must be the market price. Thus, given some specific security, its market price is a decreasing function of the aggregate liquidity needs L. This is the key feature needed in this section. The market price also happens to fall discontinuously, as L crosses ω: this is due to our particularly stark assumption regarding the uncertainty aversion of the outside investors and assuming a discontinuity at ω. This is not essential to the results, and can be relaxed, at the price of higher complexity of the analysis. The required general construction of an equilibrium is available in a working paper version of this paper. The cash-in-the-market pricing scenario of Allen and Gale (1994) or Allen and Gale (2007), chapter 4, corresponds to an extreme version of the scenario considered here, with the non-expert investors all bidding zero for all assets and where the core banks cannot raise more liquidity than ω.

3.1 The withdrawal decision of local banks.

Consider a core bank and suppose that a fraction θ of its local banks at late-consuming locations withdraw early. Suppose that aggregate liquidations are L. If $L < \omega$, the opportunity costs in terms of period-2 resources for providing one unit of resources for period-1 withdrawals is $1/\beta$. If $L \geq \omega$, the market price $\beta\underline{R}$, regardless of the security sold. The core bank will therefore sell its securities with the lowest period-2 payoff first. Suppose the core bank started initially with ξ resources. It therefore purchased $(1-\varphi r)\xi$ units of long-term securities. Given the early withdrawals, the core bank needs to raise period-1 liquidity $\ell = r\theta(1-\varphi)\xi$, and hence sell $\ell/(\beta\underline{R})$ units of its long-term securities, i.e. the fraction

$$\zeta(\theta) = \frac{1-\varphi}{1-\varphi r}\,\frac{r\theta}{\beta\underline{R}} \tag{7}$$

similar to equation (38).

Consider now one of its local banks and its beliefs $F(\cdot;s)$ about the return distributions of the securities in the portfolio of its core bank (before selling any of its securities). For ease of notation, I shall write G in place of $F(\cdot;s)$. Let

$$G^{-1}(\tau) = \sup\{R \mid G(R) < \tau\},\ \tau \in [0,1] \tag{8}$$

be the inverse function of G, see figure 2. Note that

$$E_G[R \mid R \leq G^{-1}(\zeta)] = \frac{\int_0^\zeta G^{-1}(\tau)d\tau}{\zeta} \tag{9}$$

is the expected return of all returns below the level given by $G^{-1}(\zeta)$, under the distribution G. Also note that $G^{-1}(\tau)$ is a continuous function of τ and $E_G[R \mid R \leq G^{-1}(\zeta)]$ is a continuous function of ζ.

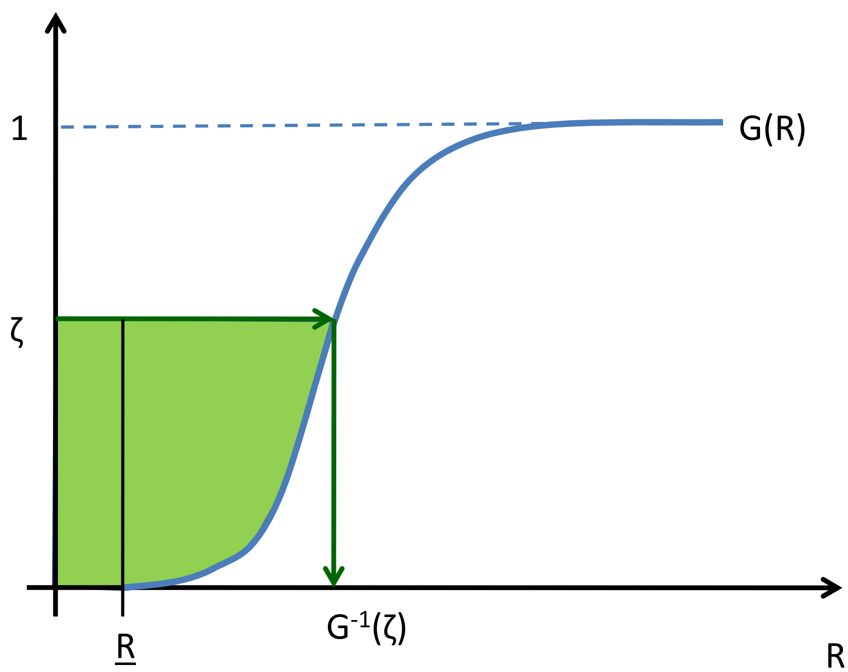

Figure 2: *The function G^{-1} and expected returns.*

From the perspective of this local bank, the period-2 opportunity costs for period-1 withdrawals are

$$\Gamma(\theta, L; G) \quad = \quad \frac{1}{\beta}\left(\mathbf{1}_{L<\omega} + \frac{E_G[R \mid R \leq G^{-1}(\zeta(\theta, L))]}{\underline{R}}\mathbf{1}_{L\geq\omega}\right) \qquad (10)$$

Proposition 1 *1. $\Gamma(\theta, L; G)$ is increasing and continuous in θ.*

2. *$\Gamma(\theta, L; G)$ is increasing in L and satisfies $\beta\Gamma(\theta, L; G) \geq 1$. More precisely, it is constant in L, except for a nonnegative jump at $L = \omega$.*

3. *Suppose that H first-order stochastically dominates G. Then*

$$\Gamma(\theta, L; G) \leq \Gamma(\theta, L; H)$$

i.e. $\Gamma(\theta, L; G)$ is increasing in G, when ordering distributions by first-order stochastic dominance.

Proof:

1. *Note that $\zeta(\theta)$ and therefore $E_G[R \mid R \leq G^{-1}(\zeta(\theta))]$ is increasing in θ. Continuity is a consequence of the continuity of $\zeta(\theta, L)$ in θ.*

2. *Note that $\underline{R}^{-1} E_G[R \mid R \leq G^{-1}(\zeta(\theta))] \geq 1$.*

3. *Define H^{-1} as the inverse of H as in 8. Since $H(R) \leq G(R)$ for all R, $H^{-1}(\tau) \geq G^{-1}(\tau)$ for all $\tau \in [0, 1]$. Equation (9) shows that $E_G[R \mid R \leq G^{-1}(\zeta)] \leq E_H[R \mid R \leq G^{-1}(\zeta)]$ and the claim follows.*

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As a result, a local bank with beliefs $G = F(\cdot; s)$ perceives the second-period payoff to be

$$c_2(0, 0; G) = E_G[R] \frac{1 - \varphi r}{1 - \varphi} \tag{11}$$

if there are no withdrawals of late-consumer local banks in period 1, i.e. if $\theta = 0$ and $L = 0$. With withdrawals of a fraction θ of late-consumer local banks and given aggregate liquidations L, the (perceived) remaining resources at period 2 per late consumer for this core bank is therefore

$$c_2(\theta, L; G) = \frac{c_2(0, 0; G) - r\theta\Gamma(\theta, L; G)}{1 - \theta} \tag{12}$$

which generalizes (39). The local bank will therefore surely opt for period-1 withdrawal, if $c_2(\theta, L; G) < r$.

Generally, $c_2(\theta, L; G)$ is not monotone in G, when ordering G according to first-order stochastic dominance: while the first term is increasing in G, the second term is now decreasing, due to the negative sign. $c_2(\theta, L; G)$ is decreasing in L and it is decreasing in θ under the mild condition (13), which generalizes (34) and which essentially assures, that no late-withdrawal local bank will be happy about other late consumer local banks withdrawing early.

Proposition 2 *1. $c_2(\theta, L; G)$ is monotonously decreasing in L. More precisely, it is constant in L, except for a nonpositive jump at $L = \omega$.*

2. $c_2(\theta, L; G)$ is continuous in θ.

3. Assume that

$$c_2(0, 0; G) < \frac{r}{\beta} \tag{13}$$

Then $c_2(\theta, L; G)$ is strictly decreasing in θ.

Proof:

1. *This follows directly from proposition 1.*

2. *Continuity follows from the continuity of $\Gamma(\theta, L; G)$ in θ.*

3. *Write $c_2(\theta, L; G)$ as*

$$c_2(\theta, L; G) = c_2(0, 0; G) - \frac{\theta}{1-\theta}\chi(\theta, L; G) \tag{14}$$

where

$$\chi(\theta, L; G) = r\Gamma(\theta, L; G) - c_2(0, 0; G) \tag{15}$$

is strictly positive and increasing in θ per (13) and proposition 1. Let $\theta_a < \theta_b$. Then

$$\begin{aligned} c_2(\theta_a, L; G) &= c_2(0, 0; G) - \frac{\theta_a}{1-\theta_a}\chi(\theta_a, L; G) \\ &> c_2(0, 0; G) - \frac{\theta_b}{1-\theta_b}\chi(\theta_a, L; G) \\ &\geq c_2(0, 0; G) - \frac{\theta_b}{1-\theta_b}\chi(\theta_b, L; G) \\ &= c_2(\theta_b, L; G) \end{aligned}$$

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3.2 Equilibrium

Assume that (13) is true for all conjectured distributions $G = F(\cdot, s)$. Therefore, if local banks opt for early withdrawals at some level of market liquidity or some fraction of other early withdrawals, they will do also for higher levels of L and θ. Let

$$S_n(\theta, L) = \{s \mid \nu(s) = n,\, c_2(\theta, L; F(\cdot, s)) < r\} \tag{16}$$

be the set of local banks with deposits at core banks n, which will surely withdraw early, if a fraction θ of depositors at core bank n do, and if there is total liquidity demand L.

Given L and a core bank n, define the mappings

$$\eta_{n,L} : [0, 1] \to [0, 1]$$

per

$$\eta_{n,L}(\theta) \quad = \quad \lambda(S_n(\theta, L))$$

where $\lambda(\cdot)$ denotes the Lebesgue measure. Intuitively, if aggregate liquidity needs are given by L and if all local banks at core bank n conjecture the fractions θ of late consumer local banks to withdraw early at that core bank, then the fractions $\eta_{n,L}(\theta)$ surely will. Fixed points of η are bank runs, where withdrawers strictly prefer to do so.

Proposition 3 *Assume that (13) is true for all conjectured distributions $G = F(\cdot, s)$.*

1. $\eta_{n,L} : [0, 1] \to [0, 1]$ *is increasing and continuous from the left, i.e. for $\theta_j \to \theta_\infty$, $\theta_j < \theta_\infty$, we have $\eta_{n,L}(\theta_j) \to \eta_{n,L}(\theta_\infty)$.*

2. *Given n, L, let $\theta_0 = 0$ and construct the sequence*

$$\theta_{j;n,L} = \eta_{n,L}(\theta_{j-1;n,L})$$

Then $\theta_{j;n,L} \to \theta_{\infty;n,L}$, which satisfies $\theta_{\infty;n,L} = \eta_{n,L}(\theta_{\infty;n,L})$. Furthermore,

$$\theta_{\infty;n,L} = \min\{\theta \mid \theta \geq \eta_{n,L}(\theta)\} \tag{17}$$

Proof:

1. *This follows from proposition 2.*

2. *The first part of the second part follows from the first part. For (17), consider any $\theta < \theta_{\infty;n,L}$. Therefore, for some j,*

$$\theta_{j-1;n,L} \leq \theta < \theta_{j;n,L} = \eta_{n,L}(\theta_{j-1;n,L}) \leq \eta_{n,L}(\theta)$$

or $\theta < \eta_{n,L}(\theta)$.

•

Given L, define

$$\overrightarrow{\theta}_{\infty;L} = (\theta_{\infty;1,L}, \ldots, \theta_{\infty;N,L})$$

Define

$$\begin{aligned} L_A &= L(\overrightarrow{\theta}_{\infty;0}) \\ L_B &= L(\overrightarrow{\theta}_{\infty;L_A}) \end{aligned}$$

If $L_A < \omega$, then $L_B = L_A$ and $\overrightarrow{\theta}_{\infty;L_A}$ is a **partial fundamental bank run**, but without systemic feature. If $L_A \geq \omega$, it still may be the case that $L_B = L_A$, i.e. there again is no systemic feature per the repercussion of aggregate liquidity needs on the individual withdrawal decisions[4]. A **systemic bank run** obtains (or is defined to be the situation), if $L_B > L_A$, i.e., more local banks withdraw, if it is known that aggregate liquidations exceed the cash in the hands of expert investors. A numerical example for a systemic bank run is provided in C.1.

[4]Here, I need $\underline{R} > 0$. If $\underline{R} = 0$, then one has cash-in-the-market pricing and a systemic bank run will always have $L = \omega$.

4 The benchmark.

Suppose (and as a benchmark for comparison), that investors are risk-neutral and that there is no adverse selection in selling the long-term securities. The analysis of the "bust" state is now a corollary to the analysis above by setting $\omega = \infty$ and using $\Gamma(\theta, L; G) = 1/\beta$ throughout. The details can be skipped, except perhaps for some useful formulas. With (12), second-period consumption will assumed to be

$$\tilde{c}_2(\theta, L; G) = \frac{c_2(0, 0; G) - r\theta/\beta}{1 - \theta} \tag{18}$$

which is monotone in G, when ordering distributions according to first-order stochastic dominance, and which does not depend on L (and where I use the $\tilde{\cdot}$ to distinguish it from the scenario above). As in (40) , a late consumer local bank will withdraw early, if $\theta \geq \tilde{\theta}^*(G)$, where

$$\tilde{\theta}^*(G) = \frac{\beta}{1 - \beta} \left(\frac{1}{r} \frac{1 - \varphi r}{1 - \varphi} E_G[R] - 1 \right) \tag{19}$$

This scenario serves as a benchmark. While there can also be a fundamental partial bank run in this case, there is no spillover to other core banks. A fundamental bank run in this scenario and the scenario with uncertainty averse investors start the same and affect the same core banks. However, a fundamental bank run with uncertainty averse investors can be systemic in the way defined and described above and therefore run considerably deeper.

5 Variant 2: adverse selection.

Consider now the variation of the model with adverse selection. Investors know the return distribution F of all securities, but not the return of any given security, while core banks do. All long-term securities are sold at the same market price p. This creates adverse selection: not only will core banks sell the securities with their worst quality first (and this happens also in the uncertainty aversion variant), but furthermore, some core banks without liquidity needs due to withdrawals may sell long-term securities of low quality, if the price is right. The latter is a key difference between the two variants. With uncertainty averse investors and sufficiently high discounting, there is no reason for "opportunistic" selling by liquid core banks[5]

Suppose that core banks with a market share μ face early withdrawals of the same[6] fraction θ of their late-consumer serving local banks, due to heterogeneous beliefs of their local banks. They need to sell a share ζ of their portfolio, where

$$\zeta = \frac{1-\varphi}{1-\varphi r}\,\frac{r\theta}{p}, \tag{20}$$

as in (45) or (38). On average, these securities pay $E_G[R \mid R \leq G^{-1}(\zeta)]$ per unit, see equation (9).

Assume that the other core banks have local banks who all correctly believe the core-bank portfolio to have securities with returns distributed according to $R \sim F(R)$. These core banks will sell long-term securities for purely opportunistic reasons, in case their price exceeds the expected return. For tractability, assume that the true portfolio F is atomless. Given a market price p for long-term securities, core banks without early withdrawals will sell all securities with $R \leq p$, i.e. sell the fraction $F(p)$.

Per rational expectations of the outside investors, the market clearing price $p = p(\theta, \mu)$ and the fraction of the portfolio $\zeta = \zeta(\theta, \mu)$ sold by the distressed core banks solve the two equations (20) and

$$p = \beta \frac{\mu \int_{\underline{R}}^{F^{-1}(\zeta)} R dF + (1-\mu) \int_{\underline{R}}^{p} R dF}{\mu\zeta + (1-\mu)F(p)} \tag{21}$$

[5]Clearly, the distinction here has been sharply drawn, for analytic purposes. It may well be that some mixture of the two variants is a better description than one of these two extreme variants.

[6]It is straightforward, but tedious to extend this to the case, where θ differs from core bank to core bank.

where (per notational convention or per calculation of the integral)

$$0 = \int_{\underline{R}}^{p(\theta,\mu)} RdF, \text{ if } p(\theta,\mu) < \underline{R}$$

and where $F^{-1}(\cdot)$ is defined as in (8). Note that the right hand side of (21) is simply the average return of the securities sold, discounted at β. Define

$$\bar{\theta} = \left(\frac{1 - r\varphi}{1 - \varphi} \right) \frac{\beta \underline{R}}{r} \tag{22}$$

as the maximal θ compatible with $\zeta \leq 1$, if $p = \beta\underline{R}$. Note that $\bar{\theta} < 1$.

Proposition 4 *1. For every $\theta \in [0, \bar{\theta}]$ and $\mu \in (0, 1]$, there is a unique solution (p, ζ) to (20,21) with $\beta\underline{R} \leq p \leq \beta R_{bust}$ and $0 \leq \zeta \leq 1$, so that $p < F^{-1}(\zeta)$.*

2. Given θ, $p(\theta, \mu)$ is a strictly increasing function in $\mu \in (0, 1]$.

Proof:

1. *Recall that the support of F is $[\underline{R}, \bar{R}]$. Define the function $\rho(p)$ per the right hand side of (21), with ζ replaced with (20). Note that $\rho(p)$ is continuous on $p \in [\beta\underline{R}, \bar{R}]$ with*

$$\rho(\beta\underline{R}) \geq \beta\underline{R}, \ \rho(\bar{R}) \leq \beta E_F[R] = \beta R_{\text{bust}} < \bar{R}$$

By the mean value theorem, there is therefore a value p with $p = \rho(p)$. Suppose that $F^{-1}(\zeta) \leq p$ at this value. Then the right hand side of (21) is not larger than βp, a contradiction. To show uniqueness, suppose to the contrary that there are two solutions, say $p_a < p_b$, together with $1 \geq \zeta_a > \zeta_b$. It is easy to see that $F(p_b)$ and $F(p_a)$ cannot both be zero: hence, $F(p_b) > 0$. Note generally that

$$\begin{aligned} \int_{\underline{R}}^{p_b} RdF - \int_{\underline{R}}^{p_a} RdF &\leq (F(p_b) - F(p_a))p_b \\ &\leq F(p_b)p_b - F(p_a)p_a \end{aligned}$$

Define the function

$$\psi(p,\zeta;\theta,\mu)=\frac{\beta\mu\int_{\underline{R}}^{F^{-1}(\zeta)}RdF+\beta(1-\mu)\int_{\underline{R}}^{p}RdF}{\mu r\theta\left(\frac{1-\varphi}{1-r\varphi}\right)+(1-\mu)F(p)p} \tag{23}$$

Note that $p_j=\rho(p_j)$ *can be rewritten as*

$$1=\psi(p_j,\zeta_j;\theta,\mu) \tag{24}$$

for $j=a,b$. *Therefore,*

$$\begin{aligned}
1 &= \psi(p_a,\zeta_a;\theta,\mu)=\frac{\beta\mu\int_{\underline{R}}^{F^{-1}(\zeta_a)}RdF+\beta(1-\mu)\int_{\underline{R}}^{p_a}RdF}{\mu r\theta\left(\frac{1-\varphi}{1-r\varphi}\right)+(1-\mu)F(p_a)p_a}\\
&> \frac{\beta\mu\int_{\underline{R}}^{F^{-1}(\zeta_a)}RdF+\beta(1-\mu)\int_{\underline{R}}^{p_a}RdF+\beta(1-\mu)(F(p_b)p_b-F(p_a)p_a)}{\mu r\theta\left(\frac{1-\varphi}{1-r\varphi}\right)+(1-\mu)F(p_a)p_a+(1-\mu)(F(p_b)p_b-F(p_a)p_a)}\\
&\geq \frac{\beta\mu\int_{\underline{R}}^{F^{-1}(\zeta_a)}RdF+\beta(1-\mu)\int_{\underline{R}}^{p_b}RdF}{\mu r\theta\left(\frac{1-\varphi}{1-r\varphi}\right)+(1-\mu)F(p_b)p_b}\\
&\geq \frac{\beta\mu\int_{\underline{R}}^{F^{-1}(\zeta_b)}RdF+\beta(1-\mu)\int_{\underline{R}}^{p_b}RdF}{\mu r\theta\left(\frac{1-\varphi}{1-r\varphi}\right)+(1-\mu)F(p_b)p_b}\\
&= \psi(p_b,\zeta_b;\theta,\mu)
\end{aligned}$$

and therefore, (24) cannot hold for p_b, *a contradiction.*

2. *Given* θ,μ, *denote the unique equilibrium with* $p(\theta,\mu)$ *and* $\zeta(\theta,\mu)$. *Let* $\zeta(p)$ *denote the expression on the right hand side of (20). The previous calculation shows more generally that*

$$\begin{aligned}
\psi(p,\zeta(p);\theta,\mu)>1 \quad &\text{for} \quad p<p(\theta,\mu) && (25)\\
\psi(p,\zeta(p);\theta,\mu)<1 \quad &\text{for} \quad p>p(\theta,\mu) && (26)
\end{aligned}$$

Consider some $\bar{\mu}$ *and write* $\bar{p}=p(\theta,\bar{\mu})$ *and* $\bar{\zeta}=\zeta(\theta,\bar{\mu})$. *Since* $\psi(\bar{p},\bar{\zeta};\theta,\bar{\mu})=1$ *and since*

$$\beta\int_{\underline{R}}^{\bar{p}}RdF<F(\bar{p})\bar{p}$$

it follows that

$$\beta \int_{\underline{R}}^{F^{-1}(\bar{\zeta})} R dF > r\theta \frac{1-\varphi}{1-r\varphi}$$

Therefore, $\psi(\bar{p}, \bar{\zeta}; \theta, \mu)$ is increasing in μ. For $\mu' > \bar{\mu}$, one therefore has $\psi(\bar{p}, \bar{\zeta}; \theta, \mu') > 1$. It follows from (25) that $p(\theta, \mu') > \bar{p}$, as claimed.

•

At the distressed core banks, local banks with beliefs G regarding their portfolio will therefore belief the opportunity costs for providing period-1 resources in terms of period-2 resources to be

$$\Gamma(\theta, \mu; G) = \frac{E_G[R \mid R \leq G^{-1}(\zeta(\theta, \mu))]}{p(\theta, \mu)} \tag{27}$$

The remaining late-consumer local banks will obtain

$$c_2(\theta, \mu; G) = \frac{1}{1-\theta} \left(\frac{x}{1-\varphi} E_G[R] - r\theta \frac{E_G[R \mid R \leq G^{-1}(\zeta(\theta, \mu))]}{p(\theta, \mu)} \right) \tag{28}$$

The analysis of the resulting equilibrium is similar to the analysis in section 3 and can therefore be omitted.

It is instructive to compare (27) to (10) for the case $\omega = 0$: the two expressions coincide iff $p(\theta, \mu) = \beta \underline{R}$. Generally, the returns are quite different. In fact,

$$\Gamma(\theta, 1; G) = \frac{E_G[R \mid R \leq G^{-1}(\zeta(\theta, \mu))]}{\beta E_F[R \mid R \leq G^{-1}(\zeta)]} \tag{29}$$

as can be seen by direct calculation. In particular, for $G = F$, I obtain

$$\Gamma(\theta, 1; F) = \frac{1}{\beta} \tag{30}$$

More generally,

Proposition 5 $\Gamma(\theta, \mu; G)$ *is decreasing in* μ.

Proof: *This is a direct consequence of (27) together with the fact that $p(\theta, \mu)$ is increasing in μ, implying that $\zeta(\theta, \mu)$ and thus $E_G[R \mid R \leq G^{-1}(\zeta)]$ are decreasing in μ.* •

I obtain the key insight that an increasing market share of distressed banks lessens rather than deepens the crisis. Furthermore, with homogeneous beliefs, $F(\cdot, s) \equiv F$, and with the market share of distressed banks approaching unity, the moral-hazard scenario turns into the standard bank run scenario considered in section B.3.

6 Some policy implications

A full discussion of the policy and welfare implications is beyond the scope here. Instead, I investigate the more modest question of the impact of certain policies for a policy maker who may be interested in learning the consequences for avoiding (or stopping) a crisis and for the government budget, given the model in this paper.

To see the key difference between the two variants considered above, suppose that all banks are distressed and need to sell an equal fraction of their portfolio to meet early withdrawals. With uncertainty averse investors, this is the scenario which will most easily lead to total market liquidations exceeding the resources of expert investors, and thereby to a systemic bank run. With adverse selection by contrast, all core banks now receive "fair value" for their assets, i.e., the situation essentially turns into a classic bank run, but only with standard discounting of assets, as the share of free-riding banks has disappeared. Therefore, the adverse selection scenario violates item six of the stylized description list in the introduction, while the scenario with uncertainty averse investors does not. For these reasons, I argue that it is more plausible to look at the 2008 financial crisis through the lense of the uncertainty averse investor scenario rather than the adverse selection scenario.

Consider, for example, a government guarantee of payoffs of the securities sold by the core banks, e.g. guaranteeing a return of at least R_{gov}. In that case, the uncertainty averse investors will pay βR_{gov} instead of $\beta \underline{R}$. In particular, if $R_{\text{gov}} = 1$, i.e. if the government guarantees that investments will not make losses, the "deep" bank run results in discounting with β throughout and turns the scenario with uncertainty averse investors into the "classic" bankrun situation of subsection 4 for the distressed core banks. The government will loose money on all securities with returns $R < R_{\text{gov}}$. Additionally, if $\beta R_{\text{gov}} > \underline{R}$, the government now creates an *additional* adverse selection problem at the core banks which are not distressed, and which now find it at their advantage to sell all assets with $\underline{R} \leq R \leq \beta R_{\text{gov}}$.

From the tax payers perspective, a more advantageous procedure in the case of uncertainty averse investors seems to be the purchase of the troubled assets outright. Consider a fixed government purchase price $p > \beta \underline{R}$ at which the government stands ready to purchase assets from the core banks. Core banks in distress will no longer sell to uninformed investors: instead, they will sell their worst assets to the government and remaining assets to expert investors, up to their resource limit. More resources will be left for period-2 withdrawals, lessening the incentives to withdraw early. Unaffected core banks will sell assets opportunistically, if $p > \underline{R}$.

The calculations now are somewhat similar to the analysis in the adverse selection framework. Given the government price p, one can calculate the expected value $\rho(p)$ of the securities obtained at that price, where $\rho(p)$ is the right hand side of (21), with ζ replaced with (20), see also the beginning of the proof of proposition 4. Assuming that the government is purchasing a large quantity of securities, the law of large number holds so that the expected value equals the safe payoff of that portfolio, per unit purchased. With that, one can then calculate the losses or gains to the tax payers at the mandated government purchase price. If $p < \rho(p)$, i.e. if the price offered by the government is below the equilibrium price of the adverse selection scenario, the government will earn a safe return above $1/\beta$. Assuming that tax payers discount the future at rate β (and that they do not suffer from some uncertainty aversion, as the government purchases this pool of assets), this is beneficial to tax payers, and it lessens the possibility for a systemic bank run.

The situation is quite different, if we are in the adverse selection scenario to begin with. In that case, the government would only find takers for its offers, if the government price is above the current market clearing price, in which case the government will make losses compared to the benchmark return of $1/\beta$.

Since I have argued that the uncertainty averse scenario is more plausible than the adverse selection scenario, the analysis here provides some support for the argument that an outright purchase of troubled assets by the government at prices above current market prices can both alleviate the financial crises as well as provide tax payers with returns above those for safe securities.

A number of private sector solutions may likewise provide reasonable avenues for resolving the crisis situation, e.g. the complete purchase of portfolios of a distressed core bank or the sale of a distressed core bank and a guarantee of its deposits through the buyer. It may be, however, that the same caution that drives uncertainty averse investors to demand steep discounts on asset backed securities might also prevent the sale of distressed financial institutions to the same investors at a price that can resolve the situation sufficiently well. Solutions that mix private sector involvement with government intervention may likewise offer specific advantages or fallacies, that can be analyzed in this context.

7 Conclusions

I have set out to provide a model of a systemic bank run delivering six features described in the introduction. I have considered two variants for outside investors, when purchasing asset-backed securities: uncertainty aversion versus adverse selection. While both variants deliver the first five points, this is only true for the sixth point with the uncertainty aversion variant. In the adverse selection case, as a larger share of financial institutions are distressed, the discounts lessen rather than rise instead.

I conclude from that that the variant with uncertainty averse investors rather than the adverse selection scenario is more suitable to analyze policy implications. For example, an outright purchase of assets at a price moderately above the market price absent government intervention may both alleviate the financial crises as well as provide tax payers with returns above those for safe securities. A number of private sector solutions may likewise provide reasonable avenues for resolving the crisis situation. Follow-up work, providing a deeper analysis of the various options and policy scenarios, is surely called for.

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Appendix

A Securitization

The model directly assumes that core banks invest in long-term securities, rather than "generating" them as asset-backed securities. It is easy to extend the model to incorporate that feature, though, and it thereby provides a rationale and reason for securitization, and this appendix provides the details. While securitization has received a substantial share of the popular blame for the financial crisis, this appendix therefore can also be viewed as providing a rationale for why securitization can be beneficial. Furthermore, this feature may prove useful for other modeling exercises.

To provide this extension, I assume that at date zero, local banks can invest in long-term projects ("mortgages") of location s or short-term securities, and they can invest in short-term securities in period 1, but they cannot invest in long-term securities. Long-term projects pay off only in period 2. I assume that long-term projects cannot be terminated ("liquidated") prematurely. I assume that local banks administer the local long-term projects, delivering their payment streams to whoever finances them originally.

Core banks invest the period-0 deposits received from local banks in local long-term projects, and turn their period-2 payments into long-term securities. In all periods, core banks can trade in short-term as well as long-term securities.

In the aggregate "boom" state, local long-term projects return $R_{\text{boom}} + \epsilon_s$, where ϵ_s is a random variable with mean zero, distributed independently and identically across locations $s \in [0, 1]$.

Long-term securities pool these risks. To describe this a bit more formally, suppose there are $m = 1, \ldots, M$ long-term securities, suppose that $(A_m)_{m=1}^{M}$ is a partition of $[0, 1]$ with each A_m having equal Lebesque measure, and suppose that the payoff for the long term security with index m is the integral of all long-term projects $s \in A_m$. The law of large numbers in Uhlig (1996) then implies a safe return. Conversely, knowning the return of the long-term securities, one might directly assume that the long-term projects return this amount plus the idiosynchratic noise ϵ_s. This structure can also be used for the "bust" episode. To make this formal structure fully consistent with the model – where a continuum of asset-backed securities is needed to e.g. generate a return distribution with a density for the adverse selection variant – it may be best to alter the model slightly, with local banks indexed by $(i, j) \in [0, 1]^2$, and with securitization of "pool j" achieved by pooling all payoffs in localities (i, j), holding j fixed, and applying the law of large numbers for each j.

B Analysis: Preliminaries

It is useful to first analyze some special cases in order to set the stage of the analysis of the bust state. The analysis of these special cases are the same, no matter which assumption has been made about the type of outside investors.

B.1 No core banks

Consider first the environment above without core banks. The investors then do not matter: they would love to short-sell the short-term securities, but they cannot do so (and that certainly seems reasonable, if one imagines the short-term securities to be Treasury bills). In that case, local banks offer contracts to their local depositors. Note that all their depositors wish to either only consume at date 1 or at date 2. Due to local Bertrand competition, the local banks will choose the deposit contract that maximizes expected utility (1).

Consider first the choice between investing everything in the long term project versus investing everything in short-term securities. In the first case, depositors only get to consume in case they turn out to be late consumers, and their ex ante utility is

$$U = \varphi u(0) + (1-\varphi)E[u(R)] \leq \varphi u(0) + (1-\varphi)u(R_{\text{boom}})$$

due to concavity of $u(\cdot)$ as well as (4). In the second case, depositors can consume in both periods, at ex ante utility equal to $u(1)$. If the choice is "either-or" and since the latter is larger than the former due to (2), local banks will only invest in short-term securities. One can view this as a version of 100% reserve banking. Note that there cannot be a bank run or financial crisis in this situation, but, as is well known and as we shall see, this solution is inefficient.

Generally,

$$U(y) = \varphi u(y) + (1-\varphi)u((1-y)R_{\text{boom}} + y)$$

is a concave function of the fraction y invested in the short-term security: the corner solution $y = 1$ obtains, if

$$(1-\varphi)R_{\text{boom}} < 1 \tag{31}$$

and otherwise one obtains an interior solution. The inefficiency still remains, see the discussion in Allen and Gale (2007), chapter 3.

B.2 Only "boom" state

To set the stage of the "bust" state analysis as well as an important benchmark, consider the situation with only a boom state. Competition drives core banks to maximize the ex-ante welfare of depositors, with local banks merely passing resources onwards. This amounts to choosing the amount x to be invested in the long-term securities, y to be invested in the short-term security and the amount z of the investment in the long-term security to be sold to outside investors at date 1 in order to solve

$$\max_{x,y,z} \varphi u(c_1) + (1-\varphi)u(c_2)$$

$$\begin{aligned} s.t. \quad & \varphi c_1 = y + \beta R_{\text{boom}} z \\ & (1-\varphi)c_2 = R_{\text{boom}}(x - z) \\ & 0 \le x,\ 0 \le y,\ x + y = 1,\ 0 \le z \le x \\ & c_2 \ge c_1 \ge 0 \end{aligned}$$

where the last constraint prevents local banks in locations with late consumers to withdraw their funds in period 1 and investing in the short security. Note that the optimal solution will have $z = 0$ due to (6): it is cheaper to deliver resources for period 1 per investing in the short-term security rather than investing it in the long-term security and selling it at a steep discount to the outside investors. With the interpretation of the sale to outside investors as the liquidation value of long-term projects, this problem is a baseline problem in the literature on banking and has been thoroughly analyzed in the literature, see e.g. Allen and Gale (2007), in particular chapter 3. A brief description of the solution is useful for the analysis below, however.

Due to (3) there will be an interior solution with $R_{\text{boom}} > c_2 > c_1 > 1$ with

$$\frac{u'(c_1)}{u'(c_2)} = R_{\text{boom}} \tag{32}$$

The period-1 withdrawals offered by the deposit contracts are

$$r = \tilde{r} = c_1 = \frac{y}{\varphi}$$

and the bank invests

$$x = 1 - \varphi r \tag{33}$$

in long term securities. With (6), $c_2 < R_{\text{boom}} < 1/\beta < c_1/\beta = r/beta$ or

$$c_2 < \frac{r}{\beta} \tag{34}$$

so that late consumers do not have an incentive to withdraw early, unless they suspect to not actually receive c_2 in period 2. Generally, this is rather far from being a sharp bound.

As is well-understood, the solution is more efficient than the solution with 100% reserve banking of subsection B.2, but potentially subject to bank runs. For example, if preferences are CRRA with an intertemporal elasticity of substitution below unity,

$$u(c) = \frac{c^{1-1/\sigma} - 1}{1 - 1/\sigma}, \quad \text{where} \quad 0 < \sigma < 1 \tag{35}$$

and if $R_{\text{boom}} > 1$, then (3) is satisfied and

$$r = \left(\varphi + (1-\varphi)R_{\text{boom}}^{\sigma-1}\right)^{-1}, \; c_1 = r, \, c_2 = R_{\text{boom}}^{\sigma} r = R_{\text{boom}} \frac{1-\varphi r}{1-\varphi} \tag{36}$$

There are perhaps two twists compared to the standard solution. First, core bank runs (i.e. local banks running on the core banks) can occur but they invoke the resale of long-term securities to outside investors at the market discount rate rather than the early termination of projects. This already could be viewed as a solution to the task set forth in the introduction of creating a bank-on-bank run in terms of marketable securities. It is obviously a rather trivial solution, as it simply amounts to one of many possible interpretations of the standard bank run model. That literature is typically silent on what it means to "liquidate" the long-term projects, and selling them at a steep discount certainly is consistent with these models.

Second, aside from liquidity provision, the core banks also offer insurance against the idiosynchratic fluctuations in the returns of long-term projects. Consider a slightly different environment, in which local depositors split into fractions φ of early consumers and $(1-\varphi)$ of late consumers at each location. The local bank may still solve a problem as above, but with the random return $R_{\text{boom}} + \epsilon_s$ in place of the safe return R_{boom}. It is obvious, that the solution involving securitization is welfare improving compared to this "local-only" solution, which exposes local depositors to additional local risks. Moreover, it is more likely to trigger "fundamental" bank runs, where long-term depositors run on the local bank, if $R_{\text{boom}} + \epsilon_s < c_1$. Indeed, absent intermediation by core banks, these fundamental bank runs are welfare-improving compared to regulating that deposit contracts need to avoid fundamental bank runs at the local level: these bank runs provide a partial substitute to the missing insurance market, see Allen and Gale (2007). Put differently, securitization improves welfare and makes the system less prone to local bank runs, but exposes it instead to the possibility of "systemic" runs on core banks and thereby to "contagion" across different locations. This interdependence has been analyzed in the literature previously, see e.g. the exposition in chapters 5 and 10 of Allen and Gale (2007), and the literature discussion there.

B.3 The "bust" state and the classic bank run case

To analyze the full model, we assume that the probability of the "bust" state is vanishingly small[7]. It therefore remains to analyze the "bust" state, fixing the first-period withdrawal r of the deposit contracts and the total investments $r\varphi$ in the short-term securities and the long-term securities $1-r\varphi$ as provided by the solution to the "boom"-only situation above.

Note first, that in the absence of a run,

$$c_{2,\text{bust}}(0) = R_{\text{bust}} \frac{(1-\varphi r)}{1-\varphi}$$

where I use the argument "(0)" to denote that the fraction zero of local banks in locations with late consumers run. Therefore, if $c_{2,\text{bust}}(0) < r$, there will be a fundamental bank run, even if core banks hold the same "market" portfolio of long-term securities and local banks believe them to do so, as insurance against the "boom-bust" aggregate uncertainty is not available. For CRRA preferences (35) and therefore (36), this will be the case if

$$R_{\text{bust}} < R_{\text{boom}}^{1-\sigma} \tag{37}$$

Suppose even further, that all long-term securities offer the return R_{bust} and that a fraction θ of all local banks serving late consumers opt for early

[7] Alternatively, assume that the "bust" state was "irrationally" ignored at the time the deposit contracts were signed.

withdrawal. The following algebra is well understood, but will be useful for comparison to the more general case. The core banks meet the additional liquidity demands by selling a fraction ζ of its long-term portfolio or $z = x\zeta$ units of its long-term securities to obtain additional liquidity ℓ, where

$$r\theta(1-\varphi) = \ell = \beta R_{\text{bust}}(1-\varphi)\zeta \tag{38}$$

The securities are discounted by outside investors at $q = \beta$ and $1/\beta$ is the opportunity cost in terms of period-2 resources for providing one unit of resources of period-1 withdrawals. This leaves the remaining late-consumer local banks with

$$\begin{aligned} c_2(\theta) &= \frac{c_{2,\text{bust}}(0) - r\theta/\beta}{1-\theta} \\ &= \frac{1}{1-\theta}\left(\frac{1-\varphi r}{1-\varphi}R_{\text{bust}} - \frac{r}{\beta}\theta\right) \end{aligned} \tag{39}$$

in period 2. Let θ^* solve $c_2(\theta) = r$,

$$\theta^* = \frac{\beta}{1-\beta}\left(\frac{1}{r}\frac{1-\varphi r}{1-\varphi}R_{\text{bust}} - 1\right) \tag{40}$$

If $\theta^* < 0$, there is a fundamental bank run: all local banks will try to withdraw early, because even if no one else did, second-period consumption would be below the promised withdrawal at date 1, $c_2(0) < r$. If $0 < \theta^* < 1$, there is scope for a Diamond-Dybvig "sunspot" bank run. If late-consumer local banks believe that the fraction of early withdrawals by late-consumer local banks exceeds θ^*, they will withdraw early too, so that $\theta = 1$ in equilibrium. If late-consumer local banks believe that the fraction of early withdrawals by late-consumer local banks is below θ^*, they will choose to wait until period 2, and $\theta = 0$ in equilibrium.

There are therefore three scenarios, namely a fundamental bank run, a a Diamond-Dybvig "sunspot" bank run and no bank run. I call these the "classic bank run" scenarios, for comparison with the more general case to be analyzed in the main part of the paper.

C Numerical examples

C.1 A numerical example for uncertainty aversion.

To provide a specific, illustrative example, suppose that $\sigma = 1/2$, $R_{\text{boom}} = 1.44$ and $\varphi = 1/7$. Equation (36) then implies

$$c_1 = r = \frac{7}{6} = 1.1666,\ x = \frac{5}{6},\ c_2 = \frac{7}{5} = 1.44\frac{35}{36}$$

Assume that $\beta = 2/3$, therefore satisfying (6). Assume that 10% of the returns are uniformly distributed on $[0.6, 1.4]$, whereas 90% are equal to 1.4 in the bust state: this is the aggregate distribution F, see figure 3. Therefore, $R_{\text{bust}} = 1.36$. Note that (37) is violated, and that therefore there is no fundamental bank run with complete information in the bust state or if the beliefs $F(\cdot, s)$ of all local banks coincide with the asset distribution.

Assume that for a fraction $(1 - \mu)$ of core banks, local banks assume the correct aggregate distribution, and will therefore not run in a fundamental bank run equilibrium. However, for the remaining fraction μ of the core banks, the local banks believe with certainty that the return is some return R, where R is randomly drawn from F. I.e., if the local banks of these core banks are enumerated $\tau \in [0; 1]$, then $\Gamma(\tau) = 0.6 + 8\tau$ for $0 \leq \tau \leq 0.1$ and $\Gamma(\tau) = 1.4$ for $\tau \geq 0.1$. As a result, the local banks are correct in aggregate, but wrong individually, see figure 4

Absent a bank run, each late consumer local bank expects a pay out of

$$c_2(0; F(\cdot; \tau)) = \Gamma(\tau)\frac{35}{36}$$

Even for the most optimistic bank, I have

$$c_2(0; F(\cdot; 1)) = 1.4\frac{35}{36} < \frac{r}{\beta} = \frac{7}{4} = 1.75$$

Therefore, the condition (13) is satisfied for all $G = F(\cdot; s)$.

Suppose first, that there are only risk neutral investors (or only expert investors), as in subsection 4. In that case, (19) can be used to calculate the

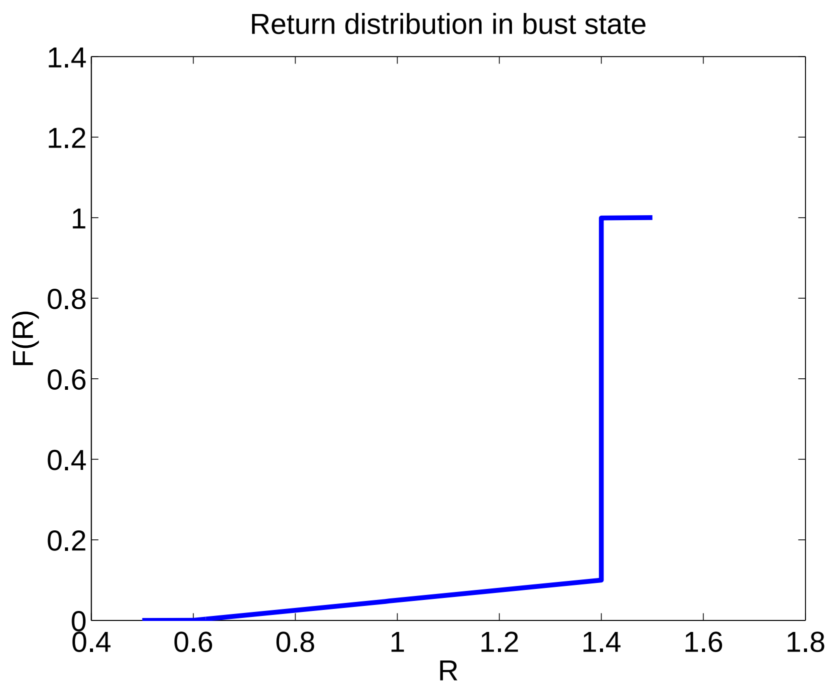

Figure 3: *Return distribution in the bust state.*

fundamental bank run, if it exists, by calculating the smallest τ so that

$$\theta_{\infty;n,0} = \tilde{\theta}^*(F(\cdot;\tau)) = \tau$$

where I have also used the notation $\theta_{\infty;n,0}$ to denote the fraction of local banks at one of the affected core banks, say with index n, if aggregate liquidity demands L are believed to be below ω (or $L = 0$, for simplicity). The solution is approximately $\theta_{\infty;n,0} = 0.0811$, i.e. 8 percent of late consumer local banks will decide to run, see figure 5.

In the scenario with uncertainty averse investors, note that

$$L = L(\theta) = r\theta(1-\varphi)\mu \tag{41}$$

so that the market price drops to the uncertainty-averse investor price $\beta\underline{R}$

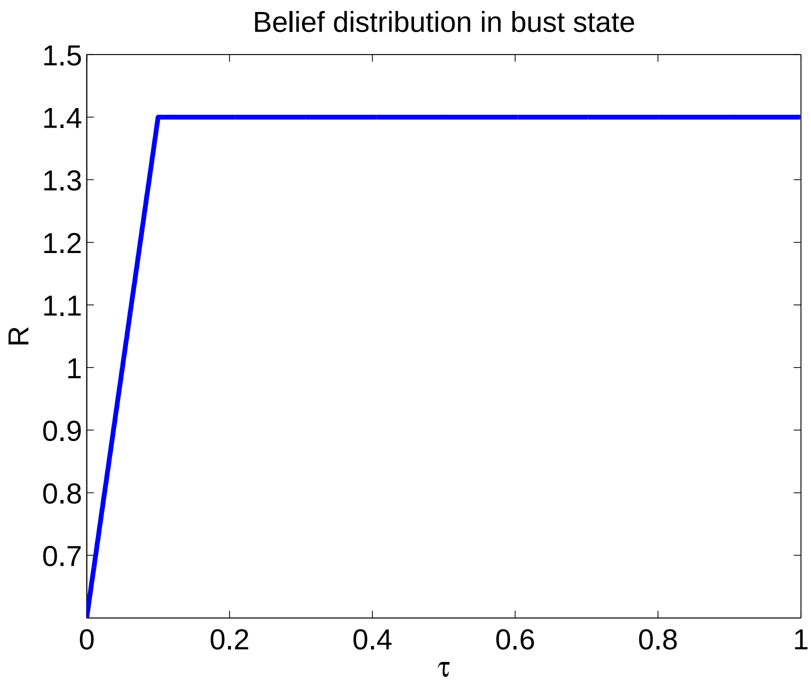

Figure 4: *Beliefs.*

as a function of θ, when θ exceeds the threshold value θ_{crit} given by

$$\theta_{\text{crit}} = \frac{\omega}{\mu r(1-\varphi)} = \frac{\omega}{\mu}$$

since my numerical values happen to imply $r(1-\varphi) = 1$. Put differently, the given expertise of outside investors will be diluted, the more core banks are affected by withdrawals, "accelerating" the bank run compared to the experts-only scenario. It is in this sense, that the bank run is systemic.

Conversely, the experts-only partial bank run described above is not an equilibrium, if $\theta_{\infty;n,0} > \theta_{\text{crit}}$ or

$$\frac{\omega}{\mu} < \theta_{\infty;n,0} r(1-\varphi) \approx 0.0811, \tag{42}$$

i.e. if the fraction of affected core banks is somewhat above 12 times the resources of the expert investors relative to the entire amount initially invested

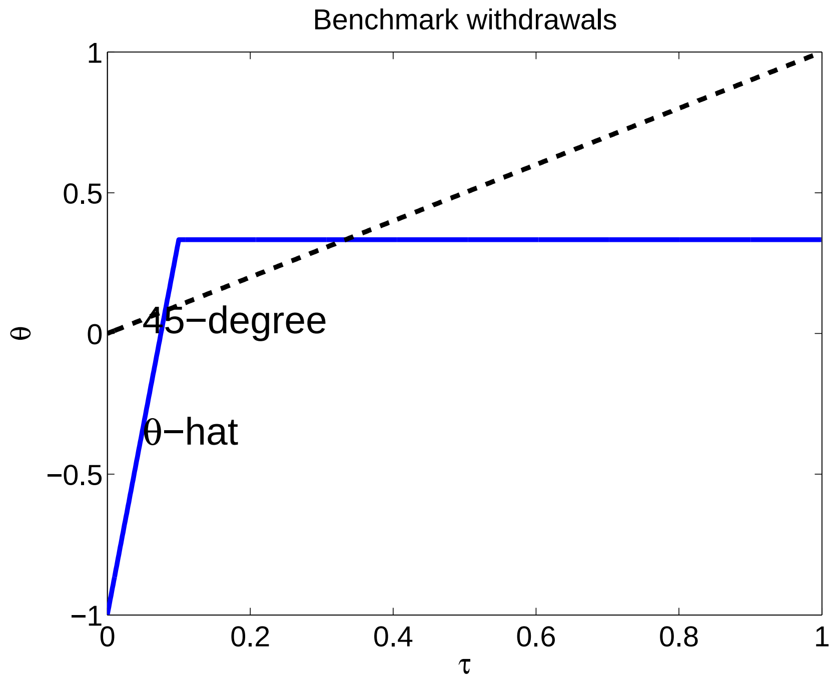

Figure 5: *Bankrun calculus when only expert investors are present.*

in all securities.

To calculate the equilibrium in that case, consider a value $L < \omega$ and a value $L > \omega$. For each value, calculate $c_2(\theta, L; F(\cdot, \theta))$. Calculate the lowest $\theta = \tau$ so that

$$c_2(\theta, L; F(\cdot, \theta)) = r \tag{43}$$

or, absent that and depending on the boundary conditions, either $\theta = \tau = 0$, if $c_2 > r$ always, or $\theta = \tau = 1$, if $c_2 < r$ always.

The resulting second-period consumption is shown in figure 6. If $L > \omega$, the graph shows that $\theta = 1$, i.e. a run on all core banks affected by doubtful local banks, as the only solution. By contrast, there are multiple solutions to (43), if $L < \omega$. Therefore, if (42) holds, a system-wide bank run on the fraction μ of the core banks, which are subject to heterogeneous beliefs by their local banks, results, while the other $1-\mu$ core banks remain unaffected (unless there is a Diamond-Dybvig sunspot-type bank run). Variations of this example can produce partial fundamental bank runs as well. Furthermore and in a generalized version of this model, if Γ varies smoothly with L, figure 6 suggests a critical value as the c_2-curve is shifted downwards with increasing L, when the equilibrium close to the small expert-only partial bank run disappears and only the system-wide bank run on the affected core banks remains.

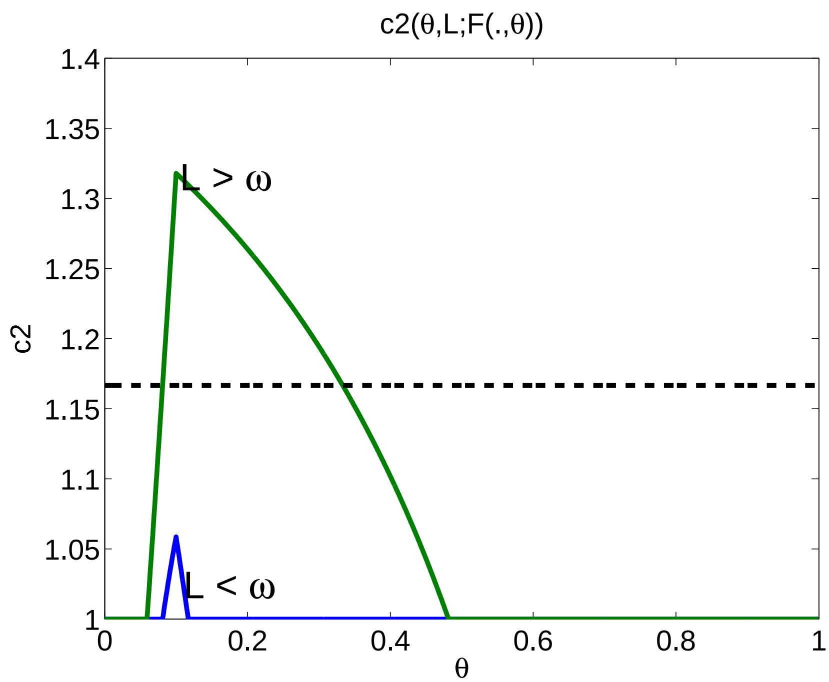

Figure 6: *Consumption of local banks that wait until the second period, assuming that all banks with $\tau < \theta$ run, and banks with $\tau > \theta$ do not. Comparison to $c_1 = r$.*

C.2 A numerical example for adverse selection.

I use the same parameterization as in subsection C.1. For low values of $\theta \leq \underline{\theta}$, the market price will be below $\underline{R} = 0.6$ and the required market discount $\Gamma(\theta, \mu; F)$ at the true distribution will equal $1/\beta$. For these low values of θ and due to the uniform distribution, the market price equals

$$p(\theta; \mu) = \beta \frac{F^{-1}(\zeta) + 0.6}{2} \tag{44}$$

Therefore, θ is low enough, iff $p(\theta; \mu) \leq 0.6$ or, equivalently, $F^{-1}(\zeta) \leq 1.2$. By the parameterization in (C.1),

$$F^{-1}(\zeta) = \min\{0.6 + 8\zeta, 1.4\}$$

Therefore, $F^{-1}(\zeta) \leq 1.2$ corresponds to $\zeta \leq 0.075$. To find p and ζ when $F^{-1}(\zeta) \leq 1.2$, I therefore need to solve

$$\zeta = \frac{r\theta}{\beta(0.6 + 4\zeta)} \left(\frac{1 - \varphi}{1 - r\varphi} \right) \tag{45}$$

Let

$$\kappa = \frac{r}{4\beta} \left(\frac{1 - \varphi}{1 - r\varphi} \right) = \frac{9}{20}$$

The solutions to (45) are therefore given by

$$\zeta = -0.075 + \sqrt{0.075^2 + \kappa\theta}$$

(where the negative root has been excluded as not sensible). Therefore, $\zeta \leq 0.075$, if

$$\theta \leq \underline{\theta} = 3 * 0.075^2 / \kappa = 0.0375.$$

For $\theta > \underline{\theta}$, the behavior of the price depends on market share of the distressed core banks. Two extreme scenarios can provide some general insights. If $\mu \to 0$, then the price will remain "stuck" at $\underline{p} = \underline{R} = 0.6$, as all remaining banks would sell arbitrarily large chunks of their worst assets otherwise. If $\mu = 1$, then discounting of future returns will remain to be done at the discount rate β. For any given μ, local banks with $c_2(\theta, \mu; G) < r$ will withdraw, see (28).