WP/16/215

IMF Working Paper

Tuning in RBC Growth Spectra

by Szilard Benk[1], Tamas Csabafi[2], Jing Dang[3], Max Gillman[4], Michal Kejak[5]

[1] International Monetary Fund (IMF).
[2] University of Missouri at St. Louis.
[3] SGCC, China.
[4] University of Missouri at St. Louis.
[5] CERGE-EI, Czech Republic.

INTERNATIONAL MONETARY FUND

WP/16/215

IMF Working Paper

Office of Executive Director

Tuning in RBC Growth Spectra

Prepared by Szilard Benk, Tamas Csabafi, Jing Dang, Max Gillman, Michal Kejak

Authorized for distribution by Ibrahim H. Canakci

September 2016

Abstract

For US postwar data, the paper explains central consumption, labor, investment and output correlations and volatilities along with output growth persistence by including a human capital investment sector and a variable physical capital utilization rate. Strong internal "amplication" results from an economy-wide productivity shock across goods and human capital investment sectors that has variances 10,000 fold smaller than in the standard RBC TFP shock. Simulated moments are compared to data moments for the business cycle, the low frequency and the Medium Cycle frequency, as well as the high frequency. A metric is provided to gauge that the results have an average of 46% deviation of simulated moments from data moments, for a broad array of targets across all windows. Within this array, key correlations have only a 15% deviation in the business cycle window, and growth persistence only an 8% deviation in the low frequency, which indicates good "propagation". Countercyclic human capital investment time and procyclic physical capital capacity utilization rates are also found as in data.

JEL Classification Numbers: E13, E32, F10, 041.

Keywords: Real Business cycles, amplification, growth persistence, Gali labor puzzle, data moments, human capital.

Authors' E-Mail Addresses: sbenk@imf.org, csabafit@umsl.edu, djruc@hotmail.com, gillmanm@umsl.edu, mkej@cerge-ei.cz

1 Introduction

Explaining real business cycle (RBC) performance without financial frictions presents a challenge in the aftermath of the Great Recession. Besides confronting the unusual post-2010 data of a below-trend recovery, RBC advancement efforts remain dispersed. King & Rebelo (2000) show how adding external margins for labor and capital utilization rates reduces the variance of the total factor productivity (TFP) shock. They argue that these margins provide better amplification with "much smaller shocks" that can be measured in a way that is similar to how Ingram et al. (1997) back out the shocks in a home production sector. Meanwhile Cogley & Nason (1995), Rotemberg & Woodford (1996), Perli & Sakellaris (1998) and Benhabib et al. (2006) address the weak internal propagation of the standard RBC model in terms of matching the data profile of a falling output growth persistence. Benhabib et al. (2006) match this profile by adding additional physical capital investment sectors with independent shocks; at the same time they find a negative labor impulse response in support of Gali (1999), who suggests that data is consistent with a negative labor response rather than the standard RBC positive TFP labor response.[1] Perli & Sakellaris (1998) adds a human capital sector to get better propagation, and Maffezzoli (2000) adds a human capital investment sector with externalities for matching international RBC data moments. DeJong & Ingram (2001) use a human capital investment sector along with two others to model a countercyclical human capital investment time, which they support with evidence that is consistent with both Perli & Sakellaris (1998) and Dellas & Sakellaris (2003). McGrattan (2015) relatedly explains the RBC "labor wedge" with labor spent in an intangible capital investment sector.

The paper explains moments of the RBC growth spectra across both business cycle and low frequencies while advancing each of the above issues: better propagation and amplification, negative labor impulse features, countercyclical human capital investment time, a labor wedge explanation using human capital time, plus in addition modeling procyclic physical capital utilization rates as in the data. It does this by taking the King & Rebelo (2000) route of adding both labor and physical capital external margins. The added labor margin comes from a second sector of human capital investment (without externalities); the added capital margin comes through a variable physical capital utilization rate for both sectors that also affects depreciation in an extension of Greenwood et al. (1988). The model then employs an economy-wide productivity shock comprised of non-independent shocks to both sectors, in contrast to much of the literature, with a high correlation between shocks and a one-third lower variance of the human capital shock relative to the goods TFP variance. The result, as compared to a King & Rebelo (2000) reduction in the TFP goods sector shock variance from 0.007 to 0.001, is a 10,000 fold reduction in the TFP shock variance for

[1] See Benhabib et al. (2006) Figures 1 and 5 for the match of output growth's autocorrelation profile and see Figure 7 for their generation of a Gali (1999) type labor impulse response.

each the goods and human capital investment sectors. The dramatic amplification results because the economy-wide shock causes the growth rate of human capital to be shocked but in a smaller magnitude than is the goods sector. Shocking the growth rate creates a permanent income effect that supplements the temporary income effect of the standard goods sector TFP shock such that both goods and human capital shocks are extremely "small" in terms of their variance while the model matches well a wide array of data moments. This includes matching consumption and physical capital investment correlations with output which the basic RBC model leaves as the consumption-output puzzle of insufficient consumption variation.[2]

Adding only the human capital investment sector and its productivity shock, but without a variable physical capital utilization rate (called Model 1), results in a goods sector "TFP" shock with a variance 400 times smaller than the baseline 0.007. Adding also the capital utilization rate (called Model 2), not only results in the TFP shock variance becoming instead 10,000 times smaller, but also creates propagation that captures the Benhabib et al. (2006) -highlighted falling growth persistence autocorrelation profile found in data. The positive economy-wide shock works by inducing reallocation initially towards scarcer human capital investment and then back to the goods sector in a dynamic fashion during which a dual Stolper & Samuelson (1941) and Rybczynski (1955) theorem results, as presented below.[3] While the standard goods sector TFP shock when taken by itself has a positive goods labor supply impulse response as in standard RBC models, the human capital TFP shock by itself induces a negative goods labor supply impulse response. Taken together in the economy-wide TFP shock, with correlation between the shocks, the labor impulse response is negative at first as suggested by Gali (1999) but then becomes positive, as found also in Benhabib et al. (2006), while matching business cycle and low frequency labor moments of the data.

Following the methodology of King et al. (1988) for extending cyclic analysis to growth spectra at the low frequency, results are presented for an extended array of the standard cyclic correlations, volatilities and the output growth autocorrelation. Results show model fit not only within the RBC window but also at the low frequency , the high frequency, as well as the Comin & Gertler (2006) "Medium Term Cycle" frequency that combines high, business cycle and low frequency windows. Using a metric based on an extension of Jermann (1998), the paper provides a numeric value of the average closeness of the model's simulated moments to those of the data, exemplifying a quantification of the success of a moment comparison that has been used since Kydland & Prescott (1982). In particular, an average deviation of the filtered simulated moments from the filtered data moments is constructed

[2]See Benhabib & Wen (2004) for an alternative approach using demand shocks.

[3]Mulligan & Sala-i Martin (1993) with a linear production function for human capital and Bond et al. (1996) with a continuous time version of our Model 1 prove related Stolper & Samuelson (1941) theorems but not duality.

on the basis of a large number of moment targets that typically come under RBC scrutiny, as augmented by this same set of moments across the other three additional frequencies (high, low and Medium). The metric is the sum of all of the normalized target distances, each of which equals the fractional moment deviation of model from data, as normalized by dividing by the total number of targets. This makes the aggregate metric the average fractional deviation of the model's moments from the data moments across the four spectra of a large moment set.

The extended Model 2 with variable capacity utilization is preferred for consistently capturing the growth persistence autocorrelation. For a broad set of 59 targets, Model 2 results show an aggregate average 46 percent deviation of model moments from data moments while Model 1 has an average 59 percent deviation. The paper also breaks down the average metric within each of the four frequencies (high, business cycle, low and Medium), for each Model 1 and 2, and separately for each correlations, volatilities and growth persistence. The model performs best in the business cycle window, especially for the fundamental consumption-output correlation and output growth persistence, but also quite well in the low and Medium frequency windows. In the business cycle window, the average aggregate metric for Model 2 is a low 15 percent deviation from the data moments. Model 2's growth persistence in the low frequency window is only an 8 percent deviation from data.

The calibration uses a systemic, combinatorial, grid search, with the reported model resulting from about $9x10^{36}$ iterations performed with massive parallel processing, with a selection made from the best (lowest) 200 average metric results. The grid search is disciplined to yield a positive-definite variance-covariance matrix for state variable convergence (Blanchard & Quah 1989) and iterative convergence between the model's TFP shock assumptions and the properties of the backed-out goods sector productivity shock (Benk et al. 2005). The data period includes the Great Recession and the paper shows how the backed-out economy wide shock compares to the standard RBC TFP shock, as in (Nolan & Thoenissen 2009). Iterative convergence ensures consistency between the assumed shocks and those backed out from data; the grid search enables comprehensive calibration space search; and the aggregate metric provides a measure of the model's performance as well as a tool to limit the focus of the calibration to those with the lowest fractional deviations.

A post-Great Recession result is that the backed-out economy-wide TFP shock of the full model rather closely tracks the standard Solow residual except after the Great Recession. Instead of falling continuously after 2010 as does the Solow residual, the model's shock begins rising as does US GDP growth, while in addition both the model's backed out shock and the US economy's GDP growth remain below trend. This reflects King & Rebelo (2000) and McGrattan (2015) criticism that the Solow residual is not an exact measure of the economy's TFP; instead the smaller shocks of a multi-sector economy, such as in this paper, may well form the basis for a better TFP measure.[4]

Relative to seminal literature, the human capital sector performs an external labor margin that builds upon Hansen (1985), Rogerson (1988), Benhabib et al. (1991), Greenwood & Hercowitz (1991), and Perli (1998). The paper differs from Comin & Gertler (2006) by using the RBC growth model of Gomme & Rupert (2007), as extended with a human capital investment shock, the existence of transition dynamics, and variable capital utilization; it also explains an extended array of moments relative to Comin & Gertler (2006). Christiano et al. (2001) also solves basic RBC puzzles, including the equity risk premium that this paper does not address, but this paper instead uses homothetic utility, production, and no adjustment cost of the physical capital stock. Meanwhile Grossman et al. (2016) focus on long term model properties with human capital, without application to data.

Section 2 describes the full model and its Stolper & Samuelson (1941) and Rybczynski (1955) duality theorems that underlie the movement of labor between sectors. Section 3 describes the calibration, the backed-out shocks and impulse responses. Section 4 presents moment results and Section 5 concludes.

2 The Model

The general model is called Model 2, in which the representative agent maximizes its expected sum of discounted utility U. The agent derives utility from consumption, c_t, leisure, x_t, and a function $s(u_t) \equiv (1-u_t)^B$, where $u_t \in [0,1)$ is the physical capital capacity utilization rate at each time period t. With $A \in R_+$, $B \in R$, and $\sigma \in R_+$, the time t period utility is given by

$$U(c_t, x_t, u_t) = \frac{\left[c_t x_t^A (1-u_t)^B\right]^{1-\sigma} - 1}{1-\sigma}, \tag{1}$$

which satisfies the necessary conditions for the existence of a balanced growth path (BGP) equilibrium.[5] The unused physical capital capacity utilization, in terms of $1-u_t$, is included in utility to be symmetric with including leisure x_t, which is the unused human capital utilization rate. The symmetry follows in the sense of DeJong et al. (1996) such that the human capital capacity utilization rate is $1-x_t$, while u_t is the physical capital utilization rate, although they exclude u_t from the utility function. The case of $B = 0$ is included as a possible special case, but in general B is allowed to be positive or negative; it ends up robustly slightly negative in the calibration, as is consistent with Otani (1996), such that

[4] King & Rebelo (2000) call the smaller resulting residuals when also using a home production sector the "Crucini residuals" (in their footnote 60); McGrattan (2015) finds adding a second investment sector "quantitatively important for analyzing U.S. aggregate fluctuations".

[5] For more, see King et al. (1988).

utility gain comes more fully utilizing physical capital. In Model 1, $(1-u_t)^B = 1$, so that the utilization rate is eliminated from the model and set implicitly to one just as when a model specification eliminates leisure time and sets labor time equal to one, in goods-only models without leisure.

The representative agent time endowment for each period t, is allocated to l_{gt}, the fraction of time spent in goods production, to l_{ht}, the fraction of time spent in human capital investment production, and to x_t, leisure:

$$1 = x_t + l_{gt} + l_{ht}. \tag{2}$$

Physical capital investment, i_{kt}, determines the capital stock k_t accumulation as in DeJong et al. (1996):

$$k_{t+1} = k_t - \delta\left(u_t\right) k_t + i_{kt}, \tag{3}$$

where $\delta\left(u_t\right)$ is a function. It is the endogenous depreciation rate of physical capital that depends on u_t. The functional form for $\delta\left(u_t\right)$ is assumed to be[6]

$$\delta\left(u_t\right) = \frac{\delta_k}{\psi} u_t^{\psi}, \tag{4}$$

with $\psi > 1$ and $\delta_k > 0$; a faster rate of utilization results in a higher rate of depreciation. It follows that $\delta'\left(u\right) > 0$ and $\delta''\left(u\right) > 0$ so that the marginal cost of utilization of the physical capital stock is increasing in the utilization rate.

Denote by y_t the real goods output that corresponds to the data notion of GDP. For the goods production function A_g is a positive factor productivity parameter, z_t^g the total factor productivity shock, v_{gt} the share of the physical capital stock being allocated to the goods sector and $v_{gt}u_tk_t$ the amount of physical capital in the goods sector that is utilized for production purposes. Let h_t denote the stock of human capital at the beginning of time period t; then $l_{gt}h_t$ represents the effective labor input, or the share of human capital used in goods production. With $\phi_1 \in [0,1]$ denoting the share of physical capital used in goods production, the output function is

$$y_t = A_g e^{z_t^g} \left(v_{gt}u_tk_t\right)^{\phi_1} \left(l_{gt}h_t\right)^{1-\phi_1}. \tag{5}$$

[6]Others with similar endogenous depreciation rate as a function of the utilization rate include Greenwood et al. (1988), DeJong et al. (1996), and Benhabib & Wen (2004).

The human capital stock is then accumulated over time according to the following standard law of motion,

$$h_{t+1} = (1 - \delta_h)h_t + i_{ht}, \tag{6}$$

where δ_h is the assumed constant depreciation rate of human capital and i_{ht} is the per period investment in human capital. For the human capital investment technology, A_h denotes a positive factor productivity parameter in the human sector; z_t^h represents the sectorial productivity shock in natural logarithms; $v_{ht} = 1 - v_{gt}$ is the remaining fraction of physical capital allocated to the human sector; and $v_{ht}u_tk_t$ is the amount of physical capital in the human sector that is utilized for human investment production. With $\phi_2 \in [0, 1]$ being the share of physical capital in the human capital investment, the production function is

$$i_{ht} = A_h e^{z_t^h}(v_{ht}u_tk_t)^{\phi_2}(l_{ht}h_t)^{1-\phi_2}. \tag{7}$$

2.1 Shock Structure

In the economy are two random shocks following first-order autoregressive processes:

the goods productivity shock z_t^g, where

$$z_t^g = \rho_g z_{t-1}^g + \varepsilon_t^g, \quad 0 < \rho_g < 1, \tag{8}$$

and the human capital investment sector productivity shock z_t^h, where

$$z_t^h = \rho_h z_{t-1}^h + \varepsilon_t^h, \quad 0 < \rho_h < 1 \tag{9}$$

and the innovations are normally distributed according to

$$\begin{pmatrix} \varepsilon_t^g \\ \varepsilon_t^h \end{pmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma}), \tag{10}$$

where the general structure of the second-order moments is the variance-covariance matrix $\mathbf{\Sigma}$, with individual variances denoted by σ_g^2 and σ_h^2. This allows for any degree of covariance between the shocks.

Given no externalities, the competitive equilibrium of the economy coincides with the result of the social planner's problem, which can be stated as[7]

$$\max_{\{c_t, l_{gt}, l_{ht}, x_t, v_{gt}, v_{ht}, u_t, k_{t+1}, h_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{[c_t x_t^A (1-u_t)^B]^{1-\sigma} - 1}{1-\sigma}, \tag{11}$$

subject to (2)-(10).

2.2 Definition of the Competitive Equilibrium

Definition 1 *A general equilibrium of this model is a set of contingent plans* $\{c_t,\ k_{t+1},\ h_{t+1},\ v_{gt},\ v_{ht},\ u_t,\ x_t,\ l_{gt},\ l_{ht}\}$ *that solve the central planner's maximization problem in (11) for the initial endowment* $\{k_0,\ h_0\}$ *and exogenous stochastic technology processes* $\{z_t^g,\ z_t^h\}$*, with initial conditions* $\{z_0^g,\ z_0^h\}$*.*

Definition 2 *A deterministic balanced growth path (BGP) equilibrium of this model is a set of paths* $\{c_t,\ k_{t+1},\ h_{t+1},\ v_{gt},\ v_{ht},\ u_t,\ x_t,\ l_{gt},\ l_{ht}\}$ *that solve the central planner's maximization problem in (11) for the initial endowment* $\{k_0,\ h_0\}$ *and exogenous technology parameters* $\{z_t^g = 0, z_t^h = 0\}$*, such that* $\{c_t, k_{t+1}, h_{t+1}\}$ *grow at a common trend, and* $\{v_{gt}, v_{ht}, u_t, x_t, l_{gt}, l_{ht}\}$ *are constant.*

Appendix A.1 presents the dynamic equilibrium conditions and Appendix A.2 the BGP equilibrium conditions.

2.3 Duality Theorems

Both a Stolper & Samuelson (1941) and a Rybczynski (1955) effect hold during factor input allocations across sectors, with duality holding between these effects in general equilibrium.[8] To illustrate this, consider letting p_{ht} denote the relative price of human capital investment to physical capital investment, where $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$. This defines it as the ratio of the shadow price of human capital investment χ_t to the shadow price of physical capital investment λ_t, where these shadow prices are given explicitly in Appendix A.1.

Proposition 3 *The sign of the derivative of* r_t^k *and* w_t *with respect to* p_{ht} *depends only on the factor intensity ranking.*

Proof. From equations (19) and (20) of Appendix A.1 and denoting the implicit factor rental prices by r_t^k and w_t, as also given in Appendix A.1 for physical capital and human capital respectively, the following relations hold between the rental prices and the relative

[7] The first order conditions of the representative agent can be found in Appendix A.1.

[8] This duality result was suggested to us by J. Benhabib.

price of capital p_{ht} :

$$r_t^k \equiv \phi_1 (z_t^h)^{\frac{\phi_1 - 1}{\phi_1 - \phi_2}} (z_t^g)^{\frac{1-\phi_2}{\phi_1 - \phi_2}} A_h^{\frac{\phi_1 - 1}{\phi_1 - \phi_2}} A_g^{\frac{1-\phi_2}{\phi_1 - \phi_2}} \left[\frac{\phi_2}{\phi_1}\right]^{\frac{\phi_2(\phi_1 - 1)}{\phi_1 - \phi_2}} \left[\frac{1-\phi_2}{1-\phi_1}\right]^{\frac{(1-\phi_2)(\phi_1 - 1)}{\phi_1 - \phi_2}} p_{ht}^{\frac{\phi_1 - 1}{\phi_1 - \phi_2}}; \quad (12)$$

$$w_t \equiv (1-\phi_1)(z_t^h)^{\frac{\phi_1}{\phi_1 - \phi_2}} (z_t^g)^{\frac{1-\phi_2}{\phi_1 - \phi_2}} A_h^{\frac{\phi_1}{\phi_1 - \phi_2}} A_g^{\frac{-\phi_2}{\phi_1 - \phi_2}} \left[\frac{\phi_2}{\phi_1}\right]^{\frac{\phi_2 \phi_1}{\phi_1 - \phi_2}} \left[\frac{1-\phi_2}{1-\phi_1}\right]^{\frac{(1-\phi_2)\phi_1}{\phi_1 - \phi_2}} p_{ht}^{\frac{\phi_1}{\phi_1 - \phi_2}}. \quad (13)$$

Given the assumption that human capital investment is relatively more intensive in human capital relative to the goods sector, so that $\phi_1 > \phi_2$, equations (12) and (13) imply $\frac{\partial r_t^k}{\partial p_{ht}} < 0$ and $\frac{\partial w_t}{\partial p_{ht}} > 0$. ■

Corollary 4 *The Stolper & Samuelson (1941) effect in the two-shock economy: as a special case of equations (8)-(10), assume identical shocks such that in log-linear (deviations from the steady state) form $\hat{z}_t^g = \hat{z}_t^h$; after such an economy-wide shock, an increase in the relative price of a sector's output will cause a relatively bigger increase in the implicit competitive price of the unit of the input that is used relatively intensively in that sector.*

Proof. Combine equations (12) and (13) and log-linearize along the models' respective steady states, with log-linearized variables denoted by a hat, to find that

$$\hat{w}_t - \hat{r}_t^k = \frac{\hat{p}_{ht} - \hat{z}_t^g + \hat{z}_t^h}{\phi_1 - \phi_2}. \quad (14)$$

Given $\phi_1 > \phi_2$ and identical shocks such that $\hat{z}_t^g = \hat{z}_t^h$, an increase in p_{ht} causes $[\hat{w}_t - \hat{r}_t^k]$ to increase, so that $\hat{w}_t$ rises more than $\hat{r}_t^k$, the human capital investment sector is relatively more factor intensive in human capital than in physical capital, and so the relative price of the more intensively used factor within the human capital sector rises when the relative price of human capital investment rises; conversely for the goods sector. ■

In a dual way and following Van Long (1992), consider the next proposition.

Proposition 5 *The Rybczynski (1955) effect: an increase in the initial allocation of a factor input in a sector will expand the output of that sector if it is more intensive in the increased input; whereas the output of the other sector more intensive in the other factor input will increase by a relatively lower quantity.*

Proof. One sector produces goods y_t (or alternatively physical capital investment) with a price normalized to unity; the second sector produces human capital investment i_{ht} at a relative price p_{ht}, in terms of the goods output. Appendix A.1 in equation 27 derives the relative price as the ratio of the marginal products with respect to human capital of each the goods and human capital investment investment sectors:

$$p_{ht} = \frac{(1-\phi_1)A_g z_t^g \left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1}}{(1-\phi_2)A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2}}. \tag{15}$$

Consider the change in human capital investment with respect to each factor of physical capital given by $v_{ht}u_t k_t$ and of human capital given by $l_{ht}h_t$, where these are notated by R_1^h and R_2^h respectively, so that $R_1^h \equiv \frac{\partial i_{ht}}{\partial v_{ht}u_t k_t}$, and $R_2^h \equiv \frac{\partial i_{ht}}{\partial l_{ht}h_t}$, and these are standard marginal products. It follows the relative price p_{ht} can be expressed in terms of only the factor prices r_t^k and w_t, and the marginal products R_1^h and R_2^h:

$$R_1^h = \phi_2 A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2 - 1} = \frac{r_t^k}{p_{ht}}, \tag{16}$$

$$R_2^h = (1-\phi_2) A_h z_t^h \left[\frac{v_{ht}u_t k_t}{l_{ht}h_t}\right]^{\phi_2} = \frac{w_t}{p_{ht}}. \tag{17}$$

Then given the assumption that $(1-\phi_2) > \phi_2$ it follows that $R_1^h < R_2^h$. This implies that in the human capital investment sector, increasing human capital by a unit will increase output of human capital investment by more than would increasing physical capital by a unit; conversely for the goods sector. ■

Proposition 6 *Define S_1^h and S_2^h as the change in the real interest rate and the wage rate with respect to a change in the relative price of human capital, so that $S_1^h \equiv \frac{\partial r_t^k}{\partial p_{ht}}$ and $S_2^h \equiv \frac{\partial w_t}{\partial p_{ht}}$. It follows that $S_1^h = \frac{r_t^k}{p_{ht}}$ and $S_2^h = \frac{w_t}{p_{ht}}$.*

Proof. See Appendix 5, equations (63) and (67). ■

Corollary 7 *Duality between Stolper-Sameulson and Rybczynski effects: The change in the output of the human capital investment sector with respect to a change in an input is equal respectively to the change in that input's implicit competitive price with respect to a change in the implicit relative price of human capital investment to goods output.*

Proof. Propositions 5 and 6 imply directly that $R_1^h = S_1^h$ and $R_2^h = S_2^h$. ■

The proposition establishes that there exists a duality between the Rybczynski (1955) and Stolper & Samuelson (1941) effects across sectors. The impulse responses below show how an equi-proportional 1% shock to both goods and human capital sectors, with the calibrated correlation between shocks, causes labor in the goods sector to fall in both Models 1 and 2, in a way consistent with the reallocation across sectors according to the change in factor prices as in the theorems stated here. The wage to interest rate ratio falls in Model 1 while it rises in Model 2 due to the ability to change the physical capital utilization margin.

However the actual Model 2 simulated shock involves a smaller human capital shock relative to the goods sector shock, rather than an equi-proportional shock. This means that the actual effect in the simulated Model 2 economy of the labor reallocations in response to the economy-wide shock is that labor at first moves towards the scarcer human capital investment sector but then moves back to the goods sector as human capital investment becomes less scarce than physical capital investment. Accordingly, the simulated moments of the model are able to capture well a broad array of data moments including the labor and output correlations at the business cycle and lower frequencies, and a countercyclic human capital investment labor at the business cycle and lower frequencies.

3 Model Simulation

By normalizing the variables that grow along the balanced growth path (BGP), and then log-linearizing all the equilibrium conditions of the two models around their normalized growth paths, two stochastic systems of linear equations result. Using the human capital stock , h_t, for normalization as in Benk et al. (2008, 2010), for both models the respective systems are solved for in terms of the state variable k_t/h_t and the two shock processes, z_t^g and z_t^h. The calibrated models are solved by the method of undetermined coefficients Uhlig (1998).[9]

3.1 The Calibration

Table 1 presents the calibrated structural and exogenous shock parameters for Model 1 and 2. Table 2 presents the calibration grid ranges, in which 5,000 steps within the ranges were employed. For Model 1, there are 53 targets including two BGP equilibrium values (balanced growth rate and leisure) as BGP targets. For Model 2, there are 59 targets, of which three are BGP equilibrium values (adding the BGP physical capital utilization rate). The high target number resulted from experiments which found a better fit with more targets, but with a diminishing return to adding targets.

The calibration methodology of Jermann (1998) is modified and combined with the shock identification scheme of Benk et al. (2005) building on Ingram et al. (1997). For the shock identification, the data period is 1959Q1 to 12015Q4 for Model 1, and 1972Q1 to 2015Q4 for Model 2, with a shorter period for Model 2 due to the lack of physical capital utilization data before 1972Q1. Please see Appendix B for a data description.

To calibrate the models, first long-run BGP targets are restricted based on US data. Targets for Model 1 are the balanced growth rate of the economy, g, and leisure time, x, set

[9] Please see Appendix D for solution methodology.

at 0.0035 and 0.5 respectively, following Gomme & Rupert (2007) These imply the weight of leisure in the agent's utility function, A, and the scale parameter of the human capital investment sector, A_h through the intratemporal and the second intertemporal margins.

For Model 2 the BGP targets are leisure time, again set at 0.5, the growth rate g being 0.0035 for the shorter data period, and the physical capital utilization rate, u, which is 0.785 as calculated from US data. Following Gomme & Rupert (2007), the long-run value of the endogenous physical capital depreciation rate is set to 0.025. These targets imply the leisure preference, A, through the first intertemporal margin in equation (33), the utilization rate's weight, B, in the utility function through the second intratemporal margin in equation (29), the human sector scale parameter, A_h, via the second intertemporal condition in (34), and the depreciation parameter of the endogenous depreciation rate, δ_k, through the physical capital law of motion in (3).

To calibrate the remaining seven structural parameters for both models, a grid in a bounded parameter space is established with lower and upper bounds for parameters as set out in Table 2.[10] For each possible combination of the grid coordinates the models are solved with iterative convergence of the backed-out shock's properties to the model's assumed shock properties, as in Benk et al. (2005). This extends the method of Jermann (1998) by iterative convergence of the shocks and a mean normalization of the distance metric to transform each individual distance measure into percentage deviations of the simulated moments from the US data targets.[11]

The resulting metric is used by limiting the examination of the results to the top 200 best (lowest measures) metric, out of $9x10^{36}$ successfully convergent runs. There are a total of 59 US data based targets for Model 2 and 53 for Model 1. The lowest obtained metric for Model 2 was 0.41, while the one presented in the Tables has a value of 0.46; this can be interpreted as on average a 46% deviation of the 59 targets from their model-achieved values.[12]

For the grid ranges of Table 2, for each calibrated parameter of Models 1 and 2, the lower bound of the discount factor β is set to 0.95 and the upper bound to 0.99. The parameter for of the constant elasticity of substitution (CES) in utility is bounded between 0.40 and 2.00 as found for example in the quarterly estimates of Hall (1988) and Mehra & Prescott (1985) respectively. The share of physical capital in the goods producing sector, ϕ_1, has a lower bound of 0.30 and upper bound of 0.40, with the range for the scale parameter of the

[10] This uses similar features to *Bayesian* estimation by setting bounds with prior information, despite not estimating the parameters within the bounds, but instead searching uniformily over a large set of possible values.

[11] The approach is alternative to use of a simulated annealing algorithm, which was also explored, but which gives a different calibration with each run. Simulated annealing is also embedded in Bayesian estimation of the calibration parameters. It gives a different calibration with each run because of its "temperature-gauge" property. Complete Matlab codes of the grid search approach as well as simulated annealing, both with iterative convergence of the model shocks to data, are available with detailed descriptions upon request.

[12] We thank Viktor Huszar, DWO LLC., for the use of a massive parallel processing system.

goods sector, A_g, residually set between 0.50 and 2.00.

In Model 1 the bounds for the constant depreciation rate of physical capital are set to 0.015 and 0.03, which coincide with annual rates of 6 percent at the lower bound and a 12 percent depreciation rate at the upper bound. In Model 2 the depreciation parameter δ_k is determined through the physical capital law of motion when given the long-run quarterly target depreciation rate of 0.025. Then the convexity parameter ψ is bounded between 2.00 and 4.00. The share of physical capital in human investment production, ϕ_2, is given a range of 0.08 and 0.29 in line with Jorgenson & Fraumeni (1991) and Jones et al. (2005). The bounds for the constant depreciation rate of human capital, δ_h, are set to 0.001 and 0.015, where estimates that serve as a basis are those of DeJong & Ingram (2001), Jorgenson & Fraumeni (1991), and Jones et al. (2005).

The range for the persistence parameters, ρ_g and ρ_h, of the goods and human capital sectorial shocks is identical with the lower bound set to 0.01 and the upper bound set to 0.99. In order to reduce computational intensity, the initial guess for each of the shock variances is set to an initial value of 0.007 as found in King and Rebelo (2000). The cross-correlation between these two sectorial shocks is given a range between -0.99 and 0.99, avoiding -1 and 1 because of the positive semi-definite requirement for solving the underlying models.

Within the defined grid, each point represents a possible combination of parameters, with all permutations investigated. Iterative convergence is imposed such that the shock parameter variance-covariance matrix that is assumed in the calibration is the same as that variance-covariance matrix of the backed-out shocks that result from use of US data. This identity between assumed shock parameters and backed-out shock parameters is implemented for each grid point following the Benk et al. (2005) methodology of using seemingly unrelated regression estimation to find the variance-covariance matrix $\mathbf{\Sigma}$ of the backed-out shocks.

Then a normalized vector distance metric is constructed for each grid point and used to select the best calibration from the entire set of grid points. The metric is constructed by using the simulation-based moment vector, denoted by Θ, along with the corresponding US data-based target moment vector, $\hat{\Theta}$. Denoting the distance metric by D, it is defined so as to give the average fractional deviation of the model moment from the data moment across all targeted moments. This is found by summing up each of the fractional deviations of model moment from data moment, and dividing by the total number of targeted moments; call the latter T. For each Model 1 and 2, with $z = 1, 2$, then the definition of D_z is $D_z \equiv \left(\sum_i \sum_j \sum_k \left| \hat{\Theta}_{ijk} - \Theta_{ijk} \right| / \left| \hat{\Theta}_{ijk} \right| \right) \Big/ T_z$, with $i = 1, 2, 3$ for the targeted moment categories of each 1) correlations, 2) volatilities, and 3) autocorrelation lags; $j = 1, ..., 5$ represents the four band-pass filtered frequencies (HF, BC, LF, MC) plus the unfiltered data used only for the autocorrelation lags (as in the literature); and $k(i, j, z)$ varies as it represents the number of targets used within each moment category and data frequency, for each of Models 1 and 2.[13]

The calibration and shock construction procedure yield a 400 times and a 10,000 times smaller shock variance for each of the two shocks pertaining to Model 1 and Model 2 respectively, as reported in Table 1, and as compared to the standard RBC 0.007 (King & Rebelo 2000). This indicates improved amplification compared to standard one-sector RBC models.

Parameter	Description	Model 1	Model 2
β	Discount Factor	0.972	0.986
σ	CES Parameter	0.850	0.412
A	Weight of Leisure	1.11	1.10
B	Weight of Capacity Util.	–	−0.159
A_g	Scale Parameter of Goods Sector	1.65.	0.80
A_h	Scale Parameter of Human Sector	0.065	0.032
ϕ_1	Physical Capital Share in Goods Production	0.319	0.36
ϕ_2	Physical Capital Share in Human Investment	0.162	0.20
δ_k	Depreciation Parameter (Physical Capital)	0.018	0.19
ψ	Convexity of Endog. Depr. Rate	–	3.34
δ_h	Depreciation Rate of Human Capital	0.010	0.001
ρ_g	Auto-correlation of TFP	0.98	0.98
ρ_h	Auto-correlation of Human Shock	0.99	0.98
σ_g^2	Variance of TFP	$1.52x10^{-4}$	$9.4x10^{-7}$
σ_h^2	Variance of Human Productivity Shock	$1.47x10^{-4}$	$3.2x10^{-7}$
$\sigma_{g,h}$	Correlation of Shock Innovations	0.994	0.995

Table 1: Model 1 and 2 calibration parameter values.

Parameter	Description	Grid Range	
		Model 1	Model 2
β	Discount Factor	0.95 − 0.99	0.95 − 0.99
σ	CES Parameter	0.40 − 2.00	0.40 − 2.00
A	Weight of Leisure	BGP^*	BGP^*
B	Weight of Capacity Util.	–	BGP^*
A_g	Scale Parameter of Goods Sector	0.50 − 2.00	0.50 − 2.00
A_h	Scale Parameter of Human Sector	BGP^*	BGP^*
ϕ_1	Physical Capital Share in Goods Production	0.30 − 0.40	0.30 − 0.40
ϕ_2	Physical Capital Share in Human Investment	0.08 − 0.29	0.08 − 0.29
δ_k	Depreciation Parameter (Physical Capital)	0.015 − 0.030	BGP^*
ψ	Convexity of Endog. Depr. Rate	–	2.00 − 4.00
δ_h	Depreciation Rate of Human Capital	0.001 − 0.015	0.001 − 0.015
ρ_g	Auto-correlation of TFP	0.01 − 0.99	0.01 − 0.99
ρ_h	Auto-correlation of Human Shock	0.01 − 0.99	0.01 − 0.99
σ_g^2	Variance of TFP	0.007(initial)	0.007(initial)
σ_h^2	Variance of Human Productivity Shock	0.007(initial)	0.007(initial)
$\sigma_{g,h}$	Correlation of Shock Innovations	(−0.99) − 0.99	(−0.99) − 0.99

Table 2: Model 1 and 2 grid search ranges. (* *BGP* refers to calibrated values for parameters obtained through use of BGP conditions).

3.2 Estimated, Backed-out, Shocks

Figures 1 and 2 graph the unfiltered goods sector shock ("TFP") series obtained from Models 1 and 2 along with the respective, traditional, Solow-residual, goods sector TFP.

[13] Alternatively, a 0.99 correlated metric is $D_{alt} = [(\hat{\Theta} - \Theta)/\hat{\Theta}]'\Omega[(\hat{\Theta} - \Theta)/\hat{\Theta}]$, where Ω is an identity matrix of the size of the number of targets k, and D_{alt} is a squared Euclidean distance; D_z in contrast is an average fractional deviation of model from data moments. D_{alt} is of interest as it is a special case of the Mahalanobis (1936) distance.

The goods TFP shock series in Models 1 and 2 are constructed as in Benk et al. (2005) by matching the implicit equilibrium solution for a set of the model's decision variables to the data for each variable in that set. Here for Model 1, the data series consist of six series: the consumption-output ratio, the investment-output ratio, labor hours, ratios of output, consumption, and investment relative to human capital. For Model 2, seven data series are used, with the utilization rate series added to the previous six, for constructing the Model 2 shocks. Each data series used for both models are for the 1972Q1 until 2015Q4 period, given the constriction of the data range that is imposed by the utilization rate data series.

The model backed-out TFP goods sector shocks are also graphed in Figures 1 and 2. The method for backing-out of each of the two Model 1 shocks is to match each of the six chosen data series to each of the six model solutions (of the matching variable, for example, consumption), in terms of the known model parameters, the state variable (k_t/h_t), and the shocks $\{z_t^g,\, z_t^h\}$. This gives six different equations, one for each of the six variables for which we have matching data, in terms of the state variable and the two shocks. We also then use US data for the state variable (k_t/h_t) data series. This leaves six equations in the two unknown shocks $\{z_t^g,\, z_t^h\}$, which is an overidentification of the two shocks. Overidentification has been found advantageous in terms of getting a relatively invariant backed-out shock relative to different combinations of the overidentifying data series that are used, as opposed to using for example any two of the data series alone to exactly identify the shocks. The estimation method for identifying the two backed-out shock series from the six over-identifying equations follows the Benk et al. (2005) method of ordinary least squares to estimate each shock at each time period using the six data points for each time period, for each shock; these six data points come from each of the over-identifying six equations. This is why more data series than two is "better" because this gives a larger "data sample" from which to estimate each shock at each point in time.

The correlations of the Model-generated, backed-out, TFP shock for the goods sector, and the standard, Solow-residual, TFP shock within each Figures 1 and 2 are 0.27 and 0.78 respectively. The high correlation in Model 2 compares to a similar magnitude found in Nolan & Thoenissen (2009), who also back out and compare their TFP model shock to the Solow residual, although using instead a different model that has a financial shock, a money supply growth shock, and a goods sector TFP shock.

Figures 1 and 2 show that the (blue) traditional TFP turns down and continues down right up to the end of 2015. The DSGE model economy-wide shocks (red lines) however fall more abruptly until 2010 and then begin turning upwards. They still lie below trend and indicate the type of slowly recovering "lost decade" that evidence on the US real GDP growth rate seems to confirm. Anemic growth is indeed a focus of much research as to whether we are in a new era of "fundamentally" stagnated growth.

Model 2 appears as the preferred model in the sense that it better tracks the traditional

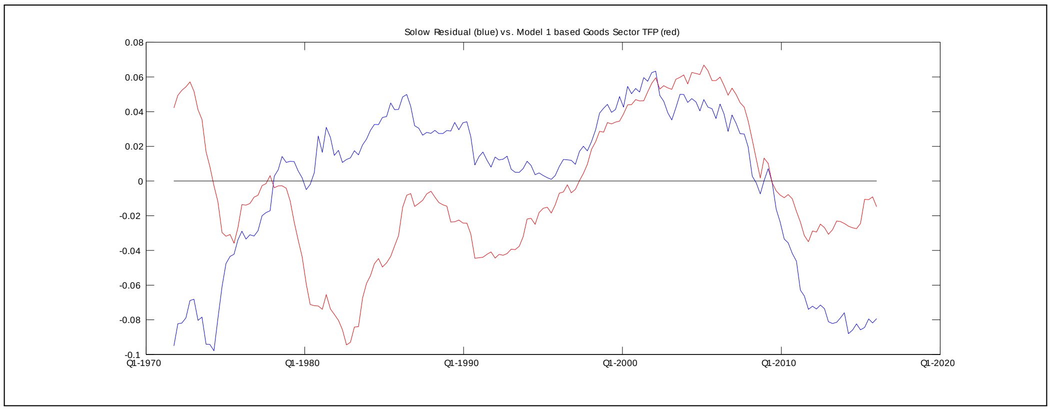

Figure 1: Total Factor Productivity - Model 1 derived (red) versus Solow residual (blue).

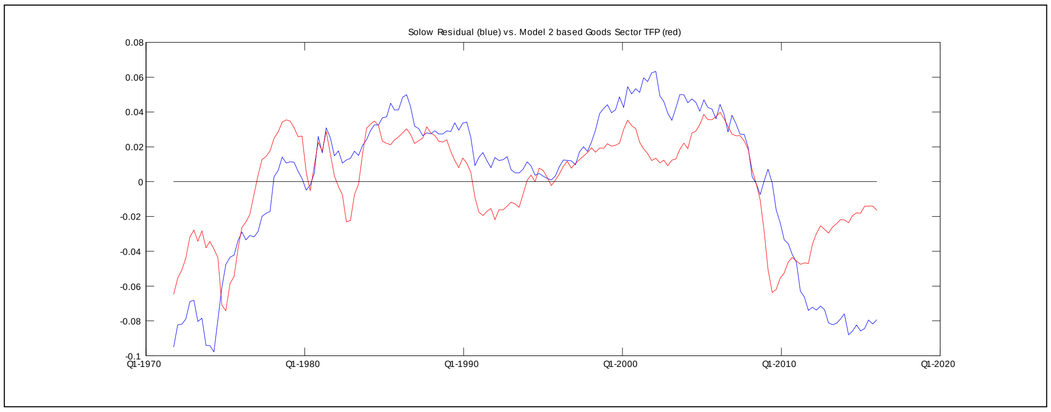

Figure 2: Total Factor Productivity - Model 2 derived (red) versus Solow residual (blue).

TFP shock for the 1972 to mid 1990s than does Model 1. And while the traditionally constructed (blue) TFP shock in contrast shows no recovery post-2010, Model 2 also is attractive in that it seems to better capture the unusual post-2010 period of below-trend recovery . For example Feenstra et al. (2015) show a positive rate of increase in TFP post 2010 but one that is well-below its historical trend, as is consistent with Model 2. [14] Also consider how the data graphed in Figure 3 shows how the median US wage growth fell sharply during the Great Recession and began steadily rising after 2010, similar to the Figure 2 post-2010 period for Model 2 (red).[15]

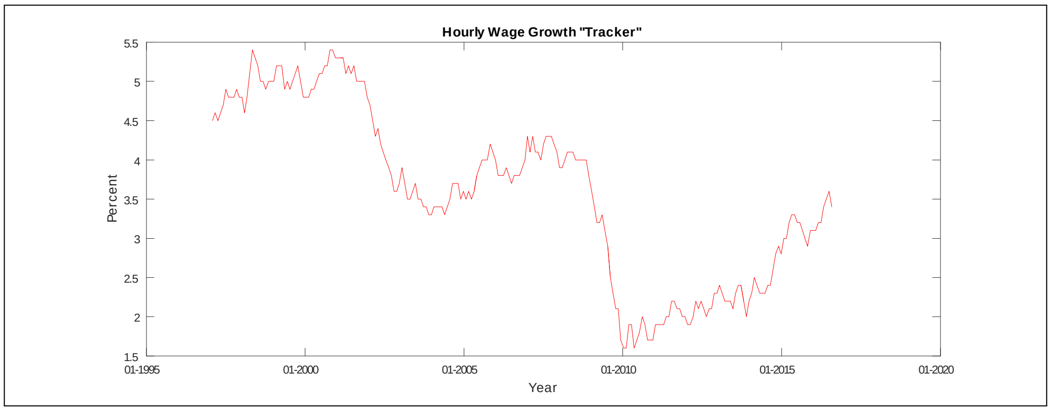

Figure 3: Wage Growth "Tracker": Three-month moving average of median hourly wage growth, 1997:3-2016:7 (Source: Federal Reserve Bank of Atlanta).

Figure 4 and 5 show the model-derived TFP series (red) and the traditional one (blue) decomposed at different frequencies using a Christiano & Fitzgerald (2003) band-pass filter.[16] A seemingly closer match to the Solow TFP by the Model 2 shock can be observed before 1990 for all but the short run frequency by examining three of the four panels: for the business cycle (upper-right), low frequency (lower left) and the Medium Cycle (lower left). Also noteworthy, the Solow residual (blue) turns upwards after 2010 in the business cycle and low frequency, although not in the unfiltered shock of Figures 1 and 2.

3.3 Impulse Responses

Figure 6 shows the impulse responses of labor in the goods sector to a standard one percent simultaneous increase in both goods sector and human capital investment sector productivity, with the shocks correlated as in the calibration (top row). For comparison, the figure

[14] "Total Factor Productivity at Constant National Prices for United States"; sourced from Feenstra et al. (2015); retrieved from FRED, https://fred.stlouisfed.org/series/RTFPNAUSA632NRUG, August 25, 2016.
[15] The Atlanta Fed "Wage Tracker" uses BLS CPS household data, and can act as a proxy of productivity.
[16] The windows for the band-pass filter frequencies are defined in detail in Section 4.

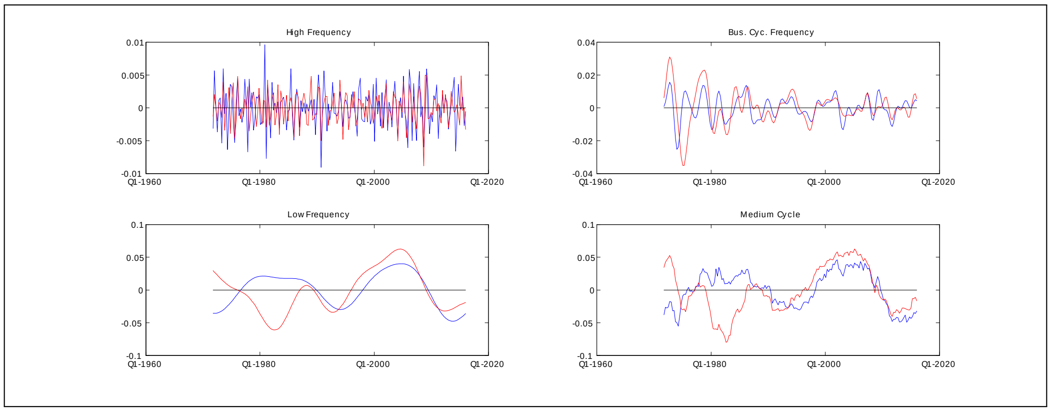

Figure 4: Total Factor Productivity - Model 1 (red) versus Solow Residual (blue).

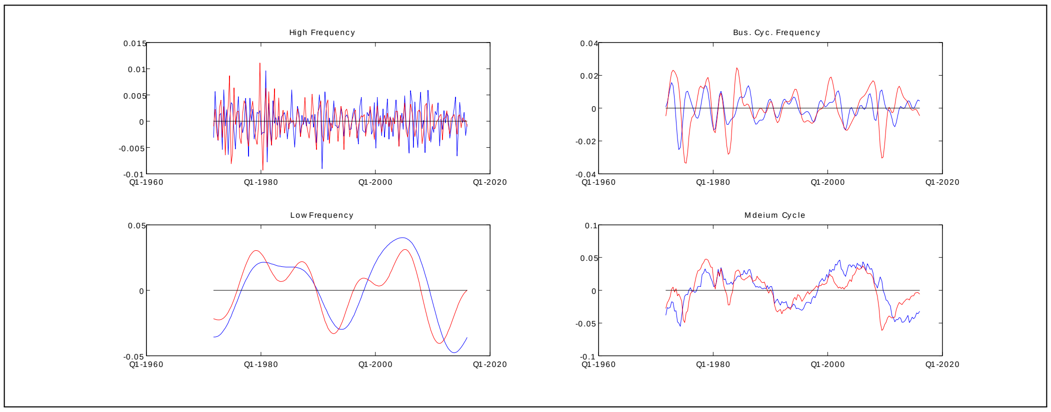

Figure 5: Total Factor Productivity - Model 2 (red) versus Solow Residual (blue).

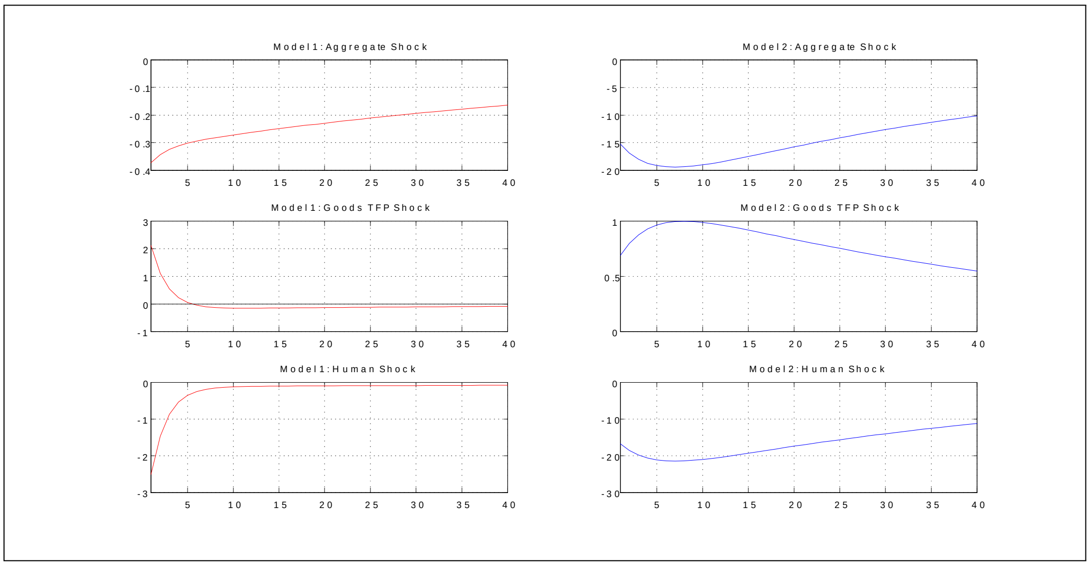

Figure 6: Goods Labor Responses in Model 1 (red) and Model 2 (blue) to an Economy-wide (top row), a goods TFP (middle row), and a human productivity shock (bottom row).

presents an uncorrelated, independent, goods sector TFP shock (middle row), and an uncorrelated, independent, human sector productivity shock (bottom row) for each Model 1 (left) and Model 2 (right). This shows how the model reflects a negative Gali (1999) type goods sector labor impulse response for the correlated equi-proportional economy-wide TFP shock (top row). This occurs even though both Models 1 and 2 have a positive labor impulse as in the standard RBC model (middle row). The negative impulse response results because of the dominating effect of the negative impulse response to labor from the human capital investment sector shock (third row), which occurs from an equi-proportional one percent increase in both sectorial productivity factors with correlation included.[17]

However, in contrast to the equi-proportional shock shown in Figure 6, the shock to the human capital sector for an economy-wide shock will be just a fraction of that of the goods sector during the actual model simulation, due to the lower numeric value of the variance of the human capital shock relative to the goods sector shock. Figure 7 shows the goods sector labor impulse response to a Model 2 economy-wide shock with a 1% goods sector positive increase and a 0.045% human capital sector positive increase, with correlation between the shocks as in the calibration. Labor at first falls in the goods sector and then rises and eventually falls back to zero. Figure 7 demonstrates how labor can at first move into the human capital sector, following Stolper-Samuelson and Rybczynski dynamics as in Section

[17] This result is sensitive to the utility parameter B, and when $B = 0$ the resulting negative labor impulse can turn positive after a few years.

2.2, when the human capital is scarce relative to physical capital, and then flows back to goods sector as the human capital becomes relatively less scarce. Such reallocation enables the model's moments to compare well to data as is reported in the next section.

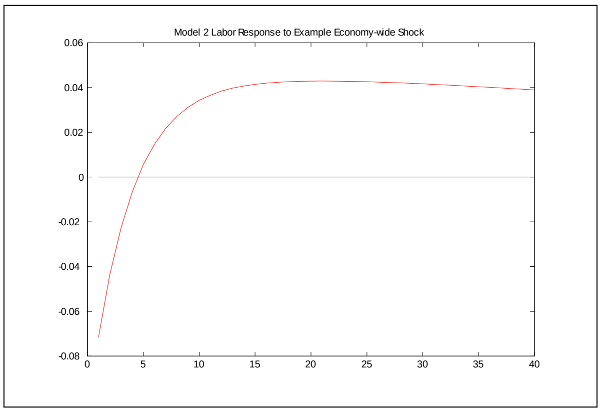

Figure 7: Goods Sector's Labor Response to an Economy-wide shock with a 1% Goods Sector TFP increase and a 0.045% Human Capital Sector productivity increase.

4 Results

Moment results based on model simulations are presented at different frequencies for key correlations with output, own volatilities, and persistence of growth rates. Using a Christiano & Fitzgerald (2003) band-pass filter, the windows are high frequency (HF: 2 - 6 quarters), business cycle frequency (BC: 6 - 32 quarters), low frequency (LF: 32 - 200 quarters), and the Comin & Gertler (2006) 'medium cycle' that combines these frequencies (MC: 2 - 200 quarters). Tables 3 - 5 report moments for US data, Model 1 and Model 2.[18]

4.1 Key Cyclic Correlations

Table 3 shows that the comovement of consumption and investment with output is closely matched by Models 1 and 2 at the business cycle frequency. Both models are able to capture a positive correlation between labor hours and output as suggested by US data at the business cycle, low frequency, and the Comin & Gertler (2006) Medium Cycle, with Model 2 closer to the data. Both models capture the positive business cycle correlation between labor hours and consumption, unlike the standard RBC model. Both models generate a

[18] Appendix C sets out the equivalence of the approach to that of Restrepo-Ochoa & Vazquez (2004); King et al. (1988) focuses on correlations of Great Ratios with output and relative volatilities of the growth rate of consumption and investment with output, which here are presented in terms of correlations between consumption and investment with output and by standard deviations of the growth rate of these key variables.

strong negative theoretical correlation between human capital investment time hours and output as suggested in DeJong et al. (1996), and as consistent with certain limited evidence. Model 2 is also able to capture the positive correlation of physical capital utilization rate and output at the business cycle frequency and the lower frequencies, although doing best at the business cycle.[19]

Variable		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$corr(c_t, y_t)$	Data	0.475	0.891	0.980	0.963
	Model 1	0.893	0.776	0.931	0.927
	Model 2	0.989	0.928	0.856	0.837
$corr(i_{kt}, y_t)$	Data	0.809	0.939	0.834	0.833
	Model 1	0.784	0.841	0.691	0.696
	Model 2	0.997	0.991	0.936	0.939
$corr(l_{gt}, y_t)$	Data	0.394	0.732	0.589	0.595
	Model 1	-0.196	0.200	0.027	0.036
	Model 2	-0.141	0.874	0.823	0.819
$corr(l_{ht}, y_t)$	Data	-	-	-	-
	Model 1	0.214	-0.016	0.131	0.111
	Model 2	0.119	-0.891	-0.833	-0.827
$corr(u_t, y_t)$	Data	0.432	0.797	0.447	0.483
	Model 1	-	-	-	-
	Model 2	0.001	0.926	0.871	0.819
$corr(c_t, l_{gt})$	Data	0.206	0.766	0.592	0.596
	Model 1	-0.077	0.672	0.362	0.319
	Model 2	-0.229	0.651	0.378	0.383

Table 3: Matching Correlations (US Data 1959Q1-2015Q4, Model 1 & 2).

4.2 Volatilities

Tables 4 show that the volatility moments of the data are captured relatively well with Model 1 being better in some cases and Model 2 in others. For example, both models are very close to the data for output growth volatility. Only the physical capacity utilization rate is too low by a full order of magnitude, this being for Model 2.

[19] Please see Appendix D.4 for a description of the simulation methodology.

Variable		High freq. 2 -6 qrs.	Bus. cyc. 6 - 32 qrs.	Low freq. 32 - 200 qrs.	Med. term 2 - 200 qrs.
$vol(g_{y,t})$	Data	0.0068	0.0064	0.0038	0.0100
	Model 1	0.0047	0.0037	0.0034	0.0068
	Model 2	0.0050	0.0043	0.0034	0.0074
$vol(g_{c,t})$	Data	0.0038	0.0036	0.0029	0.0059
	Model 1	0.0036	0.0031	0.0069	0.0079
	Model 2	0.0022	0.0017	0.0015	0.0031
$vol(g_{i_k,t})$	Data	0.0200	0.0207	0.0105	0.0302
	Model 1	0.0160	0.0150	0.0260	0.0330
	Model 2	0.0120	0.0110	0.0091	0.0190
$vol(y_t)$	Data	0.0044	0.0166	0.0469	0.0500
	Model 1	0.0034	0.0100	0.0590	0.0600
	Model 2	0.0033	0.0100	0.0360	0.0380
$vol(c_t)$	Data	0.0024	0.0097	0.0382	0.0396
	Model 1	0.0028	0.0068	0.0550	0.0550
	Model 2	0.0015	0.0038	0.0200	0.0200
$vol(i_{kt})$	Data	0.0129	0.0540	0.0912	0.1076
	Model 1	0.0110	0.0420	0.1500	0.1500
	Model 2	0.0081	0.0290	0.0910	0.0960
$vol(l_{gt})$	Data	0.0017	0.0049	0.0221	0.0227
	Model 1	0.0070	0.0112	0.0090	0.0158
	Model 2	0.0037	0.0120	0.0350	0.0370
$vol(u_t)$	Data	0.0055	0.0254	0.0318	0.0420
	Model 1	-	-	-	-
	Model 2	0.0011	0.0023	0.0039	0.0047

Table 4: Matching Volatilities (US Data 1959Q1-2015Q4, Model 1 & 2).

4.3 Persistence

Table 5 shows both Model 1 and 2 persistence through the autocorrelation profile $\rho(\cdot)$ of the unfiltered simulated model data versus the unfiltered actual data, following Benhabib et al. (2006) and the focus of Cogley & Nason (1995), for output growth, consumption growth, and physical capital investment growth, plus the goods sector labor and the physical capital capacity utilization rate. Model 1 tends to get the initial level of growth persistence, but not the autocorrelated drop-off. Model 2 better captures both the level and the drop-off across the four data autocorrelations with three lags, as can be viewed graphically as well.

Figure 8 graphs the same actual data autocorrelations and the simulated Model 1 and Model 2 autocorrelations with extension to 16 lags for A: output growth, B: consumption growth, C: physical capital investment growth, and D: goods sector labor. In contrast, traditional RBC models fail to reproduce the output growth persistence beyond the first lag, as pointed out by Benhabib et al. (2006).

Variable		Lag 1	Lag 2	Lag 3
$\rho(g_{y,t})$	Data	0.270	0.216	0.160
	Model 1	0.636	0.605	0.596
	Model 2	0.271	0.220	0.188
$\rho(g_{c,t})$	Data	0.369	0.284	0.305
	Model 1	0.631	0.610	0.608
	Model 2	0.380	0.361	0.347
$\rho(g_{ik,t})$	Data	0.264	0.177	0.082
	Model 1	0.329	0.265	0.225
	Model 2	0.282	0.213	0.170
$\rho(l_{gt})$	Data	0.987	0.975	0.962
	Model 1	0.956	0.917	0.883
	Model 2	0.993	0.983	0.971
$\rho(u_t)$	Data	0.956	0.863	0.751
	Model 1	-	-	-
	Model 2	0.956	0.919	0.887

Table 5: Simulated Autocorrelation Functions vs. Data (US Data 1959Q1 - 2015Q4).

4.4 Summary Using Metrics

Besides its use in the calibration choice, the other advantage of the distance metric is that it represents the average percentage point deviation of the simulated moments from the US data-based moments. Therefore, it allows one not only to calibrate but to also to evaluate

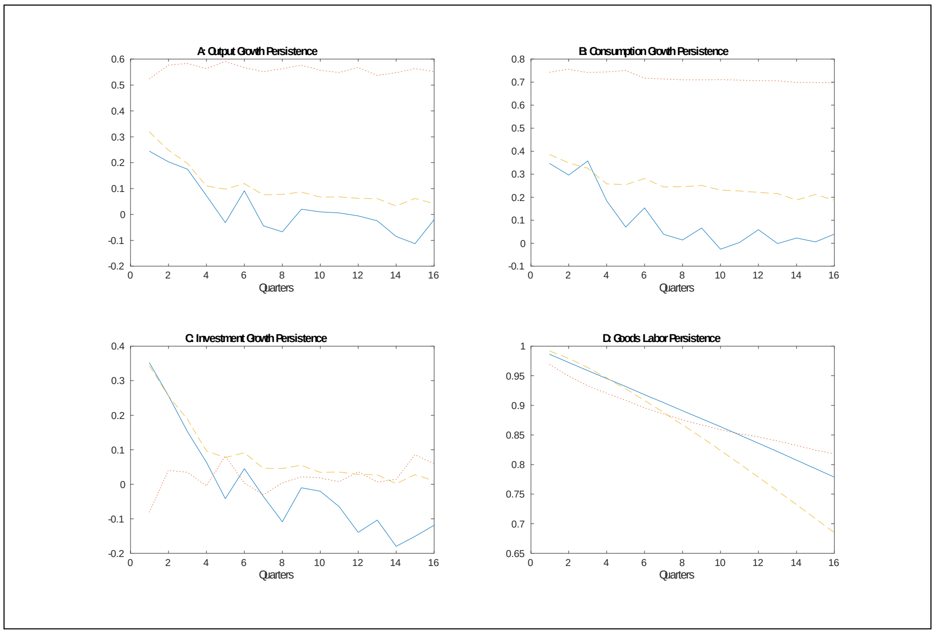

Figure 8: Autocorrelation profiles of variables for 15 quarters: US data-based, 1972Q1-2015Q4, solid blue line; Model 1 simulated data, dotted red line; Model 2 simulated data, dashed yellow line.

and compare the performances of different DSGE models relative to the data across all moments and across subsets of moments. Table 6 presents a set of moment results for the "Overall" set of moment comparisons and for subsets of each Model 1 and Model 2. Model 2 has more moments included because of capacity utilization moments.

Table 6A shows the "Overall" average metric across all moments that are reported in Tables 3, 4 and 5, except for model moments for human capital investment time which were not matched to data other than the qualitative property of being countercyclic. Adding up these moments for Model 1, there are 16 Correlation moments, 28 Volatility moments, and 12 (unfiltered) Persistence ** moments for a total of 56 targeted moments, as all of these reported model moments are targeted. Adding up the metric for each target and dividing by the number of targets gives the corresponding average metrics of 0.50 for the 16 Correlations targets, 0.53 for the 28 Volatilities targets, and 0.95 for the 12 Persistence** targets. Overall, for the 56 targets, the average Model 1 metric is 0.59. For Model 2, with the added capacity utilization moments, there are 20 Correlation moments, 32 Volatility moments, and 15 (unfiltered) Persistence ** moments for a total of 67 Overall, with corresponding average metrics respectively of 0.50, 0.51, 0.15, and 0.46.

The remaining row of Tables 6A is Persistence *, which shows the average metric of filtered data within each of the four windows of HF, BC, LF, and MC (detailed results here

TABLE 6A	Average Metric Across All Moments	
	Model 1	Model 2
Overall	0.59	0.46
Correlations	0.50	0.50
Volatilities	0.53	0.51
Persistence*	0.73	0.38
Persistence**	0.95	0.15

TABLE 6B	Average Metric Across Four Frequencies							
	Model 1 HF	Model 2 HF	Model 1 BC	Model 2 BC	Model 1 LF	Model 2 LF	Model 1 MC	Model 2 MC
Correlations	0.95	1.15	0.27	0.15	0.39	0.39	0.40	0.33
Volatilities	0.60	0.51	0.43	0.64	0.70	0.36	0.38	0.51
Persistence*	0.41	0.75	1.91	0.42	0.02	0.08	0.59	0.27

Table 6: Model 1 and 2 percentage deviation based metric for moments.

not reported). This is included in the Table (but not used to calculate the Overall metric) as an alternative measure of persistence to that of using the unfiltered data as is the focus of the literature. For this alternative, filtered, Persistence*, there are 3 autocorrelation lags for each of the 4 growth series of Model 1 (output, consumption, investment, and utilization rate). This gives 4x3=12 moment metrics within each of the four frequencies, for a total of 12x4=48 moment metrics; when all 48 are added together and divided by 48, the average metric of 0.73 results for Model 1's filtered Persistence* metric. For Model 2, also included is the capacity utilization growth with 3 lags, so as to give 5x3=15 moments that are averaged within each of the four frequencies for Model 2; this results in a total of 60 targets with an average metric of 0.38 for Model 2.

Table 6B then breaks the results down by frequency. It shows the average metric for the categories of Correlation, Volatilities, and Persistence *, each by frequency, with 12 moment metrics averaged within each frequency for Model 1 and 15 moment metrics averaged within each frequency for Model 2.

In Table 6A, Model 2 has a lower average distance metric "Overall" and for Correlations and Persistence, both unfiltered (**) and filtered (*). Here the 15% average deviation for unfiltered Persistence** is quite low. In Table 6B, noteworthy for Model 2 is a 15%, 39% and 33% average deviation of Correlation metrics in the BC, LF and MC windows, respectively, and a 36% average deviation of Volatilities in the LF window frequency. Model 2 also has lower average deviations than Model 1 of the filtered Persistence* in the BC, LF and MC windows including a 8% deviation in the LF window, while Model 1 also shows a very low 2% average deviation of filtered LF Persistence*. Model 2 then does well on Persistence Overall for both filtered and unfiltered data.[20]

[20] Model 1 and 2 were extended with a government sector and corresponding shock, in the fashion of Chari et al. (2007), but did not improve overall on the performance in terms of the distance metric; for example the government model was marginally better in BC volatilities but worse in capturing BC and LF correlations and the autocorrelation profiles of growth rates in Figure 8.

5 Conclusion

This paper shows that the model can match traditional RBC data moments for correlations, volatilities and output growth persistence, even when including the data of the Great Recession and the post-2010 below-trend recovery. In addition, physical capital utilization rate moments, human capital time's countercyclical movement, and the level and autocorrelation shape of output growth persistence are reasonably matched. The magnitude of the variance of the Model 2 economy-wide shock is more than 10000 fold smaller than the traditional RBC model, indicating strong internal propagation. The general model's resource allocation during simulation is explained using in part the demonstrated general equilibrium duality of the Stolper & Samuelson (1941) and Rybczynski (1955) theorems.

The model's backed out economy-wide productivity shock rises at a below trend rate post 2010, unlike the traditionally constructed TFP shock, but similar to Feenstra et al. (2015). The labor wedge of Chari et al. (2007) is explained in Appendix C in a manner similar McGrattan (2015). The paper presents results that extend the explanation of a broad array of moments from both the "growth cycle" spectra and the business cycle frequency.[21]

Future research includes estimating confidence intervals for the calibration methodology building on the Simulated Method of Moments (SMM) literature. Better explaining the volatility of the capacity utilization rate of physical capital and the equity premium are left for future research. Explaining the equity premium within the human capital model remains a serious challenge for reasons Li (2000) presents, but which conceivably could be addressed by accounting better for this capital using the direction given in McGrattan & Prescott (2014).

References

Baier, S., Mulholland, S., Turner, C. & Tamura, R. (2004), Income and education of the states of the United States: 1840–2000, Working Paper 2004-31, Federal Reserve Bank of Atlanta.

Barro, R. J. & Lee, J. W. (2013), 'A new data set of educational attainment in the world, 1950–2010', *Journal of Development Economics* **104**(C), 184–198.

Basu, P., Gillman, M. & Pearlman, J. (2012), 'Inflation, human capital and Tobin's q', *Journal of Economic Dynamics and Control* **36**(7), 1057–1074.

Benhabib, J., Perli, R. & Sakellaris, P. (2006), 'Persistence of business cycles in multisector real business cycle models', *International Journal of Economic Theory* **2**(3-4), 181–197.

Benhabib, J., Rogerson, R. & Wright, R. (1991), 'Homework in Macroeconomics: Household Production and Aggregate Fluctuations', *Journal of Political Economy* **99**(6), 1166–87.

Benhabib, J. & Wen, Y. (2004), 'Indeterminacy, aggregate demand, and the real business cycle', *Journal of Monetary Economics* **51**(3), 503–530.

[21] All Matlab codes are available with description in the Online Appendix.

Benk, S., Gillman, M. & Kejak, M. (2005), 'Credit Shocks in the Financial Deregulatory Era: Not the Usual Suspects', *Review of Economic Dynamics* **8**(3), 668–687.

Benk, S., Gillman, M. & Kejak, M. (2008), 'Money Velocity in an Endogenous Growth Business Cycle with Credit Shocks', *Journal of Money, Credit and Banking* **40**(6), 1281–1293.

Benk, S., Gillman, M. & Kejak, M. (2010), 'A banking explanation of the US velocity of money: 1919-2004', *Journal of Economic Dynamics and Control* **34**(4), 765–779.

Blanchard, O. J. & Quah, D. (1989), 'The Dynamic Effects of Aggregate Demand and Supply Disturbances', *American Economic Review* **79**(4), 655–73.

Bond, E. W., Wang, P. & Yip, C. K. (1996), 'A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics', *Journal of Economic Theory* **68**(1), 149–173.

Chari, V. V., Kehoe, P. J. & McGrattan, E. R. (2007), 'Business Cycle Accounting', *Econometrica* **75**(3), 781–836.

Christiano, L. J., Boldrin, M. & Fisher, J. D. M. (2001), 'Habit Persistence, Asset Returns, and the Business Cycle', *American Economic Review* **91**(1), 149–166.

Christiano, L. J. & Fitzgerald, T. J. (2003), 'The Band Pass Filter', *International Economic Review* **44**(2), 435–465.

Cogley, T. & Nason, J. M. (1995), 'Output Dynamics in Real-Business-Cycle Models', *American Economic Review* **85**(3), 492–511.

Comin, D. & Gertler, M. (2006), 'Medium-Term Business Cycles', *American Economic Review* **96**(3), 523–551.

DeJong, D. N. & Ingram, B. F. (2001), 'The Cyclical Behavior of Skill Acquisition', *Review of Economic Dynamics* **4**(3), 536–561.

DeJong, D. N., Ingram, B. F., Wen, Y. & Whiteman, C. H. (1996), Cyclical Implications of the Variable Utilization of Physical and Human Capital, Macroeconomics 9609004, EconWPA.

Dellas, H. & Sakellaris, P. (2003), 'On the cyclicality of schooling: theory and evidence', *Oxford Economic Papers* **55**(1), 148–172.

Feenstra, R. C., Inklaar, R. & Timmer, M. (2013), The Next Generation of the Penn World Table, NBER Working Papers 19255, National Bureau of Economic Research, Inc.

Feenstra, R. C., Inklaar, R. & Timmer, M. (2015), 'The Next Generation of the Penn World Table', *American Economic Review* **105**(10), 3150–3182.

Gali, J. (1999), 'Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?', *American Economic Review* **89**(1), 249–271.

Gomme, P. & Rupert, P. (2007), 'Theory, measurement and calibration of macroeconomic models', *Journal of Monetary Economics* **54**(2), 460–497.

Greene, W. H. (2003), *Econometric Analysis*, 5 edn, Prentice Hall, Upper Saddle River, NJ.

Greenwood, J. & Hercowitz, Z. (1991), 'The Allocation of Capital and Time over the Business Cycle', *Journal of Political Economy* **99**(6), 1188–214.

Greenwood, J., Hercowitz, Z. & Huffman, G. W. (1988), 'Investment, Capacity Utilization, and the Real Business Cycle', *American Economic Review* **78**(3), 402–17.

Grossman, G. M., Helpman, E., Oberfield, E. & Sampson, T. (2016), Balanced Growth Despite Uzawa, NBER Working Papers 21861, National Bureau of Economic Research, Inc.

Hall, R. E. (1988), 'Intertemporal Substitution in Consumption', *Journal of Political Economy* **96**(2), 339–357.

Hansen, G. D. (1985), 'Indivisible Labor and the Business Cycle', *Journal of Monetary Economics* **16**, 309–327.

Ingram, B. F., Kocherlakota, N. R. & Savin, N. E. (1997), 'Using theory for measurement: An analysis of the cyclical behavior of home production', *Journal of Monetary Economics* **40**(3), 435–456.

Jermann, U. J. (1998), 'Asset pricing in production economies', *Journal of Monetary Economics* **41**(2), 257–275.

Jones, L. E., Manuelli, R. E. & Siu, H. E. (2005), 'Fluctuations in Convex Models of Endogenous Growth II: Business Cycle Properties', *Review of Economic Dynamics* **8**(4), 805–828.

Jorgenson, D. & Fraumeni, B. (1991), The Output Of The Education Sector, Harvard Institute of Economic Research Working Papers 1543, Harvard - Institute of Economic Research.

King, R. G., Plosser, C. I. & Rebelo, S. T. (1988), 'Production, growth and business cycles : I. The basic neoclassical model', *Journal of Monetary Economics* **21**(2-3), 195–232.

King, R. G. & Rebelo, S. T. (2000), Resuscitating Real Business Cycles, NBER Working Papers 7534, National Bureau of Economic Research, Inc.

Kydland, F. E. & Prescott, E. C. (1982), 'Time to Build and Aggregate Fluctuations', *Econometrica* **50**(6), 1345–70.

Li, V. E. (2000), 'Household Credit and the Monetary Transmission Mechanism', *Journal of Money, Credit and Banking* **32**(3), 335–356.

Maffezzoli, M. (2000), 'Human Capital and International Real Business Cycles', *Review of Economic Dynamics* **3**(1), 137–165.

Mahalanobis, P. C. (1936), 'On the generalised distance in statistics', *Proceedings of the National Institute of Sciences of India* **2**(1), 49–55.

McGrattan, E. R. (2015), Intangible Capital and Measured Productivity, manuscript, University of Minnesota, June.

McGrattan, E. R. & Prescott, E. C. (2014), 'A Reassessment of Real Business Cycle Theory', *American Economic Review* **104**(5), 177–82.

Mehra, R. & Prescott, E. C. (1985), 'The equity premium: A puzzle', *Journal of Monetary Economics* **15**(2), 145–161.

Mulligan, C. B. & Sala-i Martin, X. (1993), 'Transitional Dynamics in Two-Sector Models of Endogenous Growth', *The Quarterly Journal of Economics* **108**(3), 739–73.

Nelson, C. R. & Plosser, C. I. (1982), 'Trends and random walks in macroeconmic time series : Some evidence and implications', *Journal of Monetary Economics* **10**(2), 139–162.

Nolan, C. & Thoenissen, C. (2009), 'Financial shocks and the US business cycle', *Journal of Monetary Economics* **56**(4), 596–604.

Otani, K. (1996), 'A Human Capital Approach to Entrepreneurial Capacity', *Economica* **63**(250), 273–289.

Perli, R. (1998), 'Increasing returns, home production and persistence of business cycles', *Journal of Economic Dynamics and Control* **22**(4), 519–543.

Perli, R. & Sakellaris, P. (1998), 'Human capital formation and business cycle persistence', *Journal of Monetary Economics* **42**(1), 67–92.

Psacharopoulos, G. (1994), 'Returns to investment in education: A global update', *World Development* **22**(9), 1325–1343.

Restrepo-Ochoa, S. I. & Vazquez, J. (2004), 'Cyclical features of the Uzawa-Lucas endogenous growth model', *Economic Modelling* **21**(2), 285–322.

Rogerson, R. (1988), 'Indivisible Labor, Lotteries and Equilibrium', *Journal of Monetary Economics* **21**, 3–16.

Rotemberg, J. J. & Woodford, M. (1996), 'Real-Business-Cycle Models and the Forecastable Movements in Output, Hours, and Consumption', *American Economic Review* **86**(1), 71–89.

Rybczynski, T. (1955), 'Factor Endowment and Relative Commodity Prices', *Economica* **22**(88), 336–341.

Stolper, W. & Samuelson, P. (1941), 'Protection and real wages', *Review of Economic Studies* **9**(1), 58–73.

Uhlig, H. (1998), A toolkit for analyzing nonlinear dynamic stochastic models easily, Discussion paper / institute for empirical macroeconomics, QM&RBC Codes, Quantitative Macroeconomics & Real Business Cycles.

Van Long, N. (1992), 'The Dynamic Rybczynski Theorem and its Dual', *Journal of Economic Integration* **7**(2), 137–150.

Appendices

A Equilibrium Condtions and the Steady State

A.1 Equilibrium Conditions

Define the Lagrange multiplier of the representative agent's budget constraint as λ_t, and that of the human capital accumulation's as χ_t. Then agent's first order conditions are the following,

$$c_t: \quad c_t^{-\sigma} x_t^{A(1-\sigma)} (1-u_t)^{B(1-\sigma)} = \lambda_t; \tag{18}$$

$$l_{gt}: \quad A c_t^{1-\sigma} x_t^{A(1-\sigma)-1} (1-u_t)^{B(1-\sigma)} = \lambda_t w_t h_t; \tag{19}$$

$$l_{ht}: \quad A c_t^{1-\sigma} x_t^{A(1-\sigma)-1} (1-u_t)^{B(1-\sigma)} = \chi_t (1-\phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} h_t; \tag{20}$$

$$\begin{aligned} u_t: \quad & B c_t^{1-\sigma} x_t^{A(1-\sigma)} (1-u_t)^{B(1-\sigma)} = \lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (v_{gt} k_t) + \\ & + \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (v_{ht} k_t) - \lambda_t \delta_k u_t^{\psi - 1} k_t; \end{aligned} \tag{21}$$

$$v_{gt}: \quad \lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (u_t k_t) = \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{(1-v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (u_t k_t); \tag{22}$$

$$\begin{aligned} k_{t+1}: \quad \lambda_t = {} & \beta E_t \lambda_{t+1} \left[1 + r_{t+1}^k u_{t+1} v_{gt+1} - \frac{\delta_k}{\psi} u_{t+1}^{\psi}\right] + \\ & + \beta E_t \chi_{t+1} \phi_2 A_h e^{z_{t+1}^h} \left[\frac{v_{ht+1} u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}}\right]^{\phi_2 - 1} (u_{t+1} v_{ht+1}); \end{aligned} \tag{23}$$

$$h_{t+1}: \quad \chi_t = \beta E_t \chi_{t+1} \left[1 + (1-\phi_2) A_h e^{z_{t+1}^h} \left[\frac{v_{ht+1} u_{t+1} k_{t+1}}{l_{ht+1} h_{t+1}}\right]^{\phi_2} l_{ht+1} - \delta_h\right] + \beta E_t \lambda_{t+1} w_{t+1} l_{gt+1};$$

$$(24)$$

where r_t^k and w_t denote the own marginal productivity conditions of physical and human capital such that $r_t^k \equiv \phi_1 A_g e^{z_t^g}(v_{gt}u_t k_t)^{\phi_1 - 1}(l_{gt}h_t)^{1-\phi_1}$ and $w_t \equiv (1-\phi_1)A_g e^{z_t^h}(v_{gt}u_t k_t)^{\phi_1}(l_{gt}h_t)^{-\phi_1}$. Also, $p_{ht} \equiv \frac{\chi_t}{\lambda_t}$ denotes the relative price of human capital in terms of consumption goods. Then the representative agent's equilibrium conditions can be stated as,

$$A_g e^{z_t^g}(v_{gt}u_t k_t)^{\phi_1}(l_{gt}h_t)^{1-\phi_1} = c_t + i_{kt}; \tag{25}$$

$$A_h e^{z_t^h}((1-v_{gt})u_t k_t)^{\phi_2}(l_{ht}h_t)^{1-\phi_2} = h_{t+1} - (1-\delta_h)h_t; \tag{26}$$

$$p_{ht} = \left[\frac{A_g}{A_h}\right]\left[\frac{e^{z_t^g}}{e^{z_t^h}}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2}\left[\frac{v_{gt}u_t k_t}{l_{gt}h_t}\right]^{\phi_1-\phi_2}; \tag{27}$$

$$\frac{A}{x_t}\frac{c_t}{h_t} = w_t; \tag{28}$$

$$\frac{B}{(1-u_t)}\frac{c_t}{k_t} = r_t - \delta_k u_t^{\psi-1}; \tag{29}$$

$$x_t = 1 - l_{gt} - l_{ht}; \tag{30}$$

$$\frac{1-\phi_1}{\phi_1}\frac{v_{gt}u_t k_t}{l_{gt}h_t} = \frac{1-\phi_2}{\phi_2}\frac{(1-v_{gt})u_t k_t}{l_{ht}h_t}; \tag{31}$$

$$i_{kt} = k_{t+1} - k_t + \frac{\delta_k}{\psi}u_t^{\psi}k_t; \tag{32}$$

$$1 = \beta E_t \left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)} \left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)} \left[1 + r_{t+1}u_{t+1} - \frac{\delta_k}{\psi} u_{t+1}^{\psi} \right] \right]; \quad (33)$$

$$\begin{aligned} 1 = \beta E_t \left(\frac{c_t}{c_{t+1}}\right) \left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)} \left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)} \frac{p_{ht+1}}{p_{ht}} \\ \left[1 + (l_{gt+1} + l_{ht+1})(1-\phi_2) A_h e^{z_t^h} \left[\frac{(1-v_{gt+1})u_{t+1}k_{t+1}}{l_{ht+1}h_{t+1}}\right]^{\phi_2} - \delta_h \right]. \end{aligned} \quad (34)$$

Equation (25) is the goods market clearing condition; equation (26) is the human capital law of motion; equation (27) defines the relative price of human capital in units of consumption goods; equation (28) is the intratemporal condition that governs the substitution between leisure and consumption; meanwhile, equation (29) is the second intratemporal condition governing the substitution between managerial capacity and consumption. Equation (30) is the time constraint; equation (31) equates weighted factor intensities across sectors; and (32) is the physical capital law of motion. Equations (33) and (34) are the inter-temporal capital efficiency conditions with respect to physical and human capital, where the capacity utilization of physical capital is the equivalent of used entrepreneurial capacity, u_t, and the capacity utilization of human capital is equivalent to total working time, $(1 - x_t)$.

The set of 10 equations in (25) - (34) and the marginal efficiency conditions fully describe Model 2. Altogether, there are 12 equations in 12 unknowns $\{k_{t+1}, h_{t+1}, i_{kt}, c_t, u_t, l_{gt}, l_{ht}, x_t, v_{gt}, p_{ht}, r_t^k, w_t\}$. Furthermore, the exogenous variables $\{z_t^g, z_t^h\}$ are governed by the AR(1) processes defined in equations (**??**), and (9).[22]

A.2 The Steady State

First, express the first order conditions for Model 2 in Appendix A.1 in terms of the variables' long-run values. Then the first order conditions become:

$$A_g(v_g uk)^{\phi_1}(l_g h)^{1-\phi_1} = c + i_k; \quad (35)$$

$$A_h((1-v_g)uk)^{\phi_2}(l_h h)^{1-\phi_2} = h(1+g) - (1-\delta_h)h; \quad (36)$$

[22] Model 1 equilibrium conditions are identical except for being stripped off of the utilization rate and there is no second intratemporal condition as in (29).

$$p_h = \left[\frac{A_g}{A_h}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2}\left[\frac{v_g uk}{l_g h}\right]^{\phi_1-\phi_2}; \tag{37}$$

$$\frac{A}{x}\frac{c}{h} = w; \tag{38}$$

$$\frac{B}{(1-u)}\frac{c}{k} = r - \delta_k u^{\psi-1}; \tag{39}$$

$$x = 1 - l_g - l_h; \tag{40}$$

$$\frac{1-\phi_1}{\phi_1}\frac{v_g uk}{l_g h} = \frac{1-\phi_2}{\phi_2}\frac{(1-v_g)uk}{l_h h}; \tag{41}$$

$$i_k = k(1+g) - k + \frac{\delta_k}{\psi}u^{\psi}k; \tag{42}$$

$$(1+g)^{\sigma} = \beta\left[1 + r^k u - \frac{\delta_k}{\psi}u^{\psi}\right]; \tag{43}$$

$$(1+g)^{\sigma} = \beta\left[1 + (l_g + l_h)(1-\phi_2)A_h\left[\frac{(1-v_g)uk}{l_h h}\right]^{\phi_2} - \delta_h\right]; \tag{44}$$

where $1+g$ is the gross balanced growth rate of the economy. Define $f_g \equiv \frac{v_g uk}{l_g h}$ and $f_h \equiv \frac{(1-v_g)uk}{l_h h}$. Then the above system can be narrowed down to 8 equations in 8 unknowns, $\{f_g,\ f_h,\ g,\ u,\ \frac{c}{k},\ l_g,\ l_h,\ p_h\}$, by also using the definitions of $r_t^k = \phi_1 A_g f_g^{\phi_1-1}$ and $w_t = (1-\phi_1)A_g f_g^{\phi_1}$, as,

$$A_g f_g^{\phi_1-1}u\left[\frac{l_g f_g}{l_g f_g + l_h f_h}\right] = \frac{c}{k} + g + \frac{\delta_k}{\psi}u^{\psi}, \tag{45}$$

$$A_h f_h^{\phi_2} l_h = g + \delta_h; \tag{46}$$

$$p_h = \left[\frac{A_g}{A_h}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2} f_g^{\phi_1-\phi_2}; \tag{47}$$

$$\frac{A}{x}\frac{c}{k}\left[\frac{l_g f_g + l_h f_h}{u}\right] = (1-\phi_1)A_g f_g^{\phi_1}; \tag{48}$$

$$\frac{B}{(1-u)}\frac{c}{k} = \phi_1 A_g f_g^{\phi_1-1} - \delta_k u^{\psi-1}; \tag{49}$$

$$\frac{1-\phi_1}{\phi_1} f_g = \frac{1-\phi_2}{\phi_2} f_h; \tag{50}$$

$$(1+g)^\sigma = \beta\left[1 + \phi_1 A_g f_g^{\phi_1-1} u - \frac{\delta_k}{\psi} u^\psi\right]; \tag{51}$$

$$(1+g)^\sigma = \beta\left[1 + (l_g + l_h)(1-\phi_2)A_h f_h^{\phi_2} - \delta_h\right]. \tag{52}$$

Given the exogenous information set of parameters (ϕ_1, ϕ_2, A_g, A_h, δ_k, ψ, δ_h, β, σ, A, B), the uniqueness of the solution to the system in (45) - (52) can be narrowed down to the uniqueness of the variables g and u. In order to show this, one can solve for f_g, f_h, $\frac{c}{k}$, l_g, l_h, p_h in terms of g and u, which leaves a system of two equations, (48) and (49), in two unknowns g and u. First, one may solve for f_g using (51) as,

$$f_g = \left[\frac{\frac{(1+g)^\sigma}{\beta} - 1 + \frac{\delta_k}{\psi}u^\psi}{\phi_1 A_g}\right]^{\frac{1}{\phi_1-1}u}. \tag{53}$$

Then f_h directly follows from (50),

$$f_h = \frac{1-\phi_1}{1-\phi_2}\frac{\phi_2}{\phi_1} f_g. \tag{54}$$

Then one can express total labor time $(l_g + l_h) \equiv D$ from equation (52):

$$D = (l_g + l_h) = \left[\frac{\frac{(1+g)^\sigma}{\beta} - 1 + \delta_h}{(1-\phi_2)A_h f_h^{\phi_2}}\right]. \tag{55}$$

To express the time shares one can express l_h in terms of g and u from equation (46) and then use the solution for total labor time in equation (55),

$$l_h = \left[\frac{g+\delta_h}{A_h}\right] f_h^{-\phi_2}, \tag{56}$$

$$l_g = D - l_h. \tag{57}$$

Next by using equation (47) and the obtained expression for f_g it follows that the relative price of human capital in terms of g and u is,

$$p_h = \left[\frac{A_g}{A_h}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2} f_g^{\phi_1-\phi_2}. \tag{58}$$

Now one can obtain an expression in g and u for c/k from equation (45),

$$\frac{c}{k} = A_g f_g^{\phi_1 - 1}\left[\frac{l_g f_g}{l_g f_g + l_h f_h}\right] - g - \frac{\delta_k}{\psi} u^\psi. \tag{59}$$

Then after substituting (53) - (59) into equation (48) and (49) one obtains a system of two highly nonlinear equations in g and u: $\Omega(g, u) = 0$. This system of two equations then can be solved numerically for the baseline calibration of parameters defined in Table 1.[23]

[23] For Model 1 the steady state solution can be obtained in a similar fashion where the solution can be narrowed down to one equation (intratemporal margin) in one unknown (the BGP growth rate).

A.3 Proof of Proposition 6

For S_1^h consider equation (22) in Appendix A.1:

$$\lambda_t \phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} (u_t k_t) = \chi_t \phi_2 A_h e^{z_t^h} \left[\frac{(1 - v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1} (u_t k_t). \tag{60}$$

Dividing both sides by λ_t and by $(u_t k_t)$ one obtains

$$\phi_1 A_g e^{z_t^g} \left[\frac{v_{gt} u_t k_t}{l_{gt} h_t}\right]^{\phi_1 - 1} = \frac{\chi_t}{\lambda_t} \phi_2 A_h e^{z_t^h} \left[\frac{(1 - v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1}. \tag{61}$$

Then using the definitions for p_{ht} and r_t^k in Appendix A one can write equation (61) as

$$r_t^k = p_{ht} \phi_2 A_h e^{z_t^h} \left[\frac{(1 - v_{gt}) u_t k_t}{l_{ht} h_t}\right]^{\phi_2 - 1}. \tag{62}$$

By taking the partial derivative of the right hand side of equation (62) with respect to $p_{ht} \equiv \chi_t / \lambda_t$ it directly follows that

$$S_1^h = \frac{\partial r_t^k}{\partial p_{ht}} = \phi_2 A_h z_t^h \left[\frac{v_{ht} u_t k_t}{l_{ht} h}\right]^{\phi_2 - 1} = \frac{r_t^k}{p_{ht}}. \tag{63}$$

For S_2^h equation the right hand side of equations (19) and (20) in Appendix A.1:

$$\lambda_t w_t h_t = \chi_t (1 - \phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} h_t. \tag{64}$$

After simplifying with h_t and dividing both sides of (64) with λ_t one gets

$$w_t = \frac{\chi_t}{\lambda_t} (1 - \phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2}; \tag{65}$$

Then using the definition of $p_{ht} \equiv \chi_t / \lambda_t$ yields

$$w_t = p_{ht} (1 - \phi_2) A_h e^{z_t^h} \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2}; \tag{66}$$

By taking the partial derivative of the right hand side of equation (66) with respect to p_{ht} it directly follows that

$$S_2^h = \frac{\partial w_t}{\partial p_{ht}} = (1 - \phi_2) A_h z_t^h \left[\frac{v_{ht} u_t k_t}{l_{ht} h_t}\right]^{\phi_2} = \frac{r_t^k}{p_{ht}}. \tag{67}$$

B Data Description

The US data used in this paper is from 1959:Q1 until 2015:Q4 except for that of the physical capital utilization rate, which is only available from 1971:Q4, and human and physical capital data, which is available only until the end of 2012. In constructing real data series for US macroeconomic variables Gomme & Rupert (2007) have been followed. Analogously to their methodology the aggregate series are constructed as:[24]

1. *Nominal Market Investment* = Non-residential Fixed Investment + Change in Private Inventories
2. *Nominal Home Investment* = Residential Fixed Investment + PCE on Durables
3. *Nominal Investment* = Nominal Home Investment + Nominal Market Investment
4. *Real Investment*= Nominal Investment / (Average Price Deflator / 100)
5. *Nominal Market Output* = Gross Domestic Product - PCE: Housing Services
6. *Nominal Private Market Output* = Nominal Market Output - Employee Compensation: Government
7. *Real Market Output* = Nominal Market Output / (Average Price Deflator / 100)
8. *Real Private Market Output* = Nominal Private Market Output / (Average Price Deflator / 100)
9. *Physical Capital Utilization Rate* = Total Capacity Utilization: Manufacturing
10. *Labor Hours* = Non-farm Business Sector: Average Weekly Hours
11. *Nominal Market Consumption* = PCE on Non-durable Goods + PCE on Services - PCE on Housing Services
12. *Real Market Consumption*= Nominal Market Consumption / (Average Price Deflator/100)
13. *Average Price Deflator* = (Implicit Price Deflator:Non-durables + Implicit Price Deflator: Services)/2

According to Gomme & Rupert (2007), output (y) is measured by real per capita GDP less real per capita Gross Housing Product as defined above. It is due to the argument that home sector production should be removed when calculating market output using the *National Income and Product Accounts* (NIPA). The price deflator is constructed by taking

[24] The raw series and the construction of the underlying data series can be found in *data.xls* included with the Matlab files upon request.

the average of the implicit price deflators on non-durables and services. Population is measured by the number of non-institutionalized persons aged over 16 years. Consumption (c) is measured by real personal expenditures on non-durables and services less Gross Housing Services. Investment is measured by the sum of real Non-residential Fixed Investment, the Change in Private Inventories, Residential Fixed Investment, and Personal Consumption Expenditures on durables. Lastly, working hours are measured by the average weekly labor hours.

The annual index of human capital per person data series is based on years of schooling [Barro & Lee (2013)], and returns to education [Psacharopoulos (1994)]. The series have been constructed by Feenstra et al. (2013) using the perpetual inventory method. Quarterly human capital data has been interpolated using the annual data of Feenstra et al. (2013) by following Baier et al. (2004) where they define the depreciation rate to human capital as the average of death rates in different age groups for which the data has been obtained from the Center for Disease Control (CDC) database. Also, for the period after 2012 the human capital data has been forecasted by fitting it to an AR1 process. The quarterly physical capital data is constructed from Bureau of Economic Analysis (BEA) annual US capital stock estimates and quarterly data on investment expenditures.

Description	Units	Seasonally Adjusted	BEA / BLS Code / Source	Frequency	Time Range
PCE on Nondurable Goods	Billions of Dollars	SAAR	DNDGRC1	Quarterly	1959Q1 - 2015Q4
PCE on Services	Billions of Dollars	SAAR	DSERRC1	Quarterly	1959Q1 - 2015Q4
PCE on Housing Services	Billions of Dollars	SAAR	DHUTRC1	Quarterly	1959Q1 - 2015Q4
Implicit Price Deflator (Nondurables)	Index 2009=100	SA	DNDGRD3	Quarterly	1959Q1 - 2015Q4
Implicit Price Deflator (Services)	Index 2009=100	SA	DSERRD3	Quarterly	1959Q1 - 2015Q4
Gross Domestic Product	Billions of Dollars	SAAR	A191RC1	Quarterly	1959Q1 - 2015Q4
Employee Compensation: Government	Billions of Dollars	SAAR	B202RC1	Quarterly	1959Q1 - 2015Q4
Civ. Noninst. Pop. 16 and over	1000s of Persons	NSA	CNP160V	Quarterly	1959Q1 - 2015Q4
Capacity Utilization: Manufacturing	Percentage	SA	TCU:MAN.	Quarterly	1971Q4 - 2015Q4
Nonfarm Bus. Sector: Avg. Weekly Hours	Index 2009=100	SA	PRS585006023	Quarterly	1959Q1 - 2015Q4
Nonres. Fixed Investment	Billions of Dollars	SAAR	A008RC1	Quarterly	1959Q1 - 2015Q4
Residential Fixed Investment	Billions of Dollars	SAAR	A011RC1	Quarterly	1959Q1 - 2015Q4
Change in Private Inventories	Billions of Dollars	SAAR	A014RC1	Quarterly	1959Q1 - 2015Q4
PCE on Durables	Billions of Dollars	SAAR	DDURRC1	Quarterly	1959Q1 - 2015Q4
Index of Human Capital per Person	Index	NSA	Penn World Tables 8.0	Annual	1950 - 2012

Table 7: Raw Data Sources.

C The Labor Wedge and the Effect of Human Capital

In Models 1 and 2 there is *equivalence* between the labor efficiency wedge as in Chari et al. (2007) and the effect of the non-market human capital investment sector in the intratemporal condition of the representative agent. To show this, first define a prototype labor wedge as in Chari et al. (2007).

Definition 8 *For a prototype exogenous growth economy as in Chari et al. (2007) without a government sector a tax like wedge between the marginal product of goods sector to labor and the marginal rate of substitution (MRS) of the consumer or agent is described by the following equation:*

$$\frac{A\tilde{c}_t}{(1-l_{gt})} = (1-\tau_{lt})\tilde{w}_t = (1-\tau_{lt})(1-\phi_1)A_g e^{z_t^g}\left[\frac{\tilde{k}_t}{z_t l_{gt}}\right]^{\phi_1} z_t; \tag{68}$$

where variables with a tilde represent normalized variables by the exogenous growth trend in the prototype exogenous economy.

Proposition 9 *The share of learning time in total time not used towards goods production drives a wedge between the MRS between non-market hours and consumption of the representative agent and the Marginal Product of Goods Labor (MPL) in the human capital normalized stationary Model 1 and 2, which is equivalent to a Chari et al. (2007) labor wedge.*

Proof. For Model 2 consider the marginal rate of substitution for the human capital normalized Model 2 [Online Appendix equation (75)] :

$$x_t = \frac{A\tilde{c}_t}{w_t}; \tag{69}$$

where variables with tildes for Model 2 represent normalized variables. Now substitute in for x_t using the time constraint in (2) and for the marginal product of goods labor using the production function specification for Model 2 in (5).The following is the resulting equation:

$$1 - l_{gt} - l_{ht} = \frac{A\hat{c}_t}{(1-\phi_1)A_g e^{z_t^g}\left[\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}}\right]^{\phi_1}} = \frac{A\hat{c}_t}{w_t}; \tag{70}$$

Dividing both sides of the above equation by the sum of non-market time, $(1 - l_{gt})$ yields an equivalent equation to (68) as

$$1 - \frac{l_{ht}}{1 - l_{gt}} = \frac{A\hat{c}_t}{(1-\phi_1)A_g e^{z_t^g}\left[\frac{v_{gt}u_t\hat{k}_t}{l_{gt}}\right]^{\phi_1}} = \frac{A\hat{c}_t}{w_t(1-l_{gt})}. \tag{71}$$

From this it directly follows that the human investment sector drives a wedge between the MRS between consumption and non-market hours and the marginal product of labor in the stationary state of Model 2. This wedge is denoted as $(1 - \eta_{1,t}) = (1 - l_{ht}/(1 - l_{gt}))$, which is equivalent to the labor wedge, $(1 - \tau_{lt})$ in equation (68). ■

Online Appendices

D Stochastic Discounting and Log-linear Solution Methodology

Such endogenous growth models exhibit non-stationary features. Endogenous variables $\{k_{t+1}, h_{t+1}, c_t, i_{kt}, i_{ht}\}$ grow with a common rate along the BGP. In order to be able to solve the model in (25) - (34), respectively, one has to rewrite the systems of equations by using the following newly defined stationary variables $g_{ht+1} \equiv \frac{h_{t+1}}{h_t}$; $\tilde{k}_t \equiv \frac{k_t}{h_t}$; $\tilde{\imath}_{kt} \equiv \frac{i_{kt}}{h_t}$;$\tilde{\imath}_{kt} \equiv \frac{i_{kt}}{h_t}$; and $\tilde{c}_t \equiv \frac{c_t}{h_t}$.

Then by using the factor reward definitions in Appendix A.1, the stationary model is log-linearized along its steady state, after which the method of undetermined coefficients is applied as in Uhlig (1998) to solve for the recursive policy functions of the models. Any variables with a hat represent the variable in log-deviation from its steady state and variables without time subscripts represent steady state values.

D.1 Model 2 - Stochastic Discounting

After normalizing the growing endogenous variables with the human capital stock, h_t, the equilibrium conditions in (25) - (34) become:

$$A_g e^{z_t^g}(v_{gt}u_t\tilde{k}_t)^{\phi_1} l_{gt}^{1-\phi_1} = \tilde{c}_t + \tilde{k}_{t+1}(1 + g_{ht+1}) - \tilde{k}_t + \frac{\delta_k}{\psi}u_t^{\psi}\tilde{k}_t; \tag{72}$$

$$A_h e^{z_t^h}((1-v_{gt})u_t\tilde{k}_t)^{\phi_2} l_{ht}^{1-\phi_2} = g_{ht+1} + \delta_h; \tag{73}$$

$$p_{ht} = \left[\frac{A_g}{A_h}\right]\left[\frac{e^{z_t^g}}{e^{z_t^h}}\right]\left[\frac{1-\phi_1}{1-\phi_2}\right]^{1-\phi_2}\left[\frac{\phi_1}{\phi_2}\right]^{\phi_2}\left[\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}}\right]^{\phi_1-\phi_2}; \tag{74}$$

$$\frac{A}{x_t}\tilde{c}_t = (1-\phi_1)A_g e^{z_t^g}\left[\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}}\right]^{\phi_1}; \tag{75}$$

$$\frac{B}{(1-u_t)}\tilde{c}_t = \left[\phi_1 A_g e^{z_t^g}\left[\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}}\right]^{\phi_1-1} - \delta_k u_t^{\psi-1}\right]\tilde{k}_t; \tag{76}$$

$$x_t = 1 - l_{gt} - l_{ht}; \tag{77}$$

$$\frac{1-\phi_1}{\phi_1}\frac{v_{gt}u_t\tilde{k}_t}{l_{gt}} = \frac{1-\phi_2}{\phi_2}\frac{(1-v_{gt})u_t\tilde{k}_t}{l_{ht}}; \tag{78}$$

$$\begin{aligned}1 = \beta E_t\left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\right)^{\sigma}\left(\frac{1}{1+g_{ht+1}}\right)^{\sigma}\left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)}\left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)}\\ \left[1+\phi_1 A_g e^{z_{t+1}^g}\left[\frac{v_{gt+1}u_{t+1}\tilde{k}_{t+1}}{l_{gt+1}}\right]^{\phi_1-1}u_{t+1} - \frac{\delta_k}{\psi}u_{t+1}^{\psi}\right];\end{aligned} \tag{79}$$

$$\begin{aligned}1 = \beta E_t\left(\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\right)\left(\frac{1}{1+g_{ht+1}}\right)^{\sigma}\left(\frac{x_{t+1}}{x_t}\right)^{A(1-\sigma)}\left(\frac{1-u_{t+1}}{1-u_t}\right)^{B(1-\sigma)}\frac{p_{ht+1}}{p_{ht}}\\ \left[1+(l_{gt+1}+l_{ht+1})(1-\phi_2)A_h e^{z_{t+1}^g}\left[\frac{(1-v_{gt+1})u_{t+1}\tilde{k}_{t+1}}{l_{ht+1}}\right]^{\phi_2} - \delta_h\right].\end{aligned} \tag{80}$$

D.2 Log-Linearized System of Model 2

Here the log-linearized system of equations is presented as implemented in Matlab. Variables with a hat represent variables transformed to log-linear deviations from their steady states. The variable $\hat{G}_{ht}$ denotes the log-linearized version of the gross growth rate of human capital.

$$0 \approx -\hat{\tilde{y}}_t + \left[\frac{\tilde{c}}{\tilde{y}}\right]\hat{\tilde{c}}_t + \left[\frac{\tilde{\imath}_k}{y}\right]\hat{\tilde{\imath}}_{kt}; \tag{81}$$

$$0 = -\hat{\tilde{y}}_t + \hat{z}_t^g + \phi_1\hat{v}_{gt} + \phi_1\hat{u}_t + \phi_1\hat{\tilde{k}}_{t-1} + (1-\phi_1)\hat{l}_{gt}; \tag{82}$$

$$0 \approx -\hat{\tilde{\imath}}_{kt} + \left[\frac{(1+g)k}{\tilde{\imath}_k}\right]\hat{\tilde{k}}_t + \left[\frac{(1+g)k}{\tilde{\imath}_k}\right]\hat{G}_{ht} + \left[\frac{\frac{\delta_k}{\psi}\tilde{k}u^{\psi} - \tilde{k}}{\tilde{\imath}_k}\right]\hat{\tilde{k}}_{t-1} + \left[\frac{\delta_k\tilde{k}u^{\psi}}{\tilde{\imath}_k}\right]\hat{u}_t \tag{83}$$

$$0 \approx -\hat{\tilde{\imath}}_{ht} + \left[\frac{(1+g)}{\tilde{\imath}_h}\right]\hat{G}_{ht} \tag{84}$$

$$0 = -\hat{\tilde{\imath}}_{ht} + \hat{z}_t^h + \phi_2\hat{v}_{ht} + \phi_2\hat{u}_t + \phi_2\hat{\tilde{k}}_{t-1} + (1-\phi_2)\hat{l}_{ht}; \tag{85}$$

$$0 \approx \hat{x}_t + \left[\frac{l_g}{x}\right]\hat{l}_{gt} + \left[\frac{l_h}{x}\right]\hat{l}_{ht}; \tag{86}$$

$$0 \approx \hat{v}_{gt} + \left[\frac{v_h}{v_g}\right]\hat{v}_{ht}; \tag{87}$$

$$0 = -\hat{\tilde{c}}_t + \hat{w}_t + \hat{x}_t; \tag{88}$$

$$0 = -\hat{\tilde{c}}_t + \hat{\tilde{k}}_{t-1} + \hat{s}_t + \left[\frac{r^k}{r^k - \delta_k u^{\psi-1}}\right]\hat{r}_t^k - \left[\frac{\delta_k(\psi-1)u^{\psi-1}}{r^k - \delta_k u^{\psi-1}}\right]\hat{u}_t \tag{89}$$

$$0 \approx \hat{s}_t + \left[\frac{u}{s}\right] \hat{u}_t; \tag{90}$$

$$0 = \hat{v}_{gt} - \hat{v}_{ht} + \hat{l}_{ht} - \hat{l}_{gt}; \tag{91}$$

$$0 = -\hat{p}_{ht} + \hat{z}_t^g - \hat{z}_t^h + [\phi_1 - \phi_2]\hat{v}_{gt} + [\phi_1 - \phi_2]\hat{u}_t + [\phi_1 - \phi_2]\hat{\tilde{k}}_{t-1} + [\phi_2 - \phi_1]\hat{l}_{gt}; \tag{92}$$

$$0 = -\hat{r}_t^k + \hat{z}_t^g + [\phi_1 - 1]\hat{v}_{gt} + [\phi_1 - 1]\hat{u}_t + [\phi_1 - 1]\hat{\tilde{k}}_{t-1} + [1 - \phi_1]\hat{l}_{gt}; \tag{93}$$

$$0 = -\hat{w}_t + \hat{z}_t^g + \phi_1 \hat{v}_{gt} + \phi_1 \hat{u}_t + \phi_1 \hat{\tilde{k}}_{t-1} - \phi_1 \hat{l}_{gt}; \tag{94}$$

$$\begin{aligned} 0 = E_t\{\sigma\hat{\tilde{c}}_t - \sigma\hat{\tilde{c}}_{t+1} - \sigma\hat{G}_{ht+1} + A(1-\sigma)\hat{x}_{t+1} - A(1-\sigma)\hat{x}_t + \frac{uB(1-\sigma)}{1-u}\hat{u}_t \\ + \left[\frac{r^k u}{1 + r^k u - \frac{\delta_k}{\psi}u^\psi}\right]\hat{v}_{gt+1} \\ + \left[\frac{r^k u - \delta_k u^\psi}{1 + r^k u - \frac{\delta_k}{\psi}u^\psi} - \frac{uB(1-\sigma)}{1-u}\right]\hat{u}_{t+1}\}; \end{aligned} \tag{95}$$

$$
\begin{aligned}
0 = E_t\{\sigma\hat{\hat{c}}_t - \sigma\hat{\hat{c}}_{t+1} - \sigma\hat{G}_{ht+1} + A(1-\sigma)\hat{x}_{t+1} - A(1-\sigma)\hat{x}_t - \frac{uB(1-\sigma)}{1-u}\hat{u}_{t+1}& \\
+ \frac{uB(1-\sigma)}{1-u}\hat{u}_t - \hat{p}_{ht}& \\
+ \left[\frac{(1-\delta_h)}{1+\frac{w}{p_h}(l_g+l_h)-\delta_h}\right]\hat{p}_{ht+1}& \\
+ \left[\frac{\frac{w}{p_h}(l_g+l_h)}{1+\frac{w}{p_h}(l_g+l_h)-\delta_h}\right]\hat{w}_{t+1}& \\
+ \left[\frac{\frac{w}{p_h}l_g}{1+\frac{w}{p_h}(l_g+l_h)-\delta_h}\right]\hat{l}_{gt+1}& \\
+ \left[\frac{\frac{w}{p_h}l_h}{1+\frac{w}{p_h}(l_g+l_h)-\delta_h}\right]\hat{l}_{ht+1}\}&
\end{aligned} \tag{96}
$$

$$
\hat{z}^g_{t+1} = \rho_g \hat{z}^g_t + \epsilon^g_{t+1}; \tag{97}
$$

$$
\hat{z}^h_{t+1} = \rho_h \hat{z}^h_t + \epsilon^h_{t+1}; \tag{98}
$$

D.3 Solution Methodology

After obtaining the log-linear systems in equations (81) to (98) the method proposed by Uhlig (1998) is applied to solve the model and obtain the recursive policy functions. The Uhlig (1998) method based on the method of undetermined coefficients following King et al. (1988) is chosen, as it is relatively simple to implement. In order to be able to apply this solution method one has to rewrite the above log-linear first order conditions in the following matrix form:

$$
Ax_t + Bx_{t-1} + Cy_t + Dz_t = 0, \tag{99}
$$

$$
E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t Lz_{t+1} + Mz_t] = 0, \tag{100}
$$

$$z_{t+1} = Nz_t + \epsilon_{t+1}, \tag{101}$$

where $E_t(\epsilon_{t+1}) = 0$; the vector x_t (size $mx1$) contains the endogenous state variables; y_t (size $nx1$) is the vector of all other endogenous variables; meanwhile, z_t (size $kx1$) is the vector of exogenous stochastic variables. It is assumed that the coefficient matrix C is of size lxn, where $l \geq n$ and of rank n. l is the number of deterministic equations, F is a coefficient matrix of size $(m+n-l)xm$, and N has only stable eigenvalues. In the underlying baseline model x_t contains the log-linear versions of $\hat{k}_t$ and $\hat{g}_{ht}$. There are two exogenous variables in z_t, namely, $\hat{z}_t^g$ and $\hat{z}_t^h$. All other six and seven endogenous variables in y_t, are $\hat{c}_t$, $\hat{u}_t$, $\hat{l}_{gt}$, $\hat{l}_{ht}$, $\hat{v}_{gt}$, $\hat{p}_{ht}$.[25]

The log-linear solution method by Uhlig (1998) is seeking to find a recursive equilibrium law of motion of the following form:

$$x_t = Px_{t-1} + Qz_t; \tag{102}$$

$$y_t = Rx_{t-1} + Sz_t. \tag{103}$$

Therefore, the underlying solution method is looking for P, Q, R, and S so that the equilibrium described by these rules is stable in nature. In the case of Model 1 and 2, $l = n$, then, [26]

(i) P must satisfy the following quadratic matrix equation:

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0. \tag{104}$$

(ii) R is given by

$$R = -C^{-1}(AP + B). \tag{105}$$

[25] The growth rate of human capital $g_{ht} \equiv h_{t+1}/h_t$ is defined as a state variable in order to satisfy the requirement that the $l \geq n$ condition is imposed by the log-linear approximation. Since it is not a state variable in the proper sense it will vanish from the recursive policy functions.

[26] For more details see Corollary 1 in Uhlig (1998).

(iii) Q satisfies

$$vec(Q) = (N'^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A))^{-1} vec((JC^{-1}D - L)N + KC^{-1}D - M), \tag{106}$$

where $vec(.)$ denotes column wise vectorization.
(iv) Lastly, S is given by

$$S = -C^{-1}(AQ + D). \tag{107}$$

In order to have a stationary recursive solution, the key is to pick up the solution for P, whose eigenvalues are both smaller than unity. Given P the solution to R, Q, and S directly follows.

D.4 Simulation Methodology

In order to characterize the cyclical and long-run components of the simulated data an asymmetric Christiano & Fitzgerald (2003) type band-pass filter is applied at different frequencies defined in Section 4. Since Model 1 and 2 are solved for human capital normalized variables we use the method described by Restrepo-Ochoa & Vazquez (2004) to construct log-level simulated series by using the model solution given by equations (102) and (103).Using the method of Restrepo-Ochoa & Vazquez (2004) non-stationary log-level series are constructed for output, consumption, and physical investment, while. Then these series are used to calculate simulated RBC correlations; volatilities; and growth persistence as it can be seen in Section 4.

Consider that n_t denotes any non-stationary variable of the model and $\tilde{n}_t \equiv n_t/h_t$. Then the logarithm of n_t can be written as

$$\log n_t = \log \tilde{n}_t + \log h_t. \tag{108}$$

Then one may observe about the growth rate of human capital, $g_{h,t} = (1 + g_t)$, that

$$\frac{h_{t+1}}{h_t} = g_h e^{g_{ht}} \approx g_h(1 + g_t), \tag{109}$$

where $g_h = 1 + g$ is the steady state value of h_{t+1}/h_t. From the log-linear solution of the model in (102) the law of motion of the log-deviations from the steady state values of

$\tilde{k}_{t+1}$ and g_{ht+1} are

$$\hat{g}_{ht} = P_{21}\hat{k}_t + Q_{21}z_t^g + Q_{22}z_t^g, \tag{110}$$

$$\hat{k}_{t+1} = P_{11}\hat{k}_t + Q_{11}z_t^g + Q_{12}z_t^g, \tag{111}$$

where P_{ij} and Q_{ij} denote the generic elements of matrices P and Q respectively and variables with hats represent variables in terms of log-deviations from their respective steady states. Next take natural logarithm of equation (109) to obtain,

$$\log h_{t+1} = \log h_t + \log g_h + \hat{g}_{ht}. \tag{112}$$

Then combining (112), (111), and (110) one can obtain an expression for the solution of the model implied synthetic human capital stock in log levels as,

$$\log h_{t+1} = \log h_t + \log g_h + P_{21}P_{11}\hat{k}_{t-1} + [P_{21}Q_{11} + Q_{21}]z_t^g + [P_{21}Q_{12} + Q_{22}]z_t^h. \tag{113}$$

From equation (113) one can observe that the synthetic time series of the human capital stock in log-levels has a unit root with a drift. From this it follows that by using equation (108) and (113) to obtain time series for consumption, investment, and output, will generate series with a unit root with a drift. This as pointed out by Restrepo-Ochoa & Vazquez (2004) is in line with evidence in the literature over the presence of a unit root in aggregate time series also highlighted by Nelson & Plosser (1982). Given the non-stationary time series obtained from the model for the log-levels of output, consumption, and investment, now the band pass filter can be applied at different frequencies as defined earlier following Basu et al. (2012), and Comin & Gertler (2006).

Then to construct the simulated growth rates for output, consumption, and physical investment the obtained non-stationary logarithmic series for output, consumption, and investment are first differenced. Construction of the growth rates this way is identical to constructing growth rates by using directly the model decision rules as suggested by King et al. (1988). This direct method is shown for consumption growth,

$$
\begin{aligned}
g_{c,t+1} &= \log c_{t+1} - \log c_t \\
&= \log \tilde{c}_{t+1} - \log \tilde{c}_t + \log h_{t+1} - \log h_t \\
&= (\log \tilde{c}_{t+1} - \log \tilde{c}) - (\log \tilde{c}_t - \log \tilde{c}) + \log \frac{h_{t+1}}{h_t} \\
&= \hat{c}_{t+1} - \hat{c}_t + (\log g_{h,t+1} - \log g_{ht}) + \log g_h \\
&= \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{ht+1} + \log g_h,
\end{aligned} \tag{114}
$$

where the key is to express any growth rate in terms of variables of the normalized stationary model, after which one can substitute out $\hat{c}_t$ and $\hat{g}_{ht}$ using the solutions for them in equations (102) and (103)

The equivalence between the two methods of obtaining the growth rates of non-stationary variables can be easily seen if one moves equation (108) one period forward to $t+1$ and then subtracts the time period t version of the same equation for consumption growth in this instance:

$$
g_{c,t} = \log c_{t+1} - \log c_t = \log \tilde{c}_{t+1} - \log \tilde{c}_t + \log h_{t+1} - \log h_t = \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{ht+1} + \log g_h; \tag{115}
$$

Equation (114) is identical to the final expression in (115) for the growth rate of consumption.

E Matlab Code Description

This section gives a detailed description of the Matlab codes used to obtain the results in the main body and appendices of this paper. The collection of the Matlab codes that produce the figures and results for this paper are available in the '*Tune in RBC Growth Spectra Matlab*' zip file .

E.1 Calibration Codes with Grid Search

The folders *Model 1 Calibration with Grid System* and *Model 2 Calibration with Grid System* contain the set of codes that perform the grid point based calibration procedure.

1. *main.m*: In this Matlab script one sets the lower and upper bounds and the number of steps between those to set up the grid point system. By running this script the calibration procedure is initiated and results are sorted in the matrix file called *Parameter_ Combinations2.m.*

2. *distance.m*: This Matlab function file defines the calibration metric, which is a weighted vector distance between a US data and a simulated data based moment vector.

3. *SUR.m*: This function file implements the extraction of shock process series using the method of Ingram et al. (1997), Benk et al. (2005, 2008, 2010), and Nolan & Thoenissen (2009); also implements the estimation of shock parameters by using the Seemingly Unrelated Regression (SUR) estimator as described in Greene (2003). For the extraction and estimation it calls the function called *solution.m* and the data file *data.xls*. The set of data used is set in this function.

4. *solution.m*: The Matlab function solves for the recursive solution of the underlying model by using the method of undetermined coefficients as in Uhlig (1998) for each iteration of the convergence and shock estimation.

5. *data.xls*: This MS Excel file contains the raw log-level US data from 1959Q1 to 2015Q4 as described in Appendix B.

6. *bpass.m*: This is a band-pass filter function file as in Christiano & Fitzgerald (2003).[27]

E.2 Calibration Codes with Simulated Annealing

The folders *Model 1 Calibration with SA* and *Model 2 Calibration with SA* contain the set of codes that perform the Simulated Annealing based calibration procedure.

1. *main.m*: In this Matlab script one sets the lower and upper bounds and initial values for each parameter. By running this script the calibration procedure is initiated.

2. *search.m*: This Matlab function file defines the calibration metric, which is a weighted vector distance between a US data and a simulated data based moment vector.

    (a) This function file initiates the shock extraction and convergence procedure using US data by calling *iteration.m*.

    (b) If shock parameter convergence occurs the *search.m* uses the estimated and convergent shock parameters to obtain simulated data and its moments.

    (c) The simulated data based moments are then used to calculate the distance between the data based moments and the simulated data based one.

3. *iteration.m*: This Matlab function file performs an iterative process as described in Benk et al. (2005, 2008, 2010). The function uses to estimate shock processes by calling the function *SUR.m*. Then the new estimates are fed back into the iterative

[27] This band pass filter Matlad function has been created by Eduard Pelz of the Federal Reserve Bank of Cleveland also available at: http://www.clev.frb.org/research/workpaper/1999/bpass.txt.

loop until the estimated parameters converge or it stops when the shock persistence parameters reach unity, e.g. explode.

4. *SUR.m*: This function file implements the extraction of shock process series using the method of Ingram et al. (1997), Benk et al. (2005, 2008, 2010), and Nolan & Thoenissen (2009); also implements the estimation of shock parameters by using the Seemingly Unrelated Regression (SUR) estimator as described in Greene (2003). For the extraction and estimation it calls the function called *solution.m* and the data file *data.mat*. The set of data used is set in this function.

5. *solution.m*: The Matlab function solves for the recursive solution of the underlying model by using the method of undetermined coefficients as in Uhlig (1998) for each iteration of the convergence and shock estimation.

6. *data.mat*: This matrix contains the raw log-level US data from 1959Q1 to 2015Q4 as described in Appendix B.

7. *bpass.m*: This is a band-pass filter function file as in Christiano & Fitzgerald (2003).[28]

E.3 Simulation Codes

The folder *Simulation Codes* contains the set of codes that carry out the simulation of the baseline Models 1 and 2.

1. *main.m*: This Matlab script produces the figures and the tables in the command window of Matlab that are equivalent to the ones in the paper body and appendices. This file also calls all simulation files to obtain the results from the baseline models of the paper and their variants.

2. *sim M1.m*: This Matlab function performs a number of tasks all related to results obtained from Model 1. These are in order:

    (a) It calculates the steady state of Model 1 and stores it in the *StSt M1* matrix.

    (b) Solves for the recursive policy functions of Model 1 using the method of undetermined coefficients as in Uhlig (1998).

    (c) Generates simulated time series for the model variables using the recursive solution of Model 1 then constructs the synthetic non-stationary log-level series of key variables and growth rates as in Restrepo-Ochoa & Vazquez (2004) and King et al. (1988).

[28] This band pass filter Matlad function has been created by Eduard Pelz of the Federal Reserve Bank of Cleveland also available at: http://www.clev.frb.org/research/workpaper/1999/bpass.txt.

(d) Calls the band-pass filter function '*bpass.m*' as described in Christiano & Fitzgerald (2003), and filters the simulated time series for key variables at four different frequencies. The frequency bands are the high frequency [2-6 quarters]; the medium or business cycle frequency [6-32 quarters]; the low frequency [32-200 quarters]; and the Comin & Gertler (2006) type medium cycle [2-200 quarters].

(e) Using the Statistical toolbox of Matlab, more specifically its *crosscorr*, *autocorr*, and *std* functions, it calculates key cross-correlations between variables, variables' auto-correlation functions, and variables' standard deviations [i.e. volatility]. The results are then stored in the *CORR M1*, *ACORR M1*, and *VOL M1* matrices.

(f) Lastly, it calculates the impulse response functions to all variables to a goods sector TFP shock stored in the *IRFa M1* matrix; to a human sector productivity shock stored in the *IRFb M1* matrix; and to an economy wide shock stored in matrix *IRFc m.*

3. *sim M2.m*: This Matlab function performs a number of tasks all related to results obtained from Model 2. These are in order:

   (a) It calculates the steady state of Model 2 and stores it in the *StSt M2* matrix.

   (b) Solves for the recursive policy functions of Model 1 using the method of undetermined coefficients as in Uhlig (1998).

   (c) Generates simulated time series for the model variables using the recursive solution of Model 2 then constructs the synthetic non-stationary log-level series of key variables and growth rates.

   (d) Calls the band-pass filter function '*bpass.m*'.

   (e) It calculates key cross-correlations between variables, variables' auto-correlation functions, and variables' standard deviations [i.e. volatility]. The result are then stored in the *CORR M2*, *ACORR M2*, and *VOL M2* matrices.

   (f) Lastly, it calculates the impulse response functions to all variables to a goods sector TFP shock stored in the *IRFa M2* matrix; to a human sector productivity shock stored in the *IRFb M2* matrix; and to an economy wide shock stored in matrix *IRFc M2*.

4. *bpass.m*: This function contains the default asymmetric band-pass filter.

5. *data moments.m*: This function imports the raw data file *data.xls* and calculates all US business cycle moments at the earlier defined four frequencies. Then it stores the correlations, auto-correlations and volatilities *CORR Data*, *ACORR Data*, and *VOL Data* respectively.

6. *parameters.m*: This files stores the baseline calibrations of Model 1 and 2 in the vectors named *Parameters M1* and *Parameters M2* respectively.

7. *solow.m*: This file extracts the Solow residual, which is then compared to an alternative measure of the goods sector technology shock.

8. *TFP M1.m*: This file extracts the goods sector TFP shock series from using the recursive solution of Model 1. It stores the TFP series in *ZG M1*, meanwhile, it calculates the auto-correlation functions of output, investment, and consumption growth after feeding the extracted TFP series back to the recursive solution of Model 1.

9. *TFP M2.m*: This file extracts the goods sector TFP shock series from using the recursive solution of Model 2. It stores the TFP series in *ZG M2*, meanwhile, it calculates the auto-correlation functions of output, investment, and consumption growth after feeding the extracted TFP series back to the recursive solution of Model 2

10. *data.xls*: This file contains raw log-level US data series from 1959Q1 to 2015Q4 as described in Appendix B. It is used to calculate data moments.