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Beyond Balanced Growth

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Beyond Balanced Growth[1]

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Abstract

Balanced growth models are commonly used in macroeconomics because they are consistent with the well-known Kaldor facts regarding economic growth. These models, however, are inconsistent with one of the most striking regularities of the growth process—the massive reallocation of labor from agriculture into manufacturing and services. This paper presents a simple model consistent with both the Kaldor facts and the dynamics of sectoral labor reallocation. The model shows that balanced growth can be consistent with structural change.

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Contents

I. INTRODUCTION

Balanced growth models are widely used in macroeconomics because they are consistent with the well-known Kaldor facts regarding economic growth. Kaldor stressed that the growth rate of output, the capital-output ratio, the real interest rate, and the labor income share are all roughly constant over time. The constancy of these "great ratios" provides a good characterization of the long-run behavior of the U.S. economy.

Just as important as these regularities stressed by Kaldor is the massive reallocation of labor from agriculture into manufacturing and services that accompanies the growth process. This reallocation process, often called "structural change," has been documented by authors such as Clark (1940), Kuznets (1957), and Chenery (1960). A few numbers help to put this phenomenon into perspective. In 1870, the U.S. share of employment in agriculture was 40 percent. One hundred years later, agriculture accounted for only 4 percent of employment. Services, which accounted for 20 percent of employment in 1870, absorbed 40 percent of the labor force by 1970. We refer to the main regularities of this reallocation process as the Kuznets facts.

The macroeconomics and growth literature, which makes heavy use of balanced growth models, generally disregards the dramatic sectoral reallocation of labor experienced by all expanding economies. In contrast, there is a literature on structural change that ignores the Kaldor facts, in part because it focuses on a longer time period for which these facts may not apply (e.g., Baumol (1967), Pasinetti (1981), Park (1995), Echevarria (1997), and Laitner (2000)).

Is there a growth model that is consistent with both the Kaldor facts and the massive sectoral labor reallocation experienced in the U.S. during the last century? At first sight the answer to this question is no. After all, one property of balanced growth models is that the fraction of capital and labor allocated to different industries remains constant over time. We show that the two sets of facts, however, can be reconciled, provided that a knife-edge condition is satisfied. The model we propose displays a *generalized balanced growth* (GBG) path—a trajectory that retains the key features of balanced growth and is consistent with the dynamics of structural change.

Balanced growth paths are easy to study. The fact that all variables grow at a constant rate transforms the system of different equations that describes the economy's competitive equilibrium into a system of static equations that can be easily solved. This system also delivers the vector of initial values for the capital stocks that is consistent with balanced growth. The generalized balanced growth path in our model shares these attractive features. The GBG path and the corresponding vector of initial conditions can be characterized analytically, even though some variables have time-varying growth rates.

Our paper is organized as follows. Section 2 summarizes briefly the main empirical facts that motivate our analysis. Section 3 presents our model. A final section discusses possible applications and extensions of our model.

II. The Empirical Facts

In addition to observing that the rate of economic expansion differs widely across countries, Kaldor proposed the following well-known set of empirical regularities:

The Kaldor Facts
1 – Per capita output grows at a rate that is roughly constant;
2 – The capital-output ratio is roughly constant;
3 – The real rate of return to capital is roughly constant;
4 – The shares of labor and capital in national income are roughly constant.

These stylized facts, which suggest that several aggregate "great ratios" evolve smoothly over time, have had an enormous impact on the construction of growth models. Are they too stylized to be facts? The first panel of Figure 1, which depicts the logarithm of per capita U.S. real GDP from 1902 to 1999 measured at an annual frequency, shows that the growth rate of output is indeed remarkably stable. This visual impression is confirmed by formal statistical tests which do not reject the hypothesis that the mean rate of growth has been the same in the first and in the second parts of the sample (see Stokey and Rebello (1995)). The second panel of Figure 1 shows the U.S. capital-output ratio for the period 1929–1998.[3] While this ratio rose during the Great Depression, it has been remarkably stable during the post-war period. The third panel of Figure 1 reports the annual real rate of return on the U.S. stock market compiled by Siegel (1995) for different time periods.[4] There is remarkably little variation in this real rate of return.[5] The last panel of Figure 1 shows that the variation in the labor income share in the period 1959–1998 is relatively small.

The data displayed in Figure 1 shows that the Kaldor facts are a good short-hand description of the U.S. growth process. But while these facts suggest that the growth process is smooth, this impression is quickly shattered once we move beyond these aggregate statistics to the simplest level of industry disaggregation.

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[3] In order to obtain the longest consistent time series for the capital stock we excluded public and residential capital.
[4] The high real return to the U.S. stock market during the 1990's does not invalidate the observation that the average real rate of return seems constant over time. Using the Ibbotson and Associates (1999) data for the real rate of return to the Standard and Poors 500 index, one cannot reject the hypothesis that the expected real rate of return has been constant across decades for the period 1930–1999. This partly reflects the fact that the high volatility of stock market returns makes it difficult to estimate the average rate of return. See Merton (1980) for a discussion of the difficulties associated with estimating expected returns.
[5] Siegel (1995) also reports the real return on bonds, which was slightly higher in the beginning of the sample than in the end. This time variation may, however, reflect the fact that the bond portfolio used in the first part of the sample comprised riskier securities (municipal and utility bonds), than the Treasury bills featured in the second part of the sample.

The Kuznets Facts

Table 1 summarizes the main elements of the structural change that has taken place in the U.S. and in other growing economies during the last 100 years.

Figure 2 illustrates the facts summarized in this table. Its first panel depicts the evolution of U.S. employment shares in Agriculture, Manufacturing, and Services from 1869 to 1998.[6] The decline in the agricultural workforce and the rise in the service sector are salient in this figure. The second panel of Figure 2 shows the evolution of aggregate consumption expenditure shares from 1940 to 1999. Over time the average U.S. consumer has increased the share of expenditure devoted to services and reduced the share devoted to agricultural products. These dynamics are associated with the rise in per capita income observed over this period. The fact that the share of expenditures devoted to the consumption of services increases, while the share devoted to agricultural goods decreases as income rises has been well documented in panel data studies of consumption patterns (Houthakker and Taylor (1970), Bils and Klenow (1998)).

Table 1. The Kuznets Facts

	Share of Total Employment	Share of Total Consumption Expenditures
Agriculture	Declines	declines
Manufacturing	stable[7]	stable
Services	Increases	increases

Are the sectoral movements documented in Figure 2 peculiar to the U.S. economy, or are they are general feature of economic development? Kongsamut et al (1999) studies both a long-run data set comprising 22 countries and a cross-section data set of 123 non-socialist countries for the period 1970–1980. Both data sets confirm the development regularities depicted in Figure 2. Growth in per capita income tends to be accompanied by a rise in services and a decline in the agricultural sector, both in terms of labor employment and relative weight in GDP.

III. The Model

Our model has three sectors of activity (agriculture, manufacturing, and services) which share production functions that are identical up to a constant of proportionality. We abstract from the presence of land and of international trade, to maintain a structure that is as close as possible to that of standard growth models.[8] We cast our model in continuous time but analogous results can be obtained in discrete time.

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[6] Construction and Mining were included in Manufacturing, while Transport and Communications were included in Services.

[7] This description reflects our focus on the last one hundred years. Prior to this period, there has been a rise in the importance of manufacturing, see Laitner (2000).

[8] For an analysis of structural change in which land plays a key role in the analysis see Goodfriend and McDermott (1995). Kongsamut (1995) finds little evidence that openness to trade influences the behavior of sectoral shares.

Production and Accumulation Technology

There are only two factors of production, capital (K_t) and labor. We normalize to one the total amount of labor available in the economy at every point in time. The production structure is as follows:

$$A_t = B_A F(\phi_t^A K_t, N_t^A X_t), \tag{1}$$

$$M_t + \dot{K}_t + \delta K_t = B_M F(\phi_t^M K_t, N_t^M X_t), \tag{2}$$

$$S_t = B_S F(\phi_t^S K_t, N_t^S X_t), \tag{3}$$

$$\phi_t^A + \phi_t^M + \phi_t^S = 1, \tag{4}$$

$$N_t^A + N_t^M + N_t^S = 1, \tag{5}$$

$$\dot{X}_t = X_t g, \tag{6}$$

$$K_0, X_0 > 0, \text{ given.} \tag{7}$$

The function $F(.,.)$ is assumed to be of class C^2, homogenous of degree one, concave, and increasing in both arguments. The variables ϕ^i and N^i denote, respectively, the fraction of capital and labor devoted to sector i. Capital and labor are freely mobile across sectors. The variable X_t denotes the level of technological progress which we assume to be labor augmenting. The existence of a balanced growth path requires this form of technical progress. Since we seek to generalize the concept of balanced growth, we retained the labor augmenting character of technical progress.

Output of the agriculture (A_t) and services (S_t) sectors can be used for consumption. The output of the manufacturing sector can be consumed (M_t) or invested ($\dot{K}_t + \delta K_t$). The assumption that only manufacturing output can be invested is consistent with the U.S. input-output tables. According to these tables the manufacturing and construction sectors produced between 90% and 93% of investment during the period 1958 to 1987.[9]

Efficiency in Production

Since capital and labor are freely mobile, an efficient allocation requires that the marginal rate of transformation be equated across the three production sectors, which implies:

$$\frac{\phi_t^A}{N_t^A} = \frac{\phi_t^M}{N_t^M} = \frac{\phi_t^S}{N_t^S} = 1. \tag{8}$$

Since the production functions of the different sectors are proportional, the relative prices of agriculture and services in terms of manufacturing goods are given by:

$$\begin{aligned} P_A &= B_M/B_A, \\ P_S &= B_M/B_S. \end{aligned}$$

[9]The U.S. input-output tables used in these calculations were published in the following issues of the Survey of Current Business: November 1964, February 1974, May 1984, April 1988, and April 1994.

Using these relative prices and the efficiency condition (8) we can rewrite the economy's resource constraint as:

$$M_t + \dot{K}_t + \delta K_t + P_A A_t + P_S S_t = B_M F(K_t, X_t). \tag{9}$$

Preferences

Given the economy's production structure, sectoral movements must originate from differences in the income elasticity of demand for the different goods. We assume that preferences are time-separable. Our momentary utility specification embeds different income elasticities in a parsimonious way that is familiar from work on the linear expenditure system[10]:

$$U = \int_0^\infty e^{-\rho t} \frac{[(A_t - \bar{A})^\beta M_t^\gamma (S_t + \bar{S})^\theta]^{1-\sigma} - 1}{1-\sigma} dt. \tag{10}$$

We assume that σ, β, γ, ρ, $\bar{A}$, $\bar{S}$, are all strictly positive and that $\beta + \gamma + \theta = 1$. These preferences imply that the income elasticity of demand is less than one for agricultural goods, equal to one for manufacturing goods, and greater than one for services. The variable $\bar{A}$ can be interpreted as the level of subsistence consumption, while $\bar{S}$ can be viewed as representing home production of services.

The Competitive Equilibrium

The competitive equilibrium for this economy coincides with the Pareto Optimal solution. This solution can be characterized by maximizing (10) subject to (9). The economy's real interest rate is given by:

$$r = B_M F_1(k, 1) - \delta, \tag{11}$$

where $k = K/X$. The optimal allocation of consumption across sectors requires:

$$\frac{P_A(A_t - \bar{A})}{\beta} = \frac{M_t}{\gamma}, \tag{12}$$

$$\frac{P_S(S_t + \bar{S})}{\theta} = \frac{M_t}{\gamma}. \tag{13}$$

The optimal path for the consumption of manufacturing goods must satisfy:

$$\frac{\dot{M}}{M} = \frac{r - \rho}{\sigma}. \tag{14}$$

A Balanced Growth Path

Suppose, for the moment that $\bar{A} = \bar{S} = 0$. It is clear from (9) that the only path along which all variables expand at a constant rate requires that A_t, M_t, S_t and K_t grow at rate g. Equations (11) and (14) determine the steady state value of k:

$$B_M F_1(k, 1) - \delta = \sigma g + \rho. \tag{15}$$

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[10]See Atkeson and Ogaki (1996) for recent empirical work using this class of preferences.

This equation has a simple interpretation. The only constant rate of growth that is feasible for this economy to adopt is g. The economy will follow a balanced growth path whenever its stock of capital is consistent with a real interest rate that leads households to choose to expand their consumption of the three goods at rate g.

A Generalized Balanced Growth Path

Let us now return to the case in which $\bar{A}$ and $\bar{S}$ are strictly positive. In this case a balanced growth path does not exist. Equations (12), (13) and (14) imply that even when the real interest rate is constant households do not choose to expand A and S at constant rates.

We will search for a path along which the real interest rate is constant. We choose this property of balanced growth as our starting point because, unlike other features of a balanced growth path (e.g. the constancy of the growth rate of output), it has clear, tractable implications in multi-sector models.

Definition *A Generalized Balanced Growth Path is a trajectory along which the real interest rate is constant.*

Equation (11) implies that k has to be constant in order for the real interest rate to be constant. The economy's resource constraint can be written as:

$$M_t + \dot{K}_t + \delta K_t + P_A A_t + P_S S_t = B_M F(k,1) X_t. \tag{16}$$

The right hand side of this equation expands at rate g. On the left hand side M_t, $\dot{K}_t$, and δK_t grow at rate g, but A_t and S_t do not. At first glance the requirement of a constant real interest rate appears incompatible with the system of differential equations that describes the competitive equilibrium. Suppose, however, that the following restriction, which we will discuss later, holds:

$$\bar{A} B_S = \bar{S} B_A. \tag{17}$$

This implies that $P_S \bar{S} - P_A \bar{A} = 0$, which allows us to re-write the resource constraint as:

$$M_t + \dot{K}_t + \delta K_t + P_A (A_t - \bar{A}) + P_S (S_t + \bar{S}) = B_M F(k,1) X_t. \tag{18}$$

Since both $A_t - \bar{A}$ and $S_t + \bar{S}$ grow at rate g, all the terms in this expression grow at a constant rate. This proves the following proposition:

Proposition *A Generalized Balanced Growth path exists whenever $\bar{A} B_S = \bar{S} B_A$. The GBG path for this model features constant relative prices, a constant aggregate labor income share, a constant growth rate for capital and aggregate output, a constant capital-output ratio, and time-varying sectoral growth rates and employment shares in the three sectors. The employment share declines in agriculture, rises in services, and is stable in manufacturing. The initial value of k consistent with the GBG path is given by equation (15).*

The growth rates of output in agriculture and services are given by:

$$\begin{aligned}\frac{\dot{A}_t}{A_t} &= g\frac{A_t - \bar{A}}{A_t}, \\ \frac{\dot{S}_t}{S_t} &= g\frac{S_t + \bar{S}}{S_t}.\end{aligned}$$

Using efficiency condition (8) we find that:

$$\begin{aligned}\dot{N}_t^A &= -g\frac{\bar{A}}{B_A X_t F(k,1)}, \\ \dot{N}_t^M &= 0, \\ \dot{N}_t^S &= g\frac{\bar{S}}{B_S X_t F(k,1)}.\end{aligned}$$

The share of labor in agriculture declines, while the share in services expands. The rates of change of both factor shares converge to zero in the long run. The two panels of Figure 3, show the evolution of the employment shares and the output share of the three sectors for a numerical example.[11] The fact that the employment shares coincide with the output shares is an implication of the production functions being identical up to a constant of proportionality.[12] As the economy grows the importance of $\bar{A}$ and $\bar{S}$ declines and the economy converges to a standard balanced growth path. This is consistent with the U.S. experience: the sectoral reallocation of labor out of agriculture has been limited since the 1970's and there has been a slowdown in the expansion of service employment.

Transition Dynamics

Analogous to a balanced growth model, this economy features some transitional dynamics whenever the initial value of k is incompatible with the GBG path. For this simple economy these transition dynamics are easy to characterize, because the model has the same stability properties as the one sector neoclassical growth model. To see this use equations (12) and (13) to substitute $A_t - \bar{A}$ and $S_t + \bar{S}$ into the utility function and in the resource constraint (9) and re-write the planning problem for this economy as:

$$\max U = \int_0^\infty e^{-\rho t}\frac{M_t^{1-\sigma} - 1}{1-\sigma}dt.,$$

[11]The parameter values used in this example are: $\bar{A} = 400$, $\bar{S} = 250$, $\beta = 0.1$, $\gamma = 0.15$, $\theta = 0.75$, $\sigma = 3$, $\rho = 0.01$, $\delta = 0.05$, $g = 0.018$, $X_0 = 100$, $B_A = 4$, $B_M = 1$, $B_S = 2.5$. The function $F(.)$ is assumed to be Cobb-Douglas with a capital share of 0.4.

[12]The requirement that the subsistence terms $\bar{A}$ and $\bar{S}$ be strictly positive to generate structural change may appear to contradict Baumol's (1967) results on unbalanced growth. His model has two sectors, manufacturing and services. His Proposition 1 implicitly assumes a momentary utility function of the form $u = S_t^{1-\gamma}M_t^{\gamma}$. Both goods are produced only with labor, and only manufacturing benefits from technical progress: $S_t = B_S N_t^S$, $M_t = B_M N_t^M X_t$. It is easy to see that this model features no structural change; it has a balanced growth path along which both N^S and N^M are constant. To produce structural change (Proposition 2) Baumol assumes that preferences or government policy somehow lead the economy to maintain a constant ratio, S_t/M_t. This obviously forces labor to flow from manufacturing into services.

subject to the constraint:

$$\frac{M_t}{\gamma} + \dot{K}_t + \delta K_t = B_M F(K_t, X_t),$$

and the initial conditions K_0 and X_0.

The GBG Restriction

The GBG path exists only when the restriction (17) holds. We are used to the many restrictions that are necessary for models to display balanced growth: labor-augmenting technical progress, isoelastic momentary utility and, in multisector models, cross-parameter restrictions similar to (17), tying together parameters of preferences and technology (see e.g. Evans, Honkapohja and Romer (1998)). It is perhaps, not surprising, that to generate richer dynamics we need an additional parameter restriction.

One interpretation of our economy is that each agent has a positive endowment of services $\bar{S}$ and a negative endowment of agricultural goods $-\bar{A}$. Restriction (17) requires that the market value of these endowments $P_S\bar{S} - P_A\bar{A}$ be equal to zero.

There is evidence of sectoral reallocation of employment out of agriculture into services for all growing countries. In contrast, evidence in favor of the Kaldor facts is less compelling for economies other than the U.S. This suggests that perhaps restriction (17) holds more closely for the U.S. economy than for other countries.

It is natural to ask what the model's implications are when equation (17) does not hold. We explore this question in the Appendix. We show that the equilibrium of an economy in which the GBG constraint does not hold converges asymptotically to the GBG path. We also characterize the behavior of the real interest rate during this adjustment process.

We have explored numerically the behavior of the real interest rate when the GBG constraint does not hold. We found that deviations from the GBG constraint that produce small changes in the pattern of labor reallocation lead to negligible movements in the real interest rate. Large violations of the GBG constraint produce moderate movements in the real interest rate but change dramatically the pattern of sectoral reallocation.[13]

IV. Extensions

The model presented here can be generalized and used in other settings. Two natural extensions are: (i) to introduce different sectoral production functions; and (ii) to allow for

[13]For example, we reduced $\bar{S}$ from 250 (the value used in the construction of Figure 3) to 200. As a consequence, the real interest rate was no longer constant but its variation was very small: it increased from 6.4% to 6.7% over a thirty year period and then slowly asymptoted to 6.4%. This reduction in $\bar{S}$ had a small impact on the pattern of labor reallocation–the fraction of total labor reallocated to the service sector fell by 3%. When we reduced $\bar{S}$ to zero the movements in the real interest rate became more pronounced–the real interest rate increased from 6.4% to 8.6% over a thirty year period and then asymptoted to 6.4%. This change in $\bar{S}$ had a large impact on sectoral reallocation patterns: the fraction of total labor reallocated toward services fell from 43% to 27%. If we fix $\bar{S}$ and lower $\bar{A}$ the effect on the behavior of the real interest rate is similar in magnitude. However, for reasons explained in the Appendix, the real interest rate falls initially instead of rising.

different sectoral rates of technical progress, so that the relative prices of the different goods can vary over time.[14] A more general version of our model can accommodate both of these phenomena (see Kongsamut et al (1999)). These more general models continue to require knife-edge conditions similar to (17) to guarantee the existence of the GBG path. However, these GBG constraints tend to be more complex and more difficult to interpret than (17).

Other long run empirical regularities can be easily accommodated within our model. Two examples are the long run decline in per capita hours worked and Wagner's law, the long run increase in the ratio of government spending to total output.

We focused on the dramatic reallocation of employment that has taken place in agriculture, manufacturing and services. But reallocation is also present within the manufacturing and the service sector. We are hopeful that the modeling techniques used in this paper will be useful in thinking about structural change at a more disaggregated level.

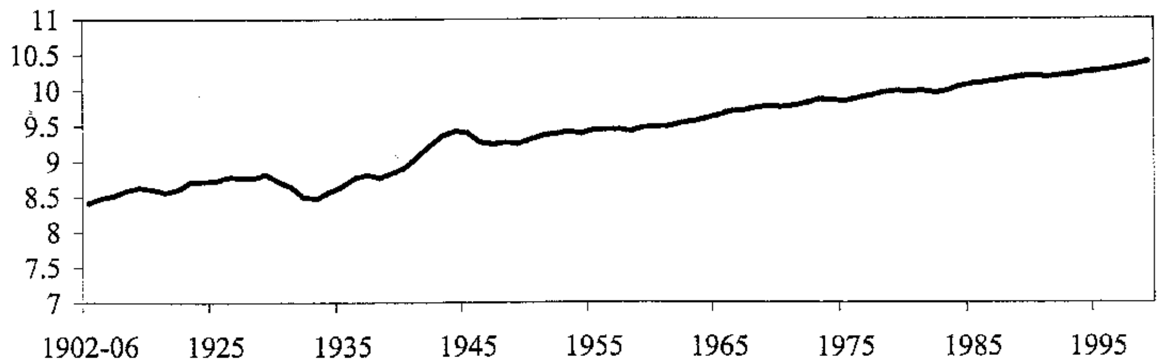
Finally, we suspect that other combinations of preferences and technologies can generate generalized balanced growth paths. Finding these is an exciting challenge to future research.

[14]Jorgenson and Gollop (1992) estimate that productivity growth has been high in agriculture and low in the service sector. However, productivity estimates are very sensitive to measurement error in the deflators used to compute real output. These deflators often fail to take into account the changes in the quality of the different goods that occur over time. Shapiro and Wilcox (1996) discuss the biases associated with increases in the quality of medical services. Hornstein and Krusell (1996) review the problems associated with productivity measurement in the service sector.

Figure 1: The Kaldor Facts

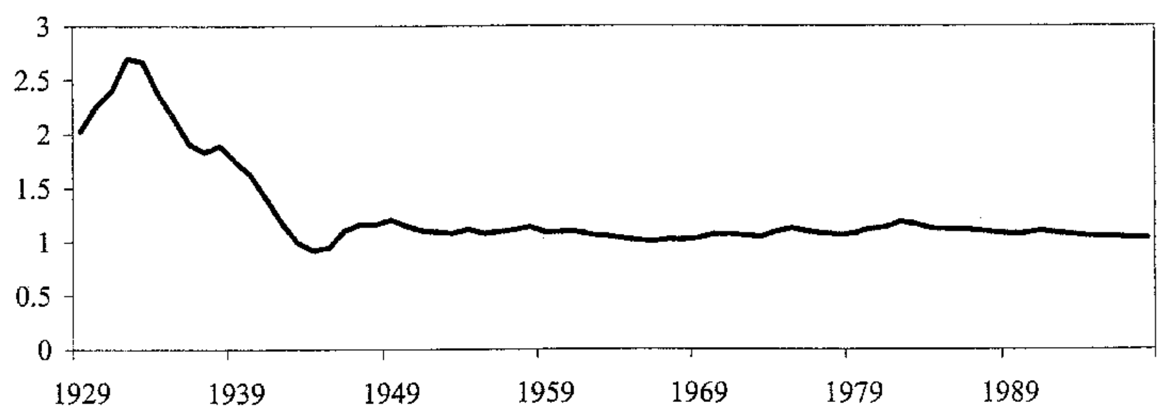
U.S. Real GDP per capita, 1902-1999

(in logarithms)



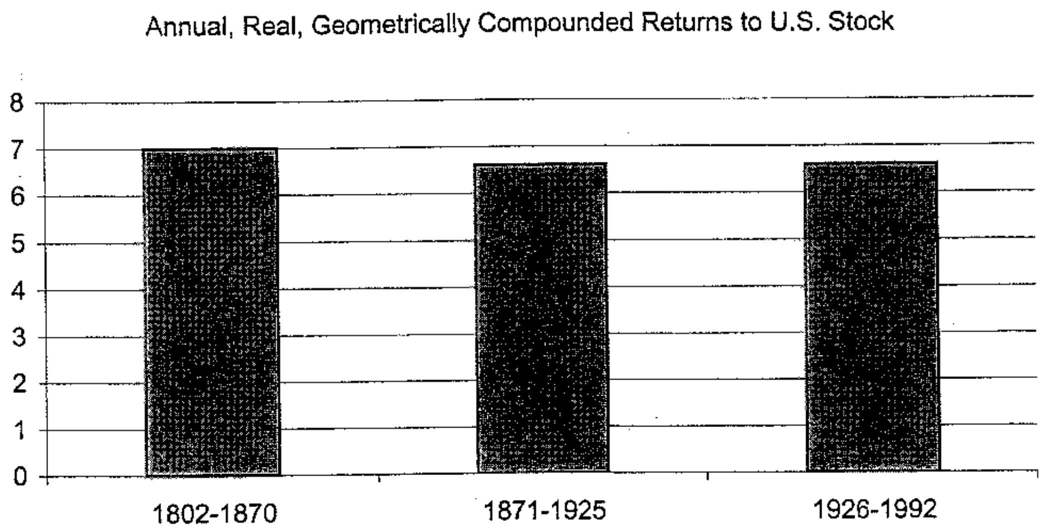
Sources: Historical Statistics of the U.S.; Economic Report of the President; and U.S. Census Bureau.

U.S. Capital-Output Ratio, 1929-1998

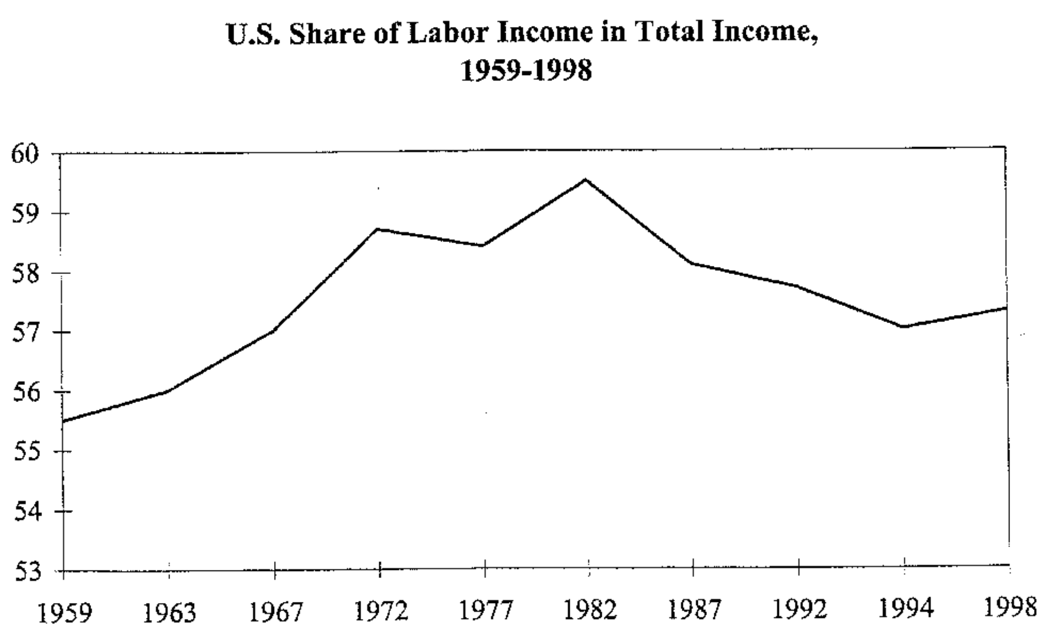


Sources: Fixed assets and consumer durable goods. Survey of current business, May 1997, April 2000.
NIPA (B.E.A. website).

Figure 1: The Kaldor Facts (cont.)

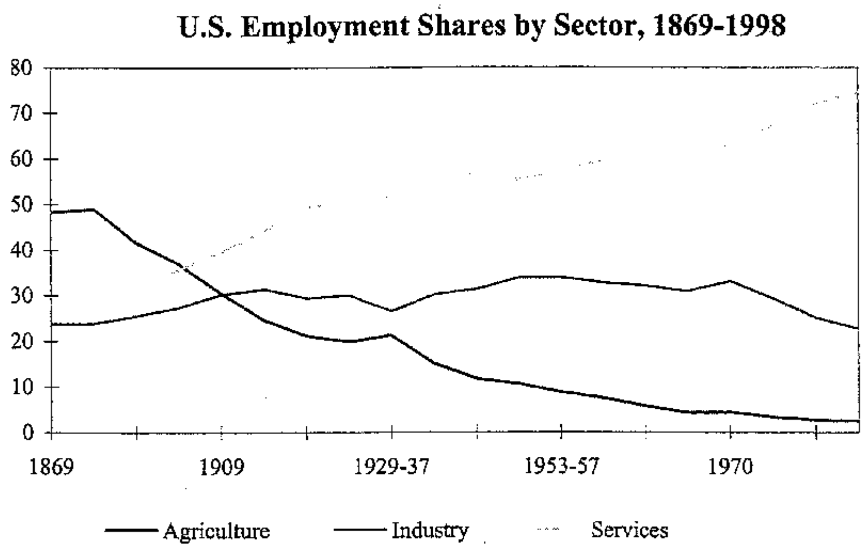


Source: Siegel (1995)

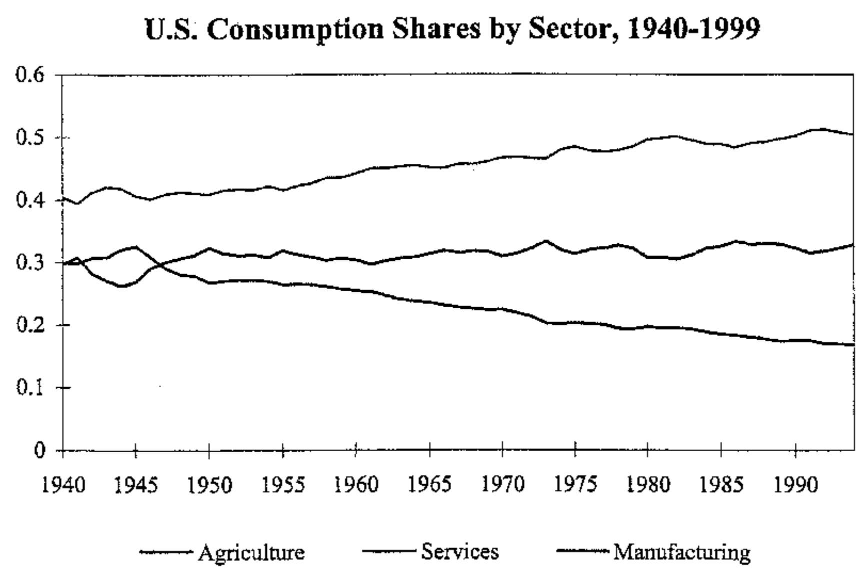


Source: Survey of Current Business, August 1996, and June 2000.

Figure 2: The Kuznets Facts

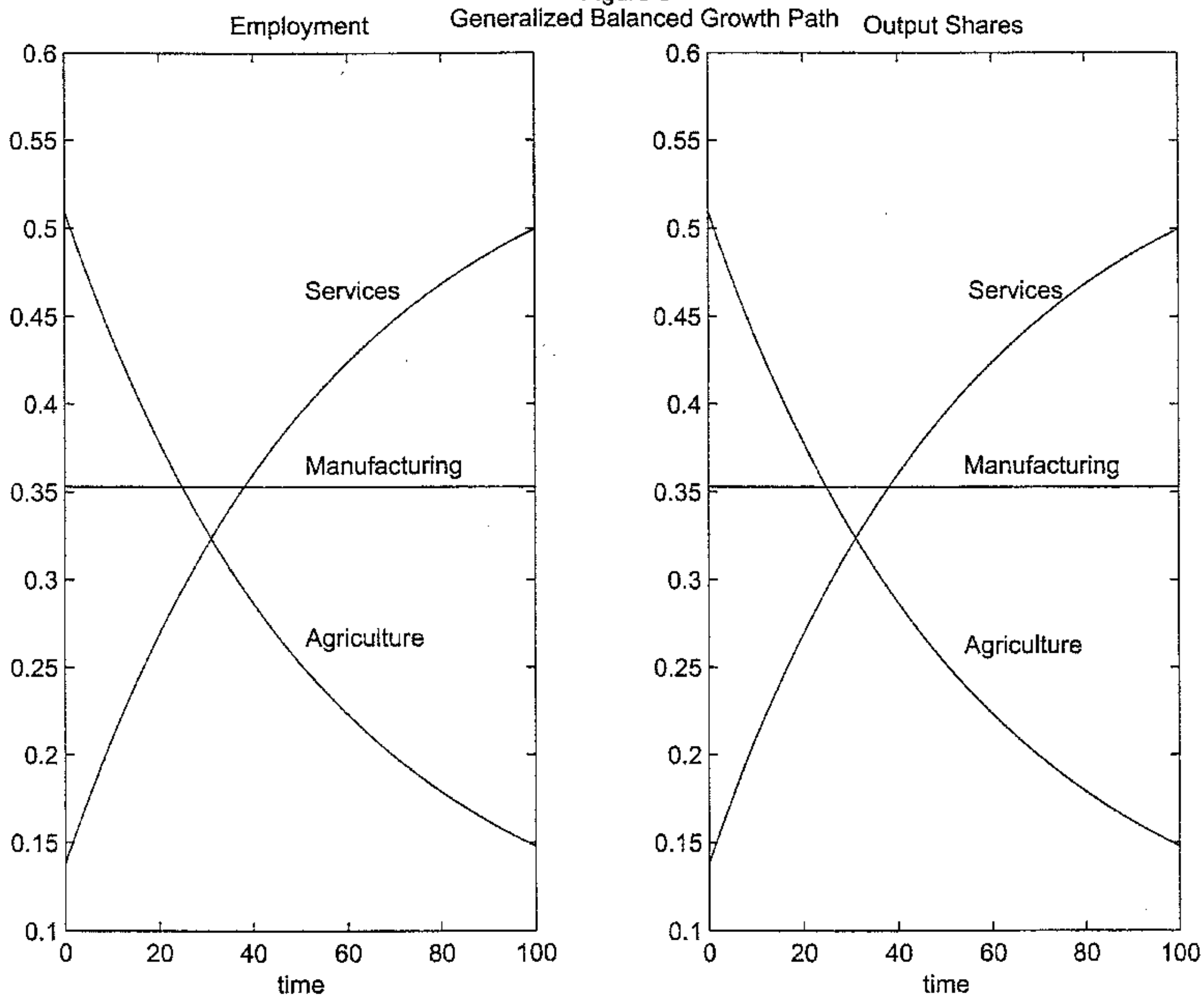


Sources: Historical Statistics of the United States, 1975 edition.
Statistical Abstract of the United States, 1999.



Source: Economic Report of the President (1990,2000).

Figure 3
Generalized Balanced Growth Path



What happens when the GBG constraint does not hold?

This appendix discusses a version of the model in which the GBG constraint does not hold–we assume that $\varepsilon = B_M(\bar{S}/B_S - \bar{A}/B_A) \neq 0$. To simplify our discussion we focus on the case of Cobb-Douglas production functions. We accomplish the following objectives: (i) we show how to use a phase diagram with moving isoclines to characterize the dynamics of the model; (ii) we show that the economy's equilibrium trajectory converges asymptotically to the GBG path; (iii) we characterize the economy's real interest rate dynamics.

Without loss of generality, let us assume that the constant ε is negative. Because the three sectors have similar production functions, we can aggregate these sectors so that the competitive equilibrium is the outcome of the following optimization problem:

$$\max \int_0^\infty \frac{M^{1-\sigma}}{1-\sigma} e^{-\rho t} dt,$$

$$\text{subject to: } \dot{K} = B_M K^\alpha X^{1-\alpha} - \frac{1}{\gamma} M - \delta K + \varepsilon. \tag{19}$$

where X grows at exogenous rate of g.

Let λ be the co-state variable associated with K. The first order conditions are:

$$M^{-\sigma} = \frac{\lambda}{\gamma},$$

$$\dot{\lambda} = \rho\lambda - \lambda\left(\alpha B_M K^{\alpha-1} X^{1-\alpha} - \delta\right).$$

The transversality condition is $\lim_{t\to\infty} K\lambda e^{-\rho t} = 0$. Define the following transformed variables: $\tilde{K} = K/X$, $\tilde{\lambda} = \lambda X^\sigma$. We can rewrite the first order conditions in terms of $\tilde{K}$ and $\tilde{\lambda}$ as follows:

$$\frac{d\tilde{K}}{dt} = B_M \tilde{K}^\alpha - \frac{1}{\gamma}\left[\frac{\tilde{\lambda}}{\gamma}\right]^{-1/\sigma} - (\delta + g)\tilde{K} + \frac{\varepsilon}{X},$$

$$\frac{d\tilde{\lambda}}{dt} = \tilde{\lambda}\left[\rho + \delta + \sigma g - \alpha B_M \tilde{K}^{\alpha-1}\right].$$

We cannot employ a standard phase diagram because X is time dependent and hence the system of differential equations is not autonomous. We can, however, use a phase diagram with moving isoclines to describe the dynamics because X depends on time in an explicit fashion, namely $X = X_0 e^{gt}$.

As an example, we use a phase diagram with moving isoclines to depict the equilibrium trajectory when $K_0/X_0 = \left[(\rho + \delta + \sigma g) / (\alpha B_M)\right]^{1/(\alpha-1)}$.

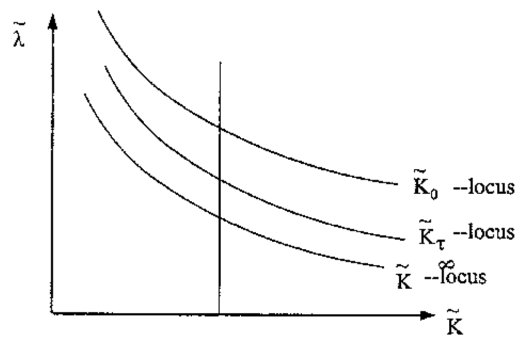


Figure 4: Time varying loci

In Figure 4, the vertical line represents the equation $d\tilde{\lambda}/dt = 0$. The downward sloping curve on the top is the locus along which $d\tilde{K}/dt = 0$ when $t = 0$. The curve in the middle represents the same locus when $t = \tau$ (we will abuse language and call it the $\tilde{K}_\tau$—locus). The curve at the bottom represents the points along which $\tilde{K}$ is constant when $t = \infty$.

At any time τ the following phase diagram indicates the directions of movement for $\tilde{K}$ and $\tilde{\lambda}$.

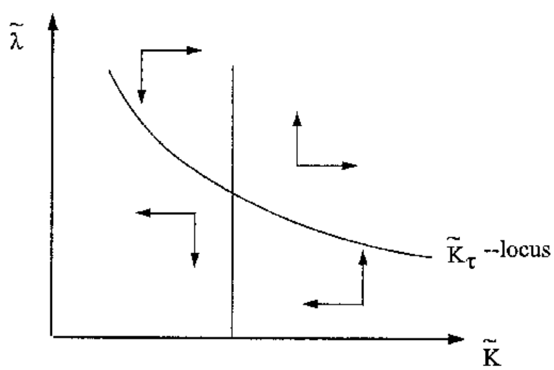


Figure 5: Directions for Immediate Movements at time τ

With the two Figures above understood, we can now draw three representative paths. The trajectory in the middle is the equilibrium trajectory.

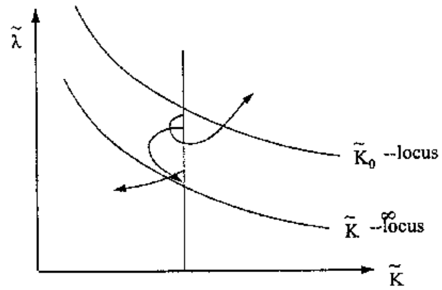


Figure 6: Representative Trajectories

To understand the top trajectory, note that the initial $(\tilde{K}, \tilde{\lambda})$ is below the $\tilde{K}_0$—locus and therefore the immediate direction of movement is toward the left. But since $(\tilde{K}, \tilde{\lambda})_0$ is close to both the $\tilde{K}_0$—locus and the vertical line, the magnitude of the movement is small. After a short while, say at $t = \tau$, $(\tilde{K}, \tilde{\lambda})_\tau$ will be above $\tilde{K}_\tau$—locus and therefore the trajectory moves down to the right. This movement continues until the path hits the vertical line. After this point the trajectory moves up to the right.

To understand the bottom trajectory, note that the initial $(\tilde{K}, \tilde{\lambda})$ is far below the $\tilde{K}_0$—locus and therefore the leftward movement is strong at the beginning. In fact, the movement is so strong that $(\tilde{K}, \tilde{\lambda})_t$ lies under $\tilde{K}_t$—locus for all t. Once the trajectory passes $\tilde{K}_\infty$—locus, there is no turning back.

The equilibrium locus is the unique path which starts at an appropriate $(\tilde{K}, \tilde{\lambda})$ so that as time elapses, $(\tilde{K}, \tilde{\lambda})_t$ comes to be above and stays above the $\tilde{K}_t$—locus and in the meantime $(\tilde{K}, \tilde{\lambda})_t$ stays to the left of the vertical line.

To see the property of the equilibrium path with ε negative, note that K_t/X_t first declines and then increases and approaches the steady state level. Therefore, the real interest rate increases first and then comes back to the original level. When ε is positive the paths of the real interest rate and of K_t/X_t are the mirror image of the paths corresponding to ε negative.

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