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A Primer for Risk Measurement of Bonded Debt from the Perspective of a Sovereign Debt Manager

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A Primer for Risk Measurement of Bonded Debt from the Perspective of a Sovereign Debt Manager

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Abstract

This paper presents some conventional and new measures of market, credit, and liquidity risks for government bonds. These measures are analyzed from the perspective of a sovereign's debt manager. In particular, it examines duration, convexity, M-square, skewness, kurtosis, and VaR statistics as measures of interest rate exposure; a VaR statistic as the prominent measure of exchange rate exposure; the balance sheet approach (or contingent claims approach), and its consequent probability of default as the most promising measure of credit risk exposure; and an elasticity approach and a VaR statistic to measure liquidity risk. Along with the formulas for the various statistics proposed, we provide simple examples of their application to some common risk valuation cases. Finally, we present an integrated approach for the simultaneous estimation of a portfolio's interest rate and exchange rate risk using the VaR methodology. The integrated approach is then extended to also include N risk factors. This approach allows us to measure the total risk of a portfolio, provided that the volatilities and correlations among the risk factors can be estimated.

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I. INTRODUCTION

Sovereign debt managers have been increasingly interested in assessing the risks associated with bond debt instruments, defined traditionally as fixed income securities that pay a fixed rate of interest until the bond matures. The value of government bonds fluctuates as market yields, exchange rates, credit and liquidity conditions change over time, thus creating opportunities for sovereign liability management operations. It is therefore essential to understand the sources of risk in government bond markets and how they affect government debt stocks.

The main sources of financial risk for sovereigns relate to the total amount, maturity structure, and currency composition of their debt stocks, as well as to the liquidity conditions in established government bond markets. In analyzing these risks, market analysts (based on the finance literature) frequently focus on market risk (interest rate risk and exchange rate risk), credit risk, and liquidity risk. In particular, measurement of credit risk, stemming from the perceived creditworthiness and potential default of a sovereign, and liquidity risk, stemming from liquidity considerations, have recently gained increased attention.

For a fixed income asset (liability), duration and convexity, along with yield, are essential metrics/variables for measuring the value of a bond to an investor or a debt manager of a sovereign. Also, these metrics/variables can be used for evaluating a portfolio of fixed income assets (liabilities). However, the use of a consistent yield calculation method is particularly important when computing the average yield of a portfolio containing a variety of bond debt instruments, while accurate computations of duration and convexity are essential for evaluating the riskiness of a bond debt portfolio.

In calculating yields, many different methods are used in the different bond markets of the world (see Appendix I). For example, to calculate the yield of a fixed-interest security, a number of methods of accrued interest calculation is used depending on the particular market. This makes direct comparison of the quoted yields for different securities across markets difficult. However, various methods of yield calculation that allow for consistent yield comparisons have been developed (e.g., Credit Suisse First Boston, 1988).

For interest rate exposure, the two most conventional measures in use are duration and convexity. These measures provide indications of the likely performance of a bond when interest rates change (e.g., parallel shifts in the yield curve). Duration measures the sensitivity of asset (liability) prices to movements in interest rates, and often involves a first-order, linear approximation to interest rate exposure. Convexity measures the curvature of the relationship between changes in asset (liability) prices and changes in interest rates.

Exchange rate exposure, as well as interest rate exposure, is frequently measured through the Value at Risk (VaR) methodology. The VaR approach determines a maximum loss that can occur in a portfolio with a certain confidence level, under nonextreme circumstances. Credit risk can be measured by a scoring model or a probabilistic model, such as the Contingent Claims Approach (CCA), which tries to determine probabilities of debt bond default from given changes in economic conditions. Liquidity risk is mostly measured by liquidity gap or liquidity elasticity analysis, supplemented by stress testing or simulation scenarios, and by VaR-type methods.

This paper describes some conventional measures of interest rate risk exposure, such as duration, convexity, M-square, skewness, kurtosis, and VaR statistics, and of exchange rate risk exposure, such as VaR statistics (Section II); of credit risk exposure, such as the probability of default and distance to distress (Section III); and liquidity exposure for a sovereign's debt portfolio, such as a liquidity elasticity measure (Section IV). In addition, the note outlines an integrated approach to market risk as opposed to individual types of risk, and provides simple application examples. The intergraded approach is then generalized for a security and a portfolio with N risk factors (Sections V and VI). We conclude by offering some remarks on the applicability of these measures (Section VII).

II. Measurement of Market Risk

The measurement of a linear exposure to a movement in an underlying risk variable has become an integral part of financial risk management. In the fixed income markets, where about three-fourths of the volatility of bond prices is explained by a common interest rate factor (Jorion and Khoury, 1966), a first-order, linear, approximation (first derivative) to the exposure of an asset to movements in interest rates is called duration, while the second-order form (second derivative) is called convexity. In the foreign exchange market, this exposure is called exchange rate exposure. When credit and liquidity considerations are prevalent, we term these exposures as credit and liquidity exposures, respectively.

A. Interest Rate Risk

Duration and convexity are considered local measures of interest rate exposure, and are valid when there is a parallel shift in the yield curve. However, they may not be valid estimates of exposure for very large rate moves (Jorion and Khoury, 1996). When very large rate moves are prevalent, stress testing or Monte Carlo simulation methods may be advisable (not covered in this section). Also, when there are twists in the yield curve, the M-square measure is recommended to be applied.

1. Duration

Duration indicates the sensitivity of a bond (or a portfolio of bonds) to changes in interest rate (parallel shifts in the spot curve). Specifically, it measures a bond's (or a portfolio of bonds') price fluctuations in response to interest rate changes. Duration is always measured in units of time. Thus, duration is defined as a weighted partial derivative:

$$D\left(duration\right) = -\frac{1}{\mathrm{P}} \times \frac{\delta \mathrm{P}}{\delta r} \qquad \text{(II.1)}$$

where:

P = the market value of a bond

$r =$ interest rates

This can be approximated (first-order approximation) as

$$\frac{\Delta \mathrm{P}}{\mathrm{P}} \cong -D\ (duration) \times \Delta r \tag{II.2}$$

This approximation states, for example, that if a bond (or a bond portfolio) has a duration of three years, the value of the bond (or the bond portfolio) will decline about 3 percent for each 1 percent increase in interest rates.

For bonds (or bond portfolios) with fixed cash flow payments (i.e., non-callable bonds), a commonly-used measure to calculate duration is the Macaulay duration (Stigum, 1990). For such bonds, duration is just the average maturity of the cash flows. Thus, the Macaulay duration is defined as the weighted average maturity, and is calculated by the weighted average of the times to each of the cash payments. The weights are the present values of the cash payments that will take place in the future. The formula of the Macaulay duration is:

$$Dm\,(Macaulay\ duration) = \frac{\sum_{t=1}^{\mathrm{T}} \mathrm{P}V(C_\tau) \times t}{\sum_{t=1}^{\mathrm{T}} \mathrm{P}V(C_t)} \tag{II.3}$$

where:

Dm = Macaulay duration, in number of periods
t = the time period (annual, semiannual, or other) that each fixed coupon or principal payment occurs
T = the number of periods to final maturity
C_t = the interest or principal payment in period t
PV (C_t) = present value of C_t

with

$$\mathrm{P}V(C_t) = \frac{C_t}{(1+y)^t} \tag{II.4}$$

where
y = the yield to maturity for the bond (see Appendix I)

and

$$\mathrm{P} = \sum_{t=1}^{\mathrm{T}} \mathrm{P}V(C_t) \tag{II.5}$$

where

P = the market value of a bond

In the Macaulay formula for duration, all present values are calculated using the yield to maturity for all cash flows that are discounted. If present values are calculated with a non-small yield to maturity, ym, for all bonds in a portfolio, the Macaulay formula should be modified to:

$$\textit{Dmo (modified duration)} = \frac{1}{1+(ym/m)} \times \frac{\sum_{t=1}^{T} \mathrm{P}V(C_t) \times t}{\sum_{t=1}^{T} PV(C_t)} \tag{II.6}$$

where

m = the frequency of compounding for the yield to maturity, i.e., if the yield to maturity is compounded quarterly, m = 4

When yields are measured using m compounding periods in a year, the resulting duration measure is expressed in number of subperiods. To convert duration to an annual measure, it should be divided by m.

The use of duration as a measure of exposure to interest rate risk relies on a number of implicit assumptions, which have financial implications. The main assumptions are that (i) cash flows are known with certainty, i.e., bonds are noncallable and default risk free; (ii) all interest rates change by the same amount (parallel shifts in the entire term structure); (iii) for Macaulay's duration, the term structure is assumed to be flat. A flat yield curve is presumed because each coupon payment is discounted at the same yield to maturity; (iv) duration is only a linear approximation of interest rate exposure and, therefore, it is valid only with infinitesimal (instantaneous) changes in yields. However, when there are large shocks to the term structure, higher-order terms in the price derivative should be incorporated into the valuation (see below convexity).

It also has been pointed out that, although long-duration bonds are more price sensitive to a given change in the yield to maturity than short-duration bonds, short-duration yields are more volatile than long-duration yields (Hull, 2000). Thus, to evaluate the riskiness of a bond, both duration and yield volatility should be considered (Yawitz and Marshall, 1981). Furthermore, since movements in the term structure are usually not parallel, duration can be considered only an approximation to the risk index of a bond.

For a portfolio of fixed income instruments, an approximation to the overall portfolio duration is a simple weighted average of the components of the portfolio durations. If x_i

represents the proportions of n different bonds in a bond debt portfolio, the portfolio duration can be approximated by:

$$Portfolio\ duration = \sum_{i=1}^{n} x_i \times D_i \tag{II.7}$$

where D_i is the duration of bond i.

In terms of the effectiveness of duration hedging over a given planning period, Bierwag, Kaufman, and Toevs (1983) have shown that the simple Macaulay duration provides the most cost effective immunization method, as compared to additive, multiplicative, and long-multiplicative process duration.

Factors Affecting Duration

In general, the duration of a bond increases with its maturity. For a bond trading at par, the duration decreases when its yield increases, while it decreases towards zero as the bond approaches maturity. Between coupon dates, the duration decreases linearly with time, but suddenly rises at a coupon date. However, for bonds trading at a discount, the duration can decrease with maturity—this decrease often starts at longer maturities. For discounted bonds, their Macaulay duration is less than its time to maturity. For example, the duration (average maturity) of a 6 percent semiannual coupon, 10 percent yield, 5 year bond trading at discount, is calculated at 4.35 years (Table 1).

For zero-coupon bonds, their Macaulay duration is equal to time to maturity (since zero coupon bonds do not pay coupons). In other words, interest rate changes after the issue date of a zero-coupon bond do not affect its value until the maturity date; they only affect the renewal of debt. For example, a zero-coupon bond with a 5-year time to maturity has a 5-year Macaulay duration. This will then mean that the value of a 5-year zero coupon bond will increase about 5 percent for each 1 percent decline in compounded interest rates (based on the approximation formula mentioned above). For consol bonds (perpetuities), the lower the yield the longer the duration.

Therefore, with regards to duration and maturity, for a zero coupon bond, duration equals maturity, while a perpetuity's duration is approximated by (1+y) / y. For coupon-paying bonds, duration is always less than maturity, but the relationship between a coupon bond's duration and its maturity is not uniform. The duration of par and premium bonds increases as maturity increases, holding coupon and yield fixed, and approaches but will always be less than the perpetuity duration. In contrast, the duration of some discount bonds can exceed that of perpetuities. With regards to duration and coupon, the duration of a bond decreases as the coupon rate increases, holding maturity and yield constant. Finally, with regard to duration and yield, the duration of a bond decreases as the yield increases, holding maturity and coupon constant (Table 1).

Duration and Borrowing Instruments

The wide use of duration as a measure of interest rate exposure derives from its simplicity to demonstrate the sensitivity of a bond debt to interest rate changes, and to provide an approximation of the magnitude of this impact. For example, a decrease in the average duration of debt from three to two months indicates that the sovereign debt's interest rate sensitivity has decreased. The approximate change in the value of the bond debt stock will, then, be a decrease by 2 percent from a 1 percentage point increase in interest rates when the duration has decreased to two months, instead of a 3 percent decrease when the duration is three months.

For a bond debt stock, its duration also indicates the time period needed for the stock to be affected by an interest rate change. It is estimated by calculating the duration of each bond in the stock and weighting it by its share in the stock portfolio. A debt stock with a longer duration is affected in a longer time period from interest rate changes and, when compared to a debt stock with a shorter duration, has more sensitivity to interest rate fluctuations.

A long duration debt stock may also imply that the share of long-term, fixed-rate instruments is high within the stock of debt. However, when the yield curve is upward-slopping, i.e., long-term interest rates are above short-term ones, borrowing with fixed-rate, long-term instruments would increase the cost of borrowing. For this reason, sovereigns monitor the duration of their debt stock and try to hold it within a given range. Thus, while the lower limit of duration restrains the volatility of debt redemptions, the upper limit constraints borrowing with higher costs, for fixed-rate instruments.

If the yield curve shifts upward and its slope remains the same, a sovereign may prefer to issue at the short end of the curve to avoid locking in high long-term rates. Meanwhile, rises in the yield curve lead to capital losses for investors with long-maturity bonds, and then investors would be better off investing in shorter-term maturities. However, even in case of an upward shift in (a positively sloped) yield curve, some countries may still wish to borrow at the long end of the curve for ensuring the availability of long-term funds (i.e., as an insurance policy).

For callable bonds, their duration (estimated by option valuation techniques, since the Macaulay formula cannot be applied because future coupon payments are not known with certainty) changes sharply as the yield is changed. A callable bond priced below par will trade like a bullet maturing on the maturity date. As the price increases above par, the bond will trade like a bullet maturing on the call date. This causes the duration to drop sharply.

For floating rate bonds, their duration is calculated to be the interest (coupon) period until the next coupon period and, therefore, the above cited Macaulay duration formula cannot be applied. For example, the duration of Treasury government bonds with quarterly coupon payments and indexed to 3-month T-bill interest rate, is 3-months at the beginning of the coupon period. If market interest rates change after the settlement of coupon interest rate, they will be reflected in the value of the bond only 3 months later.

2. Convexity

Convexity describes the curvature in the relationship between a bond's, or a portfolio of bonds', price and interest rates. When there are large yield changes, even if the term structure is flat and undergoes parallel shifts, a second-order measure of exposure may be needed to be incorporated into a bond's or a portfolio of bonds' risk valuation. Convexity is measured in periods squared. In general, convexity is defined as the a weighted second partial derivative

$$C\ (convexity) = -\frac{1}{\mathrm{P}} \times \frac{\delta^2 \mathrm{P}}{\delta r^2} = \frac{\sum_{t=1}^{\mathrm{T}} \mathrm{P}V(C_t) \times t \times (t+1)}{\sum_{t=1}^{\mathrm{T}} \mathrm{P}V(C_t)} \tag{II.8}$$

where

P = the market value of a bond
r = interest rates

This can be approximated (using a Taylor expansion with two terms or second-order approximation) as

$$\frac{\Delta \mathrm{P}}{\mathrm{P}} \cong \frac{Convexity}{2} \times \Delta r^2 - Duration \ \times \Delta r = -\left[Duration - (Convexity/2) \times \Delta r\right] \times \Delta r \tag{II.9}$$

In rearranging equation (II.9) to solve for duration, it is possible to see that convexity is a second-order effect that describes the way in which duration changes as yield changes. When the changes in yield are small, the convexity term can be ignored. However, if yield changes are not small, convexity causes duration to increase in response to a decrease in rates and to decrease in response to an increase in rates. Note that an annual measure is obtained by dividing convexity by the square of the number of compounding periods m in a year.

For noncallable bonds, convexity is a positive number, implying that the true price-yield curve lies above the duration line. That is, bond prices rise more than by duration (i.e., the linear approximation) when yields fall and decrease less than by duration when yields rise. When both duration and convexity are used together, the prediction of bond price changes is far better over a broader spectrum of rate changes (Table 2).

The convexity of a portfolio of fixed income bonds can be derived from a simple weighted average of the components of the portfolio convexity. If x_i is the proportion invested in bond i with convexity C_i, portfolio convexity can be approximated by:

$$Portfolio\ convexity = \sum_{i=1}^{\mathrm{N}} x_i \times C_i \tag{II.10}$$

Duration and convexity have traditionally been used as instruments for asset-liability management. To avoid exposure to parallel spot curve shifts, a debt manager with significant sovereign fixed-income exposure might try duration matching, i.e., to structure its assets so that their duration matches the duration of its liabilities and, thus, the two offset each other. Even more effective is duration-convexity matching, where assets are structured so that duration and convexities match – indeed, a more difficult task.

Factors Affecting Convexity

Convexity is increased by a lower coupon, a lower yield level, and longer term. Because convexity is based on measures of squared time, it increases sharply with duration. The convexity zero coupon bonds is: $T \times (T+1) / (1+y)^2$.

Positive convexity implies that prices increase at a faster rate as yields drop than prices decrease as rates rise. Thus, convexity is more desirable when the market perceives interest rate volatility to be high. In theory, bonds with a high convexity should outperform bonds with a low convexity, assuming equal duration and yield. A high-convexity bond duration will increase more in price as rates fall than a bond with low convexity. The same is true when rates rise. As duration falls more for a high-convexity bond, it becomes more defensive, outperforming a low-convexity bond. Therefore, bond portfolio managers seek for higher convexity. Note that these results hold only under the assumption of parallel shifts in the term structure. When the term structure twists, that is, changes shape instead of moving up or down, it may not be optimal to maximize convexity. For the latter case, another measure, M-square, is applied.

3. M-Square

In the traditional theory of immunization, interest rate risk is eliminated by maintaining the duration of the portfolio equal to the time horizon of the investor.[2] Then, the portfolio value cannot fall below a target value and, therefore, it is immunized against parallel shifts in the term structure. However, with twists in the yield curve, immunization can fail. Fong and Vasicek (1983 and 1984) developed a model of risk control that accounts for nonparallel shifts in the term structure. They showed that the change in the end-of-horizon value of an immunized portfolio ΔP(H) resulting from an arbitrary change in interest rates can be approximated by

$$\Delta \mathrm{P}(H) / \mathrm{P}(H) = -\mathrm{M}^2 \times \Delta_S \qquad \text{(II.11)}$$

where Δ_S is the change in the slope of the term structure and

[2] Immunization refers to the need to guarantee a minimum rate of return over a planning period.

$$M^2 = \left(1/P\right)\sum_{t=1}^{T}\left(t-H\right)^2 \times C_t \; / \; \left(1+y\right)^t \qquad \text{(II.12)}$$

This formula shows that M-square is a weighted average variance around the horizon date of the cash flows generated by the portfolio. Although Δ_S is beyond the control of the manager, M-square can be controlled. The value of Δ_S can be characterized as the twist in the term structure with M-square representing the manager's exposure to such twists. M-square is then a measure of risk for the immunized portfolio because it measures the remaining exposure of the portfolio to rate changes.

For a zero coupon bond with a maturity equal to the length of the horizon, M-square is always nonnegative and attains its lowest value, zero. This is because, for a zero coupon bond, no change in yield can affect the final value because there is no reinvestment risk and, therefore, the bond is perfectly immunized against any movement in the term structure. Note that Nawalka, Lacey, and Schneeweis (1990) have derived a closed-form for M-square, as well as for convexity, in the case of constant coupon payments.

The relationship between M-square and duration/convexity was developed by Schnabel (1990) as:

$$M^2 \cong Convexity - Duration \qquad \text{(II.13)}$$

It becomes obvious from this relation that maximizing convexity is the same as maximizing M-square for immunized portfolios. Thus, by maximizing convexity, the bond portfolio manager is also maximizing the twists in the term structure. Therefore, the manager should seek to maximize convexity when he/she expects parallel moves in the yield curve and minimize the M-square when he/she expects twists in the term structure. Conversely, a debt manager, who would need to guarantee a maximum rate of borrowing costs, he/she should seek to minimize convexity or maximize M-square when he/she expects parallel shifts or twists in the term structure, respectively.

4. Higher Moments

In practice, other statistical measures of bond price behavior are used to immunize a portfolio. The most commonly used are the skewness and kurtosis.

Skewness

In risk management, skewness (reflecting the asymmetry of a bond price distribution) indicates whether the probability of gains is similar to the probability of losses. The skewness of a sample of T bond prices is calculated as follows:

$$Sk = \frac{T}{\left(T-1\right)\times\left(T-2\right)} \times \sum_{t=1}^{T}\left[\left(\frac{P_t-\mu}{\sigma}\right)^3\right] \qquad \text{(II.14)}$$

where

μ = mean of bond prices over T
σ = standard deviation

The skewness for the normal distribution is zero. Typically, the higher the skewness value, the lower the downside risk of a portfolio (Nawrocki, 1999). In addition, the higher the negative value of skewness, the more extreme losses than gains. Therefore, portfolio managers look for portfolios with the highest possible positive skewness, while sovereign debt managers may look for portfolios with the highest negative skewness.

Kurtosis

Kurtosis reflects the extreme event of a worse-possible loss. For example, two different portfolios with the same mean, standard deviation, and skewness, but different kurtosis would tend to suffer different losses in case of extreme events, e.g., losses that only have a 1 in 1,000 chance of occurrence (every 1,000 days). In particular, the portfolio with the higher kurtosis would suffer worse losses than the portfolio with lower kurtosis. The kurtosis of a sample is calculated as follows:

$$k = \frac{\mathrm{T} \times (\mathrm{T}+1)}{(\mathrm{T}-1) \times (\mathrm{T}-2) \times (\mathrm{T}-3)} \times \sum_{t=1}^{\mathrm{T}} \left[\left(\frac{\mathrm{P}_t - \mu}{\sigma} \right)^4 \right] \tag{II.15}$$

The kurtosis for the normal distribution is 3, while "excess kurtosis" is often used in empirical work:

$$ke = \frac{\mathrm{T} \times (\mathrm{T}+1)}{(\mathrm{T}-1) \times (\mathrm{T}-2) \times (\mathrm{T}-3)} \times \sum_{t=1}^{\mathrm{T}} \left[\left(\frac{\mathrm{P}_t - \mu}{\sigma} \right)^4 \right] - 3 \times \frac{(\mathrm{T}-1)^2}{(\mathrm{T}-2) \times (\mathrm{T}-3)} \tag{II.16}$$

Distributions with a kurtosis greater than the normal distribution are said to have leptokurtosis. A high positive kurtosis indicates a greater probability of losses under extreme events. Thus, portfolio managers look for bond portfolios with lower kurtosis.

5. VaR Measures

The Value at Risk (VaR) methodology is currently considered as one of the best approaches to assess market risk (interest rate and exchange rate risk) (Marrison, 2002 - see also Appendix II). The VaR measure of market risk combines the sensitivity of a portfolio to changes in market-risk factors with the probability of a given change in these factors. That is why the Basel Committee has adopted the VaR methodology for setting the standard for the minimum amount of capital to be held against market risks. Measuring the interest rate exposure of a bonds portfolio may entail calculation of volatilities and correlations of included bonds (Jorion, 1997).

a) For the interest rate risk on a bond, VaR can be approximated by multiplying the dollar duration (i.e., duration times the current price of the bond) by the worst-case daily interest rate move. This move implies that there is a probability of only 1 percent that the change could be more than this worst case. Accordingly, this gives the value change in the worst case:

$$VaR \cong Duration \times \mathrm{P} \times \delta\, r_{worst\ case} \tag{II.17}$$

Assuming that interest rate movements have a normal probability distribution, the 1 percent worst case will correspond to 2.33 standard deviations of the daily rate movements, σ_r (Appendix II). Then, the VaR for a bond is approximately equal to the duration in dollars times 2.33 standard deviations:

$$VaR \cong Duration \times \mathrm{P} \times 2.33 \times \sigma_r \tag{II.18}$$

Thus, if the duration is 10 years, the current price of the bond \$100, and the daily standard deviation in the absolute level of interest rates is 0.2 percent, the VaR is approximately:

$$VaR \cong 10 \times \$100 \times 2.33 \times 0.002 = \$4.66$$

That is, there is 1 percent probability that the price of this bond will fall by more than \$4.66. In calculating the VaR for these bonds, the following assumptions are made: (i) the probability of the changes in interest rates is normally distributed; (ii) rates for every tenor move by the same amount (i.e., there is a parallel shift in the yield curve); and (iii) the change in the price can be well-approximated by a linear measure of duration.

b) Also, for absolute changes in interest rates (one risk factor), the VaR for a bond can be calculated using the parametric VaR approach (following the steps mentioned in Appendix II). Assuming a U.S. government bond (or any local-currency denominated government bond exposed to changes in local interest rates) with a single payment, the present value, $PV_\$$, is the cash flow, $C_\$$,at time t discounted according to the interest rate for that maturity, $r_\$$:

$$\mathrm{P}V_{\$} = \frac{C_{\$}}{(1 + r_{\$})^{t}} \tag{II.19}$$

The sensitivity of the value to changes in interest rates d_r is the derivative with respect to $r_{\$}$:

$$\frac{\delta \mathrm{P}V_{\$}}{\delta r_{\$}} = \left[\frac{-t \times C_{\$}}{(1 + r_{\$})^{t+1}}\right] = d_r \tag{II.20}$$

Then, the change in the value is the sensitivity multiplied by the change in interest rates:

$$\Delta \mathrm{P}V_{\$} = d_r \times \Delta r_{\$} \tag{II.21}$$

The standard deviation of $\mathrm{P}V_{\$}$ is then the standard deviation of the interest rates times d_r and the VaR is 2.33 times the standard deviation of value:

$$VaR = 2.33 \times d_r \times \sigma_r \tag{II.22}$$

As an example, consider a bond paying \$100 in 10 years' time, with the 10-year discount rate at 4 percent and a standard deviation in the interest rate of 0.2 percent. Then, the present value is \$68, the sensitivity is - \$650 per 100 percent increase in interest rates, and the VaR is \$3.

$$\mathrm{P}V_{\$} = \frac{100}{(1.04)^{10}} = \$68$$

$$d_r = \frac{-10 \times 100}{(1.04)^{11}} = -\$650$$

$$VaR = |\, 2.33 \times \frac{-10 \times 100}{(1.04)^{11}} \times 0.002 \,| = \$3$$

Finally, when interest rates deviate significantly from normality, the use of standard deviation multiples based on the normality assumption (such as 2.33 for the 1 percent worst case) leads to an underestimation of risk. In this case, a correction factor, δ, for the standard deviation needs to be introduced to take account of leptokurtic or "fat tailed" distributions of interest rates. The correction factor is such that δ=1 if the distribution of interest rates is normal, and δ>1 if it is leptokurtic, with δ being an increasing function of the unconditional kurtosis. Accordingly, the VaR estimate would now take into account both distributional characteristics, the standard deviation and kurtosis (see also p.12). An explicit relationship between the correction factor, δ, and kurtosis, k_i, for t-distributions has been derived by Bangia, Diebold, Schuermann, and Stroughair (1999):

$$\delta = 1.0 \div \psi \times \ln(k_i / 3) \tag{II.23}$$

where

ψ = a constant, whose value depends on the tail probability VaR measure (e.g., 1 percent). The estimate of ψ is obtained by regressing the VaR measure that incorporates the correction factor with historical VaR for the specific tail probability. For a normal distribution, $k_i = 3$ and, therefore, δ=1.

Then, equation (II.22) becomes:

$$VaR = 2.33 \times d_r \times \delta \times \sigma_r \tag{II.22a}$$

B. Exchange Rate Risk

Sovereigns with substantial portions of their debts denominated in foreign currencies assume commensurate exchange rate risk exposures—when their positions are left unhedged. Measuring the exchange rate exposure is often not an easy task, given the comovements between exchange rates and interest rates and the prevailing high correlations among bond markets. In general, VaR measures the exchange rate risk by combining the sensitivity of the portfolio to exchange rate changes and the probability of a given exchange rate change.

For absolute changes in exchange rates (one risk factor), the VaR for a bond can be calculated using the parametric VaR approach (following the steps mentioned in Appendix II). Assuming that a foreign government issues a U.S. dollar-denominated bond with a single payment, this government is now exposed to exchange rate risk due to potential changes in the local currency-dollar exchange rate (in addition to interest rate risk due to changes in U.S. interest rates). The exchange rate exposure can be measured by the corresponding VaR. To proceed, the present value of the bond in local currency, L, is the value in U.S. dollars multiplied by the exchange rate, FX:

$$\mathrm{P}V_L = FX \times \mathrm{P}V_\$ \tag{II.24}$$

$$= FX \times \frac{C_\$}{(1 + r_\$)^t} \tag{II.25}$$

The sensitivity of the value to changes in FX is the derivative with respect to FX:

$$\frac{\delta \mathrm{P}V_L}{\delta FX} = \left[\frac{C_\$}{(1 + r_\$)^t} \right] = d_{FX} \tag{II.26}$$

and, the change in value due to a change in FX is given by:

$$\Delta \mathrm{P}V_L = \Delta FX \times \frac{C_\$}{(1 + r_\$)^t} = \Delta FX \times d_{FX} \tag{II.27}$$

The standard deviation of PV_L is then the standard deviation of the exchange rates times d_{FX} and the VaR is 2.33 times the standard deviation of value:

$$VaR = 2.33 \times d_{FX} \times \sigma_{FX} \tag{II.28}$$

As an example, assume the above bond paying \$100 in 10 years' time, with the 10-year discount rate at 4 percent and a standard deviation in the interest rate of 0.2 percent. Also, assume that the exchange rate is 1.5 local currency (L) per U.S. dollar and the volatility (standard deviation) of the exchange rate is 0.03 local currency per dollar. Then, the present value is L102, the sensitivity is L68 per 100 percent increase in exchange rates, and the VaR is L4.8.

$$\mathrm{P}V_L = 1.5 \times \frac{100}{(1.04)^{10}} = L102$$

$$d_{FX} = \frac{100}{(1.04)^{10}} = L68$$

$$VaR = 2.33 \times \frac{100}{(1.04)^{10}} \times 0.03 = L4.8$$

If exchange rates deviate significantly from normality, the use of standard deviation multiples based on the normality assumption (such as 2.33 for the 1 percent worst case) leads to an underestimation of risk. In this case, a correction factor, ζ, for the standard deviation needs to be introduced to take account of leptokurtic or "fat tailed" distributions of exchange rates. The correction factor is such that ζ =1 if the distribution of exchange rates is normal, and ζ >1 if it is leptokurtic, with ζ being an increasing function of the unconditional kurtosis. Accordingly, the VaR estimate would now take into account both distributional characteristics, the standard deviation and kurtosis (see also p.12). An explicit relationship between the correction factor, ζ, and kurtosis, k_e, for t-distributions has been derived by Bangia, Diebold, Schuermann, and Stroughair (1999):

$$\zeta = 1.0 \div \phi \times \ln(k_e / 3) \tag{II.29}$$

where

ϕ = a constant, whose value depends on the tail probability VaR measure (e.g., 1 percent). The estimate of ϕ is obtained by regressing the VaR measure that incorporates the correction factor with historical VaR for the specific tail probability. For a normal distribution, $k_e = 3$ and, therefore, ζ=1.

Then, equation (II.28) becomes:

$$VaR = 2.33 \times d_{FX} \times \zeta \times \sigma_{FX} \tag{II.28a}$$

C. An Integrated Approach to Market Risk—Two Risk Factors

For absolute changes in interest rates and exchange rates (two risk factors), the parametric VaR for a bond can again be calculated following the steps mentioned in Appendix II. Assume now that a foreign government has issued a U.S. dollar-denominated bond with a single payment. This government is exposed to two risks: changes due to the U.S. dollar interest rates and changes due to the local currency-dollar exchange rate. The present value of the bond in local currency, L, is the value in U.S. dollars multiplied by the exchange rate, FX:

$$PV_L = FX \times \mathrm{P}V_\$ \tag{II.30}$$

$$= FX \times \frac{C_\$}{(1+r_\$)^t} \tag{II.31}$$

The change in value due to changes in interest rates in local currency is:

$$\frac{\delta \mathrm{P}V_L}{\delta r_\$} = FX \times \left[\frac{-t \times C_\$}{(1+r_\$)^{t+1}} \right] \tag{II.32}$$

and,

$$\Delta \mathrm{P}V_L = FX \times \left[\frac{-t \times C_\$}{(1+r_\$)^{t+1}} \right] \times \Delta r_\$ \tag{II.33}$$

To get the (linear) change in value due to a change in FX, we take first the derivative with respect to FX:

$$\frac{\delta \mathrm{P}V_L}{\delta FX} = \frac{C_\$}{(1+r_\$)^t} \tag{II.34}$$

and, the change in value due to a change in FX is given by:

$$\Delta \mathrm{P}V_L = \Delta FX \times \frac{C_\$}{(1+r_\$)^t} \tag{II.35}$$

The change in value due to both a change in interest rates and a change in FX is given by the sum of the individual changes :

$$\Delta \mathrm{P}V_L = FX \times \left[\frac{-t \times C_\$}{(1+r_\$)^{t+1}}\right] \times \Delta r_\$ + \frac{C_\$}{(1+r_\$)^t} \times \Delta FX \tag{II.36}$$

Defining the derivative with respect to U.S. dollar interest rates $d_{r,\$}$ and the derivative with respect to FX, d_{FX}, as :

$$d_{r,\$} = FX \times \frac{-t \times C_\$}{(1+r_\$)^{t+1}} \tag{II.37}$$

$$d_{FX} = \frac{C_\$}{(1+r_\$)^t} \tag{II.38}$$

we can rewrite the equation for the change in value as:

$$\Delta \mathrm{P}V_L = d_{r,\$} \times \Delta r_\$ + d_{FX} \times \Delta FX \tag{II.39}$$

The objective is to get the standard deviation of $\mathrm{P}V_L$. First, note that changes in interest rates are correlated with changes in FX. Also, assume that $d_{r,\$}$ and d_{FX} are fixed. The variance for the bond's value can be calculated as:

$$\sigma_{PV}^2 = (d_{r,\$} \times \sigma_{r,\$})^2 + (d_{FX} \times \sigma_{FX})^2 + 2 \times \rho_{r/\$,FX} \times (d_{r,\$} \times \sigma_{r,\$}) \times (d_{FX} \times \sigma_{FX}) \tag{II.40}$$

Further, as changes in interest rates, $\Delta r_\$$, and changes in FX, ΔFX, are random variables, we can estimate their variances from historical data. Thus, the variance for interest rates (using, for example, historical daily data) can be calculated as:

$$\Delta r_{\$,t} = r_{\$,t} - r_{\$,\,t-1} \tag{II.41}$$

and

$$\sigma^2_{r,\$} = \frac{1}{N-1} \times \sum_{t=1}^{N} \left(\Delta r_{\$,t} - \Delta \bar{r}_{\$} \right)^2 \tag{II.42}$$

and the variance for exchange rates (using, for example, historical daily data) as:

$$\Delta FX_t = FX_t - FX_{t-1} \tag{II.43}$$

and

$$\sigma^2_{FX} = \frac{1}{N-1} \times \sum_{t=1}^{N} \left(\Delta FX_t - \Delta \bar{FX} \right)^2 \tag{II.44}$$

The correlation can be estimated as:

$$\rho_{r/\$,FX} = \frac{1}{N-1} \times \left[\frac{\sum_{t=1}^{N} \left(\Delta r_{\$,t} - \Delta \bar{r}_{\$} \right) \times \left(\Delta FX_t - \Delta \bar{FX} \right)}{\sigma_{r,\$} \times \sigma_{FX}} \right] \tag{II.45}$$

The VaR can then be estimated as:

$$VaR = 2.33 \times \sigma_{PV}$$

$$= 2.33 \times \sqrt{\left(d_{r,\$} \times \sigma_{r,\$}\right)^2 + \left(d_{FX} \times \sigma_{FX}\right)^2 + 2 \times \rho_{r/\$,FX} \times \left(d_{r\$} \times \sigma_{r,\$}\right) \times \left(d_{FX} \times \sigma_{FX}\right)} \tag{II.46}$$

III. MEASUREMENT OF CREDIT RISK

Sovereign credit risk arises from a potential bond default, when a sovereign fails to make a scheduled payment on its bond debt. As governments accumulate more debt, the perceived ability to repay long-term debt holders becomes increasingly questionable. Some argue that for emerging market countries, the benchmark level of total debt to annual GNP is below 40 percent, while that for developed economies may exceed annual GNP (Reinhart, Rogoff and Savastano, 2003). The associated credit risk is reflected in higher yields than otherwise and in a low credit rating for some governments.

Credit Risk

In general, we consider credit risk as the risk that arises due to uncertainty in a counterparty's (i.e., creditor's) ability to meet its obligations. Depending on the type of counterparty (e.g., sovereign government, corporation, individual) and the type of obligation (e.g., government bonds, corporate bonds, derivatives transactions, lines of credit, loans), credit risk takes different forms and, therefore, is assessed and managed differently (Bank for International Settlements, 2000). Quantification of credit risk is important for assessing the likelihood of default by an obligator of investors, banks, and sovereigns. In identifying an appropriate credit risk measurement approach, entities use various types of models. Traditional models usually evaluate the expected loss on an asset or a portfolio of assets by taking into account (in a functional form) the relevant exposure (credit exposure) and uncertainty (default probability and recovery rate in the event of a default) (Culp, 2001).

In assessing the expected credit loss from a single counterparty, an institution or an investor then usually considers three factors: (i) default probability, defined as the likelihood that the counterparty will default on its obligation either over some specified horizon (e.g., a year) or over the life of the obligation. When it is calculated for a one-year horizon, this is usually called expected default frequency; (ii) credit exposure, defined as the extend of the outstanding obligation in the event of a default; and (iii) recovery rate, defined as the portion of the exposure that may be recovered (through bankruptcy proceedings or some other form of settlement) in the event of a default. Sometimes, the default probability and anticipated recovery rate are referred to as the credit quality of the obligation.

The three factors considered in the evaluation of credit risk are frequently incorporated in a formula to measure the credit risk of a portfolio of assets as follows:

$$\sum_{i=1}^{N} L_i(t)\times \max[V_i(t)-R_i(t),\, 0]+h_i\times[L_i(t+1,\ldots,T_i),\, V_i(t+1,\,\ldots,\,T_i),\, R_i(t+1,\,T_i)] \tag{III.1}$$

where

$L_i(t)$ is the risk neutral default probability or the expected default frequency of asset i at time t

$V_i(t)$ is the value of asset i at time t

$R_i(t)$ is the recovered value of asset i in case of default, including the market value of any collateral held against asset i at time t

$h_i(t)$ is a function describing the potential exposure to default of asset i

T_i is the settlement date of asset i

N is the number of assets

In the risk management framework, equation III.1 is simplified to generate the following reduced form formula:

$$\textit{Expected Loss} = \textit{Default Probability} \times \textit{Credit Exposure} \times \textit{Expected Loss Given Default} \qquad \text{(III.2)}$$

This equation (III.2) constitutes the basis for most probabilistic-type credit models used. Among the most widely-used probabilistic models are those for the estimation of expected loss, unexpected loss, the exposure at default, default probability, and the loss given default. These models use often credit VaR as a measure of exposure, and historical data on frequency of default and recovery rates, if they exist, or guess estimates.

Except for the functional-form credit models, other simpler credit evaluation approaches, like numerical scoring models and judgmental credit analysis, are often used for specific purposes. For bank loans to individuals, credit risk is typically assessed through a process of credit scoring. Prior to extending credit, a bank or other lender will obtain information about the party requesting a loan. This information might include the party's annual income, other assets, existing debts, etc. A standard formula is applied to this information to generate a number, which is called a credit score. Based upon the credit score, the lending institution will decide whether or not to extend credit. The most common numerical scoring models are the z score and loan grading models.

For businesses, large institutional counterparties, and government entities, credit risk evaluation may be complicated and credit risk is often assessed by credit analysis of the quality of a counterparty. This process entails review of information about the counterparty, including its balance sheet, income statement, recent trends in the industry, the current economic environment, etc. It might also include assessment of the exact nature of an obligation, like secured debt versus subordinated debt (as secured debt generally has higher credit quality than does subordinated debt of the same issuer). Based on the credit analysis, a credit rating is assigned to the counterparty (or the specific obligation) which can be used to make credit decisions (Saunders and Allen, 2002).

In addition to many banks, investment managers and insurance companies which prepare their own credit ratings for internal use, other firms, such as Moody's, Standard and Poor's, and Fitch, develop credit ratings for use by investors or other third parties. Institutions that have publicly traded debt often resort to one of these rating agencies to prepare credit ratings for their debt, which then distribute to investors. Some regulators also develop credit ratings, like the U.S. National Association of Insurance Commissioners which publishes credit ratings that are used for calculating capital charges for bond portfolios held by insurance companies.

Over the last decade, mark-to-market credit risk modeling, encompassing stochastic methods, has been increasingly used to quantitatively measure credit risk. This entails the use of asset value models of the Black-Scholes-Merton tradition, reduced-form and intensity models to provide a rigorous probabilistic metric of potential credit exposure, to value credit derivatives (as they represent contingent obligations), and to analyze the credit risk of portfolios of obligations for risk management or regulatory purposes (Bluhm, Overbeck, and Wagner, 2002; Schonbucher, 2003). Among the most prominent applications of this tradition is the KMV model (see below). Other commonly-used models include credit ratings-based, actuarial-type and econometric ones. Typical analytics for credit risk quantification are the mean, standard deviation, and correlations.

For estimating credit losses of financial institutions, the two most well-known credit models are JP Morgan's CreditMetrics model (JP Morgan, 1997) and the CSFP CreditRisk+ model (Crouhy, Galai, and Mark, 2000). The JP Morgan model, a ratings-based model, is based on the probability of moving from one credit quality to another, including default, within a given time horizon (usually one year). Specifically, it models the full forward distribution of the values of any bond portfolio, say, 1 year forward, where the changes in values are related to credit migration only, while interest rates are assumed to evolve in a deterministic manner. Then, this model assumes that the forward curve is constant, to isolate credit deterioration from market risk. However, most credit deterioration comes from unfavorable market moves.

The CSFP model, an actuarial-type model, focuses on default. Specifically, it adopts a Poisson process to model default for individual bonds, as defaults occur in a step-wise fashion. However, the modeling of correlation among defaults remains an issue in this approach. Furthermore, other commercial credit models are based on the structural Black-Scholes-Merton approach, like Moody's RiskCalc hybrid model, or are based on an econometric model, like McKinsey's CreditPortfolioView approach, which links macroeconomic factors to rating transition matrices. In particular, the latter is a discrete-time multi-period model, where default probabilities are conditional on macroeconomic variables, like the real growth rate, unemployment rate, level of interest rates, which to a large extent drive the credit cycle in an economy. Finally, the Basel II approach to credit risk quantification is also based on a mark-to-market framework (VaR).

The Asset Value Model

The structural Black-Scholes-Merton credit risk model is increasingly becoming a common model for assessing credit risk, typically of a corporation's debt (Duffie and Singleton, 2003). It was proposed in a Black and Scholes (1973) paper on option pricing and a more detailed paper by Merton (1974). Thus, it is also sometimes called the Merton model, or asset value model. This model, using an option pricing model of a firm's capital structure, indicates that a firm defaults when its asset value falls below its liabilities. The most popular implementation of the model is the KMV (Kealhofer, McQuown, and Vasicek) Credit Monitor model.

The model considers a corporation financed through a single debt and a single equity issue. The debt comprises a zero-coupon bond that matures at time t = t* , at which time it is to pay investors b dollars. The equity pays no dividends. An unobservable process A describes the firm's value $A^t \geq 0$ at any time t. We ascribe the outstanding debt and equity values D^t and E^t, respectively. Accordingly, at any time t

$$A^t = D^t + E^t \tag{III.3}$$

At time $t*$, when the firm's debt matures, if the A^{t^*} exceeds the bond's maturity value b, the firm will pay off the bond holders. The remaining value of the firm will belong to the equity holders, and will be equal to:

$$E^{t^*} = A^{t^*} - b \tag{III.4}$$

If at time t^*, however, A^{t^*} does not exceed the bond's maturity value b, the firm defaults on its debt. The bond holders take ownership of the firm, and the share holders are left with nothing:

$$E^{t^*} = 0 \tag{III.5}$$

Combining the above two results, we obtain a general expression for the value of the firm's stock at the maturity of its debt:

$$E^{t^*} = \max\left(A^{t^*} - b, 0\right) \tag{III.6}$$

This formula is the payoff of a call option on the firm's value A^{t^*} with strike price b. Based upon this realization, the asset value model treats the firm's equity as a call option on the value of the firm struck at the maturity value b of its debt. Using the put-call parity, we can write the firm's debt as a risk-free bond that guarantees payment of b plus a short put option on the value of the firm struck at b. Accordingly,

$$D^{t^*} = b - \max\left(b - A^{t^*}, 0\right) \tag{III.7}$$

The asset value model treats A^t just like any value of a financial instrument. It assumes that A^t follows a geometric Brownian motion with volatility σ. Further, it makes all the other simplifying assumptions of the Black-Scholes (1973) option pricing formula. Accordingly, the firm's equity can be valued at any time t as:

$$E^t = c\left(A^t, b, \sigma, r, t^* - t\right) \tag{III.8}$$

where c is the Black-Scholes formula for the value of a call option, and r is the risk-free rate. By equation III.7, we can similarly value the firm's debt at any time as:

$$D^t = be^{-rt} - \rho\left(A^t, b, \sigma, r, t^* - t\right) \qquad \text{(III.9)}$$

where ρ is the Black-Scholes formula for the value of a put. Note that we discount the payment b at the risk-free rate because that payment is risk-free in equation III.7 – we have stripped out the credit risk as a put option. In essence, we consider a credit-risky bond as a credit-riskless bond minus the option to exchange the bond for the corporation's assets in case of default.

At any time t, the distance to default for a firm's debt is defined as $\left(A^t - b\right) / \sigma$. This is a metric indicating how many standard deviations the equity holders' call option is in-the-money. The smaller the distance to default, the more likely a default is to occur. The probability of default is the probability of the call option expiring out-of-the-money. This is approximately equal to one minus the option's normalized delta (if investors were risk neutral, equality would be exact). The formula for delta can be derived from the Black-Scholes (1973) option pricing formula. To normalize that value, we divide the delta by the instrument's value.

The main shortcomings of asset value models are:

1. The assumption that the firm's debt financing consists of a one-year zero-coupon bond is, for most firms, an oversimplification.

2. The Black-Scholes (1973) simplifying assumptions are questionable in the context of corporate debt. The variance is an appropriate measure of risk only if the asset price distributions are normal, and the investor's utility function is quadratic.

3. The firm's value A^t may not be observable, which makes assigning values to it and its volatility problematic.

4. Default correlations cannot be easily measured, making aggregation of credit risk difficult.

5. For estimating credit losses, calculation of the tail risk probabilities of asymmetric, fat-tailed loss distributions has to be performed—often, not an easy task.

Despite these shortcomings, the structural approach to credit default and asset value models provide a useful context for modeling credit risk. Practical implementations of the asset value model are used by financial institutions and institutional investors to assign values to A^t and σ, and, thus, relate A^t to the observable market capitalization of the firm. An extension of this model to a sovereign has been worked out in a recent paper, by Gapen, Xiao, Lim, and Gray (2005).

Reduced-Form Models

In addition to the Black-Scholes-Merton-type structural models, market practitioners have increasingly been using reduced-form credit models and sovereign Credit Default Swap (CDS) data to measure sovereign credit risk. Among the most common reduced-form models are those introduced by Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), and Duffie and Singleton (1997 and 1999). A major advantage of these models is the relative easiness to process market information, such as term structures and risk-free rates for different risk classes and maturities, in the determination of credit risk (Bohn, 2000).

Reduced-form credit models assume that an exogenous random variable drives default, and that the probability of default over any time interval is nonzero. Default occurs when the level of such random variable undergoes a discrete shift. These models treat defaults as unpredictable Poisson events. The time at which the discrete shift occurs cannot be predicted on the basis of currently available information, i.e., the default process in these models is not determined by the value of the firm -- as in the Black-Scholes-Merton-type models, but rather as the first jump of a Poisson process. Other stochastic processes describing the term structures of CDS spreads are examined in Pan and Singleton (2005).

In general, reduced-form models assume some functional form for the rate of default and the payoff in case of default, and then calibrate those to current interest rate spreads, i.e., over a risk-free rate. Thus, these models have two key variables to estimate: (i) the rate of default and (ii) the recovery rate. The rate of default is the probability of default, i.e., the probability of a jump from no default to default (and vice versa) in a given time interval, which is often the maturity of a bond. The probability of default is determined by the respective credit rating and the holding period, with the default rate for a certain bond being estimated by the average default rate of similarly rated bonds using historical observations. The recovery rate is the amount that the bond holder will receive in case of default of the issuer, and is some percentage of the face value of the bond. Recovery rates are usually given exogenously to the model, as these models cover the time of default, i.e., the time until the first jump occurs in the Poisson process, and not the severity of loss in case of a default. The recovery rate is based on historical averages of past default experiences, determined mainly by the relevant credit ratings, the holding period, and seniority or security of bonds. The holding period plays a critical role, as the longer the maturity of a bond the higher the chance for its default.

A shortcoming of reduced-form credit models is the possible unavailability of market-priced bonds at each maturity. Therefore, sovereigns that do not have tradable bonds for the entire maturity spectrum would be difficult to calculate the fair credit spread, or default probability.

IV. MEASUREMENT OF LIQUIDITY RISK

Liquidity risk is the risk due to uncertainty in a financial entity's cash inflows to sustain its normal activities. An institution or government might lose liquidity if its counterparties avoid trading with or lending to it due to a fall in its credit rating or some other event, or if it experiences sudden unexpected cash outflows. It might also be exposed to liquidity risk if markets on which it depends on are subject to loss of liquidity. Furthermore, liquidity risk tends to compound other risks. For example, if a trading organization has a position in an illiquid asset, its limited ability to liquidate that position at short notice will compound its market risk. Or, if a firm has offsetting cash flows with two different counterparties on a given day, and the counterparty that owes it a payment defaults, the firm will have to raise cash from other sources to make the payment. If it is unable to do so, it will default too. In this case, liquidity risk is compounding credit risk.

Given its tendency to compound other risks, it is difficult to isolate liquidity risk. Except for the most simple of circumstances, comprehensive measures of liquidity risk do not exist. However, liquidity risk measurement helps ensure that an entity has a sufficient amount of internal cash available to carry on its normal operations. Certain techniques of asset-liability management can be applied to assessing liquidity risk. The most basic measure of an entity's liquidity risk is the liquidity gap. This static measure is simply the net liquid assets of an entity. A negative liquidity gap value indicates possible future liquidity problems. In essence, this simple test for liquidity risk looks at future net cash flows on a day-by-day basis, and singles out any day that has a sizeable negative net cash flow. For example, if an entity has an asset of $100 maturing in one year, and liability of $50 maturing in one month, the liquidity gap at one month is -$50.

The liquidity gap analysis can be supplemented with stress testing, i.e., looking at net cash flows on a day-by-day basis assuming that an important counterparty defaults. These analyses, however, cannot take into account contingent cash flows, such as cash flows from derivatives. If an organization's cash flows are largely contingent, liquidity risk may be assessed using some form of scenario analysis. That is, we may construct multiple scenarios for market movements and defaults over a given period of time, and assess the day-by-day cash flows under each scenario.

Another measure of an entity's liquidity risk is liquidity risk elasticity. This measures an entity's sensitivity to a change in the liquidity premium (which represents the amount of compensation required by a lender for lending to the long end of the market). For portfolios, in addition to portfolio liquidity risk reflecting the fact that the various investments held in a portfolio have different liquidity profiles, that is, some will be easier to sell than others, there is the liquidity risk reflecting changes in funding costs. A portfolio manager must then structure portfolio holdings so that not only a number of illiquid investments do not mature at the same time, but also that the impact of such changes are minimal. The same arguments apply for future contingencies that could make the liquidity structure of portfolios more risky. The liquidity risk elasticity of a portfolio of exposures is calculated according to the following formula (Culp, 2001, pp. 424-429):

$$\frac{\delta NV_t}{\delta \Xi_t} = \frac{\delta V_t}{\delta \Xi} - w \times \frac{\delta L_t}{\delta \Xi_t} \tag{IV.1}$$

where

NV_t is the current value of net assets, and

V_t and L_t are the current values of assets and liabilities, respectively

w is the proportion of liabilities funded with the assets

Ξ is the liquidity premium on the entity's funding cost, often defined as the difference between long-term and short-term nominal interest rates on the same-credit-rating yield curve for a given date. (Alternatively, liquidity premium is defined as the amount that forward interest rates exceed expected future spot – short-term – interest rates).

Note that this formula can be applied only for small changes in the funding costs, and indicates that the smaller the liquidity risk elasticity the less the liquidity risk. However, the liquidity premium depends critically on the entity. In particular, borrowing governments have revealed over time their propensity to borrow long term to lock in the interest rate costs and ensure the availability of funds. In contrast, risk-averse investors are more interested in lending short term; they tend to view long-term government securities with greater uncertainty (about interest and principal payments) and hence require compensation for the additional risk. This difference in preferences between sovereigns and investors may lead to a sizeable liquidity premium, which is an increasing function of maturity (Hicks, 1946). The liquidity premium also depends on an entity's credit rating and overall domestic and international liquidity conditions (Longstaff, 2001).

VaR Measures

Moreover, a VaR-type approach can also be used to measure liquidity risk. The liquidity-risk adjusted VaR (LVaR) uses the bid-ask spread to measure liquidity risk exposure. In addition, LVaR can be used in the context of a worst-case spread. However, as with any VaR approach, LVaR's main disadvantage is its static assessment of liquidity risk (Bangia, Diebold, Schuermann, and Stroughair, 1999; Neofotistos, 2002).

The incorporation of liquidity risk in the assessment of the overall risk of a bonded portfolio presumes that the closeout price of a bond position may be lower than the mid-price. As the bid-ask spread changes overtime, Bangia, Diebold, Schuermann, and Stroughair (1999) assume that the possible drop in the price is half the usual bid-ask spread plus the 99th percentile movement in the spread. Then, the additional loss due to the possible drop in the price is:

$$AdditionalLoss_{99\%} = 0.5 \times \overline{S} + 2.33 \times \sigma_s \tag{IV.2}$$

where

$\overline{S}$ is the average spread, and

σ_s is the standard deviation of the spread

The additional loss (IV.2) is then added to the one-day parametric VaR (II.46) to generate the liquidity-risk adjusted VaR (LVaR):

$$LVaR = VaR + AdditionalLoss_{99\%} \tag{IV.3}$$

If the distribution of spreads is not normal, the additional loss (i.e., the cost of liquidity to cover 99 percent of the spread situations) becomes:

$$AdditionalLoss - N_{99\%} = 0.5 \times \overline{S} + \alpha \times \sigma_s \tag{IV.2a}$$

where

α is a scaling factor such that a 99 percent probability coverage is achieved. In general, the greater the departure from normality, the larger the α. Also, the value of α will depend on the specific security and market.

V. An Integrated Approach to Risk Sensitivity for a Security With N Risk Factors

For a position with N risk factors, the VaR equation can be generalized as:

$$\begin{aligned} VaR = 2.33 \times \sqrt{(d_1 \times \sigma_1)^2 + (d_2 \times \sigma_2)^2 + 2 \times \rho_{1,2} \times (d_1 \times \sigma_1) \times (d_2 \times \sigma_2) + \ldots +} \\ \overline{+ (d_{N-1} \times \sigma_{N-1})^2 + (d_N \times \sigma_N)^2 + 2 \times \rho_{N-1,N} \times (d_{N-1} \times \sigma_{N-1}) \times (d_N \times \sigma_N)} \end{aligned} \tag{V.1}$$

where:

N is the total number of risk factors being used

d_N is the derivative of the portfolio's with respect to the N th risk factor $\left(\frac{\delta \mathrm{P}V}{\delta f_N}\right)$

Equivalently, equation V.1 can be rewritten in summation notation as:

$$VaR = 2.33 \times \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\rho_{i,j} \times (d_i \times \sigma_i) \times (d_j \times \sigma_j)\right]} \tag{V.2}$$

or, in matrix notation as:

$$VaR = 2.33 \times \sqrt{D \times C \times D^T} \tag{V.3}$$

where

$D = [d_{1,} \, d_s \ldots, \, d_N]$, the vector of derivatives indicating the risk factor sensitivities

$$C = \begin{bmatrix} \sigma_1 \times \sigma_1 & \rho_{1,2} \times \sigma_1 \times \sigma_2 & \cdots & \rho_{1,N} \times \sigma_1 \times \sigma_N \\ \rho_{2,1} \times \sigma_2 \times \sigma_1 & \sigma_2 \times \sigma_2 & \cdots & \rho_{2,N} \times \sigma_2 \times \sigma_N \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{N,1} \times \sigma_N \times \sigma_1 & \rho_{N,2} \times \sigma_N \times \sigma_2 & \cdots & \sigma_N \times \sigma_N \end{bmatrix}$$

covariance matrix with the variances of the state variables (interest rate, exchange rate, etc) underlying the risk factors along the diagonal and the covariances off the diagonal (the covariance matrix is symmetric), and

$\sigma_{PV}^2 = D \times C \times D^T$, the variance of the portfolio.

VI. An Integrated Approach To Risk Sensitivity For A Portfolio With N Risk Factors

For a portfolio of Z securities, each of which is affected by N risk factors, the vector of sensitivities to these risk factors for the portfolio is the sum of the vectors for the individual securities (positions). Thus,

$$D_A = [d_{A,1}, d_{A,2}, d_{A,3}, \ldots, d_{A,N}] \tag{VI.1}$$

$$D_B = [d_{B,1}, d_{B,2}, d_{B,3}, \ldots, d_{B,N}] \tag{VI.2}$$

$$D_Z = [d_{Z,1}, d_{Z,2}, d_{Z,3}, \ldots, d_{Z,N}] \tag{VI.3}$$

$$D_{Portfolio} = \left[D_A = +D_B + \ldots + D_Z\right] \tag{VI.4}$$

$$D_{Portfolio0} = [\left(d_{A,1} + d_{B,1} + \ldots + d_{Z,1}\right), \ldots, \left(d_{A,N} + d_{B,N} + \ldots + d_{N,N}\right)] \qquad \text{(VI.5)}$$

Utilizing the previous covariance matrix, C:

$$C = \begin{bmatrix} \sigma_1 \times \sigma_1 & \rho_{1,2} \times \sigma_1 \times \sigma_2 & \cdots & \rho_{1,N} \times \sigma_1 \times \sigma_N \\ \rho_{2,1} \times \sigma_2 \times \sigma_1 & \sigma_2 \times \sigma_2 & \cdots & \rho_{2,N} \times \sigma_2 \times \sigma_N \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{N,1} \times \sigma_N \times \sigma_1 & \rho_{N,2} \times \sigma_N \times \sigma_2 & \cdots & \sigma_N \times \sigma_N \end{bmatrix} \qquad \text{(VI.6)}$$

we can calculate the parametric VaR for the portfolio as:

$$VaR_{Portfolio} = 2.33 \times \sqrt{D_{Portfolio} \times C \times D^T_{Portfolio}} \qquad \text{(VI.7)}$$

This approach allows to consider all identified risks in an integrated manner, and devise a single VaR statistic for the bond portfolio. However, the contribution of each risk to the total VaR can be separated and, thus, be able to be individually managed.

VII. Epilogue

As evidenced from the above discussion, valuing the risk exposure of the debt stock of a sovereign can become a real challenge. Although duration is useful for predicting the effect of interest rate changes on the value of fixed income accounts, it should be regarded as a first-order approximation valid only for small changes in yield. Further precision can be obtained by considering convexity. If duration is set to immunize a bond portfolio, minimizing convexity will keep the portfolio duration from moving too quickly from its target value. In practice, the objective of debt managers is to maximize the duration of a debt stock, while minimizing its convexity.

While duration is considered a good first approximation to the exposure of bonds to movements in yields, control of exposure to a second factor is much harder. Proper positioning, often using maximum convexity or M-square, requires anticipating the type of movement in the term structure, either parallel or twisted. Given that the main objective of risk management is to control portfolio risk without necessarily forecasting changes in risk factors, this requirement appears unattainable. In such cases, another approach, the arbitrage pricing theory (APT), based on factor models, can be applied to bond returns. However, exposition of the APT theory is outside the scope of this report.

The VaR methodology is currently considered as the best approach to assess market risk. The VaR measure of market risk combines information on the sensitivity of the value of changes in marker-risk factors with information on the probable amount of change in these factors. It calculates the level of loss that there is only, say, 1 in 100 chance that a loss worse than the calculated VaR can occur. In essence, the VaR level is estimated based on the current value of a portfolio (position) and the calculation of the probability distribution of changes in the value over the next investment period (trading day, for trading portfolios). The estimate of the probability distribution of the price changes is based on the distribution of price changes over the last few weeks, months or years. From the probability distribution over the next investment period we can infer about the confidence level for the 99-percentile loss.

Credit risk models utilize credit scoring, probabilistic or contingent-claims frameworks to measure credit risk exposure. These models usually try to determine some form of default rates. Default entails equity holders' decision not to exercise the option to keep an entity in operation. Along these lines, KMV's approach encounters the problem of quantifying the underlying asset's value and volatility, as well as the appropriate strike price for exercising the default option. This analysis involves a forecast of the potential market value of an asset (position) or a portfolio of assets, often not encompassing correlation among credit risks. The results of asset value models have often been questioned due to the assumptions employed and the data needed to calibrate these models being nonobservable. Finally, liquidity gap or liquidity risk elasticity models or VaR-type approaches are used to measure the sufficiency of cash inflows.

Furthermore, we present an integrated approach for the simultaneous estimation of a portfolio's interest rate and exchange rate risk using the VaR methodology. Then, we extend the integrated approach to include N risk factors. This approach allows us to measure the total risk of a portfolio with different debt instruments, provided that the volatilities and correlations among the N risk factors that impact the portfolio can be estimated.

Table 1. Calculation of Duration and Convexity
(5-year, $100 face value, 6 percent semiannual coupon bond)

	Time (t) [1]	Payment [2]	(1+y)^-t [3]	[3]*[2] [4]	[4]/P [5]	[5]*[1] [6]	[6]*{1+[1]} [7]
Par Bond							
	0.5	3	0.971	2.913	0.029	0.015	0.022
	1.0	3	0.943	2.828	0.028	0.028	0.057
	1.5	3	0.915	2.745	0.027	0.041	0.103
	2.0	3	0.888	2.665	0.027	0.053	0.160
	2.5	3	0.863	2.588	0.026	0.065	0.226
	3.0	3	0.837	2.512	0.025	0.075	0.301
	3.5	3	0.813	2.439	0.024	0.085	0.384
	4.0	3	0.789	2.368	0.024	0.095	0.474
	4.5	3	0.766	2.299	0.023	0.103	0.569
	5.0	103	0.744	76.642	0.766	3.832	22.993
							25.289
Price				**100.00**			
Duration						**4.390**	
Convexity							**6.32**
Premium Bond -- spot rates = 2.5 percent							
	0.5	3	0.988	2.963	0.025	0.013	0.019
	1.0	3	0.976	2.927	0.025	0.025	0.050
	1.5	3	0.964	2.891	0.025	0.037	0.093
	2.0	3	0.952	2.855	0.025	0.049	0.147
	2.5	3	0.940	2.820	0.024	0.061	0.212
	3.0	3	0.928	2.785	0.024	0.072	0.287
	3.5	3	0.917	2.751	0.024	0.083	0.372
	4.0	3	0.906	2.717	0.023	0.093	0.467
	4.5	3	0.895	2.684	0.023	0.104	0.571
	5.0	103	0.884	91.015	0.782	3.909	23.456
							25.674
Price				**116.41**			
Duration						**4.45**	
Convexity							**6.42**
Discount Bond -- spot rates = 10.5 percent							
	0.5	3	0.951	2.854	0.034	0.017	0.026
	1.0	3	0.905	2.715	0.032	0.032	0.065
	1.5	3	0.861	2.583	0.031	0.046	0.116
	2.0	3	0.819	2.457	0.029	0.059	0.176
	2.5	3	0.779	2.337	0.028	0.070	0.244
	3.0	3	0.741	2.223	0.027	0.080	0.319
	3.5	3	0.705	2.115	0.025	0.088	0.398
	4.0	3	0.671	2.012	0.024	0.096	0.481
	4.5	3	0.638	1.914	0.023	0.103	0.566
	5.0	103	0.607	62.521	0.747	3.733	22.400
							24.790
Price				**83.73**			
Duration						**4.32**	
Convexity							**6.20**

Sources: IMF staff calculations.

Table 2. Comparison of Duration and Convexity Approximations

Yield:	9.00%
Price:	$100.00
Duration:	10.32
Convexity:	85.18

			Duration		Duration + Convexity	
Yield (%)	Change in Yield (absolute)	Actual Price ($)	Projected Price ($)	Difference (Act-Proj) ($)	Projected Price ($)	Difference (Act-Proj) ($)
4.0	-5.0	178.6	151.6	27.0	162.2	16.3
6.0	-3.0	138.6	131.0	7.6	134.8	3.8
8.0	-1.0	110.7	110.3	0.4	110.7	0.0
9.0	0.0	100.0	100.0	0.0	100.0	0.0
10.0	1.0	90.9	89.7	1.2	90.1	0.8
12.0	3.0	76.4	69.0	7.3	72.9	3.5
14.0	5.0	65.5	48.4	17.1	59.0	6.5

Sources: IMF staff calculations.

YIELD DEFINITIONS

Current yield is defined as the percent of

$$ycu = C / \mathrm{P} \tag{AI.1}$$

where

C = the constant coupon or principal payment (i.e., coupon rate times face value of a bond)

P = the current market value of a bond

T = the number of periods to final maturity

Yield to maturity is defined as the percent of

$$\text{ym (yield to maturity)} = \{[(F - \mathrm{P}) / \mathrm{T}] + C\} / [(F + \mathrm{P}) / 2] \tag{AI.2}$$

where

F = the face value of a bond at maturity

P = the market value of a bond

T = the number of periods to final maturity

C = the constant coupon or principal payment

Continuously compounded yield, at any time t, yco, is defined as

$$C_t = \mathrm{P}V(C_t) \times e^{yco \times t} \tag{AI.3}$$

where

e = 2.7182818...

yco = continuously compounded interest rate

t = numbers of years to be compounded

Compounded yield, at any time t, yc, is defined as

$$C_t = \mathrm{P}V(C_t) \times [1 + (yc / n)]^{n \times t} \tag{AI.4}$$

where

yc = constant interest rate

n = compounding frequency, the number of times per year that interest is credited, with n=1, for annual compounding, n=2, for semiannual compounding, and n=3, for monthly compounding

t = number of years to be compounded

THE VALUE-AT-RISK (VAR) METHODOLOGY

VaR has emerged as the best single risk-measurement technique available, notwithstanding its limitations to describe what happens on bad days (e.g., twice or three times a year) rather than terrible days (e.g., once every ten years) (Marrison, 2002). VaR is defined as the value that can be expected to be lost during severe, adverse market fluctuations. Typically, a severe loss is often termed as a loss that has a 1 percent chance of occurring on any given day. If we are measuring daily losses, this is equivalent to stating that, on average, VaR or more losses will be incurred on two to three days per year. VaR can be calculated for a financial instrument, a portfolio, a bank, or a sovereign. To avoid terrible days would require continued use of stress and scenario tests as a backup.

A common assumption is that movements in the market have a normal probability distribution, implying that there is a 1 percent chance that losses will be greater than 2.33 standard deviations. Then, assuming a normal distribution, the 99 percent VaR for a portfolio can be defined as:

$$VaR_T = 2.33 \times \sigma_T \qquad \text{(AII.1)}$$

where

σ is the standard deviation of the portfolio's value

T is the time period over which the standard deviation of returns is calculated

The VaR can be calculated for any time horizon. For trading operations (portfolios), a one-day horizon is typically used (often called, daily earnings at risk). In asset liability management, where the term VaR is used to refer to associated potential losses, a monthly or yearly horizon is used. In this connection, the terms credit VaR and liquidity VaR are sometimes used to describe the loss distribution from a credit portfolio or from liquidity exposure, respectively (see below). Also, the VaR probability may be higher or lower than 1 percent. A common alternative is to set the tail probability at 2.5 percent, which, assuming a normal distribution, implies 1.96 standard deviations (instead of 2.33 for the 1 percent).

For calculating the VaR for the potential losses over multiple days, a reasonable approximation to the multiday VaR is that it is equal to the one-day VaR multiplied by the square root of the number of days:

$$VaR_T = VaR_1 \times \sqrt{T} \qquad \text{(AII.2)}$$

This relationship presupposes that (i) changes in market factors are normally distributed; (ii) the one-day VaR is constant over the time period; and (iii) there is no serial correlation, i.e., the results on one day are independent of the results of the previous day.

Approaches for Calculating VaR

The VaR measure of risk calculates the level of loss that there is only, say, 1 in 100 chance that a loss worse than the calculated VaR can occur (Marrison, 2002; Linsmeier and Pearson, 1996). The VaR level is calculated by using the current holdings in a portfolio (current position in a trading portfolio) and an estimate of the probability distribution of the price changes over the next investment period (next trading day). The estimate of the probability distribution of the price changes is based on the distribution of historical price changes (over the last few weeks, months or years). From the probability distribution over the next period we can infer about the confidence level for the 99-percentile loss. There are three commonly used methods to calculate VaR: (i) parametric VaR; (ii) historical simulation; and (iii) Monte Carlo simulation.

Parametric VaR

The parametric VaR approach (also known as Linear VaR or Variance-Covariance VaR) assumes that the probability distribution of price changes is normal (and, therefore, it requires calculation of the variance and covariance parameters), and that changes in instrument values are linear with respect to changes in risk factors (the approach is linear). For bonds, for example, the sensitivity of a bond portfolio to changes in interest rates is described by duration.

The parametric VaR method employs the following steps:

1. Define the set of risk factors that will be sufficient to calculate the value of a portfolio.

2. Find the sensitivity of each instrument in the portfolio to each risk factor.

3. Get historical data on the risk factors to calculate the standard deviation of the changes and the correlations between them.

4. Estimate the standard deviation of the value of the portfolio by multiplying the sensitivities by the standard deviations, taking into account all correlations.

5. Finally, assume that the loss distribution is normally distributed and, therefore, approximate the 99 percent VaR as 2.33 times the standard deviation of the value of the portfolio.

To illustrate the calculation of the parametric VaR, consider a portfolio with two correlated instruments. The loss on the portfolio L_P is the sum of the losses on each instrument:

$$L_P = L_1 + L_2 \qquad \text{(AII.3)}$$

The standard deviation of the loss on the portfolio σ_P is :

$$\sigma_p^2 = \sigma_1^2 + \sigma_2^2 + 2 \times \rho_{1,2} \times \sigma_1 \times \sigma_2 \quad \text{(AII.4)}$$

where

σ_1 is the standard deviation of losses from instrument 1

$\rho_{1,2}$ is the correlation between losses from 1 and 2

Then, assuming normal distribution, the 99 percent VaR for the portfolio can be calculated as:

$$VaR_\rho = 2.33 \times \sigma_\rho \quad \text{(AII.5)}$$

The calculation of parametric VaR is, therefore, dependent on market data (from Reuters or Bloomberg) and portfolio information (position data). The market data is used to calculate the covariance matrix and is fed into the calculation of the derivatives vectors. The derivatives vectors for each security are usually calculated using analytic formulas. The VaR for the portfolio is then calculated by multiplying the derivative vectors with the covariance matrix. This VaR will indicate the potential loss for the portfolio in the coming day (week, month or year), and the main causes of such loss. Based on this, liability (or portfolio) management decisions will be made to bring the calculated VaR at the desired level.

Note that parametric VaR is computationally fast compared to historical or Monte Carlo simulation, but does not capture non-normality (i.e., it gives a poor description of extreme tail events, such as crises, because it assumes that the risk factors have a normal distribution) and nonlinearity (i.e., it gives a poor description of nonlinear risks when the price change is not a linear function of the change in the risk factors). Also, parametric VaR uses a covariance matrix that implicitly assumes that the correlations between risk factors is stable and constant over time.

Historical-Simulation VaR

The historical-simulation VaR approach is backward-looking by taking the market data for the last, say, 1,000 days to calculate the percent change for each risk factor. Then, each percentage change is multiplied by today's market values to present 1,000 scenarios for tomorrow's values. For each of these scenarios, the portfolio is valued using full, nonlinear pricing models. The tenth-worst day is then selected as being the 99 percent VaR (Papaioannou and Gatzonas, 2002). Note that it is required to use approximately four years of data to achieve the advantages of the historical-simulation VaR (Hendricks, 1996).

Note that historical simulation is the most simple VaR technique, and captures non-normality (i.e., it has the ability to calculate the potential changes in risk factors without assuming that they have a normal distribution with stable correlation) and nonlinearity (i.e., allows for the price change not being a linear function of the change in the risk factors). However, the results tend to be heavily influenced by the historical period used and the form of historical market movements (i.e., historical simulation will tend to reproduce past events – a crisis – in exactly the same form), thus causing mistrust in its use.

Monte Carlo Simulation VaR

Monte Carlo simulation estimates VaR by randomly creating many scenarios for future rates, using nonlinear pricing models to estimate the change in value for each scenario, and then calculating VaR according to the worst losses. The Monte Carlo approach assumes that there is a known probability distribution for the risk factors. The usual implementation of Monte Carlo assumes a stable, joint-normal distribution for the risk factors (as in parametric VaR). The analysis calculates the covariance matrix for the risk factors in the same way as in parametric VaR, but, unlike parametric VaR, it decomposes the matrix to ensure that the risk factors are correlated in each scenario (Glasserman, Heidelberger and Shahabuddin, 2001).

The scenarios start from today's market condition and go one day forward to give possible values at the end of the day. Full, non-linear pricing models are used to value the portfolio under each of the end-of-day scenarios. For bonds, nonlinear pricing means using the bond-pricing formula rather than duration. From the scenarios, VaR is selected to be the 1-percentile worst loss. For example, if 5,000 scenarios were created, the 99 percent VaR would be the fiftieth-worst result.

Decomposition of the Covariance Matrix: Two Risk Factors

The decomposition of the covariance matrix in such a way as to allow the creation of random scenarios with the same correlation as the historical market data is a difficult step. In the case of two risk factors, we can easily create correlated random numbers as follows:

1. Draw a random number z_1 from a standard normal distribution.

2. Multiply z_1 by the standard deviation of the first risk factor σ_A to create the first risk factor for that scenario, δf_A:

$$\delta f_A = z_1 \times \sigma_A \ , \ z_1 \sim \mathrm{N}(0,1) \qquad \text{(AII.6)}$$

3. Multiply z_1 by the correlation $\rho_{A,B}$.

4. Draw a second independent random number z_2 from a standard normal distribution.

5. Multiply z_2 by the root of one minus the correlation squared $\left(\sqrt{1-\rho_{A,B}^2}\right)$.

6. Add the two results together to create a random number y that has a standard deviation of one and correlation $\rho_{A,B}$ with δf_A:

$$y = z_1 \times \rho_{A,B} + z_2 \times \sqrt{1-\rho_{A,B}^2},\ z_2 \sim \mathrm{N}(0,1) \tag{AII.7}$$

7. Multiply y by the standard deviation of the second risk factor σ_B to create the second risk factor for the scenario δf_B:

$$\delta f_B = y \times \sigma_B \tag{AII.8}$$

This process can be summarized as follows:

$$\delta f_A = \sigma_A \times z_1,\ z_1 \sim \mathrm{N}(0,1) \tag{AII.9}$$

$$\delta f_B = \sigma_B \times \left(z_1 \times \rho_{A,B} + z_2 \times \sqrt{1-\rho_{A,B}^2}\right),\ z_2 \sim N(0,1) \tag{AII.10}$$

In the bond example, the created changes in the risk factors are the $\delta r_\$$ and $\delta\ FX$.

Cholesky Decomposition of the Covariance Matrix (Up to Ten Risk Factors)

When there are more than two risk factors, we create the correlation by decomposing the covariance matrix using either Cholesky decomposition (in practice, for not more than 10 risk factors in the covariance matrix) or eigenvalue decomposition (for a larger number of risk factors).

For the Cholesky decomposition, we find a matrix A (the Cholesky matrix) such that:

$$C = A^T \times A \tag{AII.11}$$

where

C is the covariance matrix
A is an upper triangular, i.e., all elements below the main diagonal are zero, and positive definite matrix.

Taking K random numbers $K_1, \ldots, K_{10}$ drawn independently from standard normal distributions, we create a vector K:

$$K = [K_1, \ldots, K_{10}] \tag{AII.12}$$

If we multiply A by K, we get a vector, L, of ten random risk factors that are correlated according to the original covariance matrix:

$$L = K \times A \tag{AII.13}$$

$$[\delta f_1, \ldots, \delta f_{10}] = [K_1, \ldots, K_{19}] \times [A]_{10\times 10} \tag{AII.14}$$

and

$$\delta f_1 = [K_1, \ldots, K_{10}] \times [A\,1]_{10\times 1} \tag{AII.15}$$

$$\delta f_2 = [K_1, \ldots, K_{10}] \times [A\,2]_{10\times 1} \tag{AII.16}$$

$$\delta f_{10} = [K_1, \ldots, K_{10}] \times [A\,10]_{10\times 1} \tag{AII.17}$$

For the Cholesky matrix to be positive definite, all eigenvalues of the covariance matrix must be positive. This implies that none of the risk factors can have a perfect correlation with another factor. In practice, this condition may not hold when constructing covariance matrices. The algorithm to find the Cholesky matrix can be found in Numerical Recipes in C (1997).

Eigenvalue Decomposition of the Covariance Matrix (Large Number of Risk Factors)

The eigenvalue decomposition is also known as principal components analysis (Johnston, 1972). It works for covariance matrices that are not positive definite and, therefore, for matrices with a large number of risk factors. However, eigenvalue decomposition may fail if negative variances are generated for some of the principal components (e.g., because of inconsistencies in the data owing to building different parts of the matrix with data from different time periods). Eigenvalue decomposition also allows to analyze the structure of the random risk factors and, thus, to identify the main drivers of risk.

For the eigenvalue decomposition, we look for two matrices, Λ and E, to satisfy the following equation:

$$C = E^T \times \Lambda \times E \tag{AII.18}$$

where

C is the covariance matrix of the risk factors

E is the matrix of eigenvectors corresponding to each eigenvalue (each column of E is one eigenvector, corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_N$) and $E^T \times E = I$. Or,

$$E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1N} \\ e_{21} & e_{22} & \cdots & e_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ e_{N1} & e_{N2} & \cdots & e_{NN} \end{bmatrix}$$

Λ is a diagonal matrix, i.e., a square matrix in which all the elements other than the main diagonal are zero, with the eigenvalues of C:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

Since Λ is diagonal, we can also decompose C as:

$$C = B^T \times B \qquad \text{(AII.19)}$$

where

$$B = \Lambda^{1/2} \times E$$

and

$$\Lambda^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_N} \end{bmatrix}$$

Computationally, we can proceed as follows:

Step 1: Since C is an NxN matrix, we can find the N eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_N$, by calculating:

determinant $(C - \lambda \times I) = 0$

Step 2: For each eigenvalue found in step 1, we can then compute the corresponding eigenvector $\underset{\sim}{E}$ from the following system of equations:

$$(C - \lambda \times I) \times \underset{\sim}{E} = \underset{\sim}{0}$$

In case of degenerate eigenvalues (i.e., two or more eigenvalues generated in step 1 have the same value), the previous system of equations becomes indetermined. This can be resolved by adding an appropriate set of constraints, such as that the eigenvectors are orthogonal which implies:

$$E_i \times E_j = |E_i| \times |E_j| \times \delta_{ij} ,$$

where δ_{ij} is the Kronecker delta, defined as

$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases},$$

and $|E_i|$ and $|E_j|$ are the modules of the vectors.

Step 3: Once the eigenvalues and the eigenvectors have been determined, we can construct the matrices E and Λ and,

Step 4: Consequently, we can calculate the matrix B such as $B = \Lambda^{1/2} E$.

Using the matrix B from the eigenvalue decomposition, we can now generate correlated random numbers. Taking N random numbers, drawn independently from standard normal distributions, we create a vector N:

$$N = [N_1, \ldots, N_N] \tag{AII.20}$$

If we multiply B by N, we get a vector, F, of random risk factors (random error components) that are correlated (weighted) according to the original covariance matrix:

$$F = N \times B \tag{AII.21}$$

$$[\delta f_1, \ldots, \delta f_N] = [N_1, \ldots, N_N] \times [B]_{N \times N} \tag{AII.22}$$

and

$$\delta f_1 = [N_1, \ldots, N_N] \times [B1]_{N \times 1} \tag{AII.23}$$

$$\delta f_2 = [N_1, \ldots, N_N] \times [B2]_{N \times 1} \tag{AII.24}$$

…

$$\delta f_N = [N_1, \ldots, N_N] \times [B\,N]_{N \times 1} \tag{AII.25}$$

Each eigenvector defines a market movement that is by definition independent of the other movements (since $E^T \times E = I$).

Note that Monte Carlo simulation captures nonlinearity (i.e., allows for the price change not being a linear function of the change in the risk factors), but does not capture non-normality (i.e., it does not have the ability to calculate the potential changes in risk factors without assuming that they have a normal or log-normal distribution) and can be computationally slow.

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