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On Myopic Equilibria in Dynamic Games with Endogenous Discounting

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Abstract

This paper derives an equilibrium for a competitive multi-stage game in which an agents' current action influences his probability of survival into the next round of play. This is directly relevant in banking, where a banks' current lending and pricing decisions determines its future probability of default.
In technical terms, our innovation is to consider a multi-stage game with endogenous discounting. An equilibrium for such a multi-stage game with endogenous discounting has not been derived before in the literature.

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Contents

I Introduction

In this article we ask what type of equilibrium behaviour results in a multi-stage game in which the players are able to influence the probability of breakdown ("default") after each stage by their own actions in that stage. This type of game corresponds to many interesting economic environments. For example, think of two commercial banks that fiercely compete in the market for customer loans. It has been widely recognized in the banking literature that increased competition can undermine prudent bank behaviour.[2] The mechanism is as follows. Competition erodes bank profits, lower profits imply lower bank charter values, and lower charter values increase moral hazard incentives for banks to make riskier loans. With sufficient competition, banks may find it worthwhile to gamble. The induced riskiness translates into higher default probabilities, or – in more game-theoretic terms – into lower continuation probabilities to reach the next time period. In effect, this simple mechanism implies that the banks' risky actions over time have endogenized the discount factor. In such a dynamic framework with endogenous discounting, it is not at all trivial what type of equilibrium behaviour will result. In this article we characterize and analyze a stationary equilibrium in a dynamic game with endogenous discounting from a game-theoretic perspective.

Our starting point is a two-player, multi-stage game with an infinite horizon in which the players face the same 'stage' game in every period, and the players' overall payoff is a discounted sum of the payoffs in every stage. We assume that the probability of reaching the next stage is determined by the players' actions in the current stage. Hence, in this game the discount factor of the players is endogenous, which effectively implies that current play has a direct impact on future per-period attainable payoffs. It is this particular feature that distinguishes our multi-stage game from the usual repeated game. Repeated games do not allow any influence of past and current play on future feasible actions or payoff functions. The 'physical environment' of our game is changing while that of a repeated game is memoryless. As a consequence, it is no longer the case that playing a Nash equilibrium of the stage game in every period constitutes a stationary subgame perfect equilibrium of the multi-stage game with endogenous discounting.

Dynamic games with infinitely many stages in complex environments generally feature a continuum of equilibria. The usual equilibrium concept to be used in such settings is the Markov perfect equilibrium, but these equilibria are typically hard to characterize.[3] Instead, we will take a simpler route. We will focus on a stationary equilibrium in which the players are myopic. Myopic play describes a situation in which the players take the future strategies of their opponents as given, irrespective of the actual history of the game. Therefore, as players do not perceive any influence on subsequent play, the 'continuation value' of the game is fixed. This allows an intuitive and easy derivation of equilibria. We find that the myopic equilibrium actions of the infinite-horizon multi-stage game are equal to the Nash equilibrium actions of some induced (one-shot) 'limit' game. The myopic equilibrium of the infinite-horizon multi-stage game corresponds to the infinite repetition of a Nash equilibrium of this limit game. In this sense, for the derivation of a stationary equilibrium, it is as though the limit game takes over the role of the stage game in a repeated game, but now

[2] See Hellman, Murdock, and Stiglitz (2000) for a dynamic model that aims to understand the interaction between financial liberalization and prudential regulation. It shows that the potential scope for gambling increases whenever the intensity of competition increases. Similar results are also obtained by, e.g., Bolt and Tieman (2004) and Keeley (1990).

[3] When studying equilibrium behaviour in complex dynamic environtments, attention is often focused on equilibria in a smaller class of so-called "Markov" strategies in which the past influences current play only though its effect on a state variable; see Fudenberg and Tirole (1991).

corrected for the endogenous impact of the discount factor on the per-period payoffs.

We also ask whether this myopic equilibrium has any intuitive appeal. Interestingly, the stationary myopic equilibrium is singled out when studying limiting equilibria of the multi-stage game with a finite horizon. In particular, we show that if the number of stages in the finite-horizon multi-stage game tends to infinity, then the unique subgame perfect equilibrium actions in (almost) every stage become arbitrarily close to the Nash equilibrium actions of the limit game. The myopic equilibrium is the only equilibrium of the infinitely many equilibria that survives this equilibrium selection mechanism. We argue that this selection is interesting and makes good sense. It is what happens if players for some reasons are not able to coordinate on a good equilibrium. We may say that playing a Nash equilibrium of the limit game in every period in a multi-stage game with endogenous discounting is the perfect analog of playing a Nash equilibrium of the stage game in every period in a repeated game.

The setup of the remainder of the paper is as follows. The next section defines the multi-stage game with infinitely many stages, in which the discount factor is endogenized. In section III, we derive the stationary myopic equilibrium. The equilibrium selection mechanism is described in section IV. Section V describes an illustrative example, and the last section concludes.

II Defining the multi-stage game with endogenous discounting

An important building block of our dynamic multi-stage game, which we dub G, is the stage game, say g, which is played in every period t, $t \geq 0$. Assume that the stage game is a (symmetric) two-player simultaneous move-game with finite action spaces A_i, $i = 1, 2$, and stage game payoff functions $\pi_i : A \to \mathbb{R}$, $i = 1, 2$, where $A = A_1 \times A_2$. The nonempty set of Nash equilibrium actions of the stage game g is denoted by $A^N(g) \subset A$.[4] For simplicity, we assume that the stage game g has a unique Nash equilibrium in pure actions a^N with corresponding payoffs $\pi^N = \pi(a^N)$. We will discuss this uniqueness restriction later on.

The players' overall payoff is a discounted sum of the payoffs in every stage. In our multi-stage game G, the discount factor represents a combination of the players' exogenous rate of time preference and an endogenous continuation probability.[5] As long as the game continues, the players' current actions determine the common probability to reach the next stage, denoted by $p_t = p(a_t)$, with $p_t \in [0, 1]$ and $a_t \in A$. In particular, assuming a common rate of time preference $r > 0$ for both players, then for $t \geq 1$ we may write the discount factor as

$$\delta_{t+1} = \frac{1}{(1+r)} \cdot p(a_t), \quad a_t \in A, \tag{1}$$

with $0 \leq \delta_t < 1$ for $t \geq 1$, and $\delta_0 = 1$. Hence, note that the discount factor that discounts the next period's payoffs to the present time will only depend on current actions.

[4] The set A^N is nonempty if the strategy space A_i is a compact and convex set of an Euclidean space, and payoff function $\pi_i(a)$ is continuous and quasi-concave on A_i.

[5] See Fudenberg and Tirole (1991) for a similar interpretation of the discount factor. They show that infinitely repeated games can represent games that terminate in finite time with probability one. Key is that the conditional probability of reaching the next period is bounded away from zero. Also, Osborne and Rubinstein (1990) analyze a bargaining model that combines risk of breakdown and time preference as driving forces to reach an agreement quickly.

To define the dynamic game, we must specify the players' strategy spaces and payoff functions. We assume that the players observe the realized actions at the end of each period. We start the game at $t = 0$, with H_0 the set of 'null' histories preceding $t = 0$, indicating that nothing has happened before $t = 0$. Denoting $A^t = A \times \cdots \times A$ the t-fold multiplication of A, then for $t \geq 1$, let history $h_t \in A^t$

$$h_t = (a_0, a_1, \ldots, a_{t-1}), \quad a_k \in A, \quad 0 \leq k \leq t-1 \tag{2}$$

be the realized choices of actions at all periods before t, and let H_t be the set of all possible period-t histories. The set H of all possible histories of the game G can be represented by the infinite union of the sets of all possible period-t histories. That is, we define

$$H = \bigcup_{t=0}^{\infty} H_t. \tag{3}$$

Strategies for the players are rules telling the players how to move at each stage for each possible history up to that stage. Since both players observe h_t, a (pure) strategy σ_i for player i in the dynamic game is a sequence of maps σ_i^t – one for each period t – that map possible period-t histories $h_t \in H_t$ to actions $a_{it} \in A_i$. Formally, for $i = 1, 2$,

$$\sigma_i : H \rightarrow A_i. \tag{4}$$

The set of all such strategies is Σ_i, and $\Sigma = \Sigma_1 \times \Sigma_2$.

Any strategy profile $\sigma = (\sigma_1, \sigma_2) \in \Sigma$ uniquely determines an outcome of the game that is valued by the players. We assume that both players maximize their (expected) present value

$$V_i(\sigma) = \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \delta(\sigma(h_s)) \right) \pi_i(\sigma(h_t)), \tag{5}$$

where $\delta(\sigma(h_0)) = \delta_0 = 1$.

Finally, if $\sigma_i \in \Sigma_i$ and $h_t \in H_t$, then σ_{it} – the continuation of σ_i after h_t – is the strategy defined by

$$\sigma_{it}(h_{\tau|t}) = \sigma_i(h_t, h_{\tau|t}), \quad \tau \geq t, \tag{6}$$

where $(h_t, h_{\tau|t}) \in H_\tau$ is the history h_t followed by history $h_{\tau|t}$ (with appropriate conventions for histories of length 0). The corresponding continuation value of the game from stage t onward is given by

$$V_i(\sigma_{it}) = \sum_{\tau=t}^{\infty} \left(\prod_{s=t}^{\tau} \delta(\sigma_{it}(h_{s|t})) \right) \pi_i(\sigma_{it}(h_{\tau|t})), \tag{7}$$

where $\delta(\sigma_{it}(h_{t|t})) = 1$ and $\pi_i(\sigma_{it}(h_{t|t})) = \pi_i(\sigma(h_t))$.

III Equilibrium analysis

In this section we will characterize a stationary myopic equilibrium of the multi-stage game, in which the discount factor is endogenized. But before we do so, we first turn to the case where the discount factor is exogenously given.

A. Equilibrium with an exogenous discount factor

The case of an exogenous discount factor takes us back to the familiar repeated games framework, in which the players play the stage game in every period. We are interested in a stationary equilibrium, where actions do not change over time. So, the corresponding strategy profile σ prescribes $\sigma_i(h) = a$, $a \in A$, for every history $h \in H$. For a fixed discount factor $\delta(a_t) = \delta$ for all $a_t \in A$, player i's present value of the infinite sequence of payoffs reduces to

$$V_i^F = \sum_{t=0}^{\infty} \delta^t \pi_i(a) = \frac{\pi_i(a)}{1-\delta}. \tag{8}$$

Focus on the unique Nash equilibrium $a^N \in A^N(g)$ of the stage game g. It is well known that playing the Nash equilibrium a^N of the stage game g in every period t is a stationary subgame perfect equilibrium of the game G under a fixed discount factor (see, e.g., Fudenberg and Tirole (1991)). Consider the next proposition.

Proposition III.1. *If a^N is a Nash equilibrium of the stage game g, then the strategy profile σ^N such that $\sigma^N(h) = a^N$, for every history $h \in H$, induces a stationary subgame perfect equilibrium of the game G with exogenous discounting.*

Under the strategy profile σ^N the future play of player j is independent of how player i plays today, so his best reply is to play to maximize his current payoff, that is, to play a_i^N given that player j plays his static Nash action.[6]

B. Equilibrium with an endogenous discount factor

In this subsection we characterize a stationary equilibrium of the game G with an endogenous discount factor $\delta(a)$, $a \in A$, where the players act myopically. Myopic play here means that the players do not perceive any influence of current actions on subsequent play.

We are interested in a stationary equilibrium, where actions do not change over time. Again, the corresponding strategy profile σ prescribes $\sigma_i(h) = a$, $a \in A$, for every history $h \in H$. Hence, the endogenous discount factor will also be constant over time, i.e., $\delta_t = \delta(a) < 1$ for all t. The present value for player i now reduces to

$$V_i^E = \sum_{t=0}^{\infty} \delta^t(a) \pi_i(a) = \frac{\pi_i(a)}{1-\delta(a)}. \tag{9}$$

Imposing constant actions over time is not innocuous in our game, since current actions are able to influence next day's attainable payoffs via the discount factor, and may therefore affect equilibrium actions over time. Fixing actions in this nonstationary environment is as though the players are committed once and for all to choosing a particular action, say (a_i^*, a_j^*), at $t = 0$, and play this action forever after. So, even when – for whatever reason – one of the players has deviated from

[6] Obviously, when the Nash equilibrium is not unique, playing any sequence of different Nash equlibria of the stage game in every period is a subgame perfect equilibrium of the (in)finitely repeated game.

playing her specified action a_i^*, her opponent will still believe that in the future she will stick to her action a_i^* again. Given the opponent's 'myopic' beliefs, in equilibrium, it is optimal for him to also stick to his action a_j^*. However, a fully rational player j may wish to change his action in the future if player i deviates from playing a_i^* today. Under myopic play, when calculating her first-order condition, player i does not take into account that player j may respond in the future. Myopic play - in the sense that players do not perceive any influence on subsequent play - allows us to derive constant equilibrium actions in the infinite-horizon multistage game.

In particular, as we will see in the next subsection, when analyzing the corresponding *finite*-horizon multistage game, subgame perfect equilibrium actions are not constant over time. They differ in every period, but they converge to constant actions as the number of stages tends to infinity. These converged actions are exactly the constant myopic equilibrium actions that we want to characterize in the *infinite-horizon* multistage game.

In our definition, the players are said to be myopic if they take the continuation value of the game as fixed.[7] Consider the next formal definition of myopic play.

Definition III.2. *Myopic play: the players take the continuation value of the game as given. That is, for a given $\sigma \in \Sigma$ we assume*

$$V_i(\sigma_{it}(h_{\tau|t})) = V_{it}, \quad \textit{for every } h_t \in H_t, \quad \tau \geq t. \tag{10}$$

In our game, definition III.2 implies that whatever the actual history of play, the players take V_i^E as given from every period onward. The first-order conditions characterize the stationary myopic equilibrium of the game G. In particular, player i's maximization problem becomes

$$\max_{a_i} \ \pi_i(a) + \delta(a) V_i^E. \tag{11}$$

As V_i^E is taken as given, the first-order condition for a maximum implies

$$\frac{\partial \pi_i(a)}{\partial a_i} + \frac{\partial \delta(a)}{\partial a_i} V_i^E = 0.$$

Using stationarity, we can insert equation (9) to get

$$\frac{\partial \pi_i(a)}{\partial a_i} + \frac{\partial \delta(a)}{\partial a_i} \frac{\pi_i(a)}{1-\delta(a)} = 0, \tag{12}$$

which is equivalent to

$$(1-\delta(a))\frac{\partial \pi_i(a)}{\partial a_i} + \pi_i(a)\frac{\partial \delta(a)}{\partial a_i} = 0. \tag{13}$$

Equation (13) characterizes the stationary myopic equilibrium of our game G yielding equilibrium actions $a^* \in A$ and corresponding equilibrium strategy profile $\sigma_i^{ME}(h) = a^*$, for every $h \in H$.

Interestingly, these myopic actions of the infinite-horizon game G are equal to the Nash equilibrium actions of a related one-shot limit game. To see this, let us now look at the one-shot game g^L that is derived from the stage game g but with modified payoff functions, given by

$$\pi_i^L(a) = \frac{\pi_i(a)}{1-\delta(a)}, \quad a \in A. \tag{14}$$

[7] Interestingly, in a repeated-game setting, Hausken (2005) also studies equilibrium behavior where players are myopic. However, in his analysis, myopia means a discount factor equal to zero, which is a different notion than the one used here.

The limit game g^L corrects the payoffs of stage game g for the endogenous impact of the disount factor. In particular, the payoffs of g^L represent the discounted sum of the per-period payoffs of g, discounted at a constant rate given that equilibrium actions are constant over time.

After some algebraic manipulation, the first-order conditions for a Nash equilibrium of g^L imply that

$$(1-\delta(a))\frac{\partial \pi_i(a)}{\partial a_i} + \pi_i(a)\frac{\partial \delta(a)}{\partial a_i} = 0. \tag{15}$$

Hence, from equations (13) and (15), it is clear that Nash equilibrium actions of the limit game g^L correspond one-to-one to myopic equilibrium actions a^* of G. That is, $a^* \in A^N(g^L)$. Consider the next proposition.

Proposition III.3. *If a^* is a Nash equilibrium of the limit game g^L, then the strategy profile σ^{ME} such that $\sigma^{ME}(h) = a^*$, for every history $h \in H$, induces a stationary myopic equilibrium of the game G with endogenous discounting.*

So, to calculate the myopic equilibrium actions of the infinite game it is sufficient to derive the Nash equilibria of the limit game. From comparing proposition III.1 with proposition III.3, we can say that playing a Nash equilibrium of the limit game in every period of the multi-stage game with endogenous discounting is the perfect analog of playing a Nash equilibrium of the stage game in every period of the repeated game with exogenous discounting.

IV Finite horizon and equilibrium selection

Dynamic games often show a continuum of equilibria. In this section we argue that our stationary myopic equilibrium survives if we consider limiting subgame perfect equilibria of the finite horizon dynamic game G^T by letting $T \to \infty$.

The finite horizon game G^T is defined in the same way as G but restricted to $T+1$ stages in which the stage game g is played, t running from 0 to T. The game G^T has a unique subgame perfect equilibrium, which can be found by backward induction.

Consider the game G^T. Obviously, in the very last stage T, the unique Nash equilibrium actions $a_T^N = a^N$ of the stage game g are played, so that $a_T^N \in A^N(g)$ with equilibrium payoffs $\bar{\pi}_i^T = \pi_i(a_T^N)$, $i = 1, 2$. In the penultimate stage $T-1$, both players correctly anticipate that Nash equilibrium actions a_T^N will be played in the next period. However, by their actions a_{T-1} in stage $T-1$ they are able to influence the last period's discount factor via $\delta_T = \delta(a_{T-1})$. In fact, since the period T outcome is known in advance in stage $T-1$, it will be subgame perfect to play Nash equilibrium actions a_{T-1}^N of a modified stage game g_{T-1} with payoff functions

$$\pi_i^{T-1}(a) = \pi_i(a) + \bar{\pi}_i^T \delta(a), \quad a \in A. \tag{16}$$

Hence, $a_{T-1}^N \in A^N(g_{T-1})$. Clearly, $a_{T-1}^N \neq a_T^N$, since the discount factor is endogenous. Similarly, in stage t, $t < T$, as part of a subgame perfect equilibrium, the players will play Nash equilibrium actions a_t^N of the modified stage game g_t with payoff functions

$$\pi_i^t(a) = \pi_i(a) + \bar{\pi}_i^{t+1}\delta(a), \quad a \in A, \tag{17}$$

with $\bar{\pi}_i^{t+1} = \pi_i^{t+1}(a_{t+1}^N)$. So, $a_t^N \in A^N(g_t)$. By backward induction this process goes on until we reach the very first stage 0, yielding Nash equilibrium actions a_0^N of the modified stage game g_0. That is, $a_0^N \in A^N(g_0)$.[8] In this way, backward induction allows us to construct a unique subgame perfect equilibrium of the finite horizon game G^T. This subgame perfect equilibrium induces an outcome path of Nash equlibrium actions $\{a_0^N, a_1^N, \cdots, a_T^N\}$.

We are interested what happens to the period-t Nash equilibrium actions a_t^N, $t = 0, \ldots, T$, as the number of periods of the finite-horizon game G^T gets large. We will show that if some stability condition is satisfied, period-t Nash equilibrium actions a_t^N converge to the Nash equilibrium actions a^* of the limit game g^L. Hence, the myopic equilibrium of the infinite-horizon game is the sole survivor when analyzing subgame perfect equilibria of the finite-horizon game in the limit. This is the message of our main result.

The proof of proposition IV.1 centers around the modified stage game. Let us take a closer look at the modified stage game g^k with payoff functions

$$\pi_i^k(a, k) = \pi_i(a) + k\delta(a), \quad a \in A, \tag{18}$$

where parameter k plays the role of future equilibrium profits. Denote the corresponding Nash equilibrium actions of g^k by $a^N(k) \in A^N(g_k)$, and define mapping $z(k)$ as follows:

$$z(k) = \pi_i^k(a^N(k), k). \tag{19}$$

In fact, the iteration process $k_{s+1} = z(k_s)$, for $s = 0, \ldots, T$, describes the evolution of the period-$(T - s)$ Nash equilibrium actions and payoffs of G^T over time. In particular, starting with $k_0 = 0$, in the first round ($s = 0$) we have

$$\begin{aligned} k_0 = 0 \;\Rightarrow\; a^N(k_0) = a_T^N \;\Rightarrow\; k_1 = \pi_i(a^N(k_0), k_0) = \bar{\pi}_i^T, \\ k_1 = \bar{\pi}_i^T \;\Rightarrow\; a^N(k_1) = a_{T-1}^N \;\Rightarrow\; k_2 = \pi_i(a^N(k_1), k_1) = \bar{\pi}_i^{T-1}, \end{aligned}$$

and so on until we arrive at $a^N(k_T) = a_0^N$. We are looking for a fixed point $k^* = z(k^*)$ that is asymptotically stable, and inducing $a^N(k^*) = a^*$.

First, for a fixed point k^* of $z(k)$ it must hold that

$$k^* = \pi_i(a^N(k^*)) + k^*\delta(a^N(k^*)), \tag{20}$$

or, equivalently,

$$k^* = \frac{\pi_i(a^N(k^*))}{1 - \delta(a^N(k^*))}. \tag{21}$$

Hence, at $k = k^*$, the Nash equilibrium actions $a^N(k^*)$ of the modified game g^{k^*} correspond to the Nash equilibrium actions of the game with payoff functions $\pi_i(a)/(1 - \delta(a))$, which is exactly the limit game g^L. Therefore, $k^* = \pi_i^L(a^*)$ and $a^N(k^*) = a^*$.

Second, to check asymptotic stability, let us look at the derivative $dz(k)/dk$; that is,

$$\frac{dz(k)}{dk} = \frac{d\pi_i^k(a^N(k), k)}{dk} = \frac{\partial \pi_i^k}{\partial a_i^N(k)} \frac{\partial a_i^N(k)}{\partial k} + \frac{\partial \pi_i^k}{\partial a_j^N(k)} \frac{\partial a_j^N(k)}{\partial k} + \frac{\partial \pi_i^k}{\partial k}. \tag{22}$$

[8] Note that as an initial condition in the recursion we set $g^T = g$.

Note that $\partial\pi_i^k/\partial k = \delta$. By the envelope theorem the first term drops out, leaving

$$\frac{dz(k)}{dk} = \frac{\partial\pi_i^k}{\partial a_j^N(k)}\frac{\partial a_j^N(k)}{\partial k} + \delta(a^N(k)). \tag{23}$$

After some straightforward manipulations, we find

$$\frac{dz(k)}{dk} = \left(-\frac{\partial^2\pi_i^k/\partial a_i \partial k}{\partial^2\pi_i^k/\partial a_i\partial a_j + \partial^2\pi_i^k/\partial^2 a_i}\right)\frac{\partial\pi_i^k}{\partial a_j^N(k)} + \delta(a^N(k)). \tag{24}$$

For asymptotic stability, we need $|dz(k)/dk| \leq 1$. Let us write $dz(k)/dk = B(a^N(k), k) + \delta(a^N(k))$, where

$$B(a^N(k), k) = \left(-\frac{\partial^2\pi_i^k/\partial a_i \partial k}{\partial^2\pi_i^k/\partial a_i\partial a_j + \partial^2\pi_i^k/\partial^2 a_i}\right)\frac{\partial\pi_i^k}{\partial k}. \tag{25}$$

Then, for the fixed point k^* of $z(k)$ to be asymptotically stable, the following condition must hold

$$|B(a^*, k^*) + \delta(a^*)| < 1. \tag{26}$$

Since $0 \leq \delta(a^*) < 1$, condition (26) holds if $|B(a^*, k^*)|$ is sufficiently small; i.e., condition (26) is satisfied if $|B(a^*, k^*)| < 1 - \delta(a^*)$. Consider the next proposition.

Proposition IV.1. *Consider the finite horizon game G^T. If stability condition (26) is satisfied, then for every $\epsilon > 0$ and for every $0 \leq t < \infty$, there exists $T > t$, such that for every $0 \leq s \leq t$ we have that $|\alpha_s^N - \alpha^*| < \epsilon$.*

In fact, since $T \to \infty$ as $t \to \infty$, from the above proposition we can easily deduce that if the number of stages of the game G^T approaches infinity, then in (almost) all periods the stage-t subgame perfect equilibrium actions a_t^N get arbitrarily close to the one-shot Nash equilibrium actions a^* of the induced limit game g^L. To put it differently, the subgame perfect equilibrium actions of the finite-horizon game G^T converge to the myopic equilibrium actions of the infinite-horizon game G.

V An illustrative example

Consider two duopolists who are engaged in Bertrand competition over time. We label this infinite-horizon game G. The stage game g is as follows. In period t, if firms 1 and 2 choose prices p_1^t and p_2^t, respectively, the quantity that consumers demand from firm i is

$$q_i(p_i^t, p_j^t) = a - p_i^t + bp_j^t,$$

where $b > 0$ reflects the extent to which firm i's product is a substitute for firm j's product. We assume that there are no fixed costs of production and that marginal costs are constant at c, where $c < a$, and that firms choose their prices simultaneously in every period t. The time-t profits to firm i when it chooses the price p_i^t and its rival chooses the price p_j^t is

$$\pi_i(p_i^t, p_j^t) = q_i(p_i^t, p_j^t)(p_i^t - c) = (a - p_i^t + bp_j^t)(p_i^t - c).$$

It is now straightforward to calculate the Nash equilibrium of the stage game g. For $0 < b < 2$, the (symmetric) Nash equilibrium prices are given by

$$p_1^N = p_2^N = \frac{a+c}{2-b}.$$

Fierce competition, driving prices down sharply, may destabilize the market as a whole and increase the risk of breakdown for the players. Thus, by the prices the firms choose over time, they are able to influence the discount factor. To keep things analytically tractable, we assume a very simple linear scheme

$$\delta_{t+1} = \delta(p_1^t, p_2^t) = \frac{p_1^t + p_2^t}{(1+r)v}, \qquad p_i^t \geq 0,\ p_1^t + p_2^t \leq v,$$

where $r > 0$ denotes the common rate of time preference, and $v \geq 1$ a scale parameter. This simple formalization tries to grasp that lower prices tend to disrupt the market.[9] We normalize a zero probability of continuation at the point where the total price $p_1^t + p_2^t$ is equal to zero. The limit game g^L has payoff functions

$$\pi_i^L(p_i, p_j) = \frac{\pi_i(p_i, p_j)}{1 - \delta(p_i, p_j)} = \frac{v(1+r)(a - p_i + bp_j)(p_i - c)}{v(1+r) - (p_i + p_j)}.$$

In solving for the Nash equilibrium of g^L, the best reply functions are given by

$$p_i^R(p_j) = v(1+r) - p_j - \sqrt{(c - v(1+r) + p_j)(a - v(1+r) + (1+b)p_j)}.$$

Naturally, the intersection of these best reply functions gives the Nash equilibrium prices (p_1^*, p_2^*) of the limit game g^L :

$$\begin{aligned} p_1^* = p_2^* = &\frac{v(1+r)(2-b) + a + c(1+b)}{2(3-b)} \\ &- \frac{\sqrt{(v(1+r)(2-b) + a + c(1+b))^2 - 4(3-b)((a+c)v(1+r) - ac)}}{2(3-b)}. \end{aligned}$$

Note that for $v \to \infty$ the endogenous effect of discounting disappears and, hence, $p_i^* \to p_i^N$.

Looking at the finite-horizon game G^T where Bertrand competition takes place for only T periods, we are interested in the Nash equilibrium prices of the modified stage game g^k with payoff functions

$$\pi_i^k(p_i, p_j, k) = \pi_i(p_i, p_j) + k\delta(p_i, p_j) = \frac{k(p_i + p_j)}{v(1+r)} + (a - p_i + bp_j)(p_i - c).$$

The best reply functions are

$$\bar{p}_i^R(p_j) = \frac{1}{2}\left(bp_j + a + c + \frac{k}{v(1+r)}\right),$$

yielding (symmetric) Nash equilibrium actions $(p_1^N(k), p_2^N(k))$ of the modified stage game g^k,

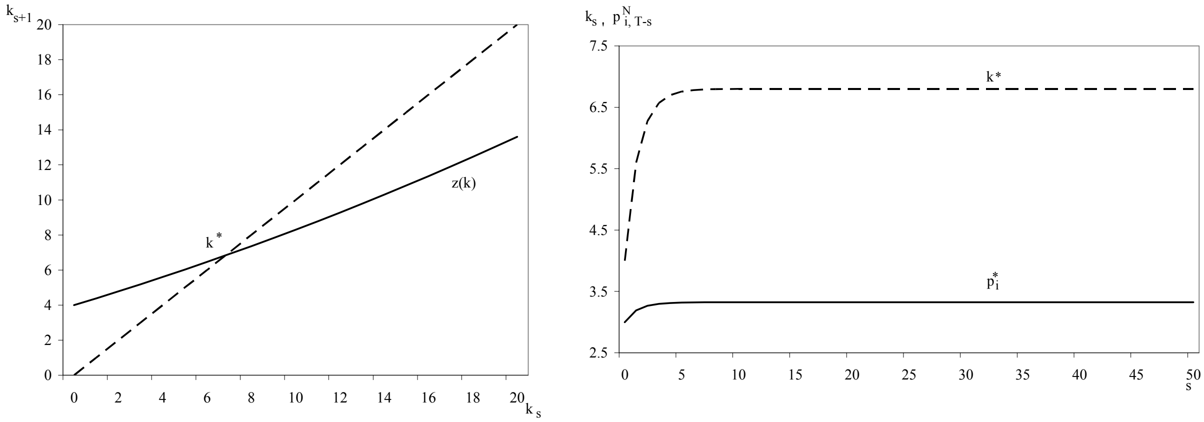
$$p_1^N(k) = p_2^N(k) = \frac{k}{(2-b)v(1+r)} + \frac{(a+c)}{2-b}.$$

For $k = 0$ we retrieve the Nash equlibrium actions of the stage game g so that $p_1^N(0) = p_2^N(0) = (a+c)/(2-b)$. The mapping $z(k) = \pi^k(p_1^N(k), p_2^N(k), k)$ can be represented by

$$z(k) = d_0 + d_1 k + d_2 k^2.$$

[9] Obviously, imposing a probability distribution to capture the endogenous effect on the discount factor would be more realistic, but it would also complicate matters in our example. See Bolt and Tieman (2004) for an application of the uniform and beta distribution.

Figure 1: Convergence to a myopic equlibrium



Note: The left panel describes the mapping $z(k)$; the right panel show the convergence toward k^* and p_i^* as a function of the number of stages of G^T.

(See the Appendix for the explicit formula.) Solving for the fixed point $k = z(k)$ gives us k^*, which verifies $(p_1^N(k^*), p_2^N(k^*)) = (p_1^*, p_2^*)$ (see Appendix). To see whether it is a stable fixed point we need to check $|dz(k)/dk| < 1$ at $k = k^*$. Following equations (24)-(26), we derive

$$B((p_1^N(k), p_2^N(k)), k) = \left(\frac{2k + b(a - c(1-b))v(1+r)}{((2-b)v(1+r))^2}\right) \times \left(\frac{1}{(2-b)v(1+r)}\right),$$

which, in absolute value, is smaller than $1 - \delta(p^N(k))$ for sufficiently large v. Hence, the iteration scheme $k_{s+1} = z(k_s)$ is asymptotically stable and converges toward k^*.

To illustrate further, let us plug in the following numerical values: $a = 2$, $b = 1$, $c = 1$, $r = 0.05$, and $v = 20$. Given these parameter values we retrieve Nash equilibrium prices $(p_1^N, p_2^N) = (3, 3)$ of the stage game g. The limit game g^L yields $(p_1^*, p_2^*) = (3.32, 3.32)$, which are the myopic equilibrium actions of the infinite horizon game G. Note that by endogenizing the discount factor, stationary equilibrium prices rise by 10 percent. From the modified stage game g^k we derive

$$p_1^N(k) = p_2^N(k) = 3 + 0.05k,$$

which induces iteration scheme $k_{s+1} = z(k_s)$, where

$$z(k) = 4.00 + 0.38k + 0.005k^2.$$

Solving for the fixed point yields $k^* = 6.80$, and $(p_1^N(k^*), p_2^N(k^*)) = (p_1^*, p_2^*) = (3.32, 3.32)$. Checking the stability condition gives $B((3.32, 3.32), 6.80) = 0.13 < 1 - \delta((3.32, 3.32)) = 0.67$. In Figure 1, the left panel shows the mapping $z(k)$ and its convergence point k^*.[10] The right panel shows the rate of convergence toward the fixed point k^* and the corresponding myopic equilibrium actions (p_1^*, p_2^*). We see that convergence is fairly rapid; already after $s = 10$ rounds time $T - s$ equilibrium actions become very close to the myopic equilibrium actions.

[10] In the example, the quadratic mapping $z(k) = k$ has obviously two roots, $k^* = 6.80$ and $k^* = 129.7$. However, this last solution is not asymptotically stable, and does therefore not correspond to a myopic equilibrium.

VI Discussion and concluding remarks

In many economic situations, agents are able to influence the probability of default by the (risky) actions they take themselves over time. But if players have a direct impact on the probability of reaching the next stage, equilibrium behaviour among these players gets more complicated. Just playing a Nash equilibrium of the stage game in every period is no longer part of a stationary subgame pefect equilibrium of the multi-stage game with endogenous discounting. In this paper, we have shown that there is an intuitive way to deal with this problem.

If players are assumed to be myopic, then stationary myopic equilibrium of the infinite-horizon multi-stage game corresponds to the infinite repetition of a Nash equilibrium of an induced, one-shot limit game. In this sense, for the derivation of a stationary equilibrium, it is as though this limit game takes over the role of the stage game in a repeated game, but now corrected for the endogenous impact of the discount factor on the per-period payoffs.

Interestingly, the stationary myopic equilibrium is the sole survivor when studying limiting subgame perfect equilibria of the multi-stage game with a finite horizon. In particular, we show that if the number of stages in the finite-horizon multi-stage game tends to infinity, then the unique subgame perfect equilibrium actions in (almost) every stage become arbitrarily close to the myopic equilibrium actions of the infinite-horizon multi-stage game. We feel that this selection mechanism is interesting and makes good sense. It is what happens if players for some reason are not able to coordinate on a good equilibrium. We may say that playing a Nash equilibrium of the limit game in every period in a multi-stage game with endogenous discounting is the perfect analog of playing a Nash equilibrium of the stage game in every period in a repeated game.

The derived myopic equilibrium seems intuitive and the equilibrium selection mechanism is appealing. But some difficulties remain. First, there is the question of uniqueness of equilibria. We assumed that the Nash equilibrium of the stage game is unique. This need not be the case. However, multiplicity of equilibria poses no real problem as long as one focuses on one particular equilibrium of the stage game. It is straightforward to show that for every Nash equilibrium of the stage game, there exists a corresponding myopic equilibrium of the dynamic game with endogenous discounting.

Second, it might be the case that although the stage game has a unique Nash equilibrium, some of the corresponding modified stage games have multiple Nash equilibria. Then, most likely, convergence towards a myopic equilibrium is lost, and in this case there seems no simple way to characterize a stationary equilibrium.

Third, we might ask what our analysis implies for the special but widely analyzed case of the stage game being a (two-by-two) bimatrix game. Interestingly, when we repeat the one-shot Prisoners' Dilemma with its unique but inefficient Nash equilibrium, it can be readily shown that by making the discount factor contingent on the four possible strategies of the stage game, the players could move to the efficient outcome after some time. Moreover, the differences in the discount factor need only be slight to trigger this switch to an efficient outcome, provided that the horizon of the game is "long" enough. However, one needs to be careful when interpretating the limit game in this bimatrix case. Evidently, the analysis gets more complicated if the bimatrix game has more than one Nash equilibrium.

In all, these above-mentioned difficulties open up an interesting avenue for further research, the more so since dynamic games in which players control their own breakdown probabilities seem to correspond to very natural economic phenomena.

Appendix

In section V, when we substitute $(p_1^N(k), p_2^N(k))$ in $\pi^k(p_1^N(k), p_2^N(k), k)$, this yields

$$z(k) = d_0 + d_1 k + d_2 k^2,$$

where

$$\begin{aligned} d_0 &= \frac{(a - c(1-b))^2}{(2-b)^2}, \\ d_1 &= \frac{a(4-b) + c(4 - b(3-b))}{v(1+r)(2-b)^2}, \text{and} \\ d_2 &= \frac{3-b}{(v(1+r)(2-b))^2}. \end{aligned}$$

Solving for the fixed point $z(k^*) = k^*$ gives

$$\begin{aligned} k^* = &- \frac{v(1+r)}{2(3-b)} \left(a(4-b) + c(4-b(3-b)) - (2-b)\left(v(2-b)(1+r)\right.\right. \\ &\left.\left. -(2-b)\sqrt{(v(1+r)(2-b) + a + b(1+c))^2 - 4(3-b)((a+c)v(1+r) - ac)}\right)\right). \end{aligned}$$

After some algebraic manipulation, substituting k^* into $p_i^N(k)$ we can show that $p_i^N(k^*) = p_i^*$.

References

Bolt, W., and A.F. Tieman, 2004, "Banking Competition, Risk, and Regulation," *Scandinavian Journal of Economics*, pp. 106, 783–804.

Fudenberg, D., and J. Tirole, 1991, *Game Theory* (Cambridge, Massachusetts: MIT Press)

Hausken, K., 2005, "The Battle of the Sexes When the Future Is Important," *Economics Letters*, 87, pp. 89–93.

Hellman, T., K. Murdock, and J. Stiglitz, 2000, "Liberalization, Moral Hazard in Banking, and Prudential Regulation: Are Capital Requirements Enough?," *American Econonomic Review*, 90, pp. 147–165.

Keeley, M. C., 1990, "Deposit Insurance, Risk, and Market Power in Banking," *American Economic Review*, 80, pp. 1183–1200.

Osborne, M., and A. Rubinstein, 1990, "Bargaining and Markets," (San Diego: Academic Press)