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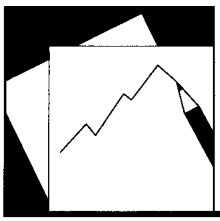

IMF Working Paper

Asset Prices in Affine Real Business Cycle Models

Aytek Malkhozov and Maral Shamloo

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Asset Prices in Affine Real Business Cycle Models*

Prepared by Aytek Malkhozov† and Maral Shamloo‡

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Abstract

We develop a tractable way to solve for equilibrium quantities and asset prices in a class of real business cycle models featuring Epstein-Zin preferences and affine dynamics for productivity growth and volatility. The method relies on log-linearization and exploits the log-normality of all the quantities. It is an easy substitute for more involved numerical techniques, such as higher order perturbation methods, and allows for easy implementation and analytical results. We show explicitly the link with perturbation techniques and find that the quantitative difference between the two is insignificant for several models of interest.

JEL Classification Numbers: C18, C61, C63, C65, C68, G12, E30.

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† McGill University. Email: aytek.malkhozov@mcgill.ca
‡ IMF Institute. Email: mshamloo@imf.org.

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I. Introduction

Recursive preferences and time variation in means and volatilities have become important features of consumption-based asset pricing literature. The introduction of these features into real business cycle (RBC) models has allowed the study of the joint behavior of real and financial variables along the business cycle. As the analysis of asset prices requires computing risk adjustments, simple log-linearization is insufficient. Furthermore, numerical methods such as value-function iteration are computationally expensive and ill-suited for problems with a large number of state variables.

In this paper we propose a simple alternative. We describe how to compute log-linearized dynamics and risk adjustments that accurately characterize asset pricing and welfare implications while retaining the computational simplicity of log-linearization methods. More specifically, we consider a standard RBC model augmented along two dimensions—recursive Epstein-Zin preferences (see, Epstein and Zin (1989), Epstein and Zin (1991), and Weil (1989)) and a general affine structure for the exogenous state variables. We show how to solve this class of models by exploiting the joint log-normality of shocks and using log-linearization. The suggested method is an easy and tractable alternative to the more common techniques. Our approach is closest to higher-order perturbation methods, where equilibrium conditions are expanded (perturbed) around a steady state using Taylor series expansions. However, whereas higher-order perturbation methods required for asset prices or models with stochastic volatility are difficult to compute, the linearity and analytical form of our suggested method is convenient and computable by hand. The suggested approximation is equivalent to a perturbation solution where higher-order terms describing the dynamics of the quantities are omitted while the key terms accounting for risk adjustments are retained. We show that the resulting approximations from the two methods are almost identical numerically for some examples of interest. We use the theoretical and implementation framework provided by Schmitt-Grohe and Uribe (2004) to do so.

We see three main advantages to our method. First, the linear structure of the solution makes it easy to describe the time-series properties of the variables of interest and carry out estimation. Second, the analytical form allows us to explicitly inspect the mechanisms behind quantity dynamics and asset prices. In particular, it is interesting to examine the effects of separating relative risk aversion from the inverse of the elasticity of inter-temporal substitution in a production economy compared to the more standard time-additive utility case. Finally, our method can deal with stochastic volatility much more easily compared to standard pertur-

bation methods. With perturbation methods at least a third-order expansion is necessary for studying stochastic volatility. We avoid the complications of higher-order exansions in our method by nesting stochastic volatility in a general affine shock structure.

Our work is related to two others that derive risk adjustments in log-linearized RBC models. Backus, Routledge, and Zin (2007) consider a problem similar to ours but without stochastic volatility. We will argue, based on the work by Caldara and others (2008), that their choice of equilibrium conditions to linearize leads to an inconsistent approximation of the value function, which is of crucial importance for models with recursive preferences.

Uhlig (2010) looks at a similar framework without stochastic volatility. The author computes risk adjustments from asset pricing equations only. We show that it is important to consistently compute the risk adjustment resulting from all the equations in the model. The failure to compute risk adjustments for quantities, in particular for models with recursive preferences, can lead to a bias in prices even if risk adjustment for prices are taken into account. For example, ignoring precautionary savings will result in a biased level of the risk-free rate if there is a preference for early resolution of uncertainty. The main contribution of our paper is to demonstrate how log-linearization and risk adjustment can be applied to solve models with stochastic vololatiliy. Solving models that feature stochastic volatility has attracted special attention. Fernandez-Villaverde and others (2009) argue that a third-order approximation is necessary when using perturbation methods. The same conclusion is reached in Malkhozov and Shamloo (2010): the third-order terms in the approximation obtained using a perturbation method capture the first-order dynamic effects of time-varying volatility. The authors use a fourth-order approximation to capture second-order dynamic effects of volatility changes as a robustness check. In this paper we can directly obtain the first-order effects of stochastic volatility on quantities and asset prices by log-linearizing the model and accounting for risk adjustments appropriately.

Furthermore, we demonstrate that standard perturbation method techniques are perfectly suitable for dealing with recursive preferences. We discuss how approximation techniques such as the one we present can be easily applied to such models. In this respect, our work is close to Rudebusch and Swanson (2008), Swanson, Anderson, and Levin (2006), Binsbergen van and others (2008), and Caldara and others (2008), who all address the specific issue of solving models with recursive preferences using perturbation methods.

The affine dynamics of exogenous state variables is central to our solution method. We build on the work by Duffie, Pan, and Singleton (2000), who introduced continuous time affine

processes as a powerful modeling tool that allows finding closed-form solutions for a number of problems in finance and economics.[1]

We demonstrate the application of our model using two examples. Both are RBC models in which agents work, consume and own assets. The first example features recursive preferences, and the second presents an application of stochastic volatility. The elements in these examples draw on several strands in the asset pricing literature. We will briefly mention a few.

The literature on asset prices in production economies has been developing rapidly. Kaltenbrunner and Lochstoer (2007), Croce (2008), and Malkhozov and Shamloo (2010) study asset prices and macroeconomic quantities in RBC-type models. This paper describes a simple and tractable way to tackle problems in this branch of literature. The two examples in our paper are very closely related to these models.

Shocks to volatility have recently emerged as an important factor in driving the business cycle. Justiniano and Primiceri (2008), Fernandez-Villaverde and others (2009), and Bloom, Floetotto, and Jaimovich (2009) show the importance of these shocks for macroeconomic quantities. While these papers argue that changes in uncertainty are most easily observable in financial markets, they do not address the issue of asset pricing implications specifically. Malkhozov and Shamloo (2010) use perturbation methods to study asset prices in a simple growth model with stochastic volatility in productivity growth. This paper offers an easy way to investigate the role of volatility shocks both for macroeconomic quantities and asset prices.

The remainder of the paper is organized as follows. Section II introduces the setup and describes the log-normal risk adjustment technique. Sections III and IV present our two examples, their calibration and the numerical comparison of the solutions obtained by the log-linear risk adjustment method with the widely used perturbation techniques. The example in Section III is a standard RBC model with Epstein-Zin preferences as well as stationary and nonstationary shocks. The example in Section IV features stochastic volatility. Section V concludes.

II. SETUP

In this section we describe our baseline real business cycle model with Epstein-Zin preferences and an affine structure of shocks. The framwework can be further extended along sev-

[1]Note that our analysis is in discrete time. For the discrete time counterpart to Duffie, Pan, and Singleton (2000) see, for example, Dai, Le, and Singleton (2010).

eral dimensions, such as flexible labour supply or capital adjustment costs, without changing the results of the following sections of the paper.

A. Preferences

The representative consumer maximises the utility function defined recursively

$$\underset{C_t}{Max} U_t$$

where

$$U_t = \left(C_t^{1-1/\psi} + \beta (E_t (U_{t+1}^{1-\gamma}))^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}}$$

Unlike CRRA utility function, Epstein-Zin recursive preferences allow us to separate the elasticity of intertemporal substitution from the coefficient of relative risk aversion (see Epstein and Zin, 1989). The parameter γ controls agents relative risk aversion and ψ his elasticity of intertemporal substitution. The standard power utility can be obtained as a special case by setting $\gamma = 1/\psi$.This separation has an important implication for the agents preferences towards the early resolution of uncertainty. In the power utility case investor is indifferent towards the timing of resolution of uncertainty, if $\gamma > 1/\psi$, $(\gamma < 1/\psi)$, an investor prefers early (late) resolution of uncertainty. Intuitively, with $\gamma > 1/\psi$ agents propensity to smooth consumption across states of the world is greater than propensity to smooth consumption across time.

It is important to include recursive preferences in our analysis. Separating the relative risk aversion parameter (γ) and the elasticity of intertemporal substitution (ψ) has been instrumental in tackling asset pricing puzzles in recent literature (see Bansal and Yaron (2004), Kaltenbrunner and Lochstoer (2007) and Croce (2008)). We show that standard macroeconomic techniques can easily handle models featuring recursive preferences and that the use of computationally expensive approaches, such as value function iterations, is not necessary. Moreover the analytical structure of the solution allows us to explicitly identify and analyse the additional effects on quantities and prices introduced by this preferecnes specification.

B. Technology

The consumption good is produced according to a Cobb-Douglas production function

$$Y_t = Z_t A_t^{1-\alpha} K_t^{\alpha}$$

where Y_t denotes output, Z_t and A_t denote the stationary and nonstationary components of the total factor productivity and K_t denotes the capital stock at time t. The law of motion of capital is given by

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

where δ is the rate of depreciation of physical capital. In addition define R_t as the marginal product of capital. It follows that:

$$R_t = \alpha \left(\frac{A_t}{K_t}\right)^{1-\alpha} + (1-\delta)$$

C. Shocks

The total factor productivity is driven by a vector of exogenous state variables u_t. We define the first two elemets of u_t to be

$$\begin{aligned} u_{t+1}^1 &= \ln A_{t+1} - \ln A_t \\ u_{t+1}^2 &= \ln Z_{t+1} \end{aligned}$$

The specification for the vector of exogenous variables is the main ingredient of our setup. Recent work suggests that changes in expectations and uncertainty about the productivity are important drivers of the business cycle (see Bloom (2009) and Bloom, Floetotto, and Jaimovich (2009)) and asset prices fluctuations (see Bansal and Yaron (2004) and Malkhozov and Shamloo (2010)). Higher dimension of the vector x_t and a very general specification for its dynamics will allow us to capture a wide range of rich information structures about productivity.

We assume discrete-time affine dynamics for exogenous variables

$$u_{t+1} = H_0 + H_1 u_t + \Sigma_t \varepsilon_{t+1}, \tag{1}$$

where H_0 and H_1 are $(n \times 1)$, n being the number of observable exogenous variables. The vector of innovations $\varepsilon_t \sim N(0, I_{n_\varepsilon})$. Furthermore, the elements of $\Sigma_t \Sigma_t^T$, the conditional variance-covariance matrix of the innovations, evolves as:

$$\left(\Sigma_t \Sigma_t^T\right)_{ij} = (G_0)_{ij} + (G_1)_{ij} u_t, \tag{2}$$

where G_0 is $(n \times n)$ and G_1 is $(n \times n \times n)$.[2]

A few comments about specification (1) are in order. First, it allows for time-varying volatility of shocks as (2). Time-varying, or stochastic, volatility has important implications for asset pricing particularly in delivering time-varying premia (see Bansal and Yaron (2004) and Malkhozov and Shamloo (2010)). It is also increasingly important for characterizing business cycle dynamics (see Bloom (2009) and Bloom, Floetotto, and Jaimovich (2009)).

Second, note that both the conditional expectation and the conditional variance-covariance matrix of u_t are affine in vector x_t itself. This affine structure is a crucial assumption which allows us to assume—and verify—that up to the first order all variables in the model are normally distributed. We will elaborate on this issue in more detail when discussing the solution method.

Finally, note that specification (1) encompasses higher-order autoregressive (AR) structures. For instance, an AR(2) process in u_t, $u_{t+1} = H_0 + H_1 u_t + H_2 u_{t-1} + \Sigma_t \varepsilon$, can be expanded as $\hat{u}_{t+1} = \hat{H}_0 + \hat{u}_t \hat{H}_1 + \varepsilon_{t+1} \hat{\Sigma}_t$, where $\hat{u}_t = \begin{bmatrix} u_t \\ u_{t-1} \end{bmatrix}$. Additional lags can be added in the same manner to the vector of state variables to account for $AR(L)$ terms, where $L > 1$.

D. Stationary Version, Equilibrium Conditions and the Solution

The model presented above features a permanent TFP shock (A_t). In order to transform the model into a stationary version, we define the scaled version of any nonstationary variable X_t as $\tilde{X}_t = \dfrac{X_t}{A_{t-1}}$. Note that the technology-adjusted value function $\tilde{V}_t = \dfrac{V_t}{A_{t-1}} = V(\tilde{K}_t, \tilde{A}_t)$ since $V(K_t, A_t)$ is homogeneous of degree 1 in A_t and K_t.

[2] Note that since G_1 is an $(n \times n \times n)$ array, $(G_1)_{ij}$ is a row vector of dimension $(1 \times n)$.

The stationary equivalent of the model is defined by equations (3) to (5) below (see Appendix A for derivation).

$$\tilde{V}_t^{1-1/\psi} = \max_{\tilde{C}_t} \left(\tilde{C}_t^{1-1/\psi} + \tilde{A}_t^{1-1/\psi} \beta (E_t(\tilde{V}_{t+1}^{1-\gamma}))^{\frac{1-1/\psi}{1-\gamma}} \right) \tag{3}$$

$$\tilde{K}_{t+1} = (1-\delta)\tilde{K}_t \tilde{A}_t^{-1} + Z_t \tilde{A}_t^{-\alpha} \tilde{K}_t^{\alpha} - \tilde{C}_t \tilde{A}_t^{-1} \tag{4}$$

$$R_t = (1-\delta) + \alpha Z_t \tilde{A}_t^{1-\alpha} \tilde{K}_t^{\alpha-1}. \tag{5}$$

The optimal policy is described by the Euler equation:

$$E_t \left[\beta \tilde{A}_t^{-1/\psi} \left(\frac{\tilde{V}_{t+1}}{E_t \left(\tilde{V}_{t+1}^{1-\gamma} \right)^{1/(1-\gamma)}} \right)^{1/\psi-\gamma} \left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1/\psi} R_{t+1} \right] = 1. \tag{6}$$

A closed form solution of (6) is often difficult (or impossible) to obtain. However, when there is no uncertainty a closed-form solution is known to exist. This particular case is referred to as the non-stochastic steady state. This is different from the stochastic steady state, where ε_t has a variance, but the model is evaluated at a point where the particular realizations of ε_t are equal to their means. Appendix B describes the non-stochastic steady state. As is customary in perturbation methods, we approximate the solution around the non-stochastic steady state. [3]

It is worth noting that Epstein-Zin preferences do not prohibit using perturbation methods. This point is also emphasized by Uhlig (2010). To deal with the term $E_t(V_{t+1}^{1-\gamma})$, one can define an additional control variable $w_t = E_t(V_{t+1}^{1-\gamma})$ and add this identity to the set of equilibrium equations. Replace $E_t(V_{t+1}^{1-\gamma})$ with w_t in all the equations and expand the equations as usual.

[3] Backus, Routledge, and Zin (2007) log-linearize a different set of conditions. Namely they choose the first-order and the envelope conditions of the dynamic programming problem. We argue that the choice of the Euler equation and the definition of the value function is the correct approach. As discussed in Caldara and others (2008) in the so-called Value Function Perturbation approach, where first-order conditions are approximated, the approximation of the value function should be an order higher than the desired approximation of the solution. Intuitively, this is because derivatives of state variables appear in the first-order conditions. Under the Equilibrium Conditions Perturbation approach, one could expand all equilibrium conditions to the same order as the desired solution. Therefore, log-linearizing the envelope condition as in Backus, Routledge, and Zin (2007) is insufficient for finding a first-order approximated solution. For instance, following the Backus, Routledge, and Zin (2007) approach will not result in a value function approximation that is homogeneous of degree one in capital and productivity. We avoid this issue by linearizing the equilibrium conditions (including the Euler equation) instead.

E. Log-Normal Risk Adjustment: An Approximation Technique

In this section we introduce an approximation technique, which we will refer to as the "log-normal risk adjustment." Denote with small letters the log of original variables, such that $x_t = \ln X_t$. Separate the variables in this model into n_x pre-determined variables denoted by vector x_t and n_y non-pre-determined variables by vector y_t.Vector y_t can include jump variables (choice variables determined at t) and exogenous random variables (innovations to shocks).[4] Note that pre-determined variables are not necessarily the set of state variables (the set of variables that uniquely define the position of the system in the state space). For instance, $\ln \tilde{A}_t$ and $\ln Z_t$ are state variables; however, in this classification they would be part of the y_t vector since they are not pre-determined.

The set of equilibrium conditions for a large group of DSGE models, including the prototype model introduced earlier, can be written in the following form:

$$f\left(x_{t+1}, x_t, y_t, E_t\left[\exp\left(\Gamma y_{t+1}\right)\right]\right) = 0, \tag{7}$$

where y_{t+1} is an $n_y \times 1$ vector and Γ is a conformable matrix of constants.[5] In the model described above, $x_t = [k_t]$ and $y_t = [c_t; v_t; u_t]'$.

We conjecture that the linearized solution will imply that y_t is a vector of conditionally normal variables. We linearize the model given this conjecture and obtain a solution for y_t which is linear in the state-variables. Given the affine structure assumed for the state variables, vector y_t will be conditionally normal, verifying our initial conjecture. We will show this in detail using our two examples.

Based on the conjecture that y_t is a vector of conditionally normal variables we can re-write equation (7) as:

$$f\left(x_{t+1}, x_t, y_t, \exp\left(\Gamma E_t\left[y_{t+1}\right] + \frac{1}{2}\Gamma\left(Var_t y_{t+1}\right)\Gamma'\right)\right) = 0. \tag{8}$$

[4] See Blanchard and Kahn (1980).

[5] Specifically, Γ will be of dimensions $n \times n_y$, where $n = n_x + n_y$, i.e. the number of equations in the system.

Linearizing equation (8) above, it is clear that by design y_t will be linear in exogenous shocks which are assumed to be normal. Therefore, our conjecture regarding conditional normality of y_t will bear out. We refer to this approximation as log-normal risk adjustment. We linearize the set of equilibrium conditions defined by f in (8).

We claim that this method captures the risk adjustment in RBC models where the stochastic and non-stochastic steady states are different. Furthermore, unlike perturbation methods this is an easy way to capture stochastic volatility. The method has the advantage that it is computationally simple and semi-analytical equations can be derived, similar to perturbation techniques. In addition, it allows us to capture the risk adjustment in all the variables without going to second- and higher-orders in the Taylor expansion of the terms, thereby making it a useful technique for understanding financial variables such as risk premia and risk-free rates, as well as utility measures that contain second-order terms. We will discuss these implications in detail in the following two examples.

III. Example 1: RBC with Recursive Preferences

In this example we develop the method explained above using a simple RBC model with Epstein-Zin preferences. We assume there is no stochastic volatility. (We will present an example with stochastic volatility in Section IV). We keep the structure of the shocks simple.

$$\begin{aligned} u_t &= \begin{bmatrix} a_t \\ z_t \end{bmatrix} \\ a_{t+1} &= (1-\rho_a)\mu + \rho_a a_t + \sigma_a \varepsilon^1_{t+1} \\ z_{t+1} &= \rho_z z_t + \sigma_z \varepsilon^2_{t+1}. \end{aligned}$$

We now need to re-write the equilibrium conditions in the same form as equation (8). The value function can be written as:

$$\begin{aligned} \widetilde{V}_t^{1-1/\psi} &= \max_{\widetilde{C}_t} \left(\widetilde{C}_t^{1-1/\psi} + \widetilde{A}_t^{1-1/\psi} \beta \left(E_t \left(\exp\left(1-\gamma\right) v_{t+1} \right) \right)^{\frac{1-1/\psi}{1-\gamma}} \right) \\ &= \max_{\widetilde{C}_t} \left(\widetilde{C}_t^{1-1/\psi} + \widetilde{A}_t^{1-1/\psi} \beta \left[\exp\left(\left(1-\gamma\right) E_t v_{t+1} + \frac{1}{2} \left(1-\gamma\right)^2 Var\left(v_{t+1}\right) \right) \right]^{\frac{1-1/\psi}{1-\gamma}} \right) \end{aligned} \quad (9)$$

where the last step follows because of our assumption that v_t is normal. We will verify this conjecture later on. Note that we have dropped the subscript t from the variance term since this model assumes constant volatility of shocks, and therefore, the variance of all variables will be time-independent.

Define *hat* variables as deviations from their non-stochastic steady state. Linearizing equation (9) around the non-stochastic steady state we obtain:

$$\zeta_1 \hat{v}_t = \zeta_2 \hat{c}_t + \zeta_3 \left(\hat{a}_t + E_t \hat{v}_{t+1} + \frac{1}{2}(1-\gamma)^2 Var(\hat{v}_{t+1}) \right), \tag{10}$$

where $\zeta_1 = \widetilde{V}^{1-\frac{1}{\psi}}, \zeta_2 = \widetilde{C}^{1-\frac{1}{\psi}}$ and $\zeta_3 = \beta \widetilde{A}^{1-\frac{1}{\psi}} \widetilde{V}^{1-\frac{1}{\psi}}$. $\widetilde{V}, \widetilde{C}$ and $\widetilde{A}$ denote the non-stochastic steady state values of those variables. Linearizing equations for $\widetilde{K}_t$ and R_t and re-writing the evolution of the shocks as deviations from their respective non-stochastic steady states we obtain:

$$\begin{aligned} \hat{k}_{t+1} &\approx k_k \hat{k}_t + k_a \hat{a}_t + k_z \hat{z}_t + k_c \hat{c}_t \\ \hat{r}_t &\approx r_{ak} \left(\hat{a}_t - \hat{k}_t \right) + r_z \hat{z}_t \\ \hat{a}_t &= \mu + \rho_A \hat{a}_{t-1} + \sigma_A \varepsilon_t^a \\ \hat{z}_t &= \rho_Z \hat{z}_t + \sigma_Z \varepsilon_t^z, \end{aligned} \tag{11}$$

where k_k, k_a, k_z and k_c are known. [6]

Finally, re-write the Euler equation (6) as $E_t(M_{t+1}R_{t+1}) = E_t[\exp(m_{t+1} + r_{t+1})] = 1$ where:

$$M_{t+1} = \beta \tilde{A}_t^{-1/\psi} \left(\frac{\tilde{V}_{t+1}}{E_t \left(\tilde{V}_{t+1}^{1-\gamma} \right)^{1/(1-\gamma)}} \right)^{1/\psi - \gamma} \left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1/\psi}.$$

Again, assuming log-normality of M_{t+1} and R_{t+1}, re-write the Euler equation and the definition of the stochastic discount factor as:

$$E_t(m_{t+1} + r_{t+1}) + \frac{1}{2} Var(m_{t+1} + r_{t+1}) = 0 \tag{12}$$

[6] $k_k = (1-\delta)\tilde{A}^{-1} + \alpha Z \tilde{A}^{-\alpha} \tilde{K}^{\alpha-1}$
$k_a = -(1-\delta)\tilde{A}^{-1} - \alpha Z \tilde{A}^{-\alpha} \tilde{K}^{\alpha-1} + \tilde{C}\tilde{K}^{-1}\tilde{A}^{-1}$
$k_z = -\tilde{C}\tilde{K}^{-1}\tilde{A}^{-1} = -k_k - k_a$
$k_c = Z\tilde{A}^{-\alpha}\tilde{K}^{\alpha-1}$

$$M_{t+1} = \beta \tilde{A}_t^{-1/\psi} \left(\frac{\tilde{V}_{t+1}}{\exp\left(E_t v_{t+1} + \frac{1}{2}(1-\gamma) Var\left(v_{t+1}\right)\right)} \right)^{1/\psi-\gamma} \left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1/\psi}.$$

Taking logs and subtracting the non-stochastic steady state yields:[7]

$$\hat{m}_{t+1} = (1/\psi - \gamma) \left(\hat{v}_{t+1} - E_t \hat{v}_{t+1} + \frac{1}{2}(1-\gamma) Var_t\left(\hat{v}_{t+1}\right) \right) - 1/\psi \hat{a}_t - 1/\psi \left(\hat{c}_{t+1} - \hat{c}_t\right).$$

Equations (10)-(12) give us three equations in three unknowns $\left(\hat{v}_t, \hat{c}_t, \text{and } \hat{k}_t\right)$ which can be solved. The system of equations can be written as:

$$\begin{bmatrix} E_t \hat{v}_{t+1} \\ E_t \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = R + W \begin{bmatrix} \hat{v}_t \\ \hat{c}_t \\ \hat{k}_t \end{bmatrix} + Q E_t \begin{bmatrix} \hat{a}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix}. \tag{13}$$

Note that R is a vector of constants. This is an important difference between our method and the standard log-linearization technique. When approximating a model to the first-order, as in log-linearization, on average variables will be at their non-stochastic steady state. This is because volatility is a second-order effect. Our suggested method on the other hand, explicitly finds the difference between the stochastic and non-stochastic steady states. This "risk-adjustment" term is summarized in vector R.

The vector R might not be important to us if the only objects of interest are the dynamic responses of variables to shocks. However, the constant term for each variable carries important economic intuition about how agents evaluate risk. For instance, the constant term in $E_t \hat{c}_{t+1}$ reflects the consumption deficit (compared to a model without risk) due to precautionary savings. More importantly, the size of the constant affects financial variables such as the unconditional mean of the risk-free rate.

The set of equations presented by (13) can be solved to obtain state-evolution and decision rules. The solution is presented in Appendix C. An alternative solution using the method of undetermined coefficients as in Campbell (1994) is presented in Appendix D.

[7] Note that v_{t+1} and $\hat{v}_{t+1}$ differ only in a constant, that is the value function in the non-stochastic steady state, and as a result $Var_t\left(v_{t+1}\right) = Var_t\left(\hat{v}_{t+1}\right)$.

A. Quantities and Prices

The log-linear structure of the model is convenient for deriving and studying the time-series properties of the variables of interest. For illustrative purposes we will focus on just some of them.

The log consumption growth is:

$$g_{t+1}^{c} = \tilde{c}_{t+1} - \tilde{c}_{t} + \tilde{a}_{t}.$$

The one-period risk-free rate is defined by:

$$\begin{aligned} r_{t}^{f} &= -\ln E_{t} M_{t+1} \\ &\approx -E_{t} m_{t+1} - \frac{1}{2} Var_{t}\left(m_{t+1}\right), \end{aligned}$$

and has the following expression:

$$r_{t}^{f} \approx -\ln \beta - \frac{1}{2}\left(1/\psi - \gamma\right)\left(\gamma - 1\right) Var_{t}\left(\tilde{v}_{t+1}\right) - \frac{1}{2} Var_{t}\left(m_{t+1}\right) + 1/\psi E_{t}\left(g_{t+1}^{c}\right).$$

Note that a first-order log-linearization would have omitted the two terms involving $Var_{t}\left(\tilde{v}_{t+1}\right)$ and $Var_{t}\left(m_{t+1}\right)$. The risk-free rate is lower in this model (compared to a deterministic model) because there is more risk: movements in realized and expected consumption growth cause a variance in the stochastic discount factor which drives down the interest rate. The intuition for why the term involving $Var\left(\tilde{v}_{t+1}\right)$ increases the risk-free rate is more subtle and comes about because of the Epstein-Zin preferences. Note that this term is only positive if $\gamma > 1/\psi$, or in other words, if the agents have a preference for early resolution of uncertainty. In this case agents would rather consume more today to resolve future uncertainty earlier. By bringing forward consumption they push up interest rates.

Returns on any asset i satisfy $E_{t}\left(\exp\left(m_{t+1} + r_{t+1}^{i}\right)\right) = 1$. Up to the first-order, all expected returns are the same $E_{t}\left(r_{t+1}^{i}\right) = -E_{t}\left(m_{t+1}\right)$. However, using log-normality to adjust for risk we can show that the risk premium of any asset i is:

$$E_{t}\left(r_{t+1}^{i} - r_{t}^{f}\right) = -Cov_{t}\left(r_{t+1}^{i}, m_{t+1}\right) - \frac{1}{2} Var_{t}\left(r_{t+1}^{i}\right),$$

where $\frac{1}{2} Var_{t}\left(r_{t+1}^{i}\right)$ is a Jensen's inequality correction term.

B. Log-Normal Risk

Define the entropy of the stochastic discount factor as $\ln E_t M_{t+1} - E_t \ln M_{t+1}$. Entropy can be interpreted as measure of market price of risk (see Alvarez and Jermann (2005)). Since M_{t+1} is log-normal, the entropy depends only on the second moment and has the following simple expression:

$$\ln E_t M_{t+1} - E_t \ln M_{t+1} = \frac{1}{2} Var_t \left(m_{t+1} \right).$$

Note that in each step we exploit the log-linear form of all the expressions of interest and the normality of the innovations to the exogenous variables.

C. Theoretical Comparison with Second-Order Perturbation Methods

The example above can also be analyzed using perturbation methods. For understanding financial variables such as risk premia and risk-free rates it is essential to use a second-order perturbation method.[8] This section is intended to compare the results obtained using the log-normal adjustment method with a perturbation method, both theoretically and numerically. We will show that log-normal risk is a truncated version of the second-order perturbation. However, the numerical results show that log-normal risk is a very close approximation to the full second-order approximation. In order to perform a second-order approximation we use the code provided by Schmitt-Grohe and Uribe (2004). We compare the results with those obtained using the log-normal risk adjustment method.

First, we consider our method with the second-order perturbation method theoretically. Consider a generic model, with Y as the set of control variables and S the set of state variables. Denote the solution to this model as:

$$Y_t = g(S_t). \tag{14}$$

The solution can be approximated to the second-order around the non-stochastic steady state, denoted by $(\bar{Y}, \bar{S})$, as follows:

$$Y_t - \bar{Y} \approx +g_s(\bar{S}) \left(S_t - \bar{S} \right) + \frac{1}{2} g_{ss}(\bar{S}) \left(S_t - \bar{S} \right)^2. \tag{15}$$

[8] Note that second-order perturbation method is sufficient for a model without stochastic volatility. However higher order perturbation methods are required for the example with stochastic volatility as we will discuss in Section (IV).

The log-normal risk solution method keeps only certain terms in the $\frac{1}{2}g_{ss}(\bar{S})\left(S_t-\bar{S}\right)^2$. In particular all the terms involving variances of state variables are kept, whereas quadratic terms in endogenous state variables (second-order dynamic terms) are omitted. The terms involving the variances of state variables are those summarizing the "risk adjustment" in control variables.[9]

The accuracy of the log-normal risk adjustment depends on the importance of the second-order dynamic terms. If these terms are negligible, the log-normal risk adjustment method provides an accurate and computationally efficient alternative to second-order perturbation methods. The next section compares a numerical calibration of the model above using second-order perturbation methods and the log-normal risk approximation.

D. Quantitative Comparison with Second-Order Perturbation Methods

In this section we present a calibration of Example 1 and compare the numerical solution found using second-order perturbation methods with the solution obtained by the log-normal risk adjustment method.

Table 1 shows a calibration of the example in Section III. We choose a monthly calibration for the model. Parameter β is chosen so that the annual rate of time preference is 0.98. The capital share in the Cobb-Douglas production is set to the common value of $1/3$. The rate of growth of the economy is determined by the trend component in the a shock, and we set μ such that the annual growth rate of the economy is 2 percent. The rate of depreciation of capital, δ, is chosen as 10 percent per annum. We let $\gamma = 5$ and the intertemporal elasticity of substitution, ψ, is chosen to be 1.5, which are both well within the range used in the literature (see, for instance, Kaltenbrunner and Lochstoer (2007)).

[9] For instance, g_{kk} or g_{kz}, second derivatives with respect to two state variables, are omitted whereas g_{ss}, the second order derivative with respect to the variance of the shocks, is included.

Table 1. Calibrating the Benchmark Parameters

This table shows the calibration values for Example 1. The calibration is based on a monthly frequency.

Parameter	Description	Value
β	Rate of time preference	0.9983
α	Cobb-Douglas share of capital	0.34
μ	Rate of growth	0.0017
δ	Rate of depreciation of capital	0.0087
γ	Risk aversion	5
ψ	Elasticity of substitution	1.5
ρ_A	Mean reversion of A shock	0.14
ρ_Z	Mean reversion of Z shock	0.0
σ_A^2	Variance of A shock	0.059
σ_Z	Variance of Z shock	0.01

Table 2 shows the values obtained for the first-order and second-order terms using a second-order perturbation method, and the log-normal risk method suggested in this paper. First note that an approximation of the form (15) for the solution to the model finds the first- and second-order derivative with respect to the state variables of the function which links the state to the control variables. These derivatives evaluated at the non-stochastic steady state are shown in the “Perturbation” columns.

The equivalent of the first-order terms in the log-normal risk approximation is the coefficients of the state variables. They are computed exactly the same way, so their values are identical to the second-order perturbation method. However, the log-normal approximation method does not include second-order terms except for the variance of the external shocks. Therefore, the equivalent of the second-order dynamic terms (those involving two state variables) in our suggested method is zero.

Finally, compare the second-order terms involving variances in both methods. These are the “risk-adjustment terms”. The risk-adjustment terms are very close in both methods—though not identical. The difference is related to the fact that second-order dynamic terms are set to zero in our method. We show this fact rigorously in Appendix F.

Table 2. Comparing Coefficients in Perturbation Method and Log-normal Risk

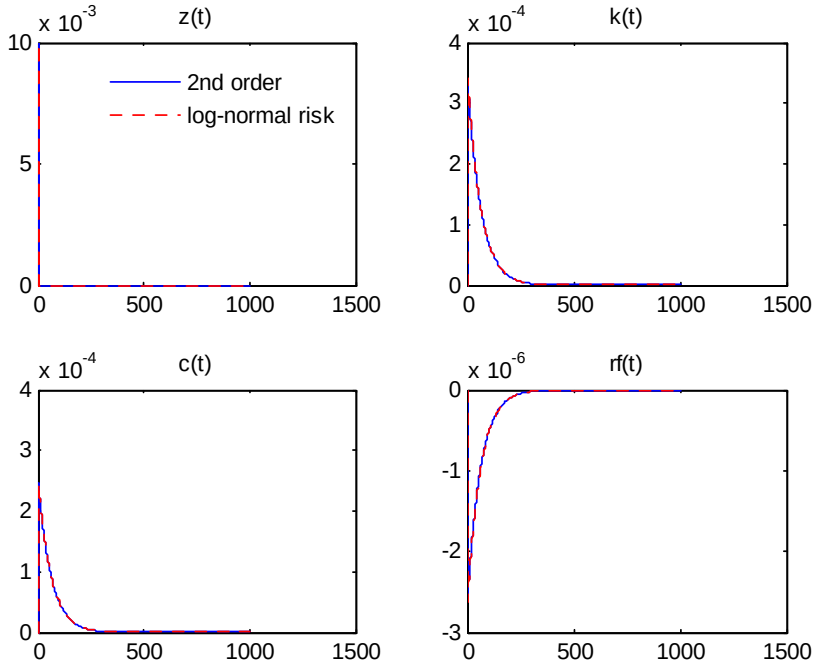
This table shows the values for the first- and second-order terms of the approximation to the model in Example 1. The "Perturbation" column shows the value for the terms obtained using a second-order perturbation method. The "LN Risk" column shows the value for the same terms obtained using the log-normal risk method.

First-Order Terms			Second-Order terms		
Parameter	Perturbation	LN Risk	Parameter	Perturbation	LN Risk
c_k	0.7146	0.7146	c_{kk}	0.0407	0
c_a	0.3291	0.3291	c_{ka}	−0.0457	0
c_z	0.0246	0.0246	c_{kz}	−0.0146	0
v_k	0.0437	0.0437	c_{aa}	0.0515	0
v_a	1.1118	1.1118	c_{az}	0.0145	0
v_z	0.0015	0.0015	c_{zz}	0.0238	0
rf_k	−0.0076	−0.0076	rf_{kk}	0.0048	0
rf_a	0.0087	0.0087	rf_{ka}	−0.0055	0
rf_z	−0.0003	−0.0003	rf_{kz}	0.0003	0
			rf_{aa}	0.0063	0
			rf_{az}	−0.0004	0
			rf_{zz}	−0.0002	0
			v_{kk}	0.0219	0
			v_{ka}	−0.0255	0
			v_{kz}	−0.0002	0
			v_{aa}	0.0296	0
			v_{az}	0.0001	0
			v_{zz}	0.0015	0
			$0.5 * c_{ss}$	−0.5814	−0.5858
			$0.5 * v_{ss}$	−8.2398	−8.2895
			$0.5 * rf_{ss}$	−0.0002	−0.0002

Figures 1 and 2 show the impulse responses of capital, consumption and risk-free rates in this model to a one-standard-deviation shock in $a(t)$ and $z(t)$. The responses obtained from each method are superimposed. All variables are calculated as percentage deviations from the *stochastic steady state*. The motivatuion for this is to concentrate on the dynamics in the two models. We will compare the stochastic steady states below.

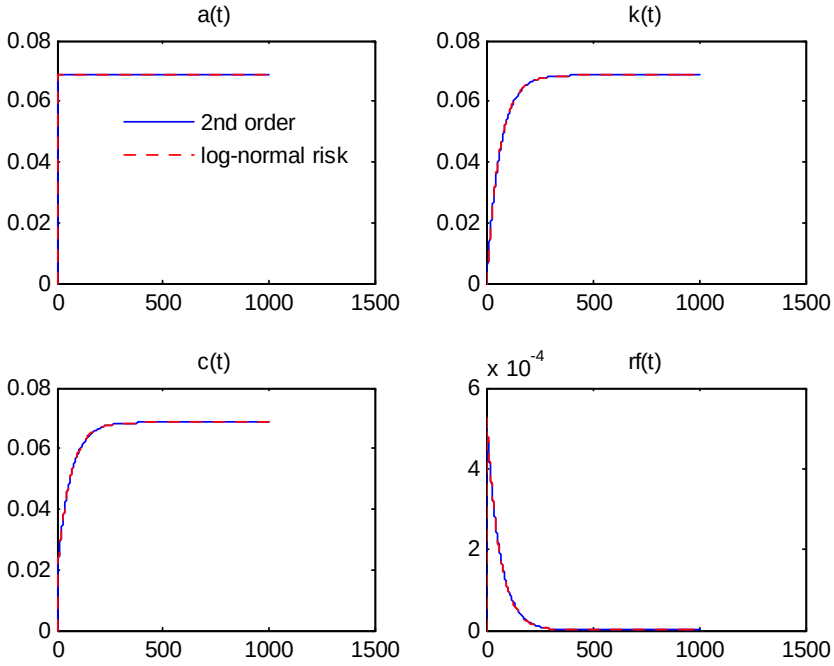
Note that the dynamic responses of capital, consumption and risk-free rates are nearly identical. This implies that the second-order dynamic terms are not quantitatively important, hence the log-normal method does a good job predicting the dynamic responses.

Figure 1. Comparison of Impulse Responses: Perturbation Method vs. Log-normal Risk



This figure shows the impulse response of the model in Example 1 to a one standard deviation shock in $z(t)$ or the stationary technology shock. The responses calculated using the second-order perturbation method and the log-normal risk adjustment method are superimposed.

Figure 2. Comparison of Impulse Responses: Perturbation Method vs. Log-normal Risk



This figure shows the impulse response of the model in Example 1 to a one standard deviation shock in $a(t)$ or the non-stationary technology shock. The responses calculated using the second-order perturbation method and the log-normal risk adjustment method are superimposed.

Table 3. Comparison of Stochastic Steady States: Perturbation Method vs. Log-normal Risk

Parameter	Second-Order Perturbation	Log-Normal Risk	% Diff
Kss	0.8666	0.8732	0.01
Css	0.0379	0.0382	0.01
$rfss$	-0.0068	-0.0069	0.01
Vss	-8.2019	-8.2513	0.01

This table shows the values of capital, consumption, risk-free rate and the value function in the stochastic steady state, computed using the second-order perturbation method and the log-normal risk adjustment method. The final column shows the percentage difference in each computed value using the two different methods.

The responses shown in Figures 1 and 2 are the deviations of each variable from its value in the stochastic steady state. Therefore, for a complete comparison we also look at the stochas-

tic steady state values using both approximation methods. From Table 3, we observe the estimated stochastic steady states calculated using a second-order perturbation method and the log-normal risk method are very close for all variables of interest. The percentage difference in the stochastic steady state between the two methods is less than 1 percent for each variable.

IV. Example 2: Stochastic Volatility in a Simple Growth Model

Now, we will consider the prototype model introduced in Section II and introduce stochastic volatility in productivity growth. We set $1/\psi = \gamma$ and assume away stationary (Z) shocks so that we can focus on the effect of stochastic volatility. The shock structure can be described as follows,

$$\begin{aligned} \hat{a}_{t+1} &= \mu + \sigma_t \varepsilon^a_{t+1} \\ \sigma^2_{t+1} &= (1-\varphi)\,\theta + \varphi \sigma^2_t + \omega \varepsilon^\sigma_{t+1}. \end{aligned}$$

where $E_t\left(\varepsilon^a_{t+1}\varepsilon^\sigma_{t+1}\right) = 0$. Define $\varepsilon_t = [\varepsilon^a_t, \varepsilon^\sigma_t]'$. In terms of our notation (1),

$$\begin{aligned} u_{t+1} &= H_0 + H_1 u_t + \Sigma_t \varepsilon_{t+1} \\ H_0 &= \begin{bmatrix} \mu \\ (1-\varphi)\,\theta \end{bmatrix} \\ H_1 &= \begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix} \\ \Sigma_t &= \begin{bmatrix} \sigma_t & 0 \\ 0 & \omega \end{bmatrix}. \end{aligned}$$

Again, the model can be written as:

$$\begin{bmatrix} E_t \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = R + W \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} + Q E_t \begin{bmatrix} \hat{a}_{t+1} \\ \sigma^2_{t+1} \end{bmatrix}.$$

Note that given the CRRA preferences, $\hat{c}_t$ describes all the information in the movements of value function, and therefore, the value function can be eliminated from the set of control variables. As a result, the solution to this model is slightly different from the previous one.

Note that because $E_t(x_{t+1}) + \frac{1}{2}Var_t(x_{t+1}) = 0$ we need to know $Var_t(x_{t+1})$ to be able to solve for x_{t+1}. Here is when the affine structure of shocks becomes important. In the previous problem we knew that if the only source of uncertainty are u_t shocks, then a linear solution would imply that x_t is a linear function of u_t and moreover that $Var_t(x_{t+1})$ is a constant. We included this constant in the R matrix above and solved for R once we found the coefficient of optimal x_t in response to u_t. We repeat the same exercise here. We know the solution is of the form:

$$\hat{c}_t = R_c + \alpha \hat{a}_t + \beta \sigma_t^2,$$

with unknown variables β and R_c. It follows that $Var_t(\hat{c}_{t+1}) = \left(\alpha^2 + \beta^2 \phi\right)\sigma_t^2 + \beta^2 \omega^2$. Furthermore, we have an equation linking $E_t(c_{t+1})$ and $Var_t(c_{t+1})$. By equating the coefficients for σ_t^2 and the constant terms we can solve for β and matrix R. As in Example 1, an alternative solution uses the method of undetermined coefficients. The solution using this method is presented in Appendix E.

Note that the conditional variance of consumption growth $Var_t(g_{t+1}^c) = Var_t(\tilde{c}_{t+1} - \tilde{c}_t - \tilde{a}_t) = c_a^2 \sigma_t^2 + c_\sigma^2 \omega^2$. It inherits the form of the productivity variance process:

$$\left(\sigma_{t+1}^c\right)^2 = (1-\varphi)\left(c_a^2 \theta + c_\sigma^2 \omega^2\right) + \varphi\left(\sigma_t^c\right)^2 + c_a^2 \omega \varepsilon_{t+1}. \tag{16}$$

This is also the form assumed by several recent consumption-based asset pricing models. See for example Bansal and Yaron (2004), Backus, Routledge, and Zin (2008) and Beeler and Campbell (2009).

A. Qualitative Comparison with Second-Order Perturbation Methods

As mentioned above, dealing with stochastic volatility with perturbation methods is not easy. To capture the full effect of stochastic volatility in perturbation methods we need to approximate the solution to the fourth-order. Therefore, we could not readily use, for instance, the software of Schmitt-Grohe and Uribe (2004). Doing a fourth-order Taylor expansion to the general form of the solution and solving for the coefficients analytically is also extremely tedious. We therefore resort to the dynare++ software, which already has this capability built in.

Programming a model, however complicated in nature, is very simple using dynare++, and it therefore enables an analysis of models that we may not have otherwise had the tools. To

be sure, there are also some disadvantages associated with using dynare++. First, it is a black box. The average user—including the authors—may not be familiar with the methods that the software uses to approximate the model and therefore may not be capable of improving on them or judging their suitability. Second, a large number of simulations are needed to obtain accurate approximations. For instance, even using 30,000 simulations our fourth-order approximation to the stochastic volatility model still is not perfectly smooth. Needless to say, running such a large number of simulations is computationally very expensive.

Comparatively, our method suggests that once the coefficients for the state-space system above are found (algebraically tedious as they may be, technically these are just quadratic equations), finding the dynamic response of the model is almost trivial. In the next section we show that this method is also very close to the response we obtain from dynare++ software.

B. Quantitative Comparison with Second-Order Perturbation Methods

There is no standard calibration for the stochastic volatility model presented above. Most stochastic volatility exercises in finance literature (such as Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2006) and Bansal, Kiku, and Yaron (2007)) assume stochastic volatility for the consumption process.[10] However, as equation (16) shows there is a direct relationship between the volatility process of consumption and that of productivity. Therefore, if we assume a particular volatility for the consumption process similar to ones in the existing asset pricing studies, the parameters of the productivity volatility process can be backed out such that they result in the desired consumption volatility. Note that our focus in this paper is the solution methodology and its accuracy, and therefore, the suitability of the calibration is of secondary concern. Malkhozov and Shamloo (2010) explore macroeconomic and asset pricing implications of these calibrations further.

Table 4 shows the calibration of the more standard parameters for Example 2. We keep them unchanged relative to the model in Example 1, except that the assumption of CRRA preferences implies that $\psi = 1/\gamma = 0.2$. Beeler and Campbell (2009) summarize some popular calibrations of the variance of the consumption growth. They are reproduced in the top panel

[10] Bloom, Floetotto, and Jaimovich (2009) has a model with stochastic volatility in consumption, but the volatility is calibrated very simply as a binary variable

of Table 5. The bottom panel gives the equivalent parameters for stochastic volatility in productivity. The examples we present in this section are based on the Bansal, Kiku, and Yaron (2006) calibration. The results for the other calibrations are similar but omitted for brevity.

Table 4. Calibrating the Benchmark Parameters

α	δ	μ	β	$\gamma = 1/\psi$
0.34	$1-0.9^{\frac{1}{12}}$	$\frac{0.02}{12}$	$0.98^{\frac{1}{12}}$	5

This table shows the calibration of the standard parameters in the model of Example 2.

Table 5. Stochastic Volatility in Productivity and Implied Stochastic Volatility in Consumption

This table shows the calibration of the stochastic volatility parameters for the model in Example 2. The top panel shows the calibration of consumption stochastic volatility according to Bansal, Kiku, and Yaron (2006, 2007) and Bansal and Yaron (2004), respectively in rows 1 to 3. The lower panel shows the calibration of the stochastic volatility in productivity which would give rise to the consumption stochastic volatilities displayed in the top panel.

Method	Parameters		
	θ^c	φ^c	ω^c
BKY1	0.00002916	0.980	0.0000068
BY	0.00006084	0.987	0.0000023
BKY2	0.00005184	0.999	0.0000028
	θ	φ	ω
BKY1	0.00006601	0.980	0.0000155
BY	0.00013870	0.987	0.0000052
BKY2	0.00011787	0.999	0.0000064

Figure 3 shows the dynare++ results from the 30,000 simulations. We approximate the model up to fourth-order to observe how the effects of stochastic volatility are captured in different orders of approximation. Figure 3 superimposes the different orders of approximations to the model. It is clear that consumption and risk-free rates do not respond to a shock to the volatility of productivity to the first-order. The second-order is purely noise. Note that we expect consumption to drop following a positive shock to volatility since consumers save more as

the amount of risk in the economy rises (this is the immediate effect). However, as volatility returns to its steady state value consumption increases; in fact, as agents initially accumulate capital to counter the effect of higher volatility, production capabilities increase and so consumption overshoots before returning to its steady state level.[11] The risk-free rate decreases as agents try to save more relative to the pre-shock levels and increases again slowly as consumption returns to its unconditional mean.

Note that the third and fourth-order approximations of the model capture the dynamics of consumption and risk-free rate responses to the shock. Whereas the first- and second-order approximations do not even qualitatively match these results; what we observe is just noise.

Figure 4 compares the fourth-order results obtained using dynare++ along with the log-normal risk responses. We observe that the responses using the log-normal risk method are very close to the fourth-order approximation responses. Evaluating the accuracy of the responses is difficult since there is no unique result obtained from dynare++ (recall that the results are the average over 30,000 simulations). However, compared to the dynare++ output, the log-normal risk results are much smoother.

C. Stochastic Volatility and Implications for Asset Prices

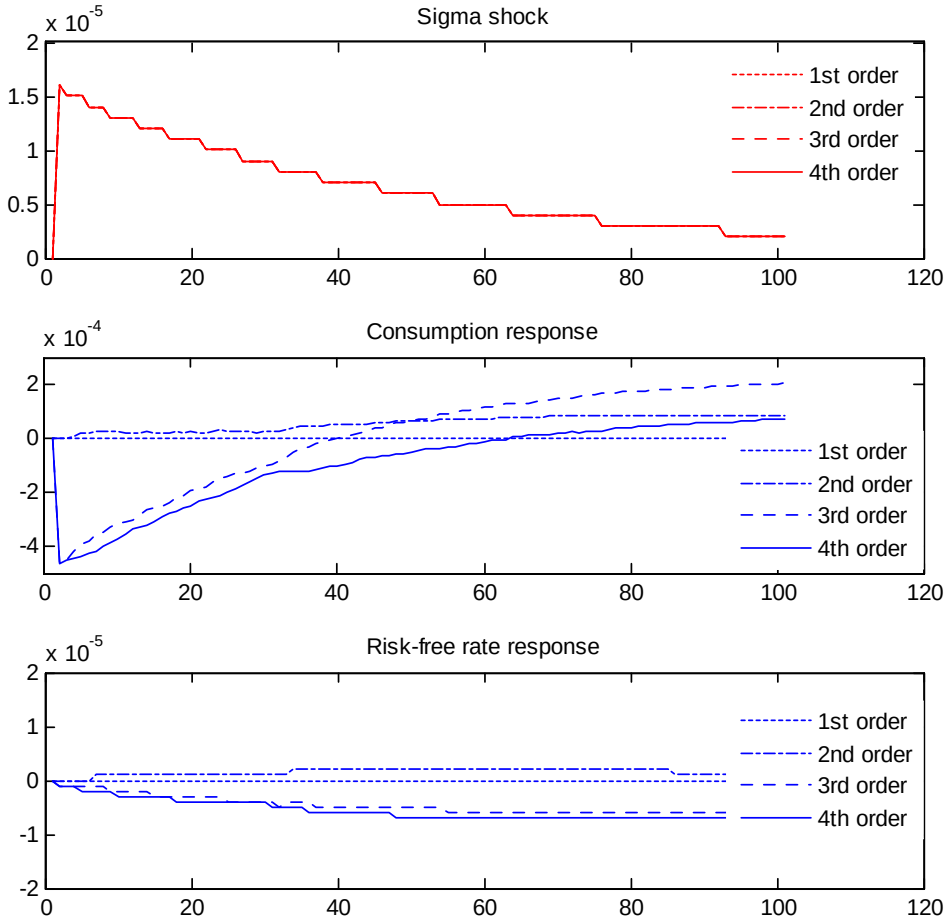
A strand of papers in the asset pricing literature claims that since stochastic volatility is an extra risk-factor in the economy, modeling it—usually as consumption stochastic volatility—will increase the premia given a certain volatility in consumption (see Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2006) and Bansal, Kiku, and Yaron (2007)). These papers assume that shocks to volatility in consumption (the stochastic volatility shock) and shocks to consumption growth are uncorrelated, and therefore, as volatility of consumption increases, consumption is reduced in order to raise precautionary savings. This suggests a negative correlation between consumption growth and consumption growth volatility.

However, if stochastic volatility in consumption is due to stochastic volatility in productivity, the mechanism suggested in Bansal and Yaron (2004) and the subsequent asset pricing

[11] Bloom, Floetotto, and Jaimovich (2009) model stochastic volatility in productivity as a two state Markov switching process and calibrate it to capture the high frequency spikes in uncertainty. In this paper we have in mind lower frequency movements in volatility and we model it as an autoregressive process.

Figure 3. Stochastic Volatility in Productivity and Implied Stochastic Volatility in Consumption

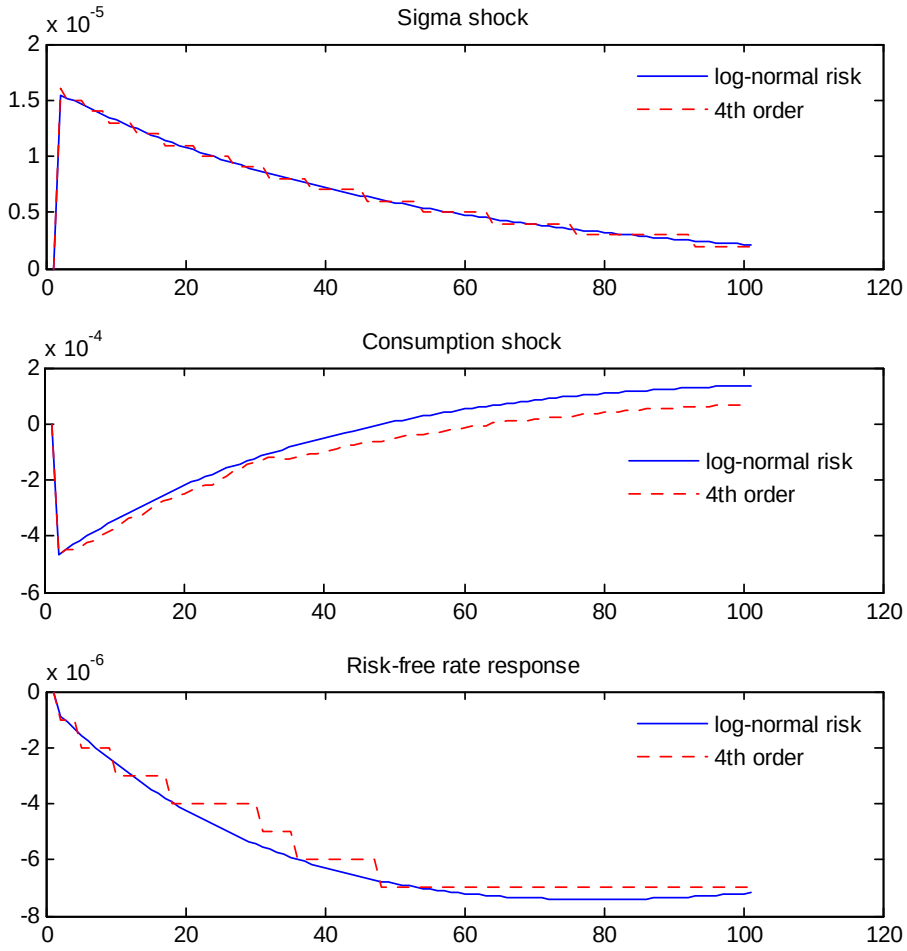
This figure shows the impulse response of consumption and risk-free rate in the model of Example 2 to a one standard deviation volatility shock. The results are obtained using dynare++ averaged over 30,000 simulations. The model is approximated to first, second, third and fourth-order and the results are superimposed.



implications are unlikely to emerge in equilibrium. The importance of the correlation between consumption growth and consumption growth variance for asset pricing has been highlighted by Backus, Routledge, and Zin (2008). Our analysis shows that even when innovations to productivity growth and productivity growth variance are uncorrelated, innovation to con-

Figure 4. Stochastic Volatility in Productivity and Implied Stochastic Volatility in Consumption

This figure shows the impulse response of consumption and risk-free rate in the model of Example 2 to a one standard deviation volatility shock. The results are obtained using the log-normal risk adjustment method (solid line) and are superimposed over the fourth approximation obtained using dynare++ (dashed line).



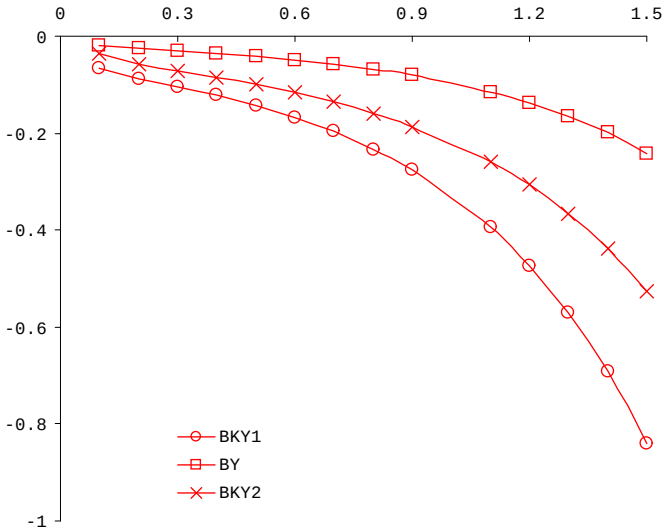
sumption growth and consumption growth variance are,

$$Cov_t(g^c_{t+1}, (\sigma^c_{t+1})^2) = c_\sigma c_x^2 \omega^2.$$

Note that this correlation is positive, implying that consumption growth increases when there

Figure 5. Consumption Growth and Variance of Consumption Growth

This figure shows the correlation between consumption growth and variance of consumption growth for different calibrations and values of ψ.



are shocks to consumption growth volatility. In other words, stochastic volatility in consumption becomes a hedge for expected consumption growth thereby reducing risk premia. The effect is the opposite of what is assumed in endowment economy asset pricing literature.

Figure 5 shows correlations for various values of parameters assuming the level of variance is equal to the long-term mean θ. Higher elasticity of intertemporal substitution implies a stronger negative correlation. Asset pricing implications of stochastic volatility of productivity are studied in detail in Malkhozov and Shamloo (2010).

V. Conclusion

We suggest a way to solve real business cycle models using approximation techniques common in asset pricing literature. Even if eventually we prefer perturbation methods for their generality, we argue that log-linearization as presented in this paper is a very convenient tool that enables correctly capturing not only the dynamics of quantities but also asset pricing and welfare implications. We show precisely how the method is related to a standard higher-order perturbation approach. Furthermore, it suggests a computationally efficient way for solving and estimating models with stochastic volatility.

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APPENDIX A. EQUILIBRIUM CONDITIONS

The value function:

$$V(K_t, A_t) = \max_{C(K_t, Z_t)} (U_t)$$

$$V_t = \max_{C(K_t, Z_t)} \left((1-\beta) C_t^{1-1/\psi} + \beta (E_t (V_{t+1}^{1-\gamma}))^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}}.$$

Define the scaled variables $\tilde{X}_t = \dfrac{X_t}{A_{t-1}}$ and the scaled value function $\tilde{V}_t = V(\tilde{K}_t, \tilde{A}_t)$. U_t, Y_t, K_{t+1} are homogeneous of degree one in A_t and K_t; therefore, the value function is homogeneous of degree one in A_t and K_t as well. In particular we can scale the problem by A_{t-1},

$$V(K_t, A_t) = A_{t-1} V(\frac{K_t}{A_{t-1}}, \frac{A_t}{A_{t-1}}).$$

In scaled variables the equilibrium conditions can be written as:

$$\begin{aligned}
\frac{V_t}{A_{t-1}} &= \max_{C(K_t, A_t)} \left(\left(\frac{C_t}{A_{t-1}} \right)^{1-1/\psi} + \frac{1}{A_{t-1}^{1-1/\psi}} \beta E_t \left(E_t (V_{t+1}^{1-\gamma}) \right)^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}} \\
\frac{V_t}{A_{t-1}} &= \max_{C(K_t, A_t)} \left(\left(\frac{C_t}{A_{t-1}} \right)^{1-1/\psi} + \frac{A_t^{1-1/\psi}}{A_{t-1}^{1-1/\psi}} \beta \left(E_t \left(\frac{V_{t+1}^{1-\gamma}}{A_t^{1-\gamma}} \right) \right)^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}} \\
\tilde{V}_t &= \max_{C(\tilde{K}_t, Z_t)} \left(\tilde{C}_t^{1-1/\psi} + \tilde{A}_t^{1-1/\psi} \beta \left(E_t (\tilde{V}_{t+1}^{1-\gamma}) \right)^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}}.
\end{aligned}$$

The first-order condition with respect to $\tilde{C}_t$ and the envelope condition are as follows:

$$\begin{aligned}
\tilde{C}_t^{-1/\psi} &= \beta \tilde{A}_t^{-1/\psi} E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{\gamma-1/\psi}{1-\gamma}} E_t \left[\tilde{V}_{t+1}^{-\gamma} \tilde{V}_{\tilde{K}t+1} \right] \\
\tilde{V}_{\tilde{K}t} &= \beta \tilde{A}_t^{-1/\psi} \tilde{V}_t^{-1/\psi} E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{\gamma-1/\psi}{1-\gamma}} E_t \left[\tilde{V}_{t+1}^{-\gamma} \tilde{V}_{Kt+1} \right] R_t.
\end{aligned}$$

Iterating the envelope condition one period forward and combining it with the first-order condition, we obtain the Euler equation for consumption and an expression for $\tilde{V}_{\tilde{K}t}$:

$$\begin{aligned}\beta E_t\left[\tilde{A}_t^{-1/\psi}E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{\gamma-1/\psi}{1-\gamma}}\tilde{V}_{t+1}^{-\gamma+1/\psi}\frac{\tilde{C}_{t+1}^{-1/\psi}}{\tilde{C}_t^{-1/\psi}}R_{t+1}\right] &= 1\\ \tilde{V}_{\tilde{K}t}-\tilde{V}_t^{1/\psi}\tilde{C}_t^{-1/\psi}R_t &= 0.\end{aligned}$$

Notice that R_t can be expressed in terms of scaled variables:

$$R_t=(1-\delta)+\alpha Z_t\tilde{A}_t^{1-\alpha}\tilde{K}_t^{\alpha-1}.$$

APPENDIX B. NON-STOCHASTIC STEADY STATE

The following relations define the equilibrium:

$$\begin{aligned}E_t\left[\beta\tilde{A}_t^{-1/\psi}E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{\gamma-1/\psi}{1-\gamma}}\tilde{V}_{t+1}^{-\gamma+1/\psi}\frac{\tilde{C}_{t+1}^{-1/\psi}}{\tilde{C}_t^{-1/\psi}}R_{t+1}\right] &= 1\\ \tilde{V}_t^{1-\frac{1}{\psi}}-\max_{C(\tilde{K}_t,Z_t)}\left(\tilde{C}_t^{1-\frac{1}{\psi}}+\tilde{A}_t^{1-1/\psi}\beta(E_t(\tilde{V}_{t+1}^{1-\gamma}))^{\frac{1-1/\psi}{1-\gamma}}\right) &= 0\\ \tilde{K}_{t+1}-(1-\delta)\tilde{K}_t\tilde{A}_t^{-1}-Z_t\tilde{A}_t^{-\alpha}\tilde{K}_t^{\alpha}+\tilde{C}_t\tilde{A}_t^{-1} &= 0.\end{aligned}$$

The non-stochastic steady state can be described by:

$$\begin{aligned}
u &= (I - H_1)^{-1} H_0 \\
\tilde{A} &= \exp(u^1) \\
Z &= \exp(u^2) \\
R &= \beta^{-1}\tilde{A}^{1/\psi} \\
\tilde{K} &= \tilde{A}\left[\frac{\beta^{-1}\tilde{A}^{1/\psi} - (1-\delta)}{\alpha Z}\right]^{\frac{1}{\alpha - 1}} \\
\tilde{Y} &= Z\tilde{A}^{1-\alpha}\tilde{K}^{\alpha} \\
\tilde{C} &= (1-\delta)\tilde{K} + \tilde{Y} - \tilde{K}\tilde{A} \\
\tilde{V} &= \tilde{C}\left(\frac{1}{1 - \beta\tilde{A}^{1-\frac{1}{\psi}}}\right)^{\frac{1}{1-1/\psi}}.
\end{aligned}$$

APPENDIX C. DIFFERENCE EQUATION SOLUTION

Equation (13) can be re-written as:

$$\begin{bmatrix} E_t\hat{v}_{t+1} \\ E_t\hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = R + P\Lambda P^{-1}\begin{bmatrix} \hat{v}_t \\ \hat{c}_t \\ \hat{k}_t \end{bmatrix} + Q'\hat{u}_t, \tag{17}$$

where Λ and P are the eigenvalues and eigenvectors associated with W. As noted above, we should find that there are m roots larger than one, associated with the jumpy variables. In this case $m = 2$, the two roots associated with $\hat{v}_t$ and $\hat{c}_t$.

First, note that $QE_t\hat{u}_{t+1} = QH_1\hat{u}_t = Q'\hat{u}_t$. Second, note that R includes constants such as $Var_t(\hat{v}_{t+1})$ and $Var_t(\hat{m}_{t+1})$ which are unknown. However, the stochastic part of the solution is independent of the constant term so we can solve the model above in two stages. First, solve the model without the R,

$$P^{-1}\begin{bmatrix} E_t\hat{v}_{t+1} \\ E_t\hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \Lambda P^{-1}\begin{bmatrix} \hat{v}_t \\ \hat{c}_t \\ \hat{k}_t \end{bmatrix} + P^{-1}Q'\hat{u}_t.$$

Redefine the system as:

$$\begin{bmatrix} E_t y_{t+1}^1 \\ E_t y_{t+1}^2 \\ y_{t+1}^3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_t^1 \\ y_t^2 \\ y_t^3 \end{bmatrix} + P^{-1}Q'\hat{u}_t.$$

For $\lambda_1 > 0$ and $\lambda_2 > 0$, the roots can be solved forward and for λ_3 it can be solved backwards yielding,

$$\begin{aligned} y_t^1 &= -\frac{\lambda_1}{\lambda_1 - \rho_A}\left(P^{-1}Q'\right)_{11}\hat{a}_t - \left(P^{-1}Q'\right)_{12}\hat{z}_t \\ y_t^2 &= -\frac{\lambda_2}{\lambda_2 - \rho_A}\left(P^{-1}Q'\right)_{21}\hat{a}_t - \left(P^{-1}Q'\right)_{22}\hat{z}_t \\ y_t^3 &= \lambda_2 y_{t-1}^3 + \left(P^{-1}Q'\right)_{31}\hat{a}_t + \left(P^{-1}Q'\right)_{32}\hat{z}_t. \end{aligned}$$

Then, find $\hat{v}_t, \hat{c}_t$ and $\hat{k}_t$ by:

$$\begin{bmatrix} \hat{v}_t \\ \hat{c}_t \\ \hat{k}_t \end{bmatrix} = P \begin{bmatrix} y_t^1 \\ y_t^2 \\ y_t^3 \end{bmatrix}. \tag{18}$$

Furthermore,

$$Var_t \begin{bmatrix} \hat{v}_{t+1} \\ \hat{c}_{t+1} \\ \hat{k}_t \end{bmatrix} = PVar_t\left(y_{t+1}\right)P',$$

with solutions for $\hat{v}_t$ and $\hat{c}_t$ at hand one can easily calculate the conditional variance of these variables at time t and thus, solve for matrix R.

APPENDIX D. EXAMPLE 1 SOLUTION: METHOD OF UNKNOWN COEFFICIENTS

The log of the stochastic discount factor is:

$$m_{t+1} = \ln\beta + (1/\psi - \gamma)\left(\tilde{v}_{t+1} - \frac{1}{(1-\gamma)}\ln E_t e^{(1-\gamma)\tilde{v}_{t+1}}\right) - 1/\psi\tilde{a}_t - 1/\psi\left(\tilde{c}_{t+1} - \tilde{c}_t\right).$$

Because of the assumed log-linearity of $\tilde{v}_t$ and the normality of shocks it can be written:[12]

$$m_{t+1} = \ln\beta + (1/\psi - \gamma)\left(\tilde{v}_{t+1} - E_t\tilde{v}_{t+1} + \frac{1}{2}(1-\gamma)Var_t(\tilde{v}_{t+1})\right) - 1/\psi\tilde{a}_t - 1/\psi(\tilde{c}_{t+1} - \tilde{c}_t).$$

The Euler equation (6) can be written as $E_t(\exp(m_{t+1} + r_{t+1}) = 1$, which again using the log-normal structure of the model implies the following condition:

$$E_t(m_{t+1} + r_{t+1}) + \frac{1}{2}Var_t(m_{t+1} + r_{t+1}) = 0.$$

Define some preliminary expressions. First,

$$-1/\psi\hat{a}_t - 1/\psi(\hat{c}_{t+1} - \hat{c}_t) + r_{t+1} =$$

$$\begin{aligned}
&-1/\psi\left[k_c c_k^2 + (k_k - 1 + \psi r_{ak}k_c)c_k + \psi r_{ak}k_k\right]\hat{k}_t \\
&-1/\psi\left[\begin{array}{c} \iota_1(I + (c_k + \psi r_{ak})k_a I - \psi r_{ak}H_1) + \iota_2((c_k + \psi r_{ak})k_z I - \psi\tau_2 H_1) \\ +c_x((-1 + (c_k + \psi r_{ak})k_c)I + H_1) \end{array}\right]\hat{u}_t \\
&-1/\psi\left[c_x\Sigma_t - \psi(r_{ak}\iota_1 + r_z\iota_2)\Sigma_t\right]\varepsilon_{t+1} \\
&-1/\psi\left[c_0(c_k + \psi r_{ak})k_c\right].
\end{aligned}$$

Next,[13]

$$Var_t(\hat{v}_{t+1}) = (v_u\Sigma_t)(v_u\Sigma_t)^T = v_u\Sigma\Sigma^T v_u^T = v_u G_0 v_u.$$

Finally,

$$Var_t(m_{t+1} + r_{t+1}) = (l\Sigma_t)(l\Sigma_t)^T = l\Sigma_t\Sigma_t^T l^T = lG_0 l^T,$$

where

$$l = (-\gamma + 1/\psi)v_u - 1/\psi(c_u - \psi(r_{ak}\iota_1 + r_z\iota_2)).$$

As the value function enters the Euler equation we need to approximate it by log-linearizing its definition (3) to complete the solution. Again using log-normality we can rewrite (3):

$$e^{(1-1/\psi)\tilde{v}_t} = e^{(1-1/\psi)\tilde{c}_t} + \beta e^{(1-1/\psi)\left(\tilde{a}_t + E_t\tilde{v}_{t+1} + \frac{1}{2}(1-\gamma)Var_t(\tilde{v}_{t+1})\right)},$$

[12] Useful result about certainty equivalence under log-normality: if $\ln x \sim N(\mu, v)$ then $\frac{1}{\alpha}\ln E(x^\alpha) - E(\ln x) = \frac{1}{2}\alpha v^2$.

[13] $\kappa_2 G_1 \kappa_2^T$ is a $(1 \times n)$ vector with k^{th} element equal to $\sum_{i,j}\kappa_{2,i}G_{1,ijk}\kappa_{2,j}$.

and linearize it

$$\zeta_1 \hat{v}_t = \zeta_2 \hat{c}_t + \zeta_3 \left(\hat{a}_t + E_t \hat{v}_{t+1} + \frac{1}{2}(1-\gamma) Var_t(\hat{v}_{t+1}) \right),$$

where

$$\begin{aligned} \zeta_1 &= \tilde{V}^{1-\frac{1}{\psi}} \\ \zeta_2 &= \tilde{C}^{1-\frac{1}{\psi}} \\ \zeta_3 &= \beta \tilde{A}^{1-\frac{1}{\psi}} \tilde{V}^{1-\frac{1}{\psi}}. \end{aligned}$$

Regrouping terms in the conditions implied by the Euler equation and the linearization of the value function definition give us the following system of equations for $(c_0, c_k, c_x, v_0, v_k, v_x)$:

$$\begin{aligned} k_c c_k + (k_k - 1 + \psi r_{ak} k_c) c_k + \psi r_{ak} k_k &= 0 \\ \iota_1 (I + (c_k + \psi r_{ak}) k_a I - \psi r_{ak} H_1) + \iota_2 ((c_k + \psi r_{ak}) k_z I - \psi \tau_2 H_1) & \\ + c_x ((-1 + (c_k + \psi r_{ak}) k_c) I + H_1) - \frac{1}{2} \psi \left((\gamma - 1/\psi)(1-\gamma) v_x G_1 v_x + l G_1 l^T \right) &= 0 \\ c_0 (c_k + \psi r_{ak}) k_c - \frac{1}{2} \psi \left((\gamma - 1/\psi)(1-\gamma)(v_x G_0 v_x + v_x G_1 v_x x) + l G_0 l^T + l G_1 l^T x \right) &= 0 \\ \zeta_1 v_k - \zeta_2 c_k - \zeta_3 v_k (k_k + k_c c_k) &= 0 \\ \zeta_1 v_x - \zeta_2 c_x - \zeta_3 \left(\iota_1 + v_k (k_a \iota_1 + k_z \iota_2 + k_c c_x) + v_x H_1 + \frac{1}{2}(1-\gamma) v_x G_1 v_x \right) &= 0 \\ \zeta_1 v_0 - \zeta_2 c_0 - \zeta_3 \left(v_0 + v_k k_c c_0 + \frac{1}{2}(1-\gamma)(v_x G_0 v_x + v_x G_1 v_x x) \right) &= 0. \end{aligned}$$

In the most general case this is a system of quadratic equations we have to solve numerically.

In Example 1, the case without stochastic volatility, the solutions for the coefficients are:

$$
\begin{aligned}
c_k &= \frac{-\left(k_k - 1 + \psi r_{ak} k_c\right) \pm \sqrt{\left(k_k - 1 + \psi r_{ak} k_c\right)^2 - 4 k_c k_k \psi r_{ak}}}{2k_c} \\
c_x &= \begin{pmatrix} \iota_1 \left(I + \left(c_k + \psi r_{ak}\right) k_a I - \psi r_{ak} H_1\right) \\ +\iota_2 \left(\left(c_k + \psi r_{ak}\right) k_z I - \psi \tau_2 H_1\right) \end{pmatrix} \left(\left(-1 + \left(c_k + \psi r_{ak}\right) k_c\right) I + H_1\right)^{-1} \\
c_0 &= \frac{\left(\frac{1}{2}\psi \left(\left(\gamma - 1/\psi\right)\left(1-\gamma\right) L_1 L_1^T + \left(\left(-\gamma + 1/\psi\right) L_1 - 1/\psi L_2\right)\left(\left(-\gamma + 1/\psi\right) L_1 - 1/\psi L_2\right)^T\right)\right)}{\left(c_k + \psi r_{ak}\right) k_c} \\
v_k &= \frac{\zeta_2 c_k}{\zeta_1 - \zeta_3 \left(k_k + k_c c_k\right)} \\
v_x &= \left(\zeta_2 c_x + \zeta_3 \left(\iota_1 + v_k \left(k_a \iota_1 + k_z \iota_2 + k_c c_x\right)\right)\right)\left(\zeta_1 I - \zeta_3 H_1\right)^{-1} \\
v_0 &= \frac{\zeta_2 c_0 + \zeta_3 \left(v_k k_c c_0 + \frac{1}{2}\left(1-\gamma\right) L_1 L_1^T\right)}{\zeta_1 - \zeta_3},
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= v_u H_2 \\
L_2 &= \left(c_u - \psi \left(r_{ak} \iota_1 + r_z \iota_2\right)\right) H_2 \\
u_t &= \begin{bmatrix} a_t \\ z_t \end{bmatrix}.
\end{aligned}
$$

APPENDIX E. EXAMPLE 2 SOLUTION: METHOD OF UNKNOWN COEFFICIENTS

The solution in Example 2, with stochastic volatility is as follows:

$$
\begin{aligned}
c_k &= \frac{-\left(k_k-1+\psi r_{ak}k_c\right)\pm\sqrt{\left(k_k-1+\psi r_{ak}k_c\right)^2-4k_ck_k\psi r_{ak}}}{2k_c}\\
c_u &= \frac{1+\left(c_k+\psi r_{ak}\right)k_a}{1-\left(c_k+\psi r_{ak}\right)k_c}\\
c_\sigma &= \frac{\frac{1}{2}\psi\left(\left(\gamma-1/\psi\right)\left(1-\gamma\right)v_x^2+\left(-1/\psi\left(c_x-\psi r_{ak}\right)+\left(-\gamma+1/\psi\right)v_x\right)^2\right)}{-1+\varphi+\left(c_k+\psi r_{ak}\right)k_c}\\
c_0 &= \frac{\frac{1}{2}\psi\left(\begin{array}{c}\left(\gamma-1/\psi\right)\left(1-\gamma\right)\left(v_\sigma^2\omega^2+v_x^2\theta\right)\\+\left(-1/\psi c_\sigma\omega+\left(-\gamma+1/\psi\right)v_\sigma\omega\right)^2\\+\left(-1/\psi\left(c_x-\psi r_{ak}\right)+\left(-\gamma+1/\psi\right)v_x\right)^2\theta\end{array}\right)}{\left(c_k+\psi r_{ak}\right)k_c}\\
v_k &= \frac{\zeta_2c_k}{\zeta_1-\zeta_3\left(k_k+k_cc_k\right)}\\
v_u &= \frac{\zeta_2c_u+\zeta_3\left(1+v_k\left(k_a+k_cc_x\right)\right)}{\zeta_1}\\
v_\sigma &= \frac{\left(\zeta_2+\zeta_3v_kk_c\right)c_\sigma+\frac{1}{2}\zeta_3\left(1-\gamma\right)v_x^2}{\zeta_1-\zeta_3\varphi}\\
v_0 &= \frac{\zeta_2c_0+\zeta_3\left(v_kk_cc_0+\frac{1}{2}\left(1-\gamma\right)\left(v_\sigma^2\omega^2+v_x^2\theta\right)\right)}{\zeta_1-\zeta_3}.
\end{aligned}
$$

APPENDIX F. RELATION TO PERTURBATION METHODS

In this appendix we show that the standard perturbation method solution and the log-normal risk adjustment approach are closely related. For models without stochastic volatility the log-linear approximation can be written as follows:

$$
\begin{aligned}
\left[g(x_t,\sigma)\right]^i &= \left[g(\bar{x},0)\right]^i+\left[g_x(\bar{x},0)\right]^i(x-\bar{x})+\frac{1}{2}\left[g^*_{\sigma\sigma}(\bar{x},0)\right]^i[\sigma][\sigma]\\
\left[h(x_t,\sigma)\right]^i &= \left[h(\bar{x},0)\right]^i+\left[h_x(\bar{x},0)\right]^i(x-\bar{x})+\frac{1}{2}\left[h^*_{\sigma\sigma}(\bar{x},0)\right]^i[\sigma][\sigma].
\end{aligned}
$$

Compared to second-order perturbation methods there are two differences. First, we drop the quadratic terms g_{xx} and h_{xx}. Second, $g_{\sigma\sigma},h_{\sigma\sigma}$ and $g^*_{\sigma\sigma},h^*_{\sigma\sigma}$ are not exactly the same.

Schmitt-Grohe and Uribe (2004) show how to compute $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$ from other first- and second-order terms. $g^*_{\sigma\sigma}$ and $h^*_{\sigma\sigma}$ can be computed in exactly the same way except for, again, ignoring a term in g_{xx}.

To illustrate this result, consider a simple model which nevertheless captures all the elements of the result. Control y_t is a function of state variables x_t and a parameter σ scaling uncertainty. State variables evolution is assumed linear and doesn't need to be approximated,

$$\begin{aligned} y_t &= g(x_t, \sigma) \\ x_{t+1} &= h_x x_t + \sigma \eta \varepsilon_{t+1}. \end{aligned}$$

The equilibrium condition which allows us to approximate g as exponential in y_{t+1} and x_{t+1} is

$$E_t\left(e^{\alpha y_{t+1} + \beta x_{t+1} + \gamma}\right) = 1.$$

At the steady state $(\sigma = 0)$,

$$\begin{aligned} e^{\alpha y + \gamma} &= 1 \\ y &= -\frac{\gamma}{\alpha}. \end{aligned}$$

Taking the first derivative with respect to x_t and evaluating it at the steady state we obtain g_x :

$$\begin{aligned} E_t\left(e^{\alpha y_{t+1} + \beta x_{t+1}} \left(\alpha g_x h_x + \beta h_x\right)\right) &= 0 \\ g_x &= \beta/\alpha. \end{aligned}$$

Similarly taking the second derivative,

$$\begin{aligned} E_t\left(e^{\alpha y_{t+1} + \beta x_{t+1}} \left(\left(\alpha g_x h_x + \beta h_x\right)^2 + \alpha g_{xx} h_x\right)\right) &= 0 \\ g_{xx} &= 0. \end{aligned}$$

Next, we take the first derivative with respect to σ and are able to verify the general result that g_σ is equal to zero:

$$E_t\left(e^{\alpha y_{t+1} + \beta x_{t+1}} \left(\alpha g_\sigma + \alpha g_x \eta \varepsilon_{t+1} + \beta \eta \varepsilon_{t+1}\right)\right) = 0.$$

Finally, we take the second derivative with respect to σ,

$$E_t\left(e^{\alpha y_{t+1}+\beta x_{t+1}}\left(\left(\alpha g_\sigma+\alpha g_x\eta\varepsilon_{t+1}+\beta\eta\varepsilon_{t+1}\right)^2+\alpha g_{\sigma\sigma}+\alpha g_{xx}\eta^2\varepsilon_{t+1}^2\right)\right)=0.$$

$g_\sigma=0$ and in our particular example $g_{xx}=0$ therefore,

$$g_{ss}=-\frac{\left(\alpha g_x+\beta\right)^2\eta^2}{\alpha}.$$

Now consider solving the model using log-normality. We assume:

$$\begin{aligned} y_t &= g_x x_t+g_0 \\ x_{t+1} &= h_x x_t+\sigma\eta\varepsilon_{t+1}. \end{aligned}$$

The equilibrium condition implies:

$$\begin{aligned} E_t\left(\alpha y_{t+1}+\beta x_{t+1}+\gamma\right)+\frac{1}{2}Var_t\left(\alpha y_{t+1}+\beta x_{t+1}+\gamma\right) &= 0 \\ \left(\alpha g_x h_x+\beta h_x\right)x_t+\alpha g_0+\gamma+\frac{1}{2}\left(\alpha g_x+\beta\right)^2\eta^2 &= 0. \end{aligned}$$

Regrouping the terms:

$$\begin{aligned} g_x &= \beta/\alpha \\ g_0 &= -\frac{\gamma+\frac{1}{2}\left(\alpha g_x+\beta\right)^2\eta^2}{\alpha}. \end{aligned}$$

We verify exactly that:

$$g_0=y+\frac{1}{2}g_{ss}.$$

This will not hold exactly if $g_{xx}\neq 0$, which in our setup would have been the case if $h_{xx}\neq 0$. In other terms we compute $g_{ss}^*=2(g_0-y)$ in the same way as g_{ss} using standard perturbation methods except for ignoring the second-order terms in g_{xx}.