CONSUMER PRICE INDEX MANUAL

Theory | 2025

International Monetary Fund | International Labour Organization
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FOREWORD

The *Consumer Price Index Manual: Theory* is the companion publication to the *Consumer Price Index Manual: Concepts and Methods.* The *Theory* publication provides a comprehensive overview of the conceptual and theoretical issues that drive the methods and practices described in the *CPI Manual*.

The chapters cover many topics. They elaborate on the theories underlying the different practices currently in use, including the four main approaches to index number theory, calculation of elementary indices, quality adjustment methods, seasonal products, durable goods, and the treatment of owner-occupied housing. This publication also discusses the chain drift problem and multilateral indices and provides empirical examples, based on actual data, for upper-level index calculation formulae.

This publication and the companion on the practice of compiling consumer price indices (CPIs) are an update of the *Consumer Price Index Manual: Theory and Practice,* published in 2004. Through the mechanism of the Inter-Secretariat Working Group on Price Statistics (IWGPS), the update has been managed by the International Monetary Fund (IMF) and jointly published by the organizations of the IWGPS: the Statistical Office of the European Union (Eurostat), the International Labour Organization (ILO), the IMF, the Organisation for Economic Co-operation and Development (OECD), the United Nations Economic Commission for Europe (UNECE), and the World Bank.

Given that many of the theoretical issues included in this volume continue to evolve, this publication will be disseminated in an electronic format only at https://www.imf.org/en/Data/Statistics/cpi-manual#companion and on the websites of the IWGPS member agencies. This will more readily facilitate chapter updates as the research on these topics continues to advance and emerging topics are included.

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PREFACE

The *Consumer Price Index Manual: Theory*, herewith referred to simply as the *Theory* publication, is the companion publication to the *Consumer Price Index Manual: Concepts and Methods.* These two volumes represent an update of the 2004 publication, *Consumer Price Index Manual: Theory and Practice*. Since 2004, methods and best practices have continued to evolve. Big data sources have become increasingly accessible, and updated calculation methods are being developed to account for these new sources. The *Theory* publication was prepared under the auspices of the Intersecretariat Working Group on Price Statistics (IWGPS), which consists of six organizations: the Statistical Office of the European Union (Eurostat), the International Labour Organization (ILO), the International Monetary Fund (IMF), the Organisation for Economic Co-operation and Development (OECD), the United Nations Economic Commission for Europe (UNECE), and the World Bank. The *Theory* publication is published jointly by the six organizations.

The IWGPS endorses this publication as a valuable overview of the main index number theories underlying the principles and recommendations used for compiling consumer price indices (CPIs). The authors of the *Theory* publication have included references to papers and books on index number theory that could be useful to compilers and data users who want to explore the subject in more depth.

The Consumer Price Index

The CPI is an index that measures the rate at which the prices of consumption goods and services are changing from one period to another. The prices are collected from shops or other retail outlets. The usual method of calculation is to take an average of the period-to-period price changes for different products, using as weights the average amounts that households spend on them. CPIs are official statistics that are usually produced by national statistical offices (NSOs), ministries of labor, or central banks.[1] They are published as quickly as possible, generally within four weeks after the reference period.

The *Theory* publication intends to benefit agencies that compile CPIs as well as the users of CPI data. It explains the economic and statistical theory on which the methods used to calculate a CPI are based. The *Manual* details the practical methods that are recommended to calculate a CPI.

A CPI is a measure of price changes of the goods and services purchased by households for their consumption. It is also widely used as a measure of inflation for the economy as a whole, partly because of the frequency and timeliness with which it is produced. It has become a key statistic used in policymaking, especially monetary policy. It is often specified in legislation and in a wide variety of contracts as the appropriate measure for adjusting payments (such as wages, rents, interest, social security, other benefits, and pensions) for the effects of inflation. It can therefore have substantial and wide-ranging financial implications for governments and businesses, as well as for households.

This publication provides compilers and users with a broader understanding of the conceptual and theoretical foundations of the methods used in practice. Calculating a CPI cannot be reduced to a simple set of rules or a standard set of procedures that can be mechanically followed in all circumstances. While there are certain general principles that may be universally applicable, the procedures followed in practice, whether they concern the collection or processing of the prices or the methods of aggregation, must take account of particular circumstances. These include the main use of the index, the nature of the markets and pricing practices within the country, and the resources available to the NSO. The *Theory* publication explains the underlying economic and theoretical concepts and principles needed to enable NSOs to make their choices in efficient and cost-effective ways and to be aware of the full implications of their choices.

The procedures used to compile CPIs are not static but continue to evolve and improve in response to several factors. First, research continually refines and strengthens the economic and statistical theory underpinning CPIs. For example, clearer insights have recently been obtained on the relative strengths and weaknesses of the various formulae and methods used to process the basic price data collected for CPI purposes. Second, recent advances in information and communications technology, such as the availability and the technical capabilities to make effective use of large-scale administrative data sets, have affected CPI methods. Both of these theoretical and data developments can impinge on all the stages of compiling a CPI. New technology can affect the methods used to collect prices and transmit them to the NSO. It can also improve the ways of processing and checking, including the methods used to adjust prices for changes in the quality of the goods and services covered. Finally, improved formulae help in calculating more accurate and reliable higher-level indices, including the overall CPI itself.

History of International Standards for CPIs

The objectives of the international standards for CPI compilation are to provide guidelines on best practices that can be used by countries when developing or revising a CPI and to promote the quality and international comparability of national CPIs. The theories and concepts discussed in this publication drive the discussions on standards and best practices.

In many countries, CPIs were first compiled mainly to be able to adjust wages to compensate for the loss of purchasing power caused by inflation. Consequently, the responsibility for compiling CPIs was often entrusted to ministries, or departments, of labor. The International Conference of Labour Statisticians (ICLS), convened by the Governing Body of the ILO, therefore provided the natural forum to discuss CPI methodology and develop guidelines.

The first international standards for CPIs were promulgated in 1925 by the Second ICLS. The first set of standards referred to "cost of living" indices rather than CPIs. A distinction is now drawn between the two different types of index. A CPI can be defined simply as measuring the change in the cost of purchasing a given "basket" of consumption goods and services, whereas a cost-of-living index is defined as measuring the change in the cost of maintaining a given standard of living or level of utility. For this reason, the Tenth ICLS in 1962 decided to adopt the more general term "consumer price index," which should be understood to embrace both concepts. There need not be a conflict between the two. As explained in the *Manual*, the best-practice methods are likely to be very similar, whichever approach is adopted.

The international standards for calculating CPIs have been revised four times in 1947, 1962, 1987, and 2003 in the form of resolutions adopted by the ICLS. The 1987 standards on CPI were followed by a manual on methods (Turvey, Ralph et. al., *Consumer Price Indices: An ILO Manual*. Geneva: International Labour Office (1989)) which provided guidance to countries on the practical application of the 1987 standards. The 1989 manual on methods was revised, expanded, and published in 2004 (*Consumer Price Index Manual: Theory and Practice*, Geneva: International Labour Office, International Labour Organization, International Monetary Fund, Organisation for Economic Co-operation and Development, Eurostat, UN Economic Commission for Europe, and World Bank (2004)).

The Background to the Present Update

Since 2004, substantial progress has been made in developing new data sources, price collection methods, and related index calculation methods. This update incorporates these developments and reflects experience gained by improving CPI compilation methods. Finally, evolving user needs and the need for greater international comparability contributed to the necessity of updating the 2004 manual.

In response to the various developments in CPI compilation methods and the emergence of new data sources, the need to update the 2004 manual was recognized and agreed to in 2014. A formal recommendation to revise the *Manual* was made at the meeting of the UNECE Expert Group on Consumer Price Indices, Geneva, May 2014, jointly organized with the ILO. The participants of this meeting noted a need for clearer, more prescriptive recommendations where research, methodological development, and practical experience support such recommendations and guidelines.

Following the 2014 meeting in Geneva, the IWGPS agreed to initiate an update of the 2004 manual with the IMF as the lead agency to manage the update. When updated, the *Manual* was split into two volumes. The first volume, *Consumer Price Index Manual: Concepts and Methods,* was endorsed by the United Nations Statistical Commission in 2020 as an international statistical standard for the compilation of the CPIs. The second volume, *Consumer Price Index Manual: Theory*, complements the first one by presenting the theories behind the methods and practices in the first volume.

The electronic version of the *Theory* publication is available on the Internet at https://www.imf.org/en/Data/Statistics/cpi-manual#companion. The *Theory* publication, available only in electronic format, will be updated as needed. This is especially true for emerging discussions and recommendations to be made by international groups reviewing the CPIs, such as the ICLS, the United Nations City Group on Price Indices (the "Ottawa Group"), and the UNECE Expert Group on Consumer Price Indices.

Comments and suggestions on the *Theory* publication are welcomed by the IWGPS and should be sent to the International Monetary Fund (e-mail: STARECPIM@imf.org). They will be considered for any future revisions.

[1] For simplicity, the *Theory* publication refers in general to NSOs as the statistical agencies responsible for compiling the CPIs.

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The IMF, the lead agency credited with updating the *CPI Manual*, deserves special thanks for dedicating resources to the review and production of this publication. The IMF Communications Department provided extensive editorial and support services for the final version of this publication.

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INTRODUCTION

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1. The Basket, Axiomatic, and Stochastic Approaches to Index Number Theory

This is a book about index number theory in general and the construction of a consumer price index (CPI) in particular. It turns out that there is no single approach to index number theory that experts agree is the "right" approach. Thus, this volume will cover the four main approaches to index number theory that are used today by national and international statistical agencies. These four main approaches are as follows:

- The fixed basket approach (and averages of fixed baskets)
- The test or axiomatic approach
- The stochastic approach
- The economic approach

In order to measure aggregate consumer price change between two periods, the *fixed basket approach* takes a "representative" basket of goods and services that households purchase in the two periods under consideration and prices out the cost of the basket using the prices of the current period for the numerator of the CPI and using the prices of the base period for the denominator of the index. This type of index dates back to the Middle Ages, but it was studied in some detail by the English economist Lowe in the early 1800s and as a result is known as Lowe Index. It is useful to introduce some notation at this point so that the basket approach can be explained more precisely. Suppose the base period is called period 0 and the current period is called period 1. Suppose that there are N goods and services that a specified group of households purchase in each period and the total quantity purchased by the households in period t of product n is q_{tn} for $t = 0,1$ and $n = 1,\ldots,N$. Suppose further that the average price for product n in period t is p_{tn} for $t = 0,1$ and $n = 1,\ldots,N$. Denote the set of period t prices as $p^t \equiv [p_{t1},\ldots,p_{tN}]$ and the corresponding set of period t quantities as $q^t \equiv [q_{t1},\ldots,q_{tN}]$ for $t = 0,1$. A possible choice of a "representative" basket of goods and services is the *base period quantity vector*, q^0. This leads to the *Laspeyres price index* $P_L(p^0,p^1,q^0) \equiv \Sigma_{n=1}^{N} p_{1n}q_{0n}/\Sigma_{n=1}^{N} p_{0n}q_{0n}$. Another possible choice of a "representative" basket of goods and services is the *current period quantity vector*, q^1. This leads to the *Paasche price index* $P_P(p^0,p^1,q^1) \equiv \Sigma_{n=1}^{N} p_{1n}q_{1n}/\Sigma_{n=1}^{N} p_{0n}q_{1n}$. Since each of these two quantity vectors is equally representative and they both measure overall inflation going from period 0 price to period 1 prices, it may make sense to take a symmetric average of these two estimates of overall inflation as our final point estimate for consumer price inflation over the two periods under consideration. This leads to the *Fisher price index*, $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0)P_P(p^0,p^1,q^1)]^{1/2}$, which is the geometric average of the Laspeyres and Paasche price indices.[1] Another variant of the fixed basket approach to index number theory is to take the fixed basket as the product by product geometric average of the quantities consumed in periods 0 and 1. Thus, the *Walsh price index* is defined as $P_W(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_{1n}(q_{0n}q_{1n})^{1/2}/\Sigma_{n=1}^{N} p_{0n}(q_{0n}q_{1n})^{1/2}$. The fixed basket approach to index number theory is explained in some detail in Chapter 2.

Note that these fixed basket indices are all *functions* of the two price vectors, p^0 and p^1, and the two quantity vectors, q^0 and q^1, that pertain to the two periods under consideration; that is, these price index functions are all of the form $P(p^0,p^1,q^0,q^1)$, where this *bilateral index number formula* is a function of the 4N variables contained in the vectors p^0,p^1,q^0,q^1.

The *axiomatic* or *test approach to index number theory* starts with a (unknown) bilateral index number formula, $P(p^0,p^1,q^0,q^1)$, and attempts to determine the functional form for the index number function by placing restrictions on the function, or in other words, asking that the index function satisfy certain tests. An example of a test is the *weak identity test*; that is, we ask that the index number function satisfy the following property: $P(p^0,p^1,q^0,q^1) = 1$ if $p^0 = p^1$ and $q^0 = q^1$. Thus, if product prices and quantities consumed are exactly the same in the two periods being compared, then the price index should be equal to 1 (indicating that there is no inflation between the two periods). Another example of a test is the *strong identity test*; that is, we ask that the index number function satisfy the following property: $P(p^0,p^1,q^0,q^1) = 1$ if $p^0 = p^1$. Thus, if product prices are exactly the same in the two periods being compared, then the price index should be equal to 1 even if the quantity vectors are different over the two periods under consideration. Another test is *linearly homogeneity in the prices of period 1*; that is, we ask that the index function $P(p^0,p^1,q^0,q^1)$ satisfy the following property: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for all numbers $\lambda > 0$. If $P(p^0,p^1,q^0,q^1)$ satisfies this property and if all prices in period 1 double, then the price level in period 1 also doubles. The test approach to index number theory is explained in some detail in Chapter 3 of this volume. It turns out that the Fisher index that emerged as a "best" index from the viewpoint of the fixed basket approach to index number theory also emerges as a "best" index from the viewpoint of the test approach.[2]

[1] Detailed references to the works of Laspeyres, Paasche, and Fisher will be found in the subsequent chapters. A similar comment applies to other authors who will be mentioned in this introductory chapter.

[2] It should be noted that the "best" index cannot be unambiguously determined. When using the basket approach, there is a problem in choosing the "best" average of the Laspeyres and Paasche indices. When using the test approach, there is a problem of deciding which tests are the most important ones for the index number formula to satisfy. Different sets

The *stochastic approach to index number theory* dates back to the work of the English economists Jevons and Edgeworth in the 1800s. This approach to index number theory will be explained in detail in Chapter 4. However, a brief outline of this approach follows.

Using the notation for prices defined earlier, the simplest example of the stochastic approach works as follows. Treat each price ratio for product n, p_{1n}/p_{0n}, as an estimate of general inflation going from period 0 to period 1.[3] Thus, a statistical model for this situation might be the following one:

$$p_{1n}/p_{0n} = \alpha + e_{tn};\ n = 1,\ldots,N, \qquad (1)$$

where α is the general measure of inflation going from period 0 to 1 and e_{tn} are independently distributed error terms with 0 means and constant variances. Now choose α as the solution to the following least squares minimization problem:

$$\min_{\alpha} \{\Sigma_{n=1}^{N} e_{tn}^{2}\} = \min_{\alpha} \{\Sigma_{n=1}^{N} [(p_{1n}/p_{0n}) - \alpha]^2\}. \qquad (2)$$

The solution to the minimization problem defined by (2) is

$$\alpha^{*} \equiv (1/N)\Sigma_{n=1}^{N} (p_{1n}/p_{0n}) \equiv P_C(p^0,p^1), \qquad (3)$$

where the *Carli index*, $P_C(p^0,p^1)$, is defined as the *arithmetic average* of the N price ratios, p_{1n}/p_{0n}. Note that the Carli index depends only on prices over the two periods under consideration in contrast to the Fisher index, which depends on both prices and quantities.

Now suppose we changed the stochastic specification from (1) to $\ln(p_{1n}/p_{0n}) = \ln\alpha + e_{tn}$ for $n = 1,\ldots,N$, where e_{tn} are the independently distributed error terms with 0 means and constant variances. Define $\beta \equiv \ln\alpha$ as the natural logarithm of α. The new least squares minimization problem is

$$\min_{\beta} \{\Sigma_{n=1}^{N} [\ln(p_{1n}/p_{0n}) - \beta]^2\}. \qquad (4)$$

The solution to the minimization problem defined by (4) is

$$\beta^{*} \equiv (1/N)\Sigma_{n=1}^{N} \ln(p_{1n}/p_{0n}). \qquad (5)$$

β^{*} is an estimator for the logarithm of the price index going from period 0 to period 1 prices. To obtain an estimator for the price index α^{**}, we need to exponentiate β^{*} to obtain α^{**}:

$$\alpha^{**} \equiv \exp[\beta^{*}] = \prod_{n=1}^{N} (p_{1n}/p_{0n})^{1/N} \equiv P_J(p^0,p^1). \qquad (6)$$

Thus, the new stochastic specification leads to a new estimator for the price index, the *geometric average* of the N individual price ratios, $P_J(p^0,p^1)$, which is the *Jevons index number formula*.

The Carli and Jevons index number formulae are examples of unweighted indices.[4] Keynes criticized these indices because they did not take into account the economic importance of each product in the budgets of the consumers of the N products. The economic importance of products can be taken into account by replacing the unweighted least squares minimization problems (2) or (4) by weighted least squares minimization problems. A measure of the economic importance of product n is its share of total consumer expenditures on the N goods and services in both periods 0 and 1. Define the inner product of the vectors p^t and q^t as $p^t \cdot q^t \equiv \Sigma_{n=1}^{N} p_{tn}q_{tn}$. The period t share of consumer expenditures on product n is defined as $s_{tn} \equiv p_{tn}q_{tn}/p^t \cdot q^t$ for $n = 1,\ldots,N$ and $t = 0,1$. Define the arithmetic average of the expenditure shares for product n over the two periods as

$$s(n) \equiv (½)s_{0n} + (½)s_{1n};\quad n = 1,\ldots,N. \qquad (7)$$

A useful weighted by economic importance version of the least squares minimization problem defined by (4) is the following weighted least squares minimization problem:

$$\min_{\beta} \{\Sigma_{n=1}^{N} s(n)[\ln(p_{1n}/p_{0n}) - \beta]^2\}. \qquad (8)$$

The solution to the minimization problem defined by (8) is

$$\beta^{**} \equiv \Sigma_{n=1}^{N} s(n)\ln(p_{1n}/p_{0n}), \qquad (9)$$

where β^{**} is an estimator for the logarithm of the price index going from period 0 to period 1 prices. To obtain an estimator for the price index α^{***}, we need to exponentiate β^{**} to obtain α^{***}:

$$\alpha^{***} \equiv \exp[\beta^{**}] = \prod_{n=1}^{N} (p_{1n}/p_{0n})^{s(n)} \equiv P_T(p^0,p^1,q^0,q^1). \qquad (10)$$

Thus, the new stochastic specification leads to a new estimator for the price index, a *share-weighted geometric average* of the N individual price ratios, $P_T(p^0,p^1,q^0,q^1)$, which is the *Törnqvist–Theil index number formula*. This index emerges as a "best" index number formula from the viewpoint of the stochastic or descriptive statistics approach to index number theory. For a more detailed description of this third approach to index number theory, see Chapter 4.

The economic approach to index number theory is the most complicated of the four main approaches to index number theory, and it is explained in detail in Chapter 5. An overview of the economic approach is presented in the following section.

2. The Economic Approach to Index Number Theory

Chapter 5 develops the economic approach to index number theory. The economic approach is based on the assumption that consumers choose their consumption bundles to maximize an index of well-being or utility subject to a budget constraint. This approach to index number theory will appear to be rather unrealistic to many price statisticians. However, it is empirically observed that consumers will purchase more of a product when its price is relatively low and less of it when its price is relatively high. Thus, as relative prices change, consumers substitute cheaper products for more expensive products in an attempt to maintain their

of admissible tests will lead to different "best" index number formulae. When using the stochastic approach, different methods of averaging the prices or different stochastic specifications of the error terms will lead to different "best" indices.

[3] It is assumed that all prices are positive in what follows.

[4] Unweighted in this context means that the price ratios are equally weighted.

standard of living. The economic approach to index number theory takes these substitution effects into account and as a result provides a more realistic measure of consumer price inflation. Moreover, the economic approach to index number theory provides economists and policy makers with (approximate) measures of consumer welfare change; that is, the economic approach to index number theory provides us not only with measures of household inflation but also with measures of real consumption. Finally, the economic approach is necessary to measure the effects of quality change and to take into account the welfare effects of new and disappearing products.

The consumer's period t budget-constrained utility maximization problem is equivalent to the problem of minimizing the cost of achieving the period t level of utility when the economic approach is used. Suppose the consumer's utility function is $f(q)$, where $q \equiv [q_1,\ldots,q_N]$ is a consumption vector. The consumer's *cost function*, $C(u,p)$, that corresponds to the given utility function $f(q)$ is defined as follows:

$$C(u,p) \equiv \min_q \{p{\cdot}q : f(q) \geq u\}, \tag{11}$$

where $p \equiv [p^1,\ldots,p_N]$ is a vector of positive prices that the consumer faces and $q \equiv [q_1,\ldots,q_N]$ is a nonnegative consumption vector. Thus, the consumer chooses the consumption bundle that minimizes the cost of achieving the target utility level u and $C(u,p)$ is the resulting minimum cost of achieving this target level of utility.

Using the notation defined in the previous section, let q^t be the consumer's observed consumption vector in period t and suppose the consumer faces the price vector p^t in period t for $t = 0,1$. The consumer's period t level of utility is $u^t \equiv f(q^t)$ for $t = 0,1$. It is assumed that the consumer minimizes the cost of achieving their period t utility level u^t in periods 0 and 1. Thus, we have the following equalities:

$$p^t{\cdot}q^t = C(u^t,p^t); \quad t = 0,1. \tag{12}$$

The *Konüs family of true cost of living indices* that provides a measure of price inflation between periods 0 and 1, $P_K(u,p^0,p^1)$, is defined for each reference utility level u using the cost function as follows:

$$P_K(u,p^0,p^1) \equiv C(u,p^1)/C(u,p^0). \tag{13}$$

Section 2 in Chapter 5 develops the properties of this family of price indices. Note that there is a possibly different true cost of living index for each choice of the reference level of utility, u. It is natural to choose u to be either $u^0 = f(q^0)$ or $u^1 = f(q^1)$ when making price comparisons between the two periods 0 and 1. This leads to the *Laspeyres type true cost of living index*, $P_K(u^0,p^0,p^1) \equiv C(u^0,p^1)/C(u^0,p^0)$, and the *Paasche type true cost of living index*, $P_K(u^1,p^0,p^1) \equiv C(u^1,p^1)/C(u^1,p^0)$.

Section 3 makes an additional assumption that $f(q)$ is a *linearly homogeneous function* so that $f(\lambda q) = \lambda f(q)$ for all numbers $\lambda > 0$. This assumption is not supported by empirical evidence using aggregate price and quantity data. But it is a very useful assumption because it leads to true cost of living indices $P_K(u,p^0,p^1)$ that are independent of the reference utility level u. It turns out that when the utility function is linearly homogeneous, the corresponding cost function $C(u,p)$ factors into the product of the utility level u times the *unit cost function* $c(p)$, which is equal to the minimum cost of achieving one unit of utility, $C(1,p)$. Thus, we have $C(u,p) = uc(p)$ and hence

$$\begin{aligned} P_K(u,p^0,p^1) &\equiv C(u,p^1)/C(u,p^0) = uc(p^1)/uc(p^0) \\ &= c(p^1)/c(p^0), \end{aligned} \tag{14}$$

where $c(p^1)/c(p^0)$ is the ratio of unit cost in period 1 to unit cost in period 0.

Chapters 2, 3, and 4 defined various bilateral index number formulae of the general form $P(p^0,p^1,q^0,q^1)$. Sections 5–9 in Chapter 5 show that various true cost of living indices of the form defined by (14) are equal to many of the bilateral index number formulae that were defined in previous chapters. These relationships are derived assuming that consumer preferences can be represented by certain specific functional forms for either the linearly homogeneous utility function $f(q)$ or the corresponding unit cost function $c(p)$. For example, if we assume that the consumer's unit cost function is a linear function of prices, so that $c(p) = \Sigma_{n=1}^{N} \alpha_n p_n$, then it can be shown that the Laspeyres price index, $P_L = p^1{\cdot}q^0/p^0{\cdot}q^0$, is *exactly* equal to the true cost of living index $c(p^1)/c(p^0)$. Thus, the Laspeyres bilateral price index is an example of an *exact index number formula*. The theory of exact index number formulae was developed by Konüs and Byushgens and Pollak. If we can find a bilateral index number formula that is exact for a unit cost function $c(p)$ that can provide a second-order Taylor series approximation to an arbitrary twice continuously differentiable unit cost function, then Diewert called the bilateral index a *superlative index*. various examples of superlative index number formulae are given in Chapter 5. It turns out that the Fisher, Törnqvist–Theil, and Walsh bilateral price indices are all superlative indices.

Section 8 in Chapter 5 shows that these three indices all approximate each other to the second order around a point where prices and quantities are equal (so that $p^0 = p^1$ and $q^0 = q^1$ at the point of approximation). This means that these three indices will tend to approximate each other reasonably well, particularly if there is no too much variation in prices and quantities going from period 0 to period 1. From the viewpoint of the basket approaches to index number theory, the Fisher and Walsh indices got good grades. The axiomatic approach to index number theory favored the Fisher index, while the stochastic approach to index number theory gave good grades to the Törnqvist–Theil index. The Walsh index is a special case of a fixed basket index or Lowe index, which is an advantage since many price statisticians prefer fixed basket indices because they are relatively easy to explain to the public. All three indices are equally good from the viewpoint of the economic approach to index number theory. Thus, it seems that it does not matter all that much on which approach to index number theory one takes: The four approaches lead to three specific index number formulae that will generate much the same answer in many situations.[5]

[5] This is true if there are no missing prices and fluctuations in prices and quantities are not too great. If there are missing prices or severe fluctuations in prices and quantities, then the three indices can differ significantly.

Section 11 of Chapter 5 discusses theoretical quantity indices that are counterparts to the Konüs family of true cost of living indices of the form $C(u,p^1)/C(u,p^0)$. The family of *Allen quantity indices* is defined for each reference price vector, p, as follows:

$$Q_A(p,q^0,q^1) \equiv C(f(q^1),p)/C(f(q^0),p). \quad (15)$$

Thus, the Allen quantity index is the ratio of the cost of achieving the period 1 level of utility, $f(q^1)$, to the cost of achieving the period 0 level of utility, $f(q^0)$, using the same reference price vector p in both the numerator and the denominator of the ratio. The two most relevant choices for the reference price vector when comparing utility levels in periods 0 and 1 are the period 0 and 1 price vectors, p^0 and p^1. This leads to the *Laspeyres type Allen index*, $Q_A(p^0,q^0,q^1)$, and to the *Paasche type Allen index*, $Q_A(p^1,q^0,q^1)$. If we make the additional assumption that the utility function is linearly homogeneous, then we have $C(f(q^t),p) = f(q^t)c(p)$ for $t = 0,1$ and the Allen quantity index simplifies to the following utility ratio:

$$\begin{aligned} Q_A(p,q^0,q^1) &\equiv C(f(q^1),p)/C(f(q^0),p) \\ &= f(q^1)c(p)/f(q^0)c(p) = f(q^1)/f(q^0). \end{aligned} \quad (16)$$

As was the case with the Konüs price index, it is possible to show that $Q_A(p,q^0,q^1) = f(q^1)/f(q^0)$ is exactly equal to various bilateral price indices of the form $Q(p^0,p^1,q^0,q^1)$[6] for certain functional forms for $f(q)$ and $Q(p^0,p^1,q^0,q^1)$.

Section 12 in Chapter 5 provides a brief discussion on the problems associated with constructing true cost of living indices when there are taste changes. The period t cost function for the consumer is the function $C^t(u,p)$ for $t = 0,1$; that is, we allow the cost function to change when going from period 0 to 1, reflecting a change in the consumer's preferences over different combinations of the N consumer goods and services. The true cost of living index using the preferences of period 0 and the reference utility level of period 0 is $C^0(u^0,p^1)/C^0(u^0,p^0)$, while the true cost of living index using the preferences of period 1 and the reference utility level of period 1 is $C^1(u^1,p^1)/C^1(u^1,p^0)$. Each of these measures is of interest, and each measure is equally valid. If we require a single estimate for real price change between the two periods, then taking the geometric average of these two estimates seems to be a reasonable procedure. The analysis in Section 12 shows how the Törnqvist–Theil index could provide an observable approximation to this average measure of price change. However, it should be noted that the results in Section 12 do not allow for completely general taste changes. The issue of taste change is of some importance since the COVID pandemic, which started in 2020, surely changed consumer preferences. In order to model the transition from pre-COVID preferences to COVID preferences while allowing for completely different preferences, it seems that econometric methods would have to be used where separate preferences are estimated for both the pre-COVID period and the COVID period. National Statistical Offices are not well equipped to undertake econometric investigations. This is an area where further research is required.

Section 13 of Chapter 5 explores the conditional cost of living concept. In this section, it is assumed that the consumer's preference function, $f(q,z)$, is defined over a vector of market goods and services q and a vector of environmental or household demographic variables, z. The consumer's minimum (market) cost of achieving the utility level u given that he or she faces the vector of market prices p is the cost $C(u,p,z) \equiv \min_q \{p \cdot q : f(q,z) \geq u\}$. Pollak's *conditional cost of living index* is the cost ratio $C(u,p^1,z)/C(u,p^0,z)$. Thus, this CPI is conditional not only on the chosen utility level, u, but it also depends on the environmental vector z. As usual, two special cases of this family of indices is of interest when comparing the prices of periods 0 and 1: the *Laspeyres type conditional index* $C(u^0,p^1,z^0)/C(u^0,p^0,z^0)$ and the *Paasche type conditional index* $C(u^1,p^1,z^1)/C(u^1,p^0,z^1)$. The main result in Section 13 shows that for a certain fairly general functional form for the conditional cost function, one can show that the Törnqvist–Theil price index $P_T(p^0,p^1,q^0,q^1)$ defined earlier is exactly equal to the geometric mean of the theoretical Laspeyres and Paasche type conditional cost of living indices; that is, we have $P_T(p^0,p^1,q^0,q^1) = \{[C(u^0,p^1,z^0)/C(u^0,p^0,z^0)][C(u^1,p^1,z^1)/C(u^1,p^0,z^1)]\}^{1/2}$.

Section 14 provides a framework for dealing with new and disappearing products in a CPI. When a new product appears for the first time, there is no price in a prior period for that product, so typically, the new product is ignored in the CPI for the period of introduction. From the viewpoint of the economic approach to index number theory, an appropriate price for the new product in the prior base period of the index is the consumer's *reservation price* for the product. It is the price that is just high enough to deter the consumer from purchasing the product during the base period. This reservation price concept was proposed by Hicks, and it is explained more fully in Section 14 of Chapter 5 and in even more detail in Chapter 8.

Section 15 of Chapter 5 introduces *household production* and the consumer's *allocation of time* into the CPI framework. Up to this point, the economic approach to index number theory assumes that the consumer chooses its consumption vector to maximize a utility function subject to a budget constraint. But in reality, consumers get utility by spending time on enjoying their purchases subject to a budget constraint and a time constraint. The addition of the time constraint to the consumer's utility maximization problem greatly complicates the construction of a CPI. Section 15 follows the path-breaking work of Becker in adding the time constraint to the consumer's utility maximization problem. This section is perhaps the most complicated section in the entire volume. National Statistical Offices have not really embraced the integration of the time constraint with the budget constraint due to the complexity of integration and perhaps due also to the difficulty in collecting data on the household's allocation of time across various activities. Section 15 does provide a framework for organizing the data and integrating the time constraint with the budget constraint.

Up to this point, the economic approach to index number theory has presented the theory as it applies to a single

[6]various bilateral quantity indices can be obtained from counterpart bilateral price indices by interchanging the role of prices and quantities. Thus the bilateral Laspeyres and Paasche quantity indices are defined as $Q_L \equiv p^0 \cdot q^1/p^0 \cdot q^0$ and $Q_P \equiv p^1 \cdot q^1/p^1 \cdot q^0$. The Fisher quantity index is defined as $Q_F \equiv [Q_L Q_P]^{1/2}$.

household. Section 16 looks at methods to aggregate over households to form what Pollak calls a *social cost of living index*. The material presented in this section assumes that price and quantity information is available for individual households. Section 16 also discusses *democratic* and *plutocratic price indices*.

Section 17 looks at the problems associated with aggregating over households in order to form measures of economy wide real consumption. The analysis in this section supports the construction of aggregate Fisher quantity indices.

Section 18 generalizes the discussion in Section 17 in order to discuss alternative measures of social welfare and the relationship of measures of income inequality to social welfare. The work of Atkinson, Kolm, Sen, Jorgenson, and Schreyer figures prominently in this section. A simple social welfare function is suggested that is equal to the product of per capita real consumption times one minus the Gini coefficient for the distribution of real income in the economy.

Section 19 addresses an important shortcoming of most of the analysis presented in Chapters 2–5 up to this point: It is usually assumed that all prices are positive in the two periods being compared. However, the shorter the length of the time period[7] and the longer the time series of prices and quantities, the greater will be the likelihood of a *lack of matching of prices*. This problem is due to the following explanatory factors:

- The existence of seasonal products that are only available in certain seasons.
- When a durable good or storable product goes on sale, consumers can purchase multiple units of the product during the sale period and then purchase zero units of the product for subsequent periods until their inventory of the product is depleted or the durable good is worn out.
- Product churn—that is, producers are constantly modifying their products and replacing "old" products with perhaps slightly different "new" products.

Section 19 discusses possible solutions to the lack of matching problem. In addition, Chapters 7–10 all deal with missing prices in more detail.

3. Elementary Indices

Chapter 6 discusses the problems associated with choosing a bilateral index number formula at the first stage of aggregation in constructing a CPI. In particular, the problem of formula choice is discussed when only price information is available. This type of index is frequently called an *elementary index*. In this case, the economic approach to index number theory cannot be applied and so only the test and stochastic approaches to index number theory can be used in this "prices only" context. Examples of elementary indices that use only price information for the two periods under consideration are the Carli, Dutot, and Jevons indices. The Carli and Jevons indices were defined earlier by (5) and (6) and were equal to the arithmetic and geometric averages of the price ratios, p_{1n}/p_{0n}. The *Dutot index* is defined using our usual notation as follows:

$$P_D(p^0,p^1) \equiv [\Sigma_{n=1}^{N} (1/N)p_{1n}]/[\Sigma_{n=1}^{N} (1/N)p^0_{\ n}]. \qquad (17)$$

Thus, the Dutot index is a ratio of the average price in period 1 to the average price in period 0. Unfortunately, this index is not invariant to changes in the units of measurement, whereas the Carli and Jevons indices are invariant. Applying the test approach to bilateral index number theory in the prices-only case leads to the Jevons index as being a "best" index.

In this chapter, it is assumed that all prices are positive, and this is a limitation of the analysis. The problems associated with missing prices will be discussed in Chapters 7 and 9.

Chapter 6 also discusses in some detail some of the problems associated with determining the *scope of an index*. For example, how exactly should a product be defined? Should the elementary aggregate have a regional or type of household dimension in addition to a product dimension? Should prices be collected from households directly or from outlets servicing households?

An annex to Chapter 6 discusses another interesting measurement problem that arises at the first stage of aggregation. Some retail outlets charge a *fixed monthly or annual fee* in order to give customers access to their products (or to give members a lower price on products). For example, telecommunications firms often charge a fixed monthly access fee that is independent of the usage of their services. The annex discusses alternative methods for treating these fixed access fees in a CPI.

4. The Chain Drift Problem and Multilateral Indices

Chapter 7 deals with possible solutions to the *chain drift problem*, which will be defined subsequently. This chapter also addresses the problems associated with missing prices.

Up to this point, we have focused on the problem of measuring consumer price inflation over the two periods, a base period 0 and a current period 1; that is, we have been discussing bilateral index number theory. In Chapter 7, the focus is on constructing an index over many periods. A simple method for adapting bilateral index number theory to the case of many periods is to choose the bilateral index number formula and fix the base period, and as new data become available, we simply compute the bilateral index linking the current period to the base period. This method generates a sequence of *fixed-base index numbers*. The problem with this strategy is that the structure of the economy changes over time with new products appearing and old products disappearing so that one is eventually forced to give up on the use of fixed-base index numbers.

An alternative to fixed-base index numbers is to use *chained indices*. Chained indices work as follows. Suppose we have chosen a suitable bilateral index number formula, say $P(p^0,p^1,q^0,q^1)$ and we want to use this formula to compute chained indices over time periods $1,2,\ldots,T$. Suppose the price and quantity vectors for period t are p^t and q^t for $t = 1,\ldots,T$. For period 1, we set the price level P^1 equal to

[7] If the time period is very short (for example, one day) hardly any purchases made by a single household will be matched over a sequence of days.

1. For period 2, we set the price level P^2 equal to the index $P(p^1,p^2,q^1,q^2)$. For period 3, we set the price level P^3 equal to $P(p^2,p^3,q^2,q^3)$ times the price level in period 2, P^2. Thus, we have $P^3 = P^2 \times P(p^2,p^3,q^2,q^3) = 1 \times P(p^1,p^2,q^1,q^2) \times P(p^2,p^3,q^2,q^3)$. Similarly, the price level in period 4 is $P^4 = P^3 \times P(p^3,p^4,q^3,q^4) = 1 \times P(p^1,p^2,q^1,q^2) \times P(p^2,p^3,q^2,q^3) \times P(p^3,p^4,q^3,q^4)$ and so on. Thus, we build up the overall price change going from period 1 to period T by multiplying together the period-to-period chain links $P(p^{t-1},p^t,q^{t-1},q^t)$ that link the prices of period t to the prices of the previous period t–1. Thus, it appears that chained indices will be more reliable than fixed-base indices when there is a great deal of product churn because the chained indices will have more matched prices on average than fixed-base indices. The use of chained indices was endorsed by Marshall many years ago.

The 2004 *Consumer Price Index Manual* endorsed the use of a superlative index number formula (like the Fisher bilateral price index) at the first stages of aggregation if price and quantity data were available, and the *Manual* also endorsed the use of chained indices. However, when this chaining strategy was implemented at the first stage of aggregation when price and quantity data were available (say at the level of a retail outlet for some class of products), it was found that the resulting indices often led to unusually low price levels as time marched on. A way of formalizing this *chain drift problem* is to look at a series of chained indices over T periods and add the data of the base period as an artificial final period $T + 1$. Ideally, we would like the resulting period $T + 1$ price level to equal the period 1 price level; that is, we would like the bilateral index number formula to satisfy Walsh's *multiperiod identity test*. In recent years, with the increasing availability of retail scanner data, statistical agencies have found that this test often fails quite spectacularly.

The chain drift problem is mainly caused by consumer stockpiling behavior. Thus, when the price of a product is particularly low, consumers tend to purchase a large amount of it (thus driving the overall price index down), but when prices return to normal in a subsequent time period, consumers purchase less than the normal amount of the product, and this means that the index does not recover to its pre-sale level. An example in Chapter 7 illustrates this phenomenon. Thus, the normal case of chain drift is downward chain drift. But upward chain drift can also occur (due to incomplete adjustment on the part of households). The case of upward chain drift is also explained in Chapter 7.

As statistical agencies and academics worked on constructing subindices of the CPI using scanner data over the past 20 years, they discovered that the use of multilateral indices could reduce chain drift. A *multilateral index* simultaneously determines price levels for a window of say T periods, so the attention shifts to the determination of price levels rather than rates of inflation over two periods (the bilateral approach to index number theory). Multilateral price indices were introduced by Gini in order to measure price levels across the different regions of Italy. Balk was an early pioneer in applying multilateral methods in the time series context. Ivancic, Diewert, and Fox stimulated general interest in the use of *rolling window multilateral index number methods* in the time series context.

Chapter 7 has treatments of several related topics:

- A comparison of likely differences in many bilateral and multilateral index number formulae is made under the assumption that there are long-run trends in prices.
- The problems raised by missing prices are discussed briefly in this chapter and in Chapter 8.
- The main multilateral indices are defined and compared, which includes the GEKS, Geary–Khamis (GK), time product dummy (TPD), and relative price similarity-linked indices.
- A new test approach to multilateral indices is developed where the objective is to construct price and quanity levels rather than a bilateral ratio of prices or quantities.
- The various indices defined in the chapter are computed for a small artificial scanner data set, which is listed in an annex to the chapter.

From the viewpoint of the axiomatic approach to multilateral index number theory, similarity-linked price indices appear to have very good properties. The basic idea behind similarity-linked price indices is as follows. If prices between the two periods are proportional, then any "reasonable" price index will be equal to the factor of proportionality. Hence, define a suitable measure of relative price dissimilarity between the prices of any two periods. The nonnegative dissimilarity measure must have the property that if prices in the two periods are proportional, then the dissimilarity measure is equal to 0. Larger measures of dissimilarity indicate larger deviations from price proportionality. When the prices of the current period become available, calculate the chosen measure of relative price dissimilarity with the prices of each prior period in a window of observations. The prior period with the lowest measure of dissimilarity is chosen as the link observation, and the bilateral Fisher index for the current period relative to the chosen link period is calculated and used to update the price level of the link period. This method of linking leads to price levels that will always satisfy Walsh's multiperiod identity test for prices. Using the predicted share measure of relative price dissimilarity is recommended because it can deal with missing prices (and zero quantities) and it penalizes a lack of matching of prices between the two periods. The SPQ similarity linking method is a bit more complicated, but it leads to quantity indices (as well as price indices) that satisfy a multiperiod identity test for quantities. However, it turns out that similarity linking works best when making international comparisons of prices or when constructing price indices for seasonal products. For "regular" products, real-time similarity linking can lead to bilateral links that are very close to chained links and thus, chain drift can still occur using similarity linking methods. It turns out that satisfying Walsh's multiperiod identity test is a necessary condition for a multilateral method to avoid chain drift but is not a sufficient condition. In order to completely rule out the possibility of chain drift, the multilateral method must satisfy Fisher's circularity test. The real-time similarity linking method can be adapted to satisfy the Circularity Test. However, all of the multilateral methods explained in Chapter 7 suffer from a common problem: they are not able to properly measure the benefits of new products. In the next chapter, the quality adjustment problem is addressed.

5. Quality Adjustment Methods

Chapter 8 presents a general framework for measuring the effects of quality change in a CPI context. Most of the existing methods for adjusting for quality change can be regarded as special cases of this framework.

Here is the basic problem: a new product suddenly appears. It could be a genuinely new product or a possibly improved version of an existing product. How can we capture the possible benefits of this new product in a CPI or in the companion index of real consumption?

It is not possible to measure the effects of quality change without using the economic approach to index number theory since we are trying to measure the benefit or utility of the new product relative to existing products. We use the utility function $f(q)$ that was defined in the beginning of Section 2. Again, we assume that the utility function is linearly homogeneous so that it satisfies the property $f(\lambda q) = f(\lambda q_1,\lambda q_2,\ldots,\lambda q_N) = \lambda f(q)$ for all numbers $\lambda > 0$. The chapter develops alternative quality adjustment methods that depend on alternative assumptions about the functional form of the utility function.

Chapter 8 studies four types of models depending on the assumptions made about $f(q)$:

- $f(q)$ is a linear function of the form $f(q) = \alpha{\cdot}q \equiv \Sigma_{n=1}^{N} \alpha_n q_n$. This class of models includes methods used by statistical agencies as well as the *time dummy hedonic regression model* studied in Chapters 6 and 7.
- $f(q)$ is again a linear function of q, but the coefficients α_n are now functions of various amounts of K price-determining characteristics, $z_1,\ldots,z_K$. Thus, we have $f(q,z) = \Sigma_{n=1}^{N} \alpha_n(z_{n1},z_{n2},\ldots,z_{nK})q_n$, where $z_{n1},z_{n2},\ldots,z_{nK}$ are the amounts of characteristic 1, 2,…,K that one unit of product n contains for $n = 1,\ldots,N$. This class of models leads to *general hedonic regression models with characteristics.*
- $f(q)$ is a *Constant Elasticity of Substitution* (CES) *utility function*. This class of utility functions includes the linear utility function defined earlier as a special case but it is more flexible; that is, it is consistent with a wider range of consumer substitution responses to changes in prices. Feenstra worked out a very elegant method for dealing with this case that does not require extensive econometric estimation; only an estimate for the elasticity of substitution is required in order to implement Feenstra's method.
- $f(q)$ is a more general functional form that allows for a wider range of consumer responses to changes in prices. This framework has been used by Hausman and Diewert and Feenstra.

The problem with the CES approach is that the reservation prices that this approach generates for missing products are infinite. Typically, it does not require an infinite price to discourage a consumer from purchasing a product. Another problem is that the elasticity of substitution must be estimated somehow and estimates tend to be quite variable. The fourth class of methods that uses more flexible functional forms generates finite reservation prices but has the disadvantage that the associated econometric estimation is quite complex and difficult to implement at scale. However, a special case of the Diewert and Feenstra methodology does have the potential to be implemented at scale. This special case also satisfies the Circularity Test and hence is not subject to chain drift.

Chapter 8 discusses some additional approaches to the treatment of quality change such as clustering.

6. Seasonal Products

Chapter 9 deals with the problem of seasonal products. A seasonal good or service has regular fluctuations in prices and quantities that are synchronized with the seasons of the year. A *strongly seasonal product* is a seasonal product that is not available in all months of the year. Thus, strongly seasonal products create a missing price problem for the seasons where the product is simply not available.

The problems associated with missing prices are addressed in Chapters 7 and 8, and so the methods for dealing with missing prices suggested in these chapters can be used to deal with missing prices in the strongly seasonal context. However, the fact that there is a degree of regularity in the appearance and disappearance of strongly seasonal products means that alternative methods for dealing with missing seasonal prices can be devised. In particular, indices that match the prices and quantities of December for the current year to the prices and quantities of December for the base year are likely to have fewer missing prices and quantities than an index that compares the prices of the current month with the prices of the previous month. In other words, year-over-year December indices and year-over-year January indices are likely to be more reliable than an index that compares January prices to February prices. Thus, Sections 2 and 3 of Chapter 9 present the algebra for constructing year-over-year indices for each month of the year. Section 2 uses carry-forward prices for any prices that are missing when a product is not present. A carry-forward price is simply the last observed price for the product that is used as an imputed price for the product when it is missing.[8] Since seasons are not perfectly synchronized with months of the year, carry-forward prices can occur in the year-over-year context.[9] In general, it is not a good idea to use carry-forward prices, particularly in conditions of general inflation (or deflation), since the carry-forward price in the inflation context will tend to give the index a downward bias. Hence, in Section 3, the various year-over-year monthly indices are recalculated using only matched prices in the two periods being compared.

Sections 4 and 5 of Chapter 9 construct annual indices in the seasonal products context. The most accurate method for constructing an annual index in this context is to treat each product in each period (month or quarter) as a separate annual product. This type of annual index was first suggested by Mudgett and Stone in separate publications. Section 4 uses the carry-forward prices constructed in Section 2 in order to calculate various Mudgett Stone annual indices, while Section 5 uses only matched prices in constructing the various annual indices. National Statistical Offices do not use the Mudgett Stone methodology when they construct annual CPIs; instead, they usually just take the arithmetic average of their monthly year-over-year indices to construct an annual

[8] A carry-forward price for the year-over-year monthly indices will in general be different from a carry-forward price for a month-to-month index. As described in the *CPI Manual* (Chapter 6), carry-forward prices are not the preferred imputation method for temporarily missing varieties.

[9] For example, weather can delay or bring forward harvests of fresh fruits and vegetables. Similarly snowfall conditions can delay the opening of ski lifts and so on. As described in the *CPI Manual* (Chapter 6), carry forward is not the recommended approach for the treatment of missing seasonal items.

index (or they take the arithmetic average of their month-to-month CPIs for the calendar year). Using our Israeli data on fresh fruits, we found that there was a substantial amount of bias using the usual method for forming annual indices compared to the Mudgett Stone method. The bias is due to the fact that expenditures on strongly seasonal products are not spread evenly over all months.

Sections 6 and 7 construct month-to-month price indices using various index number formulae. The computations in Section 6 used carry-forward prices for missing prices,[10] while the computations in Section 7 used only matched product prices. Our "best" month-to-month indices used the predicted share methodology (explained in Chapter 7) for linking the current month to the previous month that had the most similar structure of relative prices. The downward bias in using carry-forward prices (instead of matched prices) in the context of month-to-month indices was much more pronounced than it was in the context of constructing year-over-year monthly indices.

Up to this point, the various indices used monthly price and quantity data. It is of interest to use only the price data to construct various month-to-month indices. Thus, Sections 8 and 9 construct various indices such as the Carli, Dutot, and Jevons indices (and the TPD indices that use only price data). These indices that use only price information can then be compared with our "best" similarity-linked indices that used monthly price and quantity information. Section 8 used month-to-month carry-forward prices, while Section 9 used only prices that were actually observed. Section 9 also modified the predicted share relative price similarity linking methodology to the *prices-only situation*. In place of the maximum overlap bilateral Fisher index (which was used to link the current period prices to the prices of a prior period with the most similar structure of relative prices), the maximum overlap Jevons index was used to link the current period prices to the prices of a prior period. This *new prices-only multilateral method* seemed to work well using the Israeli data set, in the sense that seasonal fluctuations were muted using the prices-only predicted share indices and these indices were reasonably close to our "best" predicted share similarity-linked indices that used both price and quantity information.[11]

Section 10 looked at various annual basket indices and compared these indices to our "best" similarity-linked indices. Several of these annual basket indices captured trend inflation rather well, but the seasonal fluctuations were often very large (and in opposite directions) compared to our "best" index. It seems to be worthwhile for National Statistical Offices to invest in obtaining monthly expenditure weights in order to produce more accurate price indices for seasonal commodity groups.

Finally, in Section 11, some of the problems associated with measuring trend inflation and seasonal adjustment are discussed. The basic problem is that it is difficult to distinguish trend inflation from changes in seasonal price patterns. Thus, in Chapter 9, our focus is on obtaining measures of price change before seasonal adjustment. There are many methods suggested in the literature on how to seasonally adjust an economic time series. But all of these methods require as an input an unadjusted series. Thus, producing the best possible unadjusted series should be the main task of a National Statistical Office.

7. The Treatment of Durable Goods and Owner-Occupied Housing

Chapter 10 looks at the treatment of durable goods in a CPI. The basic problem is the following one. When a household purchases a durable good,[12] a certain price is paid for the ownership of it in the period of purchase. However, the benefits from the use of the durable good persist into the future for many periods, and thus, it does not seem to be fair to charge the entire purchase price of the durable to its period of purchase. But how exactly are we to decompose the purchase price of the durable into period-by-period charges for the use of the durable over its useful lifetime? This is the *fundamental problem of accounting*.

There is no universally agreed answer to this problem. However, there are *three main approaches* to addressing this problem:

- The *acquisitions approach*: This approach simply allocates the entire purchase price to the period of purchase.
- The *rental equivalence approach*: If rental markets for the durable good exist, then use the current period rental price as an imputed price for the use of the durable in the current period. By using the services of the durable during a period, the owner forgoes the opportunity cost of renting the services of the durable to another user.
- The *user cost approach*. This is the financial opportunity cost of using the services of the durable, which will be explained in more detail subsequently.

It is obvious that the acquisitions approach will not measure the service flow yielded by ownership of a durable good beyond the first period, and so for durables that have a long useful life (such as housing), the acquisitions approach will not be suitable for measuring the flow of utility to the consumer that using the services of the durable good generates.

The rental equivalence approach does measure the utility flow generated by using the services of the durable good, but it may fail in situations where the rental market is thin or nonexistent or heavily regulated by the government.

The user cost approach is more complicated than the first two approaches. Here is an explanation of how the approach works. Suppose that a household purchases a new unit of the durable good at the beginning of period t at price P_0^t. The 0 subscript indicates that the age of the purchased good is 0, indicating that it is a new unit of the durable good. The consumer uses the services of the durable during period t. At the end of period t (which is the beginning of period t + 1), the consumer observes that the used durable good could

[10] Section 7 uses month-to-month carry-forward prices for missing products which are different from the carry-forward prices used in Section 2. For the Israeli data set, the probability that a month-to-month price for the same product was missing turned out to be 0.447. There were very few missing prices in the year-over-year context that was used in Section 2.

[11] This new similarity-linked multilateral method that used only price information also penalized a lack of price matching.

[12] A durable good provides a flow of services to the household that persists for more than one accounting period.

be sold at price P_1^{t+1}, where the subscript 1 indicates that the durable good has been used for 1 period so it is now a secondhand good. Thus, it appears that the consumer's total cost of using the services of the durable good during period t is simply the purchase price less the selling price or $P_0^{t} - P_1^{t+1}$. But in purchasing the durable good at the beginning of period t, the household ties up its financial capital for the period, and thus there is a *financial opportunity cost of holding the durable good* over the period. This financial opportunity cost is $r^t P_0^{t}$, where r^t is the relevant interest rate that the household faces.[13] Thus, the full *user cost* is $U^t \equiv P_0^{t}(1 + r^t) - P_1^{t+1}$. This user cost formula is not the usual one used by economists. Suppose that the price of a new unit of the durable good at the beginning of period $t + 1$ is P_0^{t+1}. We can use the end of period t (or beginning of period $t + 1$) prices P_0^{t+1} and P_1^{t+1} for a new and used unit of the durable good in order to define a *depreciation rate* for the durable good, δ, defined as follows: $(1-\delta) \equiv P_1^{t+1}/P_0^{t+1}$. We can also use the new good prices P_0^{t} and P_0^{t+1} in order to define the period t *asset appreciation rate*, i^t, as follows: $(1 + i^t) \equiv P_1^{t+1}/P_0^{t+1}$. Using these definitions, we can express the end of period t price for a unit of the used durable, P_1^{t+1}, in terms of the price of a new unit of the durable at the beginning of period t, P_0^{t}, as follows:

$$P_1^{t+1} = (1-\delta)(1 + i^t)P_0^{t}. \tag{18}$$

Using (18), the user cost U^t can be obtained as follows:

$$\begin{aligned} U^t &= P_0^{t}(1 + r^t) - P_1^{t+1} \\ &= P_0^{t}(1 + r^t) - (1-\delta)(1 + i^t)P_0^{t} \\ &= [r^t - i^t + \delta(1 + i^t)]P_0^{t}. \end{aligned} \tag{19}$$

Although the user cost concept is used by many National Statistical Offices in their productivity accounts in order to measure capital services if they provide measures of the Multifactor Productivity or Total Factor Productivity for their economy, countries have been reluctant to use the user cost methodology to measure the services of consumer durables in their CPIs. The problem is that it is not straightforward to determine what exactly is the appropriate interest rate r^t, depreciation rate δ, and asset appreciation rate i^t to use in the user cost formula defined by (19). In particular, the use of ex-post asset appreciation rates i^t in formula (19) is not recommended due to the volatility in asset prices; instead, predicted or smoothed asset appreciation rates should probably be used. But this raises the question: Which of many possible methods should be used in order to smooth asset appreciation rates? It is also difficult to determine the "right" opportunity cost of financial capital r^t, and it is not easy to determine depreciation rates δ either. Nevertheless, sometimes countries are forced to use the user cost approach to value the services of owner-occupied housing (OOH) due to the lack of comparable rental markets. Making somewhat arbitrary decisions about r^t, i^t, and δ is acceptable if these decisions are explained to users. After all, user costs are routinely used by academic economists and by national statistical agencies that produce estimates of Multifactor Productivity.

National Statistical Offices for the most part just use the acquisitions approach to value the services of consumer durables in their CPIs. The exception to this rule is OOH. For the most countries, the rental equivalence approach is used, but for a few countries where rental markets are thin, a simplified user cost approach is used. Eurostat's harmonized index of consumer prices (HICP) has simply omitted the services of OOH, but this may change in the future.

Chapter 10 discusses the three main approaches to the treatment of durables in a CPI in more detail in Sections 2–5. Sections 6–8 look at different models of depreciation that are of interest if the user cost approach to the treatment of durables is implemented. Section 9 shows how the acquisitions approach will in general understate the flow of services from the use of durable goods in the national accounts.

Section 10 of Chapter 10 looks at the accounting problems that are posed by stockpiling behavior on the part of households.

Sections 11–18 deal with the complications associated with including housing services in a CPI. A main problem is that housing consists of two main assets: (i) the structure (which depreciates) and (ii) the land plot that supports the structure (which does not depreciate). Thus, it is not possible to *exactly* match the prices of the same property over time due to depreciation of the structure (and possible renovations and additions to an existing structure). Thus, we have a difficult quality adjustment problem. Hedonic regression techniques offer the best solution to these measurement problems.

Section 18 looks at a fourth approach to the treatment of OOH, namely the *payments approach*. Some of the problems associated with the use of this approach are discussed in this section.

8. Lowe, Young, and Superlative Indices: An Empirical Study for Denmark

Chapter 11 concludes this Consumer Price Index Theory volume by looking at the components of the Danish CPI and experimenting with alternative methods of aggregating the data. Thus, Statistics Denmark has provided its data for 402 monthly elementary price indices for the seven years from 2012 to 2018 along with the annual basket weights used to aggregate these elementary indices into an overall index.

This chapter computes various "practical" monthly indices such as the annual basket Lowe and Young indices along with various superlative annual indices. Thus, estimates of the amount of substitution bias in the annual indices can be computed.

An annex to the chapter explains the computation of various elementary indices and some approximate Fisher and relative price similarity-linked indices using the same data set.

[13] If the household borrows money to finance the purchase of the consumer durable, then r^t is the rate of interest that the household pays in order to secure the loan. If the household does not have to borrow funds to finance the purchase and has investments, then the relevant rate of interest r^t is the rate of return on a marginal investment made at the beginning of period t.

9. Conclusion

This volume started at the Ottawa Group meeting in Tokyo, Japan, in 2015. Thus, it has been in progress for some seven years. The authors of the various chapters are as follows: W. Erwin Diewert wrote Chapters 1–8; the authors of Chapter 9 are Diewert, Yoel Finkel, Doron Sayag, and Graham White; the authors of Chapter 10 are Diewert and Chihiro Shimizu; the authors of the main text of Chapter 11 are Martin Nielsen, Martin Larsen, and Carsten Boldsen; and the author of the annex to Chapter 11 is Diewert. The editor of the overall volume is Diewert. The authors would like to thank all of the persons who provided comments on the various chapters; these commentators are listed in a footnote at the beginning of each chapter. Needless to say, the commentators are not responsible for any remaining errors. The authors want to thank Chihiro Shimizu in particular for carefully reading the manuscript of each chapter and checking each equation for errors.

The main purpose of the volume is to provide an overview of the main index number theories that have been suggested over the past 150 years and could be useful to statisticians who construct CPIs. The volume contains hundreds of references to papers and books on index number theory that could be useful to readers who want to explore the subject more deeply. This book could also be useful to academics who wish to learn more about the problems facing price statisticians in their attempts to produce accurate CPIs. There are many unsolved problems that could be usefully studied by academics. Parts of the various chapters could be used as supplementary reading material for courses in advanced macroeconomics.[14] This book could also be useful to private sector economists who are processing micro data into aggregates to be used by management.

This volume does require some mathematical background in order to follow completely all of the various arguments made in the text. Basically, matrix algebra and advanced calculus are used in many of the chapters. Some knowledge of microeconomics is also helpful.

An important topic that is not covered in the present volume is sampling theory. The application of multilateral indices to cross-country and cross-region comparisons is also not covered but, of course, multilateral indices are studied in some detail in this volume.

[14] Another fairly recent book that could be used to teach economists and economic statisticians the fundamentals of index number theory is Bert Balk's 2008 book, *Price and Quantity Index Numbers*. His book is particularly good on the early history of index number theory and on the test and stochastic approaches to index number theory. He does not cover the economic approach.

BASIC INDEX NUMBER THEORY* 2

1. Introduction

> The answer to the question what is the Mean of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes.
>
> Francis Ysidro Edgeworth (1888; 347)

The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms produce not only commodities for final consumption but also export and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labor services, and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographic location or by the season or time of day that they are produced or consumed, then there are *billions* of commodities that are traded within each year in any advanced economy. For many purposes, it is necessary to *summarize* this vast amount of price and quantity information into a much smaller set of numbers. The question that this chapter addresses is: *How exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables?* This is the basic *index number problem*.

It is possible to pose the index number problem in the context of microeconomic theory; that is, given that we wish to implement some economic model based on producer or consumer theory, what is the "best" method for constructing a set of aggregates for the model? However, when constructing aggregate prices or quantities, other points of view (that do not rely on economics) are possible. Some of these alternative points of view will be considered in this chapter and the following two chapters. Economic approaches to index number theory will be pursued in Chapters 5 and 8.

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The index number problem can be framed as the problem of decomposing the value of a well-defined set of transactions in a period of time into an aggregate price term times an aggregate quantity term. This is the price and quantity *levels approach* to index number theory. This approach will be pursued in subsequent chapters, but there are some difficulties with the use of this approach, and so in Section 2, the problem of decomposing a *value ratio* pertaining to two periods of time into a component that measures the overall *change in prices* between the two periods (this is the *price index*) times a term that measures the overall *change in quantities* between the two periods (this is the *quantity index*) is considered. Thus, instead of attempting to construct aggregate price and quantity levels for each period, a *ratio approach* is adopted. The simplest price index is *a fixed basket type index*; that is, fixed amounts of the N quantities in the value aggregate are chosen, and then this fixed basket of quantities at the prices of period 0 and at the prices of period 1 are calculated. Thus, the fixed basket price index is simply the ratio of these two values, where the prices vary but the quantities are held fixed. Two natural choices for the fixed basket are the quantities transacted in the base period, period 0, or the quantities transacted in the current period, period 1. These two choices lead to the Laspeyres (1871) and Paasche (1874) price indices, respectively.

Unfortunately, the Paasche and Laspeyres measures of aggregate price change can differ, sometimes substantially. Thus in Section 4, taking an average of these two indices to come up with a single measure of price change is considered. In this section, it is argued that the "best" average to take is the geometric mean, which is Irving Fisher's (1922) ideal price index. In Section 5, instead of geometric average of the Paasche and Laspeyres measures of price change, taking an arithmetic average of the two baskets is considered. This fixed basket approach to index number theory leads to a price index advocated by Walsh (1901) (1921a). However, other fixed basket approaches are also possible. Instead of choosing the basket of period 0 or 1 (or an average of these two baskets), it is possible to choose a basket that pertains to an entirely different period, say period *b*. In fact, it is typical statistical agency practice to pick a basket that pertains to an entire year (or even two years) of transactions in a year prior to period 0, which is usually a month. Indices of this type, where the weight reference period differs from the price reference period, were originally proposed by Joseph Lowe (1823), and in Section 6, indices of this type will be studied.

In Section 7, another approach to the use of annual weights with monthly prices will be discussed. This approach was developed by Young (1812).

In Section 8, the advantages and disadvantages of using a *fixed-base* period in the bilateral index number comparison are considered versus always comparing the current period with the previous period, which is called the *chain system*. In the chain system, a *link* is an index number comparison of one period with the previous period. These links are multiplied together in order to make comparisons over many periods. Fixed-base or direct indices will be studied and compared to their chained counterparts in more detail in Chapter 7.[1]

In practice, CPIs are usually constructed in two or more stages of aggregation. For example, at the first stage of aggregation, subindices for various consumption categories, like food, clothing, transportation, and so on, are constructed, and then in the second stage of aggregation, an overall CPI is constructed. Does a CPI constructed in two stages coincide with a CPI constructed in a single stage? In Section 9, this question is addressed for some of the more commonly used index number formulae. In Annex 5, the consistency in aggregation of various formulae over three (or more) stages of aggregation will be discussed.

The Appendices 1–4 look at the numerical relationships between the Laspeyres, Paasche, Lowe, and Young indices.

2. The Decomposition of Value Aggregates and the Product Test

A *price index* is a measure or function that summarizes the *change* in the prices of many commodities from one situation 0 (a time period or place) to another situation 1. A *price level* can be thought of as an average of the prices pertaining to a single period. More specifically, for most practical purposes, a price index can be regarded as a weighted average of the relative prices of the commodities under consideration in the two situations. To determine a price index, it is necessary to know the following:

- which commodities or items to include in the index;
- how to determine the item prices;
- which transactions that involve these items to include in the index;
- how to determine the weights, and from which sources these weights should be drawn;
- what formula or type of mean should be used to average the selected item relative prices.

All of these price index definition questions except the last can be answered by appealing to the definition of the *value aggregate* to which the price index refers. A *value aggregate* V for a given collection of N items and transactions is computed as[2]

$$V = \Sigma_{n=1}^{N} p_n q_n, \qquad (1)$$

where p_n represents the price of the nth item in national currency units, q_n represents the corresponding quantity transacted in the time period under consideration, and the subscript n identifies the nth elementary item in the group of N items that make up the chosen value aggregate V. Considered in this definition of a value aggregate is the specification of the group of included commodities[3] (which items to include) and of the economic agents engaging in transactions involving those commodities (which transactions to include), as well as the valuation and time of recording principles motivating the behavior of the economic agents undertaking the transactions (determination of prices). The included elementary items, their valuation (p_n), the eligibility of the transactions, and the item weights (q_n) are all within the domain of definition of the value aggregate. The precise determination of p_n and q_n can be a tricky business.[4]

The value aggregate V defined by (1) referred to a certain set of transactions pertaining to a single (unspecified) time period. Now the same value aggregate for two places or time periods, periods 0 and 1, is considered. For the sake of definiteness, period 0 is called *the base period*, and period 1 is called *the current period*, and it is assumed that observations on the base period price and quantity vectors, $p^0 \equiv [p_1^0,\ldots,p_N^0]$ and $q^0 \equiv [q_1^0,\ldots,q_N^0]$, respectively, have been collected.[5] The value aggregates in the two periods are defined as

$$V^0 \equiv \Sigma_{n=1}^{N} p_n^0 q_n^0 \text{ and } V^1 \equiv \Sigma_{n=1}^{N} p_n^1 q_n^1. \qquad (2)$$

In the previous paragraph, a price index was defined as a function or measure that summarizes the *change* in the prices of the N commodities in the value aggregate from situation 0 to situation 1. In this paragraph, a *price index* $P(p^0,p^1,q^0,q^1)$ along with the corresponding *quantity index* (or *volume index*) $Q(p^0,p^1,q^0,q^1)$ is defined to be two functions of the $4N$ variables p^0,p^1,q^0,q^1 (these variables describe the prices and quantities pertaining to the value aggregate for periods 0 and 1) where these two functions satisfy the following equation:[6]

$$V^1/V^0 = P(p^0, p^1, q^0, q^1)\, Q(p^0, p^1, q^0, q^1). \qquad (3)$$

If there is only one item in the value aggregate, then the price index P should collapse down to the single price ratio p_1^1/p_1^0 and the quantity index Q should collapse down to the single quantity ratio q_1^1/q_1^0. In the case of many items, the price index P is to be interpreted as some sort of weighted average of the individual price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

Thus, the first approach to index number theory can be regarded as the problem of *decomposing* the change in a

[1] In order to compare the advantages and disadvantages of fixed-base versus chained indices, it is useful to be able to draw on other approaches to index number theory, which will be studied in Chapters 3–5.

[2] Notation: The sum of terms, $\Sigma_{n=1}^{N} p_n q_n$, will at times be written as $\Sigma_{i=1}^{N} p_i q_i$ or as $\Sigma_{k=1}^{N} p_k q_k$. In subsequent chapters, $\Sigma_{n=1}^{N} p_n q_n$ will sometimes be written as $p \cdot q$, which is called the inner product of the vectors p and q defined as $p \equiv [p_1,\ldots,p_N]$ and $q \equiv [q_1,\ldots,q_N]$.

[3] The terms "commodity," "item," and "product" will be used interchangeably in what follows. Different statistical agencies may have more specific definitions for these terms.

[4] Ralph Turvey has noted that some values may be difficult to decompose into unambiguous price and quantity components. Some examples of difficult to decompose values are bank charges, gambling expenditures, and life insurance payments. The problems associated with precisely defining p_n^t and q_n^t are discussed in some detail in Eurostat (2018). There is a great deal of valuable information in this *Manual*.

[5] Note that it is assumed that there are no new or disappearing commodities in the value aggregates. Approaches to the "new goods problem" and the problem of accounting for quality change are discussed in Chapter 8.

[6] The first person to suggest that the price and quantity indices should be jointly determined in order to satisfy equation (3) was Fisher (1911; 418). Frisch (1930; 399) called (3) the *product test*.

value aggregate, V^1/V^0, into the product of a part that is due to *price change*, $P(p^0,p^1,q^0,q^1)$, and a part that is due to *quantity change*, $Q(p^0,p^1,q^0,q^1)$. This approach to the determination of the price index is the approach that is taken in the national accounts, where a price index is used to *deflate* a value ratio in order to obtain an estimate of quantity change. Thus, in this approach to index number theory, the primary use for the price index is as a *deflator*. Note that once the functional form for the price index $P(p^0,p^1,q^0,q^1)$ is known, then the corresponding quantity or volume index $Q(p^0,p^1,q^0,q^1)$ is completely determined by P; that is, rearranging (3), we get

$$Q(p^0,p^1,q^0,q^1) = [V^1/V^0]/P(p^0,p^1,q^0,q^1). \tag{4}$$

Conversely, if the functional form for the quantity index $Q(p^0,p^1,q^0,q^1)$ is known, then the corresponding price index function $P(p^0,p^1,q^0,q^1)$ is completely determined by the quantity index function $Q(p^0,p^1,q^0,q^1)$. Thus, using this deflation approach to index number theory, separate theories for the determination of the price and quantity indices are not required: If either P or Q is determined, then the other function is implicitly determined by the product test (3).

In the next subsection, two concrete choices for the price index $P(p^0,p^1,q^0,q^1)$ are considered, and the corresponding quantity indices $Q(p^0,p^1,q^0,q^1)$ that result from using equation (4) are also calculated. These are the two choices used most frequently by national income accountants.

3. The Laspeyres and Paasche Indices

One of the easiest approaches to the determination of the price index formula was described in great detail by Lowe (1823). His approach to measuring the price change between periods 0 and 1 was to specify an approximate *representative commodity basket,*[7] which is a quantity vector $q \equiv [q_1,\ldots,q_N]$ that is representative of purchases made during the two periods under consideration, and then calculate the level of prices in period 1 relative to period 0 as the ratio of the period 1 cost of the basket, $\Sigma_{n=1}^{N} p_n^1 q_n$, to the period 0 cost of the basket, $\Sigma_{n=1}^{N} p_n^0 q_n$. This *fixed basket approach* to the determination of the price index leaves open the question as to how exactly is the fixed basket vector q to be chosen.

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the base period 0 commodity vector q^0 or the current period 1 commodity vector q^1. These two choices lead to the *Laspeyres* (1871) price index[8] P_L defined by (5) and the *Paasche* (1874) price index[9] P_P defined by (6):[10]

$$P_L(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_n^1 q_n^0 / \Sigma_{i=1}^{N} p_i^0 q_i^0; \tag{5}$$
$$P_P(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_n^1 q_n^1 / \Sigma_{i=1}^{N} p_i^0 q_i^1 . \tag{6}$$

These formulae can be rewritten in an alternative manner that is more useful for statistical agencies. Define the period *t expenditure share* on commodity n as follows:

$$s_n^t \equiv p_n^t q_n^t / \Sigma_{i=1}^{N} p_i^t q_i^t \textit{ for } n = 1,\ldots, N \textit{ and } t = 0,1. \tag{7}$$

Then the Laspeyres index (5) can be rewritten as follows:[11]

$$\begin{aligned} P_L(p^0,p^1,q^0,q^1) &\equiv \Sigma_{n=1}^{N} p_n^1 q_n^0 / \Sigma_{i=1}^{N} p_i^0 q_i^0 \\ &= \Sigma_{n=1}^{N} (p_n^1/p_n^0) p_n^0 q_n^0 / \Sigma_{i=1}^{N} p_i^0 q_i^0 \\ &= \Sigma_{n=1}^{N} (p_n^1/p_n^0) s_n^0, \end{aligned} \tag{8}$$

where the last equality follows from definitions (7).

Thus, the Laspeyres price index P_L can be written as a base period expenditure share-weighted arithmetic average of the N price ratios, p_n^1/p_n^0. The Laspeyres formula (until the recent past) has been widely used as the target index number concept for CPIs around the world. To implement it, a statistical agency needs only to collect information on expenditure shares s_n^0 for the index domain of definition for the base period 0 and then on item *prices* alone on an ongoing basis. *Thus, the Laspeyres CPI can be produced on a timely basis without having to know current period quantity information.*

The Paasche index can also be written in expenditure share and price ratio form as follows:[12]

$$\begin{aligned} P_P(p^0,p^1,q^0,q^1) &\equiv \Sigma_{n=1}^{N} p_n^1 q_n^1 / \Sigma_{i=1}^{N} p_i^0 q_i^1 \\ &= 1/[\Sigma_{i=1}^{N} p_i^0 q_i^1 / \Sigma_{n=1}^{N} p_n^1 q_n^1] \\ &= 1/[\Sigma_{i=1}^{N} (p_i^0/p_i^1) p_i^1 q_i^1 / \Sigma_{n=1}^{N} p_n^1 q_n^1] \\ &= 1/[\Sigma_{i=1}^{N} (p_i^0/p_i^1) s_i^1] \\ &= [\Sigma_{i=1}^{N} s_i^1 (p_i^1/p_i^0)^{-1}]^{-1}, \end{aligned} \tag{9}$$

[7] Lowe (1823; Annex page 95) suggested that the commodity basket vector q should be updated every five years. Lowe indices will be studied in more detail in Sections 5 and 6.

[8] This index was actually introduced and justified by Drobisch (1871a; 147) slightly earlier than Laspeyres. Laspeyres (1871; 305) in fact explicitly acknowledged that Drobisch showed him the way forward. However, the contributions of Drobisch have been forgotten for the most part by later writers because Drobisch aggressively pushed for the ratio of two unit values as being the "best" index number formula. While this formula has some excellent properties if all of the N commodities being compared have the same unit of measurement, the formula is useless when, say, both goods and services are in the index basket. Unit value price indices will be studied in more detail in subsequent chapters.

[9] Again Drobisch (1871b; 424) appears to have been the first to define explicitly and justify this formula. However, he rejected this formula in favor of his preferred formula, the ratio of unit values, and so again he did not get any credit for his early suggestion of the Paasche formula.

[10] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1, and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors, and the notation indicates that the reader is in the realm of *bilateral index number theory*; that is, the prices and quantities for a value aggregate pertaining to the *two periods* are compared.

[11] This method of rewriting the Laspeyres index (or any fixed basket index) as a share-weighted arithmetic average of price ratios was developed by Fisher (1897; 517) (1911; 397) (1922; 51) and Walsh (1901; 506) (1921a; 92). Note that this alternative formula for the Laspeyres price index requires that all base period prices be positive.

[12] This method of rewriting the Paasche index (or any fixed basket index) as a share-weighted harmonic average of the price ratios was developed by Walsh (1901; 511) (1921a; 93) and Fisher (1911; 397–398). Note that this alternative formula for the Paasche price index requires all current period prices to be positive.

where definitions (7) for $t = 1$ were used to derive this equality. Thus, the Paasche price index P_P can be written as a period 1 (or current period) expenditure share-weighted *harmonic* average of the N item price ratios p_i^1/p_i^0.[13] Note that if the statistical agency lacks timely information on quantities, then the Paasche index cannot be produced in a timely manner.

The quantity index that corresponds to the Laspeyres price index using the product test (3) is the *Paasche quantity index*; that is, if P in (4) is replaced by P_L defined by (5), then the following Paasche quantity index Q_P is obtained:

$$Q_P(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_n^1 q_n^1 / \Sigma_{i=1}^{N} p_i^1 q_i^0. \qquad (10)$$

Note that Q_P is the value of the period 1 quantity vector valued at the period 1 prices, $\Sigma_{n=1}^{N} p_n^1 q_n^1$, divided by the (hypothetical) value of the period 0 quantity vector valued at the period 1 prices, $\Sigma_{n=1}^{N} p_n^1 q_n^0$. Thus, the period 0 and 1 quantity vectors are valued at the same set of prices, the current period prices, p^1.

The quantity index that corresponds to the Paasche price index using the product test (3) is the *Laspeyres quantity index*; that is, if P in (4) is replaced by P_P defined by (6), then the following quantity index Q_L is obtained:

$$Q_L(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_n^0 q_n^1 / \Sigma_{i=1}^{N} p_i^0 q_i^0. \qquad (11)$$

Note that Q_L is the (hypothetical) value of the period 1 quantity vector valued at the period 0 prices, $\Sigma_{n=1}^{N} p_n^0 q_n^1$, divided by the value of the period 0 quantity vector valued at the period 0 prices, $\Sigma_{n=1}^{N} p_n^0 q_n^0$. Thus, the period 0 and 1 quantity vectors are valued at the same set of prices, the base period prices, p^0.

The problem with the Laspeyres and Paasche index number formulae is that they are equally plausible, but in general, they will give *different* answers. For most purposes, it is not satisfactory for the statistical agency to provide *two* answers to the question:[14] what is the "best" overall summary measure of price change for the value aggregate over the two periods in question? Thus in the following section, it is considered how "best" averages of these two estimates of price change can be constructed. Before doing this, it is asked what is the "normal" relationship between the Paasche and Laspeyres indices? Under "normal" economic conditions when the price ratios pertaining to the two situations under consideration are negatively correlated with the corresponding quantity ratios, it can be shown that the Laspeyres price index will be larger than the corresponding Paasche index.[15] In Annex 1, a precise statement of this result is presented.[16] This divergence between P_L and P_P suggests that if a *single estimate* for the price change between the two periods is required, then some sort of evenly weighted average of the two indices should be taken as the final estimate of price change between periods 0 and 1. As mentioned earlier, this strategy will be pursued in the following section. However, it should be kept in mind that usually statistical agencies will not have information on current expenditure weights, and hence averages of Paasche and Laspeyres indices can only be produced on a delayed basis (perhaps using national accounts information) or not at all.

4. The Fisher Index as an Average of the Paasche and Laspeyres Indices

As was mentioned in the previous paragraph, since the Paasche and Laspeyres price indices are equally plausible but often give different estimates of the amount of aggregate price change between periods 0 and 1, it is useful to consider taking an evenly weighted average of these fixed basket price indices as a single estimator of price change between the two periods. Examples of such *symmetric averages*[17] are the arithmetic mean, which leads to the Drobisch (1871b; 425), Sidgwick (1883; 68), and Bowley (1901; 227)[18] index, $P_D \equiv (1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher[19] (1922) ideal index, P_F, defined as follows:

$$P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)P_P(p^0,p^1,q^0,q^1)]^{1/2}. \qquad (12)$$

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; that is, in order to determine which of these fixed basket indices or which averages of them might be "best," desirable *criteria* or *tests* or *properties* are needed for the

[13] Note that the derivation in (9) shows how harmonic averages arise in index number theory in a very natural way.

[14] In principle, instead of averaging the Paasche and Laspeyres indices, the statistical agency could think of providing both (the Paasche index on a delayed basis). This suggestion would lead to a *matrix* of price comparisons between every pair of periods instead of a time series of comparisons. Walsh (1901; 425) noted this possibility: "In fact, if we use such direct comparisons at all, we ought to use all possible ones."

[15] P. Hill (1993; 383) summarized this inequality as follows: "It can be shown that relationship (13) [that is, that P_L is greater than P_P] holds whenever the price and quantity relatives (weighted by values) are negatively correlated. Such negative correlation is to be expected for price takers who react to changes in relative prices by substituting goods and services that have become relatively less expensive for those that have become relatively more expensive. In the vast majority of situations covered by index numbers, the price and quantity relatives turn out to be negatively correlated so that Laspeyres indices tend systematically to record greater increases than Paasche with the gap between them tending to widen with time."

[16] There is another way to see why P_P will often be less than P_L. If the period 0 expenditure shares s_i^0 are exactly equal to the corresponding period 1 expenditure shares s_i^1, then by Schlömilch's (1858) Inequality (see Hardy, Littlewood, and Pólya (1934; 26)), it can be shown that a weighted harmonic mean of N numbers is equal to or less than the corresponding arithmetic mean of the N numbers, and the inequality is strict if the N numbers are not all equal. If expenditure shares are approximately constant across periods, then it follows that P_P will usually be less than P_L under these conditions.

[17] For a discussion of the properties of symmetric averages, see Diewert (1993b). Formally, an average $m(a,b)$ of two numbers a and b is symmetric if $m(a,b) = m(b,a)$. In other words, the numbers a and b are treated in the same manner in the average. An example of a nonsymmetric average of a and b is $(1/4)a + (3/4)b$. In general, Walsh (1901; 105) argued for a symmetric treatment if the two periods (or countries) under consideration were to be given equal importance.

[18] Walsh (1901; 99) also suggested this index. See Diewert (1993a; 36) for additional references to the early history of index number theory.

[19] Bowley (1899; 641) appears to have been the first to suggest the use of this index. Walsh (1901; 428–429) also suggested this index while commenting on the big differences between the Laspeyres and Paasche indices in one of his numerical examples: "The figures in columns (2) [Laspeyres] and (3) [Paasche] are, singly, extravagant and absurd. But there is order in their extravagance; for the nearness of their means to the more truthful results shows that they straddle the true course, the one varying on the one side about as the other does on the other."

price index. This topic will be pursued in more detail in the next chapter, but an introduction to the test approach is provided in the present section because a test is used to determine which average of the Paasche and Laspeyres indices might be the "best."

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical consumer price index (CPI)? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*.[20] An index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1); \tag{13}$$

that is, if the period 0 and period 1 price and quantity data are interchanged and the index number formula is evaluated, then this new index $P(p^1,p^0,q^1,q^0)$ should be equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$. This is a property that is satisfied by a single price ratio, and it seems desirable that the measure of aggregate price change should also satisfy this property so that it does not matter which period is chosen as the base period. Put another way, the index number comparison between any two points of time should not depend on the choice of which period we regard as the base period: If the other period is chosen as the base period, then the new index number should simply equal the reciprocal of the original index. It should be noted that the Laspeyres and Paasche price indices *do not* satisfy this time reversal property.

Having defined what it means for a price index P to satisfy the time reversal test, it is possible to establish the following result:[21] The Fisher ideal price index defined by (12) is the *only* index that is a homogeneous[22] symmetric average of the Laspeyres and Paasche price indices, P_L and P_P, *and* satisfies the time reversal test (13). Thus, the Fisher ideal price index emerges as perhaps the "best" evenly weighted average of the Paasche and Laspeyres price indices.

It is interesting to note that this *symmetric basket approach* to index number theory dates back to one of the early pioneers of index number theory, Bowley, as the following quotations indicate:

> If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean . . . as a first approximation.
>
> Arthur. L. Bowley (1901; 227)

> When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. . . . They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found.
>
> Arthur. L. Bowley (1919; 348)[23]

The quantity index that corresponds to the Fisher price index using the product test (3) is the Fisher quantity index; that is, if P in (4) is replaced by P_F defined by (12), then the following quantity index is obtained:

$$Q_F(p^0,p^1,q^0,q^1) = [Q_L(p^0,p^1,q^0,q^1)Q_P(p^0,p^1,q^0,q^1)]^{1/2}. \tag{14}$$

Thus, the Fisher quantity index is equal to the square root of the product of the Laspeyres and Paasche quantity indices. It should also be noted that $Q_F(p^0,p^1,q^0,q^1) = P_F(q^0,q^1,p^0,p^1)$; that is, if the role of prices and quantities is interchanged in the Fisher price index formula, then the Fisher quantity index is obtained.[24]

Rather than take a symmetric average of the two basic fixed basket price indices pertaining to two situations, P_L and P_P, it is also possible to return to Lowe's basic formulation and choose the basket vector q to be a symmetric average of the base and current period basket vectors, q^0 and q^1. This approach to index number theory is pursued in the following subsection.

5. The Walsh Index and the Theory of the "Pure" Price Index

Price statisticians tend to be very comfortable with a concept of the price index that is based on pricing out a constant "representative" basket of commodities, $q \equiv (q_1,q_2,. . .,q_N)$, at the prices of period 0 and 1, $p^0 \equiv (p_1^0,p_2^0,. . .,p_N^0)$ and $p^1 \equiv (p_1^1,p_2^1,. . .,p_N^1)$ respectively. Price statisticians refer to this type of index as a *fixed basket index* or a *pure price index*,[25] and it corresponds to Knibbs' (1924; 43) *unequivocal price index*.[26] Since Lowe (1823) was the first person to describe

[20] See Diewert (1992; 218) for early references to this test. If we want the price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes, in which case satisfaction of the time reversal test may not be so important.

[21] See Diewert (1997; 138).

[22] An average or mean of two numbers a and b, $m(a,b)$, is *homogeneous* if when both numbers a and b are multiplied by a positive number λ, then the mean is also multiplied by λ; that is, m satisfies the following property: $m(\lambda a,\lambda b) = \lambda m(a,b)$. The importance of linear homogeneity will be explained in Chapter 3 when the test approach to index number theory is studied.

[23] Fisher (1911; 417–418) (1922) also considered the arithmetic, geometric, and harmonic averages of the Paasche and Laspeyres indices.

[24] Fisher (1922; 72) said that P and Q satisfied the *factor reversal test* if $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ and P and Q satisfied the product test (3) as well.

[25] See section 7 in Diewert (2001).

[26] "Suppose however that, for each commodity, $Q' = Q$, then the fraction, $\Sigma(P'Q) / \Sigma(PQ)$, namely, the ratio of aggregate value for the second unit-period to the aggregate value for the first unit-period is no longer merely a ratio of totals, it also shows unequivocally the effect of the change in price. Thus, it is an unequivocal price index for the quantitatively unchanged complex of commodities, A, B, C, and so on.

It is obvious that if the quantities were different on the two occasions, and if at the same time the prices had been unchanged, the preceding formula would become $\Sigma(PQ')/\Sigma(PQ)$. It would still be the ratio of the aggregate value for the second unit-period to the aggregate value for the first unit period. But it would be also more than this. It would show in a generalized way the ratio of the quantities on the two occasions. Thus, it is an unequivocal quantity index for the complex of commodities, unchanged as to price and differing only as to quantity.

Let it be noted that the mere algebraic form of these expressions shows at once the logic of the problem of finding these two indices is identical" (Sir George H. Knibbs (1924; 43–44)).

systematically this type of index, it is referred to as a *Lowe index*. Thus, the general functional form for the *Lowe price index* is

$$P_{Lo}(p^0,p^1,q) \equiv \Sigma_{i=1}^{N} p_i^1 q_i / \Sigma_{n=1}^{N} p_n^0 q_n = \Sigma_{i=1}^{N} s_i (p_i^1/p_i^0), \quad (15)$$

where the (hypothetical) *hybrid expenditure shares* s_i[27] corresponding to the quantity weights vector q are defined by

$$s_i \equiv p_i^0 q_i / \Sigma_{k=1}^{N} p_n^0 q_n \text{ for } i = 1,\ldots,N. \quad (16)$$

The main reason why price statisticians might prefer a member of the family of Lowe or fixed basket price indices defined by (15) is *that the fixed basket concept is easy to explain to the public*. Note that the Laspeyres and Paasche indices are special cases of the pure price concept if we choose $q = q^0$ (which leads to the Laspeyres index) or if we choose $q = q^1$ (which leads to the Paasche index).[28] The practical problem of picking q remains to be resolved, and that problem is addressed in this section.

It should be noted that Walsh (1901; 105) (1921a) also saw the price index number problem in this framework:

> Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period, and there is a second period which is compared with it. Price variations have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period? Or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods.
>
> Correa Moylan Walsh (1921a; 90)

Walsh's suggestion will be followed here, and thus the ith quantity weight, q_i, is restricted to be an average or *mean* of the base period quantity q_i^0 and the current period quantity for commodity i q_i^1, say $m(q_i^0,q_i^1)$, for $i = 1,2,\ldots,N$.[29] Under this assumption, the Lowe price index (15) becomes

$$P_{Lo}(p^0,p^1,q^*) \equiv \Sigma_{n=1}^{N} p_n^1 q_n^* / \Sigma_{n=1}^{N} p_n^0 q_n^*, \quad (17)$$

where $q_n^* \equiv m(q_n^0,q_n^1)$ for $n = 1,\ldots,N$ and $m(x,y)$ is an average or mean of positive numbers x and y.

In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (17). For the first such test or property, we ask that P_{Lo} satisfy the *time reversal test* (13). Under this hypothesis, it can be shown that the mean function m must be a *symmetric mean*;[30] that is, m must satisfy the following property $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$. This assumption still does not pin down the functional form for the pure price index defined by (17). For example, the function $m(a,b)$ could be the *arithmetic mean*, $(½)a + (½)b$, in which case (17) reduces to the *Marshall* (1887) *Edgeworth* (1925) *price index* P_{ME}, which was the pure price index preferred by Knibbs (1924; 56):

$$P_{ME}(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_n^1 [(½)q_n^0 + (½)q_n^1] / \Sigma_{n=1}^{N} p_n^0 [(½)q_n^0 + (½)q_n^1]. \quad (18)$$

On the other hand, the function $m(a,b)$ could be the *geometric mean*, $(ab)^{1/2}$, in which case (17) reduces to the *Walsh* (1901; 398) (1921a; 97) *price index*, P_W[31]:

$$P_W(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} p_n^1 [q_n^0 q_n^1]^{1/2} / \Sigma_{n=1}^{N} p_n^0 [q_n^0 q_n^1]^{1/2}. \quad (19)$$

There are many other possibilities for the mean function m, including the mean of order r, $[(½)a^r + (½)b^r]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index defined by (17), it is necessary to impose at least one additional test or axiom on $P_{Lo}(p^0,p^1,q^*)$.

There is a potential problem with the use of the Edgeworth Marshall price index (18) that has been noticed in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (18), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country. In technical terms, the Edgeworth Marshall formula is not homogeneous of degree 0 in the components of both q^0 and q^1.[32] To prevent this problem from occurring in the use of the pure price index defined by (17), it is asked that $P_{Lo}(p^0,p^1,q^*)$ satisfy the following *invariance to proportional changes in current quantities test:*[33]

$$P_{Lo}(p^0,p^1,m(q_1^0,\lambda q_1^1),\ldots,m(q_N^0,\lambda q_N^1)) = P_{Lo}(p^0,p^1,m(q_1^0,q_1^1),\ldots,m(q_N^0,q_N^1)) \text{ for all } \lambda > 0. \quad (20)$$

[27] Fisher (1922; 53) used the terminology "weighted by a hybrid value" while Walsh (1932; 657) used the term "hybrid weights."

[28] Note that the ith share defined by (16) in this case is the hybrid share $s_i \equiv p_i^0 q_i^1 / \Sigma_{j=1}^{N} p_j^0 q_j^1$, which uses the prices of period 0 and the quantities of period 1.

[29] Note that we have chosen the mean function $m(q_i^0,q_i^1)$ to be the same for each item i. We assume that $m(a,b)$ has the following two properties: $m(a,b)$ is a positive and continuous function, defined for all positive numbers a and b and $m(a,a) = a$ for all $a > 0$.

[30] See Section 7 of Diewert (2001) for a proof. For more on symmetric means, see Diewert (1993b; 361).

[31] Walsh endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance" (C.M. Walsh (1921a; 103)). His formula 6 is P_W defined by (19) and his 9 is the Fisher ideal defined by (12). The *Walsh quantity index*, $Q_W(p^0,p^1,q^0,q^1)$, is defined as $P_W(q^0,q^1,p^0,p^1)$; that is, the role of prices and quantities in definition (19) is interchanged. If the Walsh quantity index is used to deflate the value ratio, an implicit price index is obtained, which is Walsh's formula 8.

[32] Thus, using (4), the companion quantity index defined by (4) will not be homogeneous of degree 1 in the components of the vector q^1 and homogeneous of degree –1 in the components of q^0.

[33] This is the terminology used by Diewert (1992; 216). Vogt (1980) was the first to propose this test. If this test holds, then the corresponding implicit quantity index defined by (4) will be linearly homogeneous in the components of q^1, which is a desirable property for a quantity index.

The two tests, the time reversal test (13) and the linear homogeneity invariance test (20), enable one to determine the precise functional form for the pure price index $P_{Lo}(p^0,p^1,q^*)$ defined by (17): the pure price index $P_{Lo}(p^0,p^1,q^*)$ must be the Walsh index P_W defined by (19).[34]

In order to be of practical use by statistical agencies, an index number formula must be able to be expressed as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the N price ratios, p_i^1/p_i^0. The Walsh price index defined by (19) can be rewritten in this format:

$$\begin{aligned} P_W(p^0,p^1,q^0,q^1) &\equiv \Sigma_{n=1}^N p_n^1(q_n^0q_n^1)^{1/2} / \Sigma_{j=1}^N p_j^0(q_j^0q_j^1)^{1/2} \\ &= \Sigma_{n=1}^N [p_n^1/(p_n^0p_n^1)^{1/2}]\,(s_n^0s_n^1)^{1/2} / \Sigma_{j=1}^N [p_j^0/(p_j^0p_j^1)^{1/2}]\,(s_j^0s_j^1)^{1/2} \\ &= \Sigma_{n=1}^N (s_n^0s_n^1)^{1/2}\,[p_n^1/p_n^0]^{1/2} / \Sigma_{j=1}^N (s_j^0s_j^1)^{1/2}\,[p_j^0/p_j^1]^{1/2}\,. \end{aligned} \tag{21}$$

The approach taken to index number theory in this section was to consider averages of various fixed basket type price indices. The first approach was to take an even-handed average of the two primary fixed basket indices: the Laspeyres and Paasche price indices. These two primary indices are based on pricing out the baskets that pertain to the two periods (or locations) under consideration. Taking an average of them led to the Fisher ideal price index P_F defined by (12). The second approach was to average the basket quantity weights and then price out this average basket at the prices pertaining to the two situations under consideration. This approach led to the Walsh price index P_W defined by (19). Both of these indices can be written as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the N price ratios, p_i^1/p_i^0. Assuming that the statistical agency has information on these three sets of variables, which index should be used? Experience with normal time series data at higher levels of aggregation has shown that these two indices will not differ substantially, and thus it is a matter of indifference which of these indices is used in practice.[35] Both of these indices are examples of *superlative indices*, which will be defined in Chapter 5. However, note that both of these indices treat the data pertaining to the two situations in a *symmetric* manner. P. Hill (1988) commented on superlative price indices and the importance of a symmetric treatment of the data as follows:

> Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great.
>
> P. Hill (1993; 384)

6. The Lowe Index with Monthly Prices and Annual Base Year Quantities

It is now necessary to discuss a major practical problem with the theory of basket-type indices. Up to now, it has been assumed that the quantity vector $q \equiv (q_1,q_2,...,q_N)$ that appeared in the definition of the Lowe index, $P_{Lo}(p^0,p^1,q)$, defined by (15), is either the base period quantity vector q^0 or the current period quantity vector q^1 or an average of these two quantity vectors. In fact, in terms of actual statistical agency practice, the quantity vector q is frequently taken to be an *annual quantity vector* that refers to a *base year*, b say, that is prior to the base period for the prices, period 0. Typically, a statistical agency will produce a consumer price index at a monthly or quarterly frequency, but for the sake of definiteness, a monthly frequency will be assumed in what follows. Thus, a typical price index will have the form $P_{Lo}(p^0,p^t,q^b)$, where p^0 is the price vector pertaining to the base period month for prices, month 0, p^t is the price vector pertaining to the current period month for prices, month t, and q^b is a reference basket quantity vector that refers to the base year b, which is equal to or prior to month 0.[36] Note that this Lowe index $P_{Lo}(p^0,p^t,q^b)$ is *not* a true Laspeyres index (because the annual quantity vector q^b is not equal to the monthly quantity vector q^0 in general).[37]

The question is: "Why do statistical agencies *not* pick the reference quantity vector q in the Lowe formula to be the monthly quantity vector q^0 that pertains to transactions in month 0 (so that the index would reduce to an ordinary Laspeyres price index)?" There are two main reasons why this is not done:

- Most economies are subject to seasonal fluctuations, and so picking the quantity vector of month 0 as the reference quantity vector for all months of the year would not be representative of transactions made throughout the year.
- Monthly household quantity or expenditure weights are usually collected by the statistical agency using a household expenditure survey with a relatively small sample. In practice, it is prohibitively expensive for NSOs to draw samples large enough to support the derivation of monthly quantity or expenditure weights. Due to these

[34] See Section 7 in Diewert (2001).

[35] Diewert (1978; 887–889) showed that these two indices will approximate each other to the second order around an equal price and quantity point. Thus, for normal time series data where prices and quantities do not change much going from the base period to the current period, the indices will approximate each other quite closely. However, if scanner data from retail outlets or from individual households are used at the first stage of aggregation, and the price and quantity data are very volatile, then second-order approximations may not be very accurate and the Walsh and Fisher indices may differ substantially. As will be seen in Chapter 3, the Fisher index may be preferred over the Walsh index because of its better axiomatic properties.

[36] Month 0 is called the price reference period and year b is called the weight reference period.

[37] Triplett (1981; 12) defined the Lowe index, calling it a Laspeyres index, and calling the index that has the weight reference period equal to the price reference period, a pure Laspeyres index. However, Balk (1980; 69) asserted that although the Lowe index is of the fixed-base type, it is not a Laspeyres price index. Triplett also noted the hybrid share representation for the Lowe index defined by (15) and (16). Triplett noted that the ratio of two Lowe indices using the same quantity weights was also a Lowe index. Baldwin (1990; 255) called the Lowe index an *annual basket index*.

budgetary constraints, it is standard practice to average these monthly expenditure or quantity weights over an entire year (or in some cases, over several years) in an attempt to reduce these sampling errors.

- Monthly household quantity or expenditure weights for month 0 are generally not available in month 1.

The index number problems that are caused by seasonal monthly weights will be studied in more detail in Chapter 9. For now, it can be argued that the use of annual weights in a monthly index number formula is simply a method for dealing with poor estimates of monthly quantities or for dealing with the seasonality problem.[38] However, it should be noted that the use of annual weights in a monthly CPI is not consistent with the economic approach to index number theory.[39]

One problem with using annual weights corresponding to a perhaps distant year in the context of a monthly CPI must be noted at this point: If there are systematic (but divergent) trends in commodity prices and households increase their purchases of commodities that decline (relatively) in price and decrease their purchases of commodities that increase (relatively) in price, then the use of distant quantity weights will tend to lead to an upward bias in this Lowe index compared to one that used more current weights. This observation suggests that statistical agencies should strive to get up-to-date weights on an ongoing basis.

It is useful to explain how the annual quantity vector q^b could be obtained from monthly expenditures on each commodity during the chosen base year b. Let the month m expenditure of the reference population in the base year b for commodity i be $v_i^{b,m}$, and let the corresponding price and quantity be $p_i^{b,m}$ and $q_i^{b,m}$, respectively. Of course, value, price, and quantity for each commodity are related by the following equations:

$$v_i^{b,m} = p_i^{b,m} q_i^{b,m};\ i = 1,\ldots,N;\ m = 1,\ldots,12. \qquad (22)$$

For each commodity i, an estimate for the annual total quantity, q_i^b, can be obtained by price deflating monthly values and summing over months in the base year b as follows:

$$q_i^b \equiv \Sigma_{m=1}^{12} v_i^{b,m}/p_i^{b,m} = \Sigma_{m=1}^{12} q_i^{b,m};\ i = 1,\ldots,N, \qquad (23)$$

where (22) was used to derive the second equation in (23). In practice, these equations will be evaluated using aggregate expenditures over closely related commodities, and the price $p_i^{b,m}$ will be the month m price index for this elementary commodity group i in year b relative to the first month of year b.

For some purposes, it is also useful to have annual prices by commodity to match up with the annual quantities defined by (23). Following national income accounting conventions, a reasonable[40] price p_i^b to match up with the annual quantity q_i^b is the value of total consumption of commodity i in year b divided by q_i^b. Thus, we have

$$\begin{aligned} p_i^b &\equiv \Sigma_{m=1}^{12} v_i^{b,m}/\Sigma_{m=1}^{12} q_i^{b,m}\ i = 1,\ldots,N \\ &= \Sigma_{m=1}^{12} v_i^{b,m}/\Sigma_{m=1}^{12} [v_i^{b,m}/p_i^{b,m}] \text{ using (22)} \\ &= \Sigma_{m=1}^{12} [s_i^{b,m}(p_i^{b,m})^{-1}]^{-1}, \end{aligned} \qquad (24)$$

where the share of annual expenditure on commodity i in month m of the base year b is

$$s_i^{b,m} \equiv v_i^{b,m}/\Sigma_{k=1}^{12} v_i^{b,k};\ i = 1,\ldots,N;\ m = 1,\ldots,12. \qquad (25)$$

Thus, the annual base year price for commodity i, p_i^b, turns out to be a monthly expenditure-weighted *harmonic mean* of the monthly prices for commodity i in the base year, $p_i^{b,1}$, $p_i^{b,2},\ldots, p_i^{b,12}$.

Using the annual commodity prices for the base year defined by (24), a vector of these prices can be defined as $p^b \equiv [p_1^b,\ldots,p_N^b]$. Using this definition, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ can be expressed as a ratio of two Laspeyres indices, where the price vector p^b plays the role of base period prices in each of the two Laspeyres indices:

$$\begin{aligned} P_{Lo}(p^0,p^t,q^b) &\equiv \Sigma_{i=1}^N p_i^t q_i^b/\Sigma_{i=1}^N p_i^0 q_i^b \\ &= [\Sigma_{i=1}^N p_i^t q_i^b/\Sigma_{i=1}^N p_i^b q_i^b]/[\Sigma_{i=1}^N p_i^0 q_i^b/\Sigma_{i=1}^N p_i^b q_i^b] \\ &= \Sigma_{i=1}^N s_i^b\,(p_i^t/p_i^b)/\Sigma_{i=1}^N s_i^b\,(p_i^0/p_i^b) \\ &= P_L(p^b,p^t,q^b)/P_L(p^b,p^0,q^b), \end{aligned} \qquad (26)$$

where the *base year expenditure shares* are defined as $s_i^b \equiv p_i^b q_i^b/\Sigma_{n=1}^N p_n^b q_n^b$ and the Laspeyres formula P_L was defined by (5). Thus, the preceding equation shows that the Lowe monthly price index comparing the prices of month 0 to those of month t using the quantities of base year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the Laspeyres index that compares the prices of month t to those of year b, $P_L(p^b,p^t,q^b)$, divided by the Laspeyres index that compares the prices of month 0 to those of year b, $P_L(p^b,p^0,q^b)$. Note that the Laspeyres index in the numerator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity i in month t, p_i^t, with the corresponding annual average prices in the base year b, p_i^b. The Laspeyres index in the denominator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity i in month 0, p_i^0, with the corresponding annual average prices in the base year b, p_i^b.

There is another convenient formula for evaluating the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, and that is to use the hybrid weights

[38] In fact, the use of the Lowe index $P_{Lo}(p^0,p^t,q^b)$ in the context of seasonal commodities corresponds to Bean and Stine's (1924; 31) Type A index number formula. Bean and Stine made three additional suggestions for price indices in the context of seasonal commodities. Their contributions will be evaluated in Chapter 9.

[39] Thus, if one takes the economic approach to index number theory, then the use of annual weights will lead to a certain amount of substitution bias; see Chapter 7 for details.

[40] Hence these annual commodity prices are essentially unit value prices. Under conditions of high inflation, the annual prices defined by (24) may no longer be "reasonable" or representative of prices during the entire base year because the expenditures in the final months of the high inflation year will be somewhat artificially blown up by general inflation. Under these conditions, the annual prices and annual commodity expenditure shares should be interpreted with caution. For more on dealing with situations where there is high inflation within a year, see Hill (1996).

formula (15). In the present context (assuming that all prices in the base period are positive), the formula becomes

$$P_{Lo}(p^0,p^t,q^b) \equiv \Sigma_{i=1}^N p_i^t q_i^b / \Sigma_{n=1}^N p_n^0 q_n^b$$
$$= \Sigma_{i=1}^N (p_i^t/p_i^0) p_i^0 q_i^b / \Sigma_{n=1}^N p_n^0 q_n^b = \Sigma_{i=1}^N (p_i^t/p_i^0) s_i^{0b}, \qquad (27)$$

where the hybrid weights s_i^{0b} using the prices of month 0 and the quantities of year b are defined by

$$s_i^{0b} \equiv p_i^0 q_i^b / \Sigma_{n=1}^N p_n^0 q_n^b = (p_i^0/p_i^b) p_i^b q_i^b / \Sigma_{n=1}^N (p_n^0/p_n^b) p_n^b q_n^b;\ i = 1,\ldots,N. \qquad (28)$$

The second equation in (28) shows how the base year expenditures, $p_i^b q_i^b$, can be multiplied by the commodity price indices, p_i^0/p_i^b, in order to calculate the hybrid shares.

There is one additional formula for the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, that will be exhibited. Note that the Laspeyres decomposition of the Lowe index defined by the third line in (26) involves the very long-term price relatives, p_i^t/p_i^b, which compare the prices in month t, p_i^t, with the possibly distant base year prices, p_i^b, and that the hybrid share decomposition of the Lowe index defined by the last equality in (27) involves the long-term monthly price relatives, p_i^t/p_i^0, which compare the prices in month t, p_i^t, with the base month prices, p_i^0. Both of these formulae are not satisfactory in practice because of the problem of sample attrition: Each month, a substantial fraction of commodities disappears from the marketplace, and thus it is useful to have a formula for updating the previous month's price index using just month-over-month price relatives. In other words, long-term price relatives disappear at a rate that is too large in practice to base an index number formula on their use. The Lowe index for month $t + 1$, $P_{Lo}(p^0,p^{t+1},q^b)$, can be written in terms of the Lowe index for the prior month t, $P_{Lo}(p^0,p^t,q^b)$, and an *updating factor* as follows:

$$P_{Lo}(p^0,p^{t+1},q^b) \equiv \Sigma_{i=1}^N p_i^{t+1} q_i^b / \Sigma_{n=1}^N p_n^0 q_n^b$$
$$= [\Sigma_{i=1}^N p_i^t q_i^b / \Sigma_{n=1}^N p_n^0 q_n^b][\Sigma_{i=1}^N p_i^{t+1} q_i^b / \Sigma_{n=1}^N p_n^t q_n^b]$$
$$= P_{Lo}(p^0,p^t,q^b)\,[\Sigma_{i=1}^N p_i^{t+1} q_i^b / \Sigma_{n=1}^N p_n^t q_n^b]$$
$$= P_{Lo}(p^0,p^t,q^b)[\Sigma_{i=1}^N (p_i^{t+1}/p_i^t) p_i^t q_i^b / \Sigma_{n=1}^N p_n^t q_n^b]\ \textit{if all}\ p_i^t > 0$$
$$= P_{Lo}(p^0,p^t,q^b)\Sigma_{i=1}^N (p_i^{t+1}/p_i^t) s_i^{tb}, \qquad (29)$$

where the hybrid weights s_i^{tb} are defined by

$$s_i^{tb} \equiv p_i^t q_i^b / \Sigma_{n=1}^N p_n^t q_n^b;\ i = 1,\ldots,N. \qquad (30)$$

Thus, the required updating factor, going from month t to month $t + 1$, is the chain link index $\Sigma_{i=1}^N s_i^{tb}(p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{tb} corresponding to month t and base year b.[41]

The Lowe index $P_{Lo}(p^0,p^t,q^b)$ can be regarded as an approximation to the ordinary Laspeyres index, $P_L(p^0,p^t,q^0)$, that compares the prices of the base month 0, p^0, to those of month t, p^t, using the quantity vector of month 0, q^0, as weights. It turns out that there is a relatively simple formula that relates these two indices.[42] In order to explain this formula, it is first necessary to make a few definitions. Define the *nth price relative* between month 0 and month t as

$$r_n \equiv p_n^t/p_n^0;\ n = 1,\ldots,N. \qquad (31)$$

The ordinary Laspeyres price index, going from month 0 to t, can be defined in terms of these price relatives as follows:

$$P_L(p^0,p^t,q^0) \equiv \Sigma_{n=1}^N p_n^t q_n^0 / \Sigma_{i=1}^N p_i^0 q_i^0$$
$$= \Sigma_{n=1}^N (p_n^t/p_n^0) p_n^0 q_n^0 / \Sigma_{i=1}^N p_i^0 q_i^0 = \Sigma_{n=1}^N s_n^0 r_n \equiv r^*, \qquad (32)$$

using definitions (7) and (31) in order to derive the penultimate equality.

Define the nth quantity relative to t_n as the ratio of the quantity of commodity n used in the base year b, q_n^b, to the quantity used in month 0, q_n^0, as follows:

$$t_n \equiv q_n^b/q_n^0;\ n = 1,\ldots,N. \qquad (33)$$

The Laspeyres quantity index, $Q_L(q^0,q^b,p^0)$, that compares quantities in year b, q^b, to the corresponding quantities in month 0, q^0, using the prices of month 0, p^0, as weights can be defined as a weighted average t^* of the quantity ratios t_n as follows:

$$Q_L(q^0,q^b,p^0) \equiv \Sigma_{n=1}^N p_n^0 q_n^b / \Sigma_{i=1}^N p_i^0 q_i^0$$
$$= \Sigma_{n=1}^N p_n^0 q_n^0 (q_n^b/q_n^0) / \Sigma_{i=1}^N p_i^0 q_i^0$$
$$= \Sigma_{n=1}^N s_n^0 t_n \text{ using (7) and (33)}$$
$$\equiv t^*. \qquad (34)$$

The relationship between the Lowe index $P_{Lo}(p^0,p^t,q^b)$ that uses the quantities of year b as weights to compare the prices of month t to month 0 and the corresponding ordinary Laspeyres index $P_L(p^0,p^t,q^0)$ that uses the quantities of month 0 as weights is the following one:[43]

$$P_{Lo}(p^0,p^t,q^b) \equiv \Sigma_{n=1}^N p_n^t q_n^b / \Sigma_{n=1}^N p_n^0 q_n^b$$
$$= P_L(p^0,p^t,q^0) + \Sigma_{n=1}^N (r_n - r^*)(t_n - t^*) s_n^0 / Q_L(q^0,q^b,p^0). \qquad (35)$$

Thus, the Lowe price index using the quantities of year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\Sigma_{n=1}^N (r_n - r^*)(t_n - t^*) s_n^0$ between the price relatives $r_n \equiv p_n^t/p_n^0$ and the quantity relatives $t_n \equiv q_n^b/q_n^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^b,p^0)$ between month 0 and base year b.

Formula (35) shows that the Lowe price index will coincide with the Laspeyres price index if the covariance or correlation between the month 0 to t price relatives $r_n \equiv p_n^t/p_n^0$ and the month 0 to year b quantity relatives $t_n \equiv q_n^b/q_n^0$ is zero. Note that this covariance will be zero under three different sets of conditions:

[41] If one or more of p_i^{tb} are equal to 0, then define the link factor by $\Sigma_{i=1}^N p_i^{t+1} q_i^b / \Sigma_{n=1}^N p_n^t q_n^b$.

[42] In what follows, it is assumed that all prices and quantities in month 0 are positive.

[43] See Annex 2 for the derivation of this formula.

- If the month t prices are proportional to the month 0 prices so that all $r_n = r^*$
- If the base year b quantities are proportional to the month 0 quantities so that all $t_n = t^*$
- If the distribution of the relative prices r_n is independent of the distribution of the relative quantities t_n

The first two conditions are unlikely to hold empirically, but the third is possible, at least approximately, if consumers do not systematically change their purchasing habits in response to changes in relative prices.

If this covariance in (35) is negative, then the Lowe index will be less than the Laspeyres, and finally, if the covariance is positive, then the Lowe index will be greater than the Laspeyres index. Although the sign and magnitude of the covariance term, $\Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)$, is ultimately an empirical matter, it is possible to make some reasonable conjectures about its likely sign. If the base year b precedes the price reference month 0 *and there are long-term trends in prices*, then it is likely that this covariance is *positive*, and hence this implies that the Lowe index will exceed the corresponding Laspeyres price index;[44] that is,

$$P_{Lo}(p^0,p^t,q^b) > P_L(p^0,p^t,q^0). \qquad (36)$$

To see why this covariance is likely to be positive, suppose that there is a long-term upward trend in the price of commodity n so that $r_n - r^* \equiv (p_n^t/p_n^0) - r^*$ is positive. With normal consumer substitution responses,[45] q_n^t/q_n^0 less an average quantity change of this type is likely to be negative, or, upon taking reciprocals, q_n^0/q_n^t less an average quantity change of this (reciprocal) type is likely to be positive. But if the long-term upward trend in prices has persisted back to the base year b, then $t_n - t^* \equiv (q_n^b/q_n^0) - t^*$ is also likely to be positive. Hence, the covariance will be positive under these circumstances. Moreover, the more distant the base year b is from the base month 0, the bigger the residuals $t_n - t^*$ will likely be and the bigger will be the positive covariance. Similarly, the more distant the current period month t is from the base period month 0, the bigger the residuals $r_n - r^*$ will likely be and the bigger will be the positive covariance. *Thus, under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Lowe index will normally be greater than the corresponding Laspeyres index.*[46]

The Paasche index between months 0 and t is defined as follows:

$$P_P(p^0,p^t,q^t) \equiv \Sigma_{n=1}^{N} p_n^t q_n^t / \Sigma_{i=1}^{N} p_i^0 q_i^t . \qquad (37)$$

As was discussed in Section 4, a reasonable target index to measure the price change going from month 0 to t is some sort of symmetric average of the Paasche index $P_P(p^0,p^t,q^t)$ defined by (37) and the corresponding Laspeyres index, $P_L(p^0,p^t,q^0)$ defined by (32). Using the results in Annex 1, the relationship between the Paasche and Laspeyres indices can be written as follows:

$$P_P(p^0,p^t,q^t) \equiv \Sigma_{n=1}^{N} p_n^t q_n^t / \Sigma_{n=1}^{N} p_n^0 q_n^t = P_L(p^0,p^t,q^0) + \Sigma_{n=1}^{N} (r_n - r^*)(u_n - u^*) s_n^0 / Q_L(q^0,q^t,p^0), \qquad (38)$$

where the price relatives $r_n \equiv p_n^t/p_n^0$ are defined by (31) and their share-weighted average r^* by (32) and the u_n, u^* and Q_L are defined as follows:

$$u_n \equiv q_n^t/q_n^0;\ n = 1,,,.N, \qquad (39)$$

$$u^* \equiv \Sigma_{n=1}^{N} s_n^0 u_n \equiv Q_L(q^0,q^t,p^0), \qquad (40)$$

and the month 0 expenditure shares s_i^0 are defined by (7). Thus, u^* is equal to the Laspeyres quantity index between months 0 and t. This means that the Paasche price index that uses the quantities of month t as weights, $P_P(p^0,p^t,q^t)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a weighted covariance term $\Sigma_{n=1}^{N} (r_n - r^*)(u_n - u^*) s_n^0$ between the price relatives $r_n \equiv p_n^t/p_n^0$ and the corresponding quantity relatives $u_n \equiv q_n^t/q_n^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^t,p^0)$ between month 0 and month t.

Although the sign and magnitude of the covariance term, $\Sigma_{n=1}^{N} (r_n - r^*)(u_n - u^*) s_n^0$, is again an empirical matter, it is possible to make a reasonable conjecture about its likely sign. If *there are long-term trends in prices and consumers respond normally to price changes in their purchases*, then it is likely that that this covariance is *negative* and hence the Paasche index will be less than the corresponding Laspeyres price index; that is,

$$P_P(p^0,p^t,q^t) < P_L(p^0,p^t,q^0). \qquad (41)$$

To see why this covariance is likely to be negative, suppose that there is a long-term upward trend in the price of commodity n[47] so that $r_n - r^* \equiv (p_n^t/p_n^0) - r^*$ is positive. With normal consumer substitution responses, q_n^t/q_n^0 less an average quantity change of this type is likely to be negative. Hence, $u_n - u^* \equiv (q_n^t/q_n^0) - u^*$ is likely to be negative. Thus, the covariance will be negative under these circumstances.

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[44] It is also necessary to assume that households have normal substitution effects in response to these long-term trends in prices; that is, if a commodity increases (relatively) in price, its consumption will decline (relatively) and if a commodity decreases relatively in price, its consumption will increase relatively.

[45] Walsh (1901; 281–282) was well aware of consumer substitution effects as can be seen in the following comment, which noted the basic problem with a fixed basket index that uses the quantity weights of a single period: "The argument made by the arithmetic averagist supposes that we buy the same quantities of every class at both periods in spite of the variation in their prices, which we rarely, if ever, do. As a rough proposition, we—a community—generally spend more on articles that have risen in price and get less of them, and spend less on articles that have fallen in price and get more of them."

[46] If expression (26) is substituted into the left-hand side of (36), the resulting inequality becomes $P_L(p^b,p^t,q^b) > P_L(p^b,p^0,q^b)P_L(p^0,p^t,q^0)$. Thus the Laspeyres index from b to t is bigger than the Laspeyres index from b to 0 multiplied by the Laspeyres index from 0 to t. This is not surprising since the Laspeyres index from b to t continues using period b weights until period t. On the right side of the inequality, period b weights are only used until period 0 when period 0 weights are introduced, which will reflect any substitution households that may have made from b to 0. This point was noted by Carsten Boldsen.

[47] The reader can carry through the argument if there is a long-term relative decline in the price of the ith commodity. The argument required to obtain a negative covariance requires that there be some differences in the long-term trends in prices; that is, if all prices grow (or fall) at the same rate, we have price proportionality and the covariance will be zero.

Moreover, the more distant is the base month 0 from the current month t, the bigger in magnitude the residuals $u_n - u^*$ will likely be and the bigger in magnitude will be the negative covariance.[48] Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_n - r^*$ will likely be and the bigger in magnitude will be the covariance. *Thus, under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Laspeyres index will be greater than the corresponding Paasche index*, with the divergence likely growing as month t becomes more distant from month 0.

Putting the arguments in the previous paragraphs together, it can be seen that under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Lowe price index between months 0 and t will exceed the corresponding Laspeyres price index, which in turn will exceed the corresponding Paasche price index; that is, under these hypotheses,

$$P_{Lo}(p^0,p^t,q^b) > P_L(p^0,p^t,q^0) > P_P(p^0,p^t,q^t). \qquad (42)$$

Thus, if the long-run target price index is an average of the Laspeyres and Paasche indices, it can be seen that the Laspeyres index will have an *upward bias* relative to this target index and the Paasche index will have a *downward bias.* In addition, *if the base year b is prior to the price reference month, month 0, then the Lowe index will also have an upward bias relative to the Laspeyres index and hence also to the target index*.

7. The Young Index

Recall the definitions for the base year quantities, q_n^b, and the base year prices, p_n^b, given by (23) and (24). The *base year expenditure shares* can be defined in the usual way as follows:

$$s_n^b \equiv p_n^b q_n^b / \Sigma_{i=1}^N p_i^b q_i^b;\ n = 1,\ldots,N. \qquad (43)$$

Define the vector of base year expenditure shares in the usual way as $s^b \equiv [s_1^b,\ldots,s_N^b]$. These base year expenditure shares were used to provide an alternative formula for the base year b Lowe price index going from month 0 to t defined in (26) as $P_{Lo}(p^0,p^t,q^b) = [\Sigma_{i=1}^N s_i^b(p_i^t/p_i^b)]/[\Sigma_{i=1}^N s_i^b(p_i^0/p_i^b)]$. Rather than using this index as their target index, some statistical agencies use the following related index, which also uses base year expenditure shares as weights:[49]

$$P_Y(p^0,p^t,s^b) \equiv \Sigma_{i=1}^N s_i^b(p_i^t/p_i^0). \qquad (44)$$

This type of index was first defined by the English economist Arthur Young (1812).[50] Note that there is a change in focus when the Young index is used compared to the other indices proposed earlier in this chapter. Up to this point, the indices proposed have been of the fixed basket type (or averages of such indices) where a *commodity basket* that is somehow representative of the two periods being compared is chosen and then "purchased" at the prices of the two periods and the index is taken to be the ratio of these two costs. On the other hand, for the Young index, one instead chooses *representative expenditure shares* that pertain to the two periods under consideration and then uses these shares to calculate the overall index as a share-weighted average of the individual price ratios, p_i^t/p_i^0. Note that this share-weighted average of price ratios view of index number theory is a bit different from the view taken at the beginning of this chapter, which viewed the index number problem as the problem of decomposing a value ratio into the product of two terms, one of which expresses the amount of price change between the two periods and the other which expresses the amount of quantity change.[51] However, the two approaches are not necessarily inconsistent; the weighted average of price ratios approach to index number theory generates a price index, and the companion quantity index can always be generated using the product test (see equation (4)).

Statistical agencies sometimes regard the Young index defined earlier as an approximation to the Laspeyres price index $P_L(p^0,p^t,q^0)$. Hence, it is of interest to see how the two indices compare. Defining the long-term monthly price relatives going from month 0 to t as $r_i \equiv p_i^t/p_i^0$, and using definitions (32) and (44) leads to the following formula:

$$\begin{aligned} P_Y(p^0,p^t,s^b) - P_L(p^0,p^t,q^0) &= \Sigma_{i=1}^N s_i^b(p_i^t/p_i^0) - \Sigma_{i=1}^N s_i^0(p_i^t/p_i^0) \\ &= \Sigma_{i=1}^N [s_i^b - s_i^0] r_i \\ &= \Sigma_{i=1}^N [s_i^b - s_i^0][r_i - r^*], \end{aligned} \qquad (45)$$

where $r_i \equiv p_i^t/p_i^0$ for $i = 1,\ldots,N$ and $r^* \equiv \Sigma_{i=1}^N s_i^0(p_i^t/p_i^0)$. The last equality follows from the line above since $\Sigma_{i=1}^N [s_i^b - s_i^0] r^* = [1 - 1] r^* = 0$. Thus, the Young index $P_Y(p^0,p^t,s^b)$ is equal to the Laspeyres index $P_L(p^0,p^t,q^0)$ plus the *covariance* between the difference in the annual shares pertaining to year b and the month 0 shares, $s_i^b - s_i^0$, and the deviations of the relative prices from their mean, $r_i - r^*$.

It is no longer possible to guess at what the likely sign of the covariance term is. The question is no longer whether the *quantity* demanded goes down as the price of commodity i goes up (the answer to this question is usually yes)

[48] However, $Q_L = u^*$ may also be growing in magnitude so the net effect on the divergence between P_L and P_P is ambiguous.

[49] We require all prices in the base period to be positive in order for the Young index to be well defined.

[50] The attribution of this formula to Young was given by Walsh (1901; 536) (1932; 657).

[51] Fisher's 1922 book is famous for developing the value ratio decomposition approach to index number theory, but his introductory chapters took the share-weighted average point of view: "An index number of prices, then shows the *average percentage change* of prices from one point of time to another" (Irving Fisher (1922; 3)). Fisher went on to note the importance of economic weighting: "The preceding calculation treats all the commodities as equally important; consequently, the average was called 'simple.' If one commodity is more important than another, we may treat the more important as though it were two or three commodities, thus giving it two or three times as much 'weight' as the other commodity" (Irving Fisher (1922; 6)). Walsh (1901; 430–431) considered both approaches: "We can either (1) draw some average of the total money values of the classes during an epoch of years, and with weighting so determined employ the geometric average of the price variations [ratios]; or (2) draw some average of the mass quantities of the classes during the epoch, and apply to them Scrope's method." Scrope's method is the same as using the Lowe index. Walsh (1901; 88–90) consistently stressed the importance of weighting price ratios by their economic importance (rather than using equally weighted averages of price relatives).

but the new question is: does the *share* of expenditure go down as the price of commodity i goes up? The answer to this question depends on the elasticity of demand for the product.

However, let us provisionally assume both that there are long-run trends in commodity prices and that if the trend in prices for commodity i is above the mean, then the expenditure share for the commodity trends *down* (and vice versa). Thus, we are assuming high elasticities or very strong substitution effects. Assuming also that the base year b is prior to month 0, then under these conditions, suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very elastic consumer substitution responses, s_i will tend to decrease relatively over time, and since s_i^b is assumed to be prior to s_i^0, s_i^0 is expected to be less than s_i^b or $s_i^b - s_i^0$ will likely be positive. Thus, the covariance is likely to be *positive* under these circumstances. *Hence, with long-run trends in prices and very elastic responses of consumers to price changes, the Young index is likely to be greater than the corresponding Laspeyres index.*

Assume that there are long-run trends in commodity prices. If the trend in price for commodity i is above the mean, then suppose that the expenditure share for the commodity trends *up* (and vice versa). Thus, we are assuming low elasticities or very weak substitution effects. Assume also that the base year b is prior to month 0, and suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very inelastic consumer substitution responses, s_i will tend to increase relatively over time, and since s_i^b is assumed to be prior to s_i^0, it will be the case that s_i^0 is greater than s_i^b or $s_i^b - s_i^0$ is negative. Thus, the covariance is likely to be *negative* under these circumstances. *Hence, with long-run trends in prices and very inelastic responses of consumers to price changes, the Young index is likely to be less than the corresponding Laspeyres index.*

The previous two paragraphs indicate that, a priori, it is not known what the likely difference between the Young index and the corresponding Laspeyres index will be. If elasticities of substitution are close to one, then the two sets of expenditure shares, s_i^b and s_i^0, will be close to each other and the difference between the two indices will be close to 0. However, if monthly expenditure shares have strong seasonal components (or if there are missing products for some months for whatever reason), then the annual shares s_i^b could differ substantially from the monthly shares s_i^0.

It is useful to have a formula for updating the previous month's Young price index using just month-over-month price relatives. The Young index for month $t + 1$, $P_Y(p^0,p^{t+1},s^b)$, can be written in terms of the Young index for month t, $P_Y(p^0,p^t,s^b)$, and an updating factor as follows:

$$\begin{aligned} P_Y(p^0,p^{t+1},s^b) &\equiv \Sigma_{i=1}^N s_i^b(p_i^{t+1}/p_i^0) \\ &= P_Y(p^0,p^t,s^b)[\Sigma_{i=1}^N s_i^b(p_i^{t+1}/p_i^0)/\Sigma_{n=1}^N s_n^b(p_n^t/p_n^0)] \\ &= P_Y(p^0,p^t,s^b)[\Sigma_{i=1}^N p_i^b q_i^b(p_i^t/p_i^0)(p_i^{t+1}/p_i^t)/\Sigma_{n=1}^N p_n^b q_n^b(p_n^t/p_n^0)] \\ &= P_Y(p^0,p^t,s^b)[\Sigma_{i=1}^N s_i^{b0t}(p_i^{t+1}/p_i^t)], \end{aligned} \quad (46)$$

where the hybrid weights s_i^{b0t} are defined as follows:

$$\begin{aligned} s_i^{b0t} &\equiv p_i^b q_i^b(p_i^t/p_i^0)/\Sigma_{n=1}^N p_n^b q_n^b(p_n^t/p_n^0) \\ &= s_i^b(p_i^t/p_i^0)/\Sigma_{n=1}^N s_n^b(p_n^t/p_n^0);\ i = 1,\ldots,N. \end{aligned} \quad (47)$$

Thus, the hybrid weights s_i^{b0t} can be obtained from the base year weights s_i^b by updating them; that is, by multiplying them by the price relatives (or indices at higher levels of aggregation), p_i^t/p_i^0. Thus, the required updating factor, going from month t to month $t + 1$, is the chain link index, $\Sigma_{i=1}^N s_i^{b0t}(p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{b0t} defined by (47). Note that we require the period t prices, p_i^t, to be positive in order to ensure that the link factor is well defined.

Even if the Young index provides a close approximation to the corresponding Laspeyres index, it is difficult to recommend the use of the Young index as a final estimate of the change in prices going from period 0 to t, just as it was difficult to recommend the use of the Laspeyres index as the *final* estimate of inflation going from period 0 to t. Recall that the problem with the Laspeyres index was its lack of symmetry in the treatment of the two periods under consideration; that is, using the justification for the Laspeyres index as a good fixed basket index, there was an identical justification for the use of the Paasche index as an equally good fixed basket index to compare prices in periods 0 and t. The Young index suffers from a similar lack of symmetry with respect to the treatment of the base period. The problem can be explained as follows. The Young index, $P_Y(p^0,p^t,s^b)$, defined by (44) calculates the price change between months 0 and t, treating month 0 as the base. But there is no particular reason to necessarily treat month 0 as the base month other than convention. Hence, if we treat month t as the base and use the same formula to measure the price change from month t back to month 0, the index $P_Y(p^t,p^0,s^b) = \Sigma_{i=1}^N s_i^b(p_i^0/p_i^t)$ would be appropriate. This estimate of price change can then be made comparable to the original Young index by taking its reciprocal, leading to the following *rebased Young index,*[52] $P_Y^*(p^0,p^t,s^b)$, defined as

$$P_Y^*(p^0,p^t,s^b) \equiv 1/\Sigma_{i=1}^N s_i^b(p_i^0/p_i^t) = [\Sigma_{i=1}^N s_i^b(p_i^t/p_i^0)^{-1}]^{-1}. \quad (48)$$

Thus, the rebased Young index, $P_Y^*(p^0,p^t,s^b)$, that uses the current month as the base period is a *share-weighted harmonic mean* of the price relatives going from month 0 to month t, whereas the original Young index, $P_Y(p^0,p^t,s^b)$, is a *share-weighted arithmetic mean* of the same price relatives.

Fisher argued that an index number formula should give the same answer no matter which period is chosen as the base:

> Either one of the two times may be taken as the "base." Will it make a difference which is chosen? Certainly, it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one

[52] Using Fisher's (1922; 118) terminology, $P_Y^*(p^0,p^t,s^b) \equiv 1/[P_Y(p^t,p^0,s^b)]$ is the *time antithesis* of the original Young index, $P_Y(p^0,p^t,s^b)$.

point of comparison and the other point, *no matter which of the two is taken as the base*.

Irving Fisher (1922; 64)

The problem with the Young index is that not only does it not coincide with its rebased counterpart, but also there is a definite inequality between the two indices, namely

$$P_Y^*(p^0,p^t,s^b) \le P_Y(p^0,p^t,s^b) \quad (49)$$

with a strict inequality provided that the period t price vector p^t is not proportional to the period 0 price vector p^0.[53] Thus, a statistical agency that uses the direct Young index $P_Y(p^0,p^t,s^b)$ will generally show a higher inflation rate than a statistical agency that uses the same raw data but uses the rebased Young index, $P_Y^*(p^0,p^t,s^b)$.

The inequality (49) does not tell us by how much the Young index will exceed its rebased time antithesis. However, in Annex 3, it is shown that to the accuracy of a certain second-order Taylor series approximation, the following relationship holds between the direct Young index and its time antithesis:

$$P_Y(p^0,p^t,s^b) = P_Y^*(p^0,p^t,s^b) + P_Y(p^0,p^t,s^b)var(e), \quad (50)$$

where $var(e)$ is defined as

$$var(e) \equiv \Sigma_{n=1}^N s_n^b[e_n - e^*]^2. \quad (51)$$

The deviations e_n are defined by $1 + e_n = r_n/r^*$ for $n = 1,\ldots,N$, where r_n and their weighted mean r^* are defined as follows:

$$r_n \equiv p_n^t/p_n^0;\ n = 1,\ldots,N; \quad (52)$$

$$r^* \equiv \Sigma_{n=1}^N s_n^b r_n = P_Y(p^0,p^t,s^b). \quad (53)$$

The weighted mean of the e_n is defined as e^*:

$$e^* \equiv \Sigma_{n=1}^N s_n^b e_n, \quad (54)$$

which turns out to 0. Hence the more dispersion there is in the price relatives p_n^t/p_n^0, to the accuracy of a second-order approximation, the more the direct Young index will exceed its counterpart that uses month t as the initial base period rather than month 0.

Given two a priori equally plausible index number formulae that give different answers, such as the Young index and its time antithesis, Fisher (1922; 136) generally suggested taking the geometric average of the two indices,[54] and a benefit of this averaging is that the resulting formula will satisfy the time reversal test. Thus, rather than using *either* the base period 0 Young index, $P_Y(p^0,p^t,s^b)$, *or* the current period t Young index, $P_Y^*(p^0,p^t,s^b)$, which is always below the base period 0 Young index, if there is any dispersion in relative prices, it seems preferable to use the following index, which is the *geometric average* of the two alternatively based Young indices:[55]

$$P_Y^{**}(p^0,p^t,s^b) \equiv [P_Y(p^0,p^t,s^b)P_Y^*(p^0,p^t,s^b)]^{1/2}. \quad (55)$$

If the base year shares s_i^b happen to coincide with both the month 0 and month t shares, s_i^0 and s_i^t, respectively, it can be seen that the time-rectified Young index $P_Y^{**}(p^0,p^t,s^b)$ defined by (55) will coincide with the Fisher ideal price index between months 0 and t, $P_F(p^0,p^t,q^0,q^t)$.[56] Note also that the index P_Y^{**} defined by (55) can be produced on a timely basis by a statistical agency since it does not depend on quantity information for months 0 and t. However, this point illustrates the problem with using out-of-date base year shares (or annual quantities) as weights for monthly prices: The base year shares may not be representative for the actual expenditure shares (or quantities) for month 0 and the subsequent months. Thus in general, the use of the Fisher or Walsh indices is recommended over the use of indices that rely on annual baskets of a prior year. However, this recommendation is tempered by the fact that the statistical agency may not be able to obtain information on current period quantities or expenditures in a timely fashion, and thus it may be necessary to use indices that do not depend on the availability of current information on expenditures or quantities.

[53] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901;517) or Fisher (1922; 383–384). This inequality is a special case of Schlömilch's (1858) Inequality; see Hardy, Littlewood, and Polyá (1934; 26). Walsh (1901; 330–332) explicitly noted the inequality (49) and also noted that the corresponding geometric average would fall between the harmonic and arithmetic averages. Walsh (1901; 432) computed some numerical examples of the Young index and found big differences between it and his "best" indices, even using weights that were representative for the periods being compared. Recall that the Lowe index becomes the Walsh index when geometric mean quantity weights are chosen and so the Lowe index can perform well when representative weights are used. This is not necessarily the case for the Young index, even using representative weights. Walsh (1901; 433) summed up his numerical experiments with the Young index as follows: "In fact, Young's method, in every form, has been found to be bad."

[54] "We now come to a third use of these tests, namely, to 'rectify' formulae, i.e., to derive from any given formula which does not satisfy a test another formula which does satisfy it; . . . . This is easily done by 'crossing,' that is, by averaging antitheses. If a given formula fails to satisfy Test 1 [the time reversal test], its time antithesis will also fail to satisfy it; but the two will fail, as it were, in opposite ways, so that a cross between them (obtained by *geometrical* averaging) will give the golden mean which does satisfy" (Irving Fisher (1922; 136)). Actually, the basic idea behind Fisher's rectification procedure was suggested by Walsh, who was a discussant for Fisher (1921) when Fisher gave a preview of his 1922 book: "We merely have to take any index number, find its antithesis in the way prescribed by Professor Fisher, and then draw the geometric mean between the two" (Correa Moylan Walsh (1921b; 542)).

[55] This index is a base year-weighted counterpart to an equally weighted index proposed by Carruthers, Sellwood, and Ward (1980; 25) and Dalén (1992; 140) in the context of elementary index number formulae. See Chapter 6 for further discussion of this unweighted index.

[56] However, if there are systematic trends in shares, then s_i^b will not coincide with s_i^0 and s_i^t, and it is likely that the rectified Young index will differ from the Fisher index since the base year shares will not in general be representative for the shares for months 0 and t.

8. Fixed-Base versus Chained Indices

In this section,[57] the merits of using the chain system for constructing price indices in the time series context versus using the fixed-base system are discussed.

The chain system[58] measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one-period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus, if the bilateral price index is P, the chain system generates the following pattern of price levels for the first three periods:[59]

$$1,\ P(p^0,p^1,q^0,q^1),\ P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2). \qquad (56)$$

On the other hand, the fixed-base system of price levels using the same bilateral index number formula P simply computes the level of prices in period t relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus, the fixed-base pattern of price levels for periods 0,1, and 2 is[60]

$$1,\ P(p^0,p^1,q^0,q^1),\ P(p^0,p^2,q^0,q^2). \qquad (57)$$

Note that in both the chain system and the fixed-base system of price levels defined by (56) and (57), the base period price level is set to 1. The usual practice in statistical agencies is to set the base period price level equal to 100. If this is done, then it is necessary to multiply each of the numbers in (56) and (57) by 100.

Because of the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), many statistical agencies loosely base their CPI on the use of the Laspeyres formula (5) and the fixed-base system. Therefore, it is of some interest to look at some of the possible problems associated with the use of fixed-base Laspeyres indices.

The main problem with the use of fixed-base Laspeyres indices is that the period 0 fixed basket of commodities that is being priced in period t can often be quite different from the period t basket. Thus, if there are systematic *trends* in at least some of the prices and quantities[61] in the index basket, the fixed-base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed-base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$.[62] This means that both indices are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

The fixed-base Laspeyres quantity index cannot be used forever: Eventually, the base period quantities q^0 are so far removed from the current period quantities q^t that the base must be changed. Chaining is merely the limiting case where the base is changed for each period.[63]

A main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indices.[64] These two indices each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration, and it could be expected that a single-point estimate of the aggregate price change should lie between these two estimates. Thus, the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth."[65]

P. Hill (1993; 388), drawing on the earlier research of Szulc (1983) and P. Hill (1988; 136–137), noted that it is not appropriate to use the chain system when prices oscillate (or "bounce" to use Szulc's (1983; 548) term). This phenomenon can occur in the context of regular seasonal fluctuations or in the context of price wars or highly discounted sale prices. However, in the context of roughly monotonically changing prices and quantities, P. Hill (1993; 389) recommended the use of chained symmetrically weighted indices. The Fisher and Walsh indices are examples of symmetrically weighted indices.

It is possible to be a bit more precise under what conditions one should chain or not chain. Basically, *one should chain if the prices and quantities pertaining to adjacent periods are more similar than the prices and quantities of more distant period***s,** since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link.[66] Of course, one needs a measure of how similar

[57] This section is largely based on the work of P. Hill (1988) (1993; 385–390).

[58] The chain principle was introduced independently into the economics literature by Lehr (1885; 45–46) and Marshall (1887; 373). Both authors observed that the chain system would mitigate the difficulties due to the introduction of new commodities into the economy, a point also mentioned by P. Hill (1993; 388). Fisher (1911; 203) introduced the term "chain system."

[59] Let the value of transactions in period t be $V^t \equiv \Sigma_{n=1}^{N} p_n^t q_n^t$ for $t = 0,1,2$. Then the period t quantity aggregates that correspond to the price levels defined by (56) are equal to the following expressions: $Q^0 \equiv V^0$; $Q^1 \equiv V^1/P(p^0,p^1,q^0,q^1)$, and $Q^2 \equiv V^2/P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

[60] The period t quantity aggregates that correspond to the price levels defined by (57) are equal to the following expressions: $Q^0 \equiv V^0$; $Q^1 \equiv V^1/P(p^0,p^1,q^0,q^1)$ and $Q^2 \equiv V^2/P(p^0,p^2,q^0,q^2)$.

[61] Examples of rapidly downward trending prices and upward trending quantities are computers, electronic equipment of all types, internet access, and (quality-adjusted) telecommunication charges.

[62] Note that $P_L(p^0,p^t,q^0,q^t)$ will equal $P_P(p^0,p^t,q^0,q^t)$ if *either* the two quantity vectors q^0 and q^t are proportional *or* the two price vectors p^0 and p^t are proportional. Thus, in order to obtain a difference between the Paasche and Laspeyres indices, nonproportional movements in *both* prices and quantities are required.

[63] Regular seasonal fluctuations can cause monthly or quarterly data to "bounce" using the term due to Szulc (1983) and chaining bouncing data can lead to a considerable amount of index "drift"; that is, if after 12 months, prices and quantities return to their levels of a year earlier, then a chained monthly index will usually not return to unity. Hence, the use of chained indices for "noisy" monthly or quarterly data is not recommended. The chain drift problem will be discussed in more detail in Chapter 7.

[64] See Diewert (1978; 895) and P.Hill (1988) (1993; 387–388). Another main advantage of using chained indices is that chaining will in general increase the number of matched prices in situations where there is a considerable amount of product turnover.

[65] However, if the underlying data are very volatile, then chaining may not reduce the spread between the Paasche and Laspeyres indices. In this case, the methods based on multilateral index number theory should be used; see Chapter 7.

[66] Walsh, in discussing whether fixed-base or chained index numbers should be constructed, took for granted that the precision of all reasonable bilateral index number formulae would improve, provided that the two periods or situations being compared were more similar and hence, for this reason, favored the use of chained indices: "The question is really, in which of the two courses [fixed-base or chained index numbers] are we likely to gain greater exactness in the comparisons

the prices and quantities are pertaining to two periods. The similarity measures could be *relative* ones or *absolute* ones. In the case of absolute comparisons, two vectors of the same dimension are similar if they are identical and dissimilar otherwise. In the case of relative comparisons, two vectors are similar if they are proportional and dissimilar if they are nonproportional.[67] Once a similarity measure has been defined, the prices and quantities of each period can be compared to each other using this measure, and a "tree" or path that links all of the observations can be constructed where the most similar observations are compared with each other using a bilateral index number formula.[68] R. Hill (1995) defined the price structures between the two countries to be more dissimilar the bigger the spread between P_L and P_P is—that is, the bigger is max $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that it could be the case that $P_L = P_P$ (so that the Hill measure would register a maximal degree of similarity), but p^0 could be very different from p^t. Thus, there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into R. Hill's (1999a) (1999b) (2001) (2009) spanning tree algorithm for linking observations.

The method of linking observations explained in the previous paragraph based on the similarity of the price and quantity structures of any two observations may not be practical in a statistical agency context since the addition of a new period may lead to a reordering of the previous links. However, as will be seen in Chapter 7, it is possible to come up with a similarity linking method that does not involve changing index values for prior periods.

Some index number theorists have objected to the chain principle on the grounds that it has no counterpart in the spatial context:

> They [chain indices] only apply to intertemporal comparisons, and in contrast to direct indices they are not applicable to cases in which no natural order or sequence exists. Thus the idea of a chain index for example has no counterpart in interregional or international price comparisons, because countries cannot be sequenced in a "logical" or "natural" way (there is no $k + 1$ nor $k–1$country to be compared with country k).
>
> Peter von der Lippe (2001; 12)[69]

This is of course correct, but the approach of Robert Hill does lead to a "natural" set of spatial links. Applying the same approach to the time series context will lead to a set of links between periods that may not be month to month, but it will in many cases justify year-over-year linking of the data pertaining to the same month. This problem will be addressed in Chapters 7 and 9.

It is of some interest to determine if there are index number formulae that give the same answer when either the fixed-base or chain system is used. Comparing the sequence of chain indices defined by (56) to the corresponding fixed-base indices, it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

$$P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2). \qquad (58)$$

If an index number formula P satisfies (58), then P satisfies the *circularity test*.[70]

If it is assumed that the index number formula P satisfies certain properties or tests in addition to the circularity test,[71] then Funke, Hacker, and Voeller (1979) showed that P must have the following functional form due originally to Konüs and Byushgens[72] (1926; 163–166):[73]

$$P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} (p_n^{\,1}/p_n^{\,0})^{\alpha_n}, \qquad (59)$$

actually made? Here the probability seems to incline in favor of the second course; for the conditions are likely to be less diverse between two contiguous periods than between two periods say fifty years apart" (Correa Moylan Walsh (1901; 206)). Walsh (1921a; 84–85) later reiterated his preference for chained index numbers. Fisher also made use of the idea that the chain system would usually make bilateral comparisons between price and quantity data that were more similar and hence the resulting comparisons would be more accurate: "The index numbers for 1909 and 1910 (each calculated in terms of 1867–1877) are compared with each other. But direct comparison between 1909 and 1910 would give a different and more valuable result. To use a common base is like comparing the relative heights of two men by measuring the height of each above the floor, instead of putting them back to back and directly measuring the difference of level between the tops of their heads" (Irving Fisher (1911; 204)). "It seems, therefore, advisable to compare each year with the next, or, in other words, to make each year the base year for the next. Such a procedure has been recommended by Marshall, Edgeworth and Flux. It largely meets the difficulty of non-uniform changes in the Q's, for any inequalities for successive years are relatively small" (Irving Fisher (1911; 423–424)).

[67] Diewert (2009) took an axiomatic approach to defining various indices of absolute and relative dissimilarity. Measures of relative price similarity or dissimilarity will be discussed in Chapter 7.

[68] Fisher (1922; 271–276) hinted at the possibility of using spatial linking; that is, linking countries that are similar in structure. However, the modern literature has grown due to the pioneering efforts of R. Hill (1995) (2009). R. Hill (1995) used the spread between the Paasche and Laspeyres price indices as an indicator of similarity and showed that this criterion gives the same results as a criterion that looks at the spread between the Paasche and Laspeyres quantity indices.

[69] It should be noted that von der Lippe (2001; 56–58) was a vigorous critic of all index number tests based on symmetry in the time series context, although he was willing to accept symmetry in the context of making international comparisons. "But there are good reasons *not* to insist on such criteria in the *intertemporal* case. When no symmetry exists between 0 and t, there is no point in interchanging 0 and t" (Peter von der Lippe (2001; 58)).

[70] The test was named after Fisher (1922; 413) and the concept was originally proposed by Westergaard (1890; 218–219).

[71] The additional tests are as follows: (i) positivity and continuity of $P(p^0,p^1,q^0,q^1)$ for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1; (ii) the identity test; (iii) the commensurability test; (iv) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree one in the components of p^1 and (v) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree zero in the components of q^1. These tests will be explained in Chapter 3.

[72] Konüs and Byushgens showed that the index defined by (59) is exact for Cobb–Douglas (1928) preferences; see also Pollak (1983; 119–120). The concept of an exact index number formula will be explained in Chapter 5.

[73] This result can be derived using results in Eichhorn (1978; 167–168) and Vogt and Barta (1997; 47). A simple proof can be found in Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*. . . ." Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. . . . Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

where the N constants α_n satisfy the following conditions:

$$\Sigma_{n=1}^{N} \alpha_n = 1 \text{ and } \alpha_n > 0 \textit{ for } n = 1,\ldots,N. \qquad (60)$$

Thus, under very weak regularity conditions, the only price index satisfying the circularity test (and the additional tests listed above in footnote 72) is a weighted geometric average of all the individual price ratios, the weights being constant through time.[74]

An interesting special case of the family of indices defined by (59) occurs when the weights α_i are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

$$P_J(p^0,p^1) \equiv \prod_{n=1}^{N} (p_n^1/p_n^0)^{1/N}. \qquad (61)$$

The problem with the indices defined by Konüs and Byushgens and Jevons is that the individual price ratios, p_n^1/p_n^0, have weights (either α_n or $1/N$) that are *independent* of the economic importance of commodity n in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity n consumed or the expenditures on commodity n during the two periods. Hence, these indices are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share or quantity information is available.

These results indicate that it is not useful to ask that the price index P satisfies the circularity test *exactly*. However, it is of some interest to find index number formulae that satisfy the circularity test to some degree of *approximation*, since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed-base systems. Irving Fisher (1922; 284) found that deviations from circularity using his data set and the Fisher ideal price index P_F defined by (12) were quite small. This relatively high degree of correspondence between fixed-base and chain indices has been found to hold for other symmetrically weighted formulae like the Walsh index P_W defined by (19).[75] Thus, in most time series applications of index number theory where the base year in fixed-base indices is changed every five years or so, it will not matter very much whether the statistical agency uses a fixed-base price index or a chain index, provided that a symmetrically weighted formula is used.[76] This of course depends on the length of the time series considered and the degree of variation in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less will be the correspondence.[77]

[74]This result will be discussed in more detail in Chapter 3.

[75]See, for example, Diewert (1978; 894). Walsh (1901; 424 and 429) found that his three preferred formulae all approximated each other very well as did the Fisher ideal for his artificial data set.

[76]More specifically, most superlative indices (which are symmetrically weighted) will usually satisfy the circularity test to a high degree of approximation in the time series context using aggregated data. See Chapter 5 for the definition of a superlative index. It is worth stressing that fixed-base Paasche and Laspeyres indices are very likely to diverge considerably over a five-year period if computers (or any other commodity that has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration. See Chapters 7 and 11 for some empirical evidence on the divergence between the Laspeyres and Paasche indices.

[77]Again, see Szulc (1983) and P.Hill (1988). This topic will be discussed in more detail in Chapters 7 and 11.

It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist index P_T.[78] The natural logarithm of this index is defined as follows:

$$lnP_T(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} \tfrac{1}{2}(s_n^{\,0} + s_n^{\,1})ln(p_n^1/p_n^0), \qquad (62)$$

where the period t expenditure shares $s_n^{\,t}$ are defined by (7). Alterman, Diewert, and Feenstra (1999; 61) showed that if the logarithmic price ratios $\ln(p_n^t/p_n^{t-1})$ trended linearly with time t and the expenditure shares $s_n^{\,t}$ also trended linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly.[79] Since many economic time series on prices and quantities satisfy these assumptions approximately, then under these conditions, the Törnqvist index P_T will satisfy the circularity test approximately. As will be seen in Chapter 7, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indices, so that for many economic time series (with smooth trends), all three of these symmetrically weighted indices will satisfy the circularity test to a high-enough degree of approximation that it will not matter whether we use the fixed-base or chain principle.[80]

Walsh (1901; 401) (1921a; 98) (1921b; 540) introduced the following useful variant of the circularity test:

$$1 = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)\ldots$$
$$P(p^{T-1},p^T,q^{T-1},q^T)P(p^T,p^0,q^T,q^0). \qquad (63)$$

The motivation for this test is explained as follows. Use the bilateral index formula $P(p^0,p^1,q^0,q^1)$ to calculate the change in prices going from period 0 to 1; use the same formula evaluated at the data corresponding to periods 1 and 2, $P(p^1,p^2,q^1,q^2)$, to calculate the change in prices going from period 1 to 2, . . . ; use $P(p^{T-1},p^T,q^{T-1},q^T)$ to calculate the change in prices going from period T–1 to T; introduce an artificial period $T + 1$ that has exactly the price and quantity of the initial period 0; and use $P(p^T,p^0,q^T,q^0)$ to calculate the change in prices going from period T to 0. Finally, multiply all of these indices together, and since we end up where we started, then the product of all of these indices should ideally be one. Diewert (1993a; 40) called this test a *multiperiod identity test*.[81] Note that if $T = 2$ (so that the number of periods is 3 in total), then Walsh's test reduces to Fisher's (1921; 534) (1922; 64) *time reversal test*.[82]

[78]This formula was implicitly introduced in Törnqvist (1936) and explicitly defined in Törnqvist and Törnqvist (1937).

[79]This result will be proved in Chapter 7. This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert, and Feenstra (1999; 65).

[80]However, if the smooth trends assumption is violated to a considerable degree or if there are a substantial number of new and disappearing products, then this result will not hold as will be seen in Chapter 7. If prices and quantities are subject to big fluctuations, then it will be necessary to move to a multilateral index; see Chapter 7. Note that with new and disappearing products, fixed-base indices can only be used if the base is changed frequently.

[81]Walsh (1921a; 98) called his test the *circular test* but since Fisher also used this term to describe his transitivity test defined earlier by (58), it seems best to stick to Fisher's terminology since it is well established in the literature.

[82]Walsh (1921b; 540–541) noted that the time reversal test was a special case of his circularity test.

Walsh (1901; 423–433) showed how his circularity test could be used in order to evaluate how "good" any bilateral index number formula was. What he did was invent artificial price and quantity data for five periods and he added a sixth period that had the data of the first period. He then evaluated the right-hand side of (63) for various formulae, $P(p^0,p^1,q^0,q^1)$, and determined how far from unity the results were. His "best" formulae had products that were close to one.[83]

This same framework is often used to evaluate the efficacy of chained indices versus their direct counterparts. Thus, if the right-hand side of (63) turns out to be different from unity, the chained indices are said to suffer from "chain drift." If a formula does suffer from chain drift, it is sometimes recommended that fixed-base indices be used in place of chained ones. However, this advice, if accepted, would *always* lead to the adoption of fixed-base indices, provided that the bilateral index formula satisfies the identity test, $P(p^0,p^0,q^0,q^0) = 1$. But at the first level of aggregation, there will be tremendous product turnover in most economies. Under these conditions, the adoption of a fixed-base index would soon lead to a lack of matching of the products, and the resulting fixed-base indices would lose their relevance. Thus, it is not recommended that Walsh's circularity test be used to decide whether fixed-base or chained indices should be calculated. However, it is fair to use Walsh's circularity test as he originally used it—that is, as an approximate method for deciding how "good" a particular index number formula is. **In order to decide whether to chain or use fixed-base indices, one should decide on the basis of how similar the observations being compared are and choose the method that will best link up the most similar observations.** The question of when to chain and when not to will be discussed in more detail in Chapter 7.

9. Two-Stage Aggregation versus Single-Stage Aggregation

Does a Laspeyres or Paasche or Fisher index that is constructed in two stages equal the corresponding index that is constructed in a single stage? This question is addressed in the present section. In practice, it is a big advantage to be consistent in aggregation because consistency in aggregation allows the production of an index to be decentralized.

Suppose that the price and quantity data for period t, p^t, and q^t can be written in terms of M subvectors as follows:

$$p^t = [p^{t1},p^{t2},\ldots,p^{tM}];\ q^t = [q^{t1},q^{t2},\ldots,q^{tM}];\ t = 0,1, \qquad (64)$$

where the dimensionality of the subvectors p^{tm} and q^{tm} is $N(m)$ for $m = 1,2,\ldots,M$, with the sum of the dimensions $N(m)$ equal to N. These subvectors correspond to the price and quantity data for subcomponents of an overall CPI for period t. For the first stage of aggregation, construct subindices for each of these components going from period 0 to 1. For the base period, set the aggregate price level for each of these subcomponents, say $P_m^{\ 0}$ for $m = 1,2,\ldots M$, equal to 1 and set the corresponding base period subcomponent quantities, say $Q_m^{\ 0}$ for $m = 1,2,\ldots,M$, equal to the base period value of consumption for that subcomponent for $m = 1,2,\ldots,M$:

$$P_m^{\ 0} \equiv 1;\ Q_m^{\ 0} \equiv \Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m};\ m = 1,\ldots,M. \qquad (65)$$

Now use the *Laspeyres formula* in order to construct a period 1 price for each subcomponent, say $P_m^{\ 1}$ for $m = 1,2,\ldots,M$, of the CPI. Since the dimensionality of the subcomponent vectors, p^{tm} and q^{tm}, differs from the dimensionality of the complete period t vectors of prices and quantities, p^t and q^t, it is necessary to use different symbols for these subcomponent Laspeyres indices, say $P_L^{\ m}$ for $m = 1,2,\ldots M$. Thus, the period 1 subcomponent prices are defined as follows:

$$\begin{aligned} P_m^{\ 1} &\equiv P_L^{\ m}(p^{0m},p^{1m},q^{0m},q^{1m}) \\ &\equiv \Sigma_{i=1}^{N(m)}\, p_i^{\ 1m}q_i^{\ 0m}/\Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m};\ m = 1,\ldots,M. \end{aligned} \qquad (66)$$

Once the period 1 prices for the M subindices have been defined by (66), then the corresponding subcomponent period 1 quantities $Q_m^{\ 1}$ for $m = 1,2,\ldots,M$ can be defined by deflating the period 1 subcomponent values $\Sigma_{i=1}^{N(m)}\, p_i^{\ 1m}q_i^{\ 1m}$ by the period 1 price levels, $P_m^{\ 1}$:

$$Q_m^{\ 1} \equiv \Sigma_{i=1}^{N(m)}\, p_i^{\ 1m}q_i^{\ 1m}/P_m^{\ 1};\ m = 1,\ldots,M. \qquad (67)$$

Now we define the period 0 and 1 *subcomponent price level vectors* P^0 and P^1 as follows:

$$P^0 \equiv [P_1^{\ 0},P_2^{\ 0},\ldots,P_M^{\ 0}] \equiv 1_M;\ P^1 \equiv [P_1^{\ 1},P_2^{\ 1},\ldots,P_M^{\ 1}], \qquad (68)$$

where 1_M denotes a vector of ones of dimension M and the components of P^1 are defined by (67). The period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

$$Q^0 \equiv [Q_1^{\ 0},Q_2^{\ 0},\ldots,Q_M^{\ 0}];\ Q^1 \equiv [Q_1^{\ 1},Q_2^{\ 1},\ldots,Q_M^{\ 1}], \qquad (69)$$

where the components of Q^0 are defined by definitions (65) and the components of Q^1 are defined by definitions (67). The price and quantity vectors in (68) and (69) represent the results of the first-stage aggregation. Now use these vectors as inputs into the second-stage aggregation problem; that is, apply the Laspeyres price index formula using the information in (68) and (69) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second-stage aggregation problem have dimension M instead of the single-stage formula that utilized vectors of dimension N, a different symbol is required for the new Laspeyres index, which we choose to be $P_L^{\ *}$. Thus, the Laspeyres price index computed in two stages is denoted as $P_L^{\ *}(P^0,P^1,Q^0,Q^1)$. This index is defined as follows:

$$\begin{aligned} P_L^{\ *}(P^0,P^1,Q^0,Q^1) &\equiv \Sigma_{m=1}^{M}\, P_m^{\ 1}Q_m^{\ 0}/\Sigma_{m=1}^{M}\, P_m^{\ 0}Q_m^{\ 0} \\ &= \Sigma_{m=1}^{M}\, P_m^{\ 1}[\Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m}]/ \\ &\quad \Sigma_{m=1}^{M}\, [\Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m}] \text{ using (65)} \\ &= \Sigma_{m=1}^{M}\, [\Sigma_{i=1}^{N(m)}\, p_i^{\ 1m}q_i^{\ 0m}/ \\ &\quad \Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m}][\Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m}]/ \\ &\quad \Sigma_{m=1}^{M}\, \Sigma_{i=1}^{N(m)}\, p_i^{\ 0m}q_i^{\ 0m} \text{ using (66)} \end{aligned}$$

[83] This is essentially a variant of the methodology that Fisher (1922; 284) used to check how well various formulae corresponded to his version of the circularity test.

$$= \Sigma_{m=1}{}^{M} \Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{0m} /$$
$$\Sigma_{m=1}{}^{M} \Sigma_{i=1}{}^{N(m)} p_i{}^{0m} q_i{}^{0m}$$
$$\equiv P_L(p^0,p^1,q^0,q^1), \tag{70}$$

where $P_L(p^0,p^1,q^0,q^1)$ is the overall Laspeyres price index calculated in a single stage. Thus, *the two-stage Laspeyres index exactly equals the single-stage Laspeyres index*:[84]

$$P_L{}^*(P^0,P^1,Q^0,Q^1) = P_L(p^0,p^1,q^0,q^1). \tag{71}$$

Recall that (26) established that the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, was equal to the ratio of two Laspeyres indices, $P_L(p^b,p^t,q^b)/P_L(p^b,p^0,q^b)$. Thus, the two-stage aggregation result (71) for the Laspeyres formula implies that the Lowe index is also consistent in aggregation.[85]

Does the same two-stage aggregation result hold for the Paasche index? The single-stage Paasche index is defined as

$$P_P(p^0,p^1,q^0,q^1) \equiv \Sigma_{m=1}{}^{M} \Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} /$$
$$\Sigma_{m=1}{}^{M} \Sigma_{i=1}{}^{N(m)} p_i{}^{0m} q_i{}^{1m}. \tag{72}$$

The Paasche subaggregate price and quantity levels for period 0 are still defined by (65). However, the period 1 subcomponent Paasche price levels are defined as follows:

$$P_m{}^1 \equiv P_P{}^m(p^{0m},p^{1m},q^{0m},q^{1m}) \equiv \Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} /$$
$$\Sigma_{i=1}{}^{N(m)} p_i{}^{0m} q_i{}^{1m};\ m = 1,\ldots,M. \tag{73}$$

Using definitions (73) for the period 1 price levels $P_m{}^1$, the Paasche period 1 subaggregate quantity levels are defined by definitions (67). The Paasche price index computed in two stages is denoted as $P_P{}^*(P^0,P^1,Q^0,Q^1)$ and defined as follows:

$$P_P{}^*(P^0,P^1,Q^0,Q^1) \equiv \Sigma_{m=1}{}^{M} P_m{}^1 Q_m{}^1 / \Sigma_{m=1}{}^{M} P_m{}^0 Q_m{}^1$$
$$= \Sigma_{m=1}{}^{M} P_m{}^1 [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} / P_m{}^1] /$$
$$\Sigma_{m=1}{}^{M} P_m{}^0 [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} / P_m{}^1] \text{ using (67)}$$
$$= \Sigma_{m=1}{}^{M} [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m}] /$$
$$\Sigma_{m=1}{}^{M} [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} / P_m{}^1] \text{ using (65)}$$
$$= \Sigma_{m=1}{}^{M} [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m}] / \Sigma_{m=1}{}^{M} [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} /$$
$$(\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m} / \Sigma_{i=1}{}^{N(m)} p_i{}^{0m} q_i{}^{1m})] \text{ using (73)}$$
$$= \Sigma_{m=1}{}^{M} [\Sigma_{i=1}{}^{N(m)} p_i{}^{1m} q_i{}^{1m}] / \Sigma_{m=1}{}^{M} [\Sigma_{i=1}{}^{N(m)} p_i{}^{0m} q_i{}^{1m}]$$
$$= P_P(p^0,p^1,q^0,q^1) \text{ using (72).} \tag{74}$$

Thus, *the two-stage Paasche index exactly equals the single-stage Paasche index*.[86]

Definitions (65)–(69) can be used to construct first-stage subaggregates for any index number formula except that in definition (66) replace $P_m{}^1 \equiv P_L{}^m(p^{0m},p^{1m},q^{0m},q^{1m})$ or $P_m{}^1 \equiv P_P{}^m(p^{0m},p^{1m},q^{0m},q^{1m})$ by $P_m{}^1 \circ P^m(p^{0m},p^{1m},q^{0m},q^{1m})$, where $P^m(p^{0m},p^{1m},q^{0m},q^{1m})$ can represent any bilateral index number formula.

Suppose that the Fisher or Törnqvist formula is used at each stage of the aggregation; that is, in equations (66), suppose that the Laspeyres formula $P_L{}^m(p^{0m},p^{1m},q^{0m},q^{1m})$ is replaced by the Fisher formula $P_F{}^m(p^{0m},p^{1m},q^{0m},q^{1m})$ (or by the Törnqvist formula $P_T{}^m(p^{0m},p^{1m},q^0m,q^1m)$) and in equation (70), $P_L{}^*(P^0,P^1,Q^0,Q^1)$ is replaced by $P_F{}^*(P^0,P^1,Q^0,Q^1)$ (or by $P_T{}^*(P^0,P^1,Q^0,Q^1)$) and $P_L(p^0,p^1,q^0,q^1)$ is replaced by $P_F(p^0,p^1,q^0,q^1)$ (or by $P_T(p^0,p^1,q^0,q^1)$). Then the two-stage aggregation equality does not hold for these index number formulae. It can be shown that, in general,

$$P_F{}^*(P^0,P^1,Q^0,Q^1) \neq P_F(p^0,p^1,q^0,q^1)$$
$$\textit{and } P_T{}^*(P^0,P^1,Q^0,Q^1) \neq P_T(p^0,p^1,q^0,q^1). \tag{75}$$

However, even though the Fisher and Törnqvist formulae are not *exactly* consistent in aggregation, it can be shown that these formulae are *approximately* consistent in aggregation. More specifically, it can be shown that the two-stage Fisher formula $P_F{}^*$ and the single-stage Fisher formula P_F in (75), both regarded as functions of the 4N variables in the vectors p^0,p^1,q^0,q^1, approximate each other to the second order around a point where the two price vectors are equal (so that $p^0 = p^1$) and where the two quantity vectors are equal (so that $q^0 = q^1$) and a similar result holds for the two-stage and single-stage Törnqvist indices in (75).[87] Thus, for normal time series data, single-stage and two-stage Fisher and Törnqvist indices will usually be numerically very close.[88]

[84] Balk (1996; 362) (2008; 106–107) established this two-stage consistency in aggregation result for both the Laspeyres and Paasche indices. Blackorby and Primont (1980; 88) established the result for the Laspeyres index.
[85] This result was established in Eurostat (2018; 173).
[86] For additional results on consistency on aggregation over three or more stages of aggregation, see Annex 5. For further materials on the problem of consistency in aggregation, see the references in Blackorby and Primont (1980), Diewert (1978) (1980), and Balk (1996).

[87] See Diewert (1978; 889). In fact, these derivative equalities are still true, provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.
[88] For an empirical comparison of the four indices, see Diewert (1978; 894–895). For the Canadian consumer data considered there, the chained two-stage Fisher in 1971 was 2.3228 and the corresponding chained two-stage Törnqvist was 2.3230, the same values as for the corresponding single-stage indices. Additional empirical results will be exhibited in subsequent chapters.

Annex 1 The Relationship between the Paasche and Laspeyres Indices

Recall the notation used in Section 2. Define the nth relative price or price relative r_n and the nth quantity relative t_n as follows:

$$r_n \equiv p_n^1/p_n^0;\ t_n \equiv q_n^1/q_n^0;\ n = 1,\ldots,N. \qquad (A1.1)$$

Using formula (8) for the *Laspeyres price index* P_L and definitions (A1.1), we have

$$P_L = \Sigma_{n=1}^{N} r_n s_n^0 \equiv r^*; \qquad (A1.2)$$

that is, we define the "average" price relative r^* as the base period expenditure share-weighted average of the individual price relatives, r_i.

The *Laspeyres quantity index*, $Q_L(q^0,q^1,p^0)$, that compares quantities in month 1, q^1, to the corresponding quantities in month 0, q^0, using the prices of month 0, p^0, as weights can be defined as a weighted average of the quantity ratios t_n as follows:

$$Q_L(q^0,q^1,p^0) = \Sigma_{n=1}^{N} s_n^0 t_n \equiv t^*. \qquad (A1.3)$$

Before we compare the Paasche and Laspeyres price indices, we need to undertake a preliminary computation using these definitions of r_n and t_n. Define the *weighted covariance* between the r_n and t_n as follows:

$$\begin{aligned}
\mathrm{Cov}(r,t,s^0) &\equiv \Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)s_n^0 \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0 - \Sigma_{n=1}^{N} r_n t^* s_n^0 - \Sigma_{n=1}^{N} r^* t_n s_n^0 + \Sigma_{n=1}^{N} r^* t^* s_n^0 \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0 - t^*\Sigma_{n=1}^{N} r_n s_n^0 - r^*\Sigma_{n=1}^{N} t_n s_n^0 + r^* t^*\Sigma_{n=1}^{N} s_n^0 \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0 - t^*\Sigma_{n=1}^{N} r_n s_n^0 - r^*\Sigma_{n=1}^{N} t_n s_n^0 + r^* t^* \\
&\quad \text{using } \Sigma_{n=1}^{N} s_n^0 = 1 \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0 - t^* r^* - r^* t^* + r^* t^* \text{ using (A1.2) and (A1.3)} \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0 - t^* r^*. \qquad (A1.4)
\end{aligned}$$

Rearranging (A1.4) leads to the following *covariance identity*:[89]

$$\Sigma_{n=1}^{N} r_n t_n s_n^0 = \Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)s_n^0 + r^* t^*. \qquad (A1.5)$$

Using formula (6) for the Paasche price index P_P, we have

$$\begin{aligned}
P_P &\equiv \Sigma_{n=1}^{N} p_n^1 q_n^1/\Sigma_{i=1}^{N} p_i^0 q_i^1 \\
&= \Sigma_{n=1}^{N} r_n t_n p_n^0 q_n^0/\Sigma_{i=1}^{N} t_i p_i^0 q_i^0 \text{ using definitions (A1.1)} \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0/\Sigma_{i=1}^{N} t_i s_i^0 \\
&= \Sigma_{n=1}^{N} r_n t_n s_n^0/t^* \text{ using definition (A1.3)} \\
&= [\{\Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)s_n^0\} + r^* t^*]/t^* \text{ using (A1.5)} \\
&= [\Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)s_n^0/t^*] + r^* \\
&= [\Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)s_n^0/Q_L(q^0,q^1,p^0)] \\
&\quad + P_L(p^0,p^1,q^0), \qquad (A1.6)
\end{aligned}$$

where the last equality follows from definitions (A1.2) and (A1.3). Taking the difference between P_P and P_L and using (A1.6), we have

$$P_P - P_L = \Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)s_n^0/Q_L(q^0,q^1,p^0). \qquad (A1.7)$$

Thus, the difference between the Paasche and Laspeyres price indices is equal to the covariance between the price ratios, $r_n = p_n^1/p_n^0$, and the corresponding quantity ratios, $t_n = q_n^1/q_n^0$, divided by the (positive) Laspeyres quantity index, $Q_L(q^0,q^1,p^0)$. If this covariance is negative, which is the usual case in the consumer context, then P_P will be less than P_L.

[89] The analysis in this annex was performed by Bortkiewicz (1923; 374–375).

Annex 2 The Relationship between the Lowe and Laspeyres Indices

We shall use the same notation for the long-term monthly price relatives $r_n \equiv p_n^{\,t}/p_n^{\,0}$ that was used in Annex 1. However, we shall change the definition of the t_n in order to relate the base year annual quantities $q_n^{\,b}$ to the base month quantities $q_n^{\,0}$:

$$t_n \equiv q_n^{\,b}/q_n^{\,0};\ n = 1,\ldots,N. \qquad \text{(A2.1)}$$

We also define a new *Laspeyres quantity index* $Q_L(q^0,q^b,p^0)$, which compares the base year quantity vector q^b to the base month quantity vector q^0, using the price weights of the base month p^0, as follows:

$$\begin{aligned} Q_L(q^0,q^b,p^0) &\equiv \Sigma_{n=1}^{N}\, p_n^{\,0}q_n^{\,b}/\Sigma_{i=1}^{N}\, p_i^{\,0}q_i^{\,0} \\ &= \Sigma_{n=1}^{N}\, p_n^{\,0}q_n^{\,0}(q_n^{\,b}/q_n^{\,0})/\Sigma_{i=1}^{N}\, p_i^{\,0}q_i^{\,0} \\ &= \Sigma_{n=1}^{N}\, s_n^{\,0}(q_n^{\,b}/q_n^{\,0}) \text{ using definitions (7)} \\ &= \Sigma_{n=1}^{N}\, s_n^{\,0}t_n \text{ using definitions (A2.1)} \\ &\equiv t^*. \qquad \text{(A2.2)} \end{aligned}$$

Using definition (26) in the main text, the *Lowe index* comparing the prices in month t to those of month 0, using the quantity weights of the base year b, is equal to

$$\begin{aligned} P_{Lo}(p^0,p^t,q^b) &\equiv \Sigma_{n=1}^{N}\, p_n^{\,t}q_n^{\,b} \,/\, \Sigma_{n=1}^{N}\, p_n^{\,0}q_n^{\,b} \qquad \text{(A2.3)} \\ &= \Sigma_{n=1}^{N}\, p_n^{\,t}\, t_nq_n^{\,0} \,/\, \Sigma_{n=1}^{N}\, p_n^{\,0}\, t_nq_n^{\,0} \text{ using definitions (A2.1)} \\ &= \Sigma_{n=1}^{N}\, r_np_n^{\,0}\, t_nq_n^{\,0} \,/\, \Sigma_{n=1}^{N}\, p_n^{\,0}\, t_nq_n^{\,0} \text{ using definitions (A1.1)} \\ &= \Sigma_{n=1}^{N}\, r_nt_n\, s_n^{\,0} \,/\, \Sigma_{n=1}^{N}\, t_n\, s_n^{\,0} \text{ using definitions (7)} \\ &= \Sigma_{n=1}^{N}\, r_nt_n\, s_n^{\,0}/t^* \text{ using (A2.2)} \\ &= [\{\Sigma_{n=1}^{N}\, (r_n - r^*)(t_n - t^*)s_n^{\,0}\} + r^*t^*]/t^* \text{ using the identity (A1.5)} \\ &= [\Sigma_{n=1}^{N}\, (r_n - r^*)(t_n - t^*)s_n^{\,0}/t^*] + r^* \\ &= [\Sigma_{n=1}^{N}\, (r_n - r^*)(t_n - t^*)s_n^{\,0}/t^*] + P_L(p^0,p^t,q^0) \text{ using definition (A1.2)} \\ &= [\mathrm{Cov}(r,t,s^0)/Q_L(q^0,q^b,p^0)] + P_L(p^0,p^t,q^0), \end{aligned}$$

where the last equality follows from definitions (A1.4) and (A2.2). Subtracting the Laspeyres price index relating the prices of month t to those of month 0, $P_L(p^0,p^t,q^0)$, from both sides of (A2.3) leads to the following relationship of this monthly Laspeyres price index to its Lowe counterpart:

$$\begin{aligned} &P_{Lo}(p^0,p^t,q^b) - P_L(p^0,p^t,q^0) \qquad \text{(A2.4)} \\ &= \Sigma_{n=1}^{N}\, (r_n - r^*)(t_n - t^*)s_n^{\,0}/Q_L(q^0,q^b,p^0) \\ &= \mathrm{Cov}(r,t,s^0)/Q_L(q^0,q^b,p^0). \end{aligned}$$

Annex 3 The Relationship between the Young Index and Its Time Antithesis

Recall that the direct Young index, $P_Y(p^0,p^t,s^b)$, was defined by (44) and its time antithesis, $P_Y^{\,*}(p^0,p^t,s^b)$, was defined by (48). Define the nth relative price between months 0 and t as

$$r_n \equiv p_n^{\,t}/p_n^{\,0};\ n = 1,\ldots,N, \qquad \text{(A3.1)}$$

and define the weighted average (using the base year weights $s_i^{\,b}$) of r_n as

$$r^* \equiv \Sigma_{n=1}^{N}\, s_n^{\,b}r_n, \qquad \text{(A3.2)}$$

which equals the direct Young index, $P_Y(p^0,p^t,s^b)$. Define the deviation e_n of r_n from their weighted average r^* using the following equations:

$$r_n = r^*(1 + e_n);\ n = 1,\ldots,N. \qquad \text{(A3.3)}$$

If equations (A3.3) are substituted into equation (A3.2), the following equation is obtained:

$$\begin{aligned} r^* &= \Sigma_{n=1}^{N}\, s_n^{\,b}r^*(1 + e_n) \\ &= r^* + r^*\Sigma_{n=1}^{N}\, s_n^{\,b}e_n, \qquad \text{(A3.4)} \end{aligned}$$

since $\Sigma_{n=1}^{N}\, s_n^{\,b} = 1$. Thus,

$$e^* \equiv \Sigma_{n=1}^{N}\, s_n^{\,b}e_n = 0. \qquad \text{(A3.5)}$$

Thus, the weighted mean e^* of the deviations e_n equals 0.

As the direct Young index, $P_Y(p^0,p^t,s^b)$, and its time antithesis, $P_Y^{\,*}(p^0,p^t,s^b)$, can be written as functions of r^*, the annual share weights $s_n^{\,b}$ and the deviations of the price relatives e_n from their weighted mean are as follows:

$$\begin{aligned} P_Y(p^0,p^t,s^b) &= r^*; \qquad \text{(A3.6)} \\ P_Y^{\,*}(p^0,p^t,s^b) &= [\Sigma_{n=1}^{N}\, s_n^{\,b}\{r^*(1 + e_n)\}^{-1}]^{-1}; \\ &= r^*[\Sigma_{n=1}^{N}\, s_n^{\,b}(1 + e_n)^{-1}]^{-1}. \qquad \text{(A3.7)} \end{aligned}$$

Now regard $P_Y^{\,*}(p^0,p^t,s^b)$ as a function of the vector of deviations, $e \equiv [e_1,\ldots,e_N]$, say $P_Y^{\,*}(e)$. The second-order Taylor series approximation to $P_Y^{\,*}(e)$ around the point $e = 0_N$ is given by the following expression:[90]

$$\begin{aligned} P_Y^{\,*}(e) &\approx r^* + r^*\Sigma_{n=1}^{N}\, s_n^{\,b}e_n \\ &\quad + r^*\Sigma_{n=1}^{N}\, \Sigma_{i=1}^{N}\, s_n^{\,b}s_i^{\,b}\, e_ne_i - r^*\Sigma_{n=1}^{N}\, s_n^{\,b}[e_n]^2 \\ &= r^* + r^*[0] + r^*\Sigma_{n=1}^{N}\, [\Sigma_{i=1}^{N}\, s_n^{\,b}\, e_n]s_i^{\,b}e_i \\ &\quad - r^*\Sigma_{n=1}^{N}\, s_n^{\,b}[e_n - e^*]^2 \text{ using (A3.5)} \\ &= r^* + r^*\Sigma_{n=1}^{N}\, [0]s_i^{\,b}e_i - r^*\Sigma_{n=1}^{N}\, s_n^{\,b}[e_n - e^*]^2 \text{ using (A3.5)} \\ &= r^* - r^*var(e), \qquad \text{(A3.8)} \end{aligned}$$

[90] This type of second-order approximation was developed by Dalén (1992; 143) for the case $r^* =1$ and by Diewert (1995; 29) for the case of a general r^*.

where the weighted sample variance of the vector e of price deviations is defined as

$$var(e) \equiv \Sigma_{n=1}^{N} s_n^{\ b}[e_n - e^*]^2. \quad \text{(A3.9)}$$

Using $P_Y(p^0,p^t,s^b) = r^*$, rearranging (A3.8) gives us the following approximate relationship between the direct Young index $P_Y(p^0,p^t,s^b)$ and its time antithesis $P_Y^{\ *}(p^0,p^t,s^b)$, to the accuracy of a second-order Taylor series approximation about a price point where the month t price vector is proportional to the month 0 price vector:

$$P_Y(p^0,p^t,s^b) \approx P_Y^{\ *}(p^0,p^t,s^b) + P_Y(p^0,p^t,s^b)var(e). \quad \text{(A3.10)}$$

Thus, to the accuracy of a second-order approximation, the direct Young index will *exceed* its time antithesis by a term equal to the direct Young index times the weighted variance of the deviations of the price relatives from their weighted mean. Thus, the bigger the dispersion in relative prices, the more the direct Young index will exceed its time antithesis.

Annex 4 The Relationship between the Lowe Index and the Young Index

This chapter has indicated that the Laspeyres, Paasche, and Fisher indices are preferred target indices because they weight prices by the most relevant quantity vectors for making overall price comparisons between any two periods; that is, they use the quantity vectors that are equal or proportional to actual consumption for the two periods in the comparison. However, often national statistical offices cannot collect current period expenditure or quantity information, and so their options are limited to a choice between the Lowe and the Young index.[92] This choice is based on the fact that they have limited resources to conduct a household expenditure survey, and in some instances, there can be a five- to ten-year time lapse between the survey periods. The question to be addressed here is: "Which of these two indices is the preferred option under these circumstances?"

Recall that the *Young index* between periods 0 and t, $P_Y(p^0,p^t,s^b)$, was defined by (44), where p^0 and p^t are the price vectors for periods 0 and t and s^b is the vector of expenditure share weights for a previous period (usually a year prior to month 0). For convenience, we repeat this definition here:

$$P_Y(p^0,p^t,s^b) \equiv \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ t}/p_n^{\ 0}). \quad \text{(A4.1)}$$

The Young index between the base period b for the weights and the base period 0 for the monthly prices is defined as follows:

$$P_Y(p^b,p^0,s^b) \equiv \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ 0}/p_n^{\ b}). \quad \text{(A4.2)}$$

Using definition (26) in the main text, the *Lowe index* comparing the prices in month t to those of month 0, using the quantity weights q^b of the base year b, is equal to

$$\begin{aligned}
P_{Lo}(p^0,p^t,q^b) &\equiv \Sigma_{n=1}^{N} p_n^{\ t}q_n^{\ b} / \Sigma_{n=1}^{N} p_n^{\ 0}q_n^{\ b} \\
&= \Sigma_{n=1}^{N} (p_n^{\ t}/p_n^{\ 0})p_n^{\ 0}q_n^{\ b} / \Sigma_{n=1}^{N} p_n^{\ 0}q_n^{\ b} \quad \text{(A4.3)} \\
&= \Sigma_{n=1}^{N} (p_n^{\ t}/p_n^{\ 0})p_n^{\ b}q_n^{\ b}(p_n^{\ 0}/p_n^{\ b}) / \Sigma_{n=1}^{N} p_n^{\ b}q_n^{\ b}(p_n^{\ 0}/p_n^{\ b}) \\
&= \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ t}/p_n^{\ 0})(p_n^{\ 0}/p_n^{\ b}) / \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ 0}/p_n^{\ b}) \\
&\qquad \text{using } s_n^{\ b} \circ p_n^{\ b}q_n^{\ b}/p^b \cdot q^b \\
&= \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ t}/p_n^{\ 0})(p_n^{\ 0}/p_n^{\ b}) / P_Y(p^b,p^0,s^b) \\
&\qquad \text{using definition (A4.2)} \\
&= \Sigma_{n=1}^{N} s_n^{\ b}r_n t_n / P_Y(p^b,p^0,s^b) \\
&\qquad \text{defining } r_n \equiv p_n^{\ t}/p_n^{\ 0};\ t_n \equiv p_n^{\ 0}/p_n^{\ b} \\
&= [\Sigma_{n=1}^{N} s_n^{\ b}(r_n - r^*)(t_n - t^*) + r^*t^*] / P_Y(p^b,p^0,s^b) \\
&\qquad \text{using the identity (A1.5)} \\
&= [\mathrm{Cov}(r,t,s^b) + P_Y(p^0,p^t,s^b)P_Y(p^b,p^0,s^b)] / P_Y(p^b,p^0,s^b) \\
&= [\mathrm{Cov}(r,t,s^b)/P_Y(p^b,p^0,s^b)] + P_Y(p^0,p^t,s^b),
\end{aligned}$$

since $r^* \equiv \Sigma_{n=1}^{N} s_n^{\ b}r_n = \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ t}/p_n^{\ 0}) = P_Y(p^0,p^t,s^b)$ is the Young index going from period 0 to t and $t^* \equiv \Sigma_{n=1}^{N} s_n^{\ b}t_n = \Sigma_{n=1}^{N} s_n^{\ b}(p_n^{\ 0}/p_n^{\ b}) = P_Y(p^b,p^0,s^b)$ is the Young index going from period b to 0. The *weighted covariance* between the vectors of relative prices r and t is defined as

$$\begin{aligned} \mathrm{Cov}(r,t,s^b) &\equiv \Sigma_{n=1}^{N} s_n^{\,b}(r_n - r^*)(t_n - t^*) \quad \text{(A4.4)} \\ &= \Sigma_{n=1}^{N} s_n^{\,b}[(p_n^{\,t}/p_n^{\,0}) - r^*][(p_n^{\,0}/p_n^{\,b}) - t^*] \\ &= \Sigma_{n=1}^{N} s_n^{\,b}[(p_n^{\,t}/p_n^{\,0}) - P_Y(p^0,p^t,s^b)][(p_n^{\,0}/p_n^{\,b}) - P_Y(p^b,p^0,s^b)]. \end{aligned}$$

If there are *diverging long-run trends in prices*, we would expect $\mathrm{Cov}(r,t,s^b)$ to be positive; that is, if product n has an increasing price (relative to other products) over the entire period running from period 0 to t, then $(p_n^{\,t}/p_n^{\,0}) - r^*$ and $(p_n^{\,0}/p_n^{\,b}) - t^*$ will both be positive; if product n has a decreasing price (relative to other products) over the entire period, then $(p_n^{\,t}/p_n^{\,0}) - r^*$ and $(p_n^{\,0}/p_n^{\,b}) - t^*$ will both be negative. Thus, the covariance will be positive in either case. Under these conditions, the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, will exceed the corresponding Young index, $P_Y(p^0,p^t,s^b)$, using (A4.3). Since both the Lowe and Young index will both tend to be above our preferred target index (the Fisher index), the national statistical office would come closer to the target index by using the Young index over the corresponding Lowe index.

Annex 5 Three-Stage Aggregation

Suppose that we have price and quantity data for two periods that are classified by three distinct categories. For example, commodities may be classified by the type of product or service at the first level of aggregation, by the type of outlet or household at the second level of aggregation, and by their location or region at the third level of aggregation. We suppose that the first classification has N categories, the second has M categories, and the third has K categories. Denote the period t price, quantity, and value transacted for the category indexed by k, m, and n by $p_{kmn}^{\,t}$, $q_{kmn}^{\,t}$, and $v_{kmn}^{\,t} \equiv p_{kmn}^{\,t} q_{kmn}^{\,t}$, respectively for $t = 1$, t, $k = 1,\ldots,K$; $m = 1,\ldots,M$ and $n = 1,\ldots,N$.[91] Here we will show that the Laspeyres (1871) and Paasche (1874) indices are consistent in aggregation if they are constructed in a three-stage aggregation procedure. As was seen in Section 9, this consistency in aggregation property for the Laspeyres and Paasche indices is well known if there are two stages of aggregation, but it does not seem to be well known for three or more stages of aggregation. In this annex, we extend the results to show that the Lowe (1823) and Young (1812) indices are also consistent in aggregation over two or three stages of aggregation.

Conditional on k and m (choices of the last two categories), we can calculate the aggregate value of transactions over the third category for period t, $v_{km}^{\,t}$, as follows:

$$v_{km}^{\,t} \equiv \Sigma_{n=1}^{N} v_{kmn}^{\,t} > 0;\ t = 0,1;\ k = 1,\ldots,K;\ m = 1,\ldots,M. \quad \text{(A5.1)}$$

The *overall Laspeyres price index* that compares the prices of period 0 to period 1 is defined as follows:

$$\begin{aligned} P_L^{\,1} &\equiv \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\,1} q_{kmn}^{\,0} / \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\,0} q_{kmn}^{\,0}; \\ &= \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\,1} q_{kmn}^{\,0} / \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} v_{km}^{\,0}, \quad \text{(A5.2)} \end{aligned}$$

where we have used definitions (A5.1) to derive the second line of (A5.2).

The same data will be used to aggregate in three stages: first aggregate over the n category; then in the second stage, aggregate over the m category; and in the third stage, aggregate over the k category. Thus, in the first stage of aggregation, a family of KM Laspeyres indices will be constructed where we condition on categories k and m and construct a *conditional Laspeyres index* for period t, $P_{Lkm}^{\,1}$, that aggregates over the last category. Thus, construct the

[91] It is not necessary to assume that all prices and quantities be positive. However, we do require that each $v_{km}^{\,t}$ be positive for all k, m, and t; see definitions 1. At the first stage of aggregation, it is likely that many commodities will not be transacted in both periods under consideration. In this case, prices and quantities for the missing products can be set equal to 0. However, if a commodity is transacted in one period but not the other, then there can be a problem. In general, bilateral price indices are not meaningful (or well defined) unless there are positive matching prices in the two periods being compared. Thus, suppose $p_{kmn}^{\,1} > 0$, $q_{kmn}^{\,1} > 0$ and $q_{kmn}^{\,t} = 0$. In order to obtain a meaningful price index that compares prices in period t to prices in period 1, it will be necessary to either artificially set $q_{kmn}^{\,1}$ equal to 0 or provide an artificial positive imputed price for $p_{kmn}^{\,t}$.

following period *t first stage of aggregation Laspeyres price indices*:

$$P_{Lkm}{}^{t} \equiv \Sigma_{n=1}{}^{N} p_{kmn}{}^{t}q_{kmn}{}^{0}/\Sigma_{n=1}{}^{N} p_{kmn}{}^{0}q_{kmn}{}^{0}; \quad t = 0,1; k = 1,...,K; m = 1,...,M$$
$$= \Sigma_{n=1}{}^{N} p_{kmn}{}^{t}q_{kmn}{}^{0}/v_{km}{}^{0}, \qquad \text{(A5.3)}$$

where the second line follows from definitions (A5.1). Using definitions (A5.3) when $t = 0$, we see that the following equations hold:

$$P_{Lkm}{}^{0} = 1; k = 1,...,K; m = 1,...,M. \qquad \text{(A5.4)}$$

Define the period t quantity or volume index $Q_{km}{}^{t}$ that pairs up with the period t price index $P_{Lkm}{}^{t}$ defined by (A5.3) as the period t transaction value over n (conditional on choosing categories k and m), $v_{km}{}^{t}$, divided by $P_{Lkm}{}^{t}$:

$$Q_{km}{}^{t} \equiv v_{km}{}^{t}/P_{Lkm}{}^{t}; t = 0,1; k = 1,...,K; m = 1,...,M$$
$$= v_{km}{}^{t}/[\Sigma_{n=1}{}^{N} p_{kmn}{}^{t}q_{kmn}{}^{0}/v_{km}{}^{0}], \qquad \text{(A5.5)}$$

where the second line follows from (A5.3). Using definitions (A5.1) and (A5.5), we get

$$Q_{km}{}^{0} = v_{km}{}^{0}; k = 1,...,K; m = 1,...,M. \qquad \text{(A5.6)}$$

For our second stage of the three-stage aggregation procedure, we will aggregate over the second category using the Laspeyres price and quantity indices, $P_{Lkm}{}^{t}$ and $Q_{km}{}^{t}$, defined by (A5.3) and (A6.6), as our basic building blocks. Thus, define the *conditional on k Laspeyres price index* for period t, $P_{Lk}{}^{t}$, as follows:

$$P_{Lk}{}^{t} \equiv \Sigma_{m=1}{}^{M} P_{Lkm}{}^{t}Q_{km}{}^{0}/\Sigma_{m=1}{}^{M} P_{Lkm}{}^{0}Q_{km}{}^{0}; \quad t = 0,1; k = 1,...,K \qquad \text{(A5.7)}$$
$$= \Sigma_{m=1}{}^{M} [\Sigma_{n=1}{}^{N} p_{kmn}{}^{t}q_{kmn}{}^{0}/v_{km}{}^{0}]v_{km}{}^{0}/\Sigma_{m=1}{}^{M} v_{km}{}^{0} \quad \text{using (A5.3) and (A5.6)}$$
$$= [\Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{t}q_{kmn}{}^{0}]/\Sigma_{m=1}{}^{M} v_{km}{}^{0}.$$ [92]

Using (A5.1) and (A5.7) when t is set equal to 0, we find that the following equalities hold:

$$P_{Lk}{}^{0} = 1; k = 1,...,K. \qquad \text{(A5.8)}$$

The Laspeyres price index $P_{Lk}{}^{t}$ defined by (A5.7) applies to the conditional on k expenditures $\Sigma_{m=1}{}^{M} v_{km}{}^{t} = \Sigma_{m=1}{}^{M} [\Sigma_{n=1}{}^{N} v_{kmn}{}^{t}]$. Thus, we define the companion quantity or volume index that matches up with $P_{Lk}{}^{t}$ defined by (7) as follows:

$$Q_{k}{}^{t} \equiv \Sigma_{m=1}{}^{M} v_{km}{}^{t}/P_{Lk}{}^{t}; t = 0,1; k = 1,...,K$$
$$= \Sigma_{m=1}{}^{M} v_{km}{}^{t}/[\Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{t}q_{kmn}{}^{0}/\Sigma_{m=1}{}^{M} v_{km}{}^{0}], \qquad \text{(A5.9)}$$

where the second line follows from definitions (A5.7). When $t = 1$, it can be seen that definitions (A5.1) and (A5.9) imply the following equations:

$$Q_{k}{}^{0} = \Sigma_{m=1}{}^{M} v_{km}{}^{0}; k = 1,...,K. \qquad \text{(A5.10)}$$

Our third and final stage of aggregation is to use the prices and quantities defined by (A5.7)–(A5.10) for $t = 0,1$ to form a Laspeyres index that aggregates over the k classification. The final *three stages of aggregation Laspeyres price index* $P_{L}{}^{1*}$ are defined as follows:

$$P_{L}{}^{1*} \equiv \Sigma_{k=1}{}^{K} P_{Lk}{}^{1}Q_{k}{}^{0}/\Sigma_{k=1}{}^{K} P_{Lk}{}^{0}Q_{k}{}^{0}; \qquad \text{(A5.11)}$$
$$= \Sigma_{k=1}{}^{K} [\Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{0}/\Sigma_{i=1}{}^{M} v_{ki}{}^{0}][\Sigma_{j=1}{}^{M} v_{kj}{}^{0}]/\Sigma_{k=1}{}^{K}[\Sigma_{m=1}{}^{M} v_{km}{}^{0}] \quad \text{using (A5.7) and (A5.10)}$$
$$= \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{0}/\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} v_{km}{}^{0}$$
$$= P_{L}{}^{1} \quad \text{using definition (A5.2).}$$

Thus, the Laspeyres price index constructed in three stages is equal to the corresponding single-stage Laspeyres price index. The same method of proof can be used to show that the Laspeyres index constructed in four or more stages of aggregation is equal to the single-stage Laspeyres index.

The above proof can be modified to show that the single-stage Paasche index is equal to its counterpart Paasche index constructed in two or three stages.

We now consider the consistency in aggregation properties of the Lowe (1823) index. The situation is a bit more complex than the framework that was described above in that *three* periods are involved in a comparison of prices between the two periods. Thus, let $q_{kmn}{}^{b}$ be the quantity transacted in the quantity base period b for the commodity category indexed by k, m, and n. The *Lowe index* that compares the prices of period 1, $p_{kmn}{}^{1}$, with the prices of period 0, $p_{kmn}{}^{0}$, is $P_{Lo}{}^{1}$, defined as follows:

$$P_{Lo}{}^{1} \equiv \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{0}q_{kmn}{}^{b};$$
$$= \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} v_{kmn}{}^{0}b, \qquad \text{(A5.12)}$$

where the *hybrid expenditure weights* using the prices of period 0 and the quantities of period b for commodity category indexed by k, m, and n are defined as follows:

$$v_{kmn}{}^{0}b \equiv p_{kmn}{}^{0}q_{kmn}{}^{b}; k = 1,...,K; m = 1,...,M; n = 1,...,N. \qquad \text{(A5.13)}$$

For each k and m, define the *period 0 hybrid conditional on k and m total expenditure* on commodities indexed by n as follows:[93]

$$v_{km}{}^{0}b \equiv \Sigma_{n=1}{}^{N} v_{kmn}{}^{0}b; k = 1,...,K; m = 1,...,M. \qquad \text{(A5.14)}$$

[92] If $K = 1$, then it can be verified that (A5.7) establishes the consistency in aggregation of the Laspeyres price index over two stages of aggregation.

[93] We assume that $v_{km}{}^{0}b > 0$ for $k = 1,...,K$ and $m = 1,...,M$.

Substituting (A5.14) into definition (A5.12), we see that the Lowe index for period 1 relative to period 0 can be written as follows:

$$P_{Lo}^{\ 1} = \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} v_{km}^{\ 0} b. \qquad \text{(A5.15)}$$

These data will be used to aggregate in three stages: first aggregate over the *n* category: then in the second stage, aggregate over the *m* category; and in the third stage, aggregate over the *k* category. Thus, in the first stage of aggregation, a family of KM Lowe indices will be constructed, where we condition on categories *k* and *m* and construct a *conditional Lowe index* for period *t*, $P_{Lokm}^{\ t}$, that aggregates over the *n* category. We now compare the prices of period 1 to the prices of period 0 using the Lowe formula. Thus, construct the following period 1 *first stage of aggregation Lowe price indices*:

$$\begin{aligned} P_{Lokm}^{\ 1} b &\equiv \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / \Sigma_{n=1}^{N} p_{kmn}^{\ 0} q_{kmn}^{\ b}; \\ & k = 1,\ldots,K;\ m = 1,\ldots,M \\ &= \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / v_{km}^{\ 0} b, \qquad \text{(A5.16)} \end{aligned}$$

where the second line follows from definitions (A5.13) and (A5.14). These conditional Lowe indices can act as period 1 Lowe conditional price levels. The corresponding period 0 Lowe conditional price levels are defined as follows:

$$\begin{aligned} P_{Lokm}^{\ 0} b &= \Sigma_{n=1}^{N} p_{kmn}^{\ 0} q_{kmn}^{\ b} / \Sigma_{n=1}^{N} p_{kmn}^{\ 0} q_{kmn}^{\ b} = 1; \\ & k = 1,\ldots,K;\ m = 1,\ldots,M. \qquad \text{(A5.17)} \end{aligned}$$

It is not obvious how to define the subaggregate quantity $Q_{km}^{\ 0} b$ that should match up with the Lowe subaggregate price index for period 0, $P_{Lokm}^{\ 0} b$. In order to achieve consistency in aggregation for the Lowe index, we will set the subaggregate hybrid value for period 0, $v_{km}^{\ 0} b$ equal to subaggregate price $P_{Lokm}^{\ 0} b$ times subaggregate quantity $Q_{km}^{\ 0} b$. Thus, we have the following definitions:

$$\begin{aligned} Q_{km}^{\ 0} b &\equiv v_{km}^{\ 0} b / P_{Lokm}^{\ 0} b;\ k = 1,\ldots,K;\ m = 1,\ldots,M \\ &= v_{km}^{\ 0} b, \qquad \text{(A5.18)} \end{aligned}$$

where the second line follows from (A5.17).

For our second stage of the three-stage aggregation procedure, we aggregate over the second category using the Lowe price and quantity subindices, $P_{Lokm}^{\ 0} b$, $P_{Lokm}^{\ 1} b$, and $Q_{km}^{\ 1} b$, defined by (A5.16)–(A5.18) as our basic building blocks. Thus, we define the *conditional on k Lowe price index* for period 1 relative to period 0, $P_{Lok}^{\ 1} b$, as follows:

$$\begin{aligned} P_{Lok}^{\ 1} b &\equiv \Sigma_{m=1}^{M} P_{Lokm}^{\ 1} b Q_{km}^{\ 0} b / \Sigma_{m=1}^{M} P_{Lokm}^{\ 0} b Q_{km}^{\ 0} b;\ k = 1,\ldots,K \\ &= \Sigma_{m=1}^{M} [\Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / v_{km}^{\ 0} b] v_{km}^{\ 0} b / \Sigma_{m=1}^{M} v_{km}^{\ 0} b \\ & \text{using (A5.16) and (A5.18)} \\ &= \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / \Sigma_{m=1}^{M} v_{km}^{\ 0} b.^{94} \qquad \text{(A5.19)} \end{aligned}$$

We treat $P_{Lok}^{\ 1} b$ as period 1 conditional on *k* Lowe price levels. The corresponding period 0 Lowe conditional on *k* price levels are defined as follows:

$$\begin{aligned} P_{Lok}^{\ 0} b &\equiv \Sigma_{m=1}^{M} P_{Lokm}^{\ 0} b Q_{km}^{\ 0} b / \Sigma_{m=1}^{M} P_{Lokm}^{\ 0} b Q_{km}^{\ 0} b \\ &= 1;\ k = 1,\ldots,K. \qquad \text{(A5.20)} \end{aligned}$$

Total hybrid expenditures on category *k* goods and services for period 0, $v_k^{\ 0} b$,[95] are defined as follows:

$$\begin{aligned} v_k^{\ 0} b &\equiv \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} v_{kmn}^{\ 0} b \\ &= \Sigma_{m=1}^{M} v_{km}^{\ 0} b \text{ using definitions (A5.14).} \qquad \text{(A5.21)} \end{aligned}$$

Define the period 0 *Lowe quantity subaggregate for category* *k*, $Q_k^{\ 0} b$, as period 0 hybrid expenditures on the category, $v_k^{\ 0} b$, deflated by the subaggregate Lowe price index for category *k* in period 0, $P_{Lok}^{\ 0} b$:

$$\begin{aligned} Q_k^{\ 0} b &\equiv v_k^{\ 0} b / P_{Lokm}^{\ 0} b;\ k = 1,\ldots,K; \\ &= v_k^{\ 0} b \text{ using (A5.20)} \\ &= \Sigma_{m=1}^{M} v_{km}^{\ 0} b \text{ using (A5.21).} \qquad \text{(A5.22)} \end{aligned}$$

Our third and final stage of aggregation is to use the prices and quantities defined by (A5.19), (A5.20), and (A5.22) to form a Lowe index that aggregates over the *k* classification. This is referred to as the *three stages of aggregation Lowe price index* $P_{Lo}^{\ 1*}$ defined as follows:

$$\begin{aligned} P_{Lo}^{\ 1*} &\equiv \Sigma_{k=1}^{K} P_{Lok}^{\ 1} b Q_k^{\ 0} b / \Sigma_{k=1}^{K} P_{Lok}^{\ 0} b Q_k^{\ 0} b; \\ &= \Sigma_{k=1}^{K} [\Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / \Sigma_{j=1}^{M} v_{kj}^{\ 0} b][\Sigma_{m=1}^{M} v_{km}^{\ 0} b] / \\ & [\Sigma_{k=1}^{K} \Sigma_{m=1}^{M} v_{km}^{\ 0} b] \text{ using (A5.19) and (A5.22)} \\ &= \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / \\ & [\Sigma_{k=1}^{K} \Sigma_{m=1}^{M} v_{km}^{\ 0} b] \textit{ cancelling terms} \\ &= \Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} p_{kmn}^{\ 1} q_{kmn}^{\ b} / \\ & [\Sigma_{k=1}^{K} \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} v_{kmn}^{\ 0} b] \text{ using (A5.14)} \\ &= P_{Lo}^{\ 1} \text{ using (A5.12).} \qquad \text{(A5.23)} \end{aligned}$$

Thus, the Lowe price index constructed in three stages is equal to the corresponding single-stage Lowe price index. The same method of proof can be used to show that the Lowe index constructed in four or more stages of aggregation is equal to the single-stage Lowe index.

We turn to the Young index and its consistency in aggregation properties. Let $p_{kmn}^{\ b}$, $q_{kmn}^{\ b}$, and $v_{kmn}^{\ b} \equiv p_{knm}^{\ b} q_{kmn}^{\ b}$ be the price, quantity, and transaction value for commodity class indexed by *k*, *m*, and *n* for the base period *b* for $k = 1,\ldots,K$, $m = 1,\ldots,M$, and $n = 1,\ldots,N$. As usual, $p_{kmn}^{\ t}$ is the price of the commodity that is indexed by categories *k*, *m*, and *n* in period *t* for $t = 0,1$. The base period expenditure share for commodity *k*, *m*, and *n*, $s_{kmn}^{\ b}$, is defined as follows:

$$s_{kmn}^{\ b} \equiv v_{kmn}^{\ b} / \Sigma_{r=1}^{K} \Sigma_{s=1}^{M} \Sigma_{t=1}^{N} v_{rst}^{\ b};\ k = 1,\ldots,K;$$

[94] If $K = 1$, then it can be verified that (A5.19) establishes the consistency in aggregation of the Lowe price index over two stages of aggregation.

[95] This is the cost of purchasing the base period basket of commodities that are in category *k* using the prices of period 0 and the quantities of the base period *b*.

$$m = 1,\ldots,M;\ n = 1,\ldots,N. \qquad (A5.24)$$

The period 1 Young index $P_Y^{\,1}$ that compares the prices of period 1 with the prices of period 0 is the following share-weighted average of the price ratios $p_{kmn}^{\,1}/p_{kmn}^{\,1}$:[96]

$$P_Y^{\,1} \equiv \Sigma_{k=1}^{\;K} \Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}[p_{kmn}^{\;1}/p_{kmn}^{\;0}]. \qquad (A5.25)$$

We will compare this single-stage Young index with a corresponding Young index that aggregates the price ratios in three stages. For the first stage of aggregation, we need to define the following *conditional shares* that condition on k and m and aggregate over n, $s_{km}^{\;b}$:

$$s_{km}^{\;b} \equiv \Sigma_{n=1}^{\;N} s_{kmn}^{\;b};\ k = 1,\ldots,K;\ m = 1,\ldots,M. \qquad (A5.26)$$

The *first-stage conditional Young indices*, $P_{Ymn}^{\;1}$, that compare the prices of the commodities in the class of commodities indexed by k and m for period 1 relative to period 0 are defined as follows:

$$P_{Ykm}^{\;1} \equiv \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}[p_{kmn}^{\;1}/p_{kmn}^{\;0}]/s_{km}^{\;b};$$
$$k = 1,\ldots,K;\ m = 1,\ldots,M. \qquad (A5.27)$$

These conditional Young indices can act as period 1 Young conditional price levels. The corresponding period 0 Young conditional price levels are defined as follows:

$$P_{Ykm}^{\;1} \equiv \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}[p_{kmn}^{\;0}/p_{kmn}^{\;0}]/s_{km}^{\;b} = 1;$$
$$k = 1,\ldots,K;\ m = 1,\ldots,M. \qquad (A5.28)$$

The second stage of aggregation uses the prices defined by (A5.27) and (A5.28) and the shares defined by (A5.26). Thus, define the *second-stage conditional Young indices*, $P_{Yk}^{\;t}$, that condition on expenditures in the k category and compare the aggregate prices defined by (A5.27) for period 1 to their counterparts in period 0:[97]

$$P_{Yk}^{\;1} \equiv \Sigma_{m=1}^{\;M} s_{km}^{\;b}[P_{Ykm}^{\;1}/P_{Ykm}^{\;0}]/\Sigma_{m=1}^{\;M} s_{km}^{\;b};\ k = 1,\ldots,K$$
$$= \Sigma_{m=1}^{\;M} s_{km}^{\;b}\{\Sigma_{n=1}^{\;N} s_{kmn}^{\;b}[p_{kmn}^{\;1}/p_{kmn}^{\;0}]/s_{km}^{\;b}\}/\Sigma_{m=1}^{\;M} s_{km}^{\;b} \text{ using (A5.27) and (A5.28)}$$
$$= \Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}(p_{kmn}^{\;1}/p_{kmn}^{\;0})/\Sigma_{m=1}^{\;M} s_{km}^{\;b} \text{ canceling terms}$$
$$= \Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}(p_{kmn}^{\;1}/p_{kmn}^{\;0})/\Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b} \text{ using (A5.26).} \qquad (A5.29)$$

This conditional on k Young indices can act as period 1 Young conditional on k price levels. The corresponding period 0 Young conditional on k price levels are defined as follows:

$$P_{Yk}^{\;0} \equiv \Sigma_{m=1}^{\;M} s_{km}^{\;b}[P_{Ykm}^{\;0}/P_{Ykm}^{\;0}]/\Sigma_{m=1}^{\;M} s_{km}^{\;b} = 1;$$
$$k = 1,\ldots,K. \qquad (A5.30)$$

Finally, in order to implement the third stage of aggregation, we define aggregate *shares that condition on expenditure category* k, $s_k^{\;b}$:

$$s_k^{\;b} \equiv \Sigma_{m=1}^{\;M} s_{km}^{\;b};\ k = 1,\ldots,K = \Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b} \qquad (A5.31)$$

where the second line follows from definitions (A5.26). Note that these shares sum to one:

$$\Sigma_{k=1}^{\;K} s_k^{\;b} = \Sigma_{k=1}^{\;K} \Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b} = 1. \qquad (A5.32)$$

The final stage of aggregation is to aggregate over the k classification. The three-stage Young index that compares the prices of period 1 to period 0 is $P_Y^{\;1*}$, which is defined as follows:

$$P_Y^{\;1*} \equiv \Sigma_{k=1}^{\;K} s_k^{\;b}[P_{Yk}^{\;1}/P_{Yk}^{\;0}]$$
$$= \Sigma_{k=1}^{\;K} s_k^{\;b} P_{Yk}^{\;1} \text{ using (A5.30)}$$
$$= \Sigma_{k=1}^{\;K} s_k^{\;b}\,[\Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}(p_{kmn}^{\;1}/p_{kmn}^{\;0})/\Sigma_{m=1}^{\;M} s_{km}^{\;b}] \text{ using (A5.29)}$$
$$= \Sigma_{k=1}^{\;K} s_k^{\;b}\,[\Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}(p_{kmn}^{\;1}/p_{kmn}^{\;0})/s_k^{\;b}] \text{ using (A5.31)}$$
$$= \Sigma_{k=1}^{\;K} \Sigma_{m=1}^{\;M} \Sigma_{n=1}^{\;N} s_{kmn}^{\;b}(p_{kmn}^{\;1}/p_{kmn}^{\;0}) \text{ canceling terms}$$
$$= P_Y^{\;1} \text{ using (A5.25).} \qquad (A5.33)$$

Thus, the Young price index constructed in three stages is equal to the corresponding single-stage Young price index. The same method of proof can be used to show that the Young index constructed in four or more stages of aggregation is equal to the single-stage Young index.

References

Alterman, William F., W. Erwin Diewert, and Robert C. Feenstra. 1999. *International Trade Price Indexes and Seasonal Commodities*. Washington, DC: Bureau of Labor Statistics,

Baldwin, Andrew 1990. "Seasonal Baskets in Consumer Price Indexes." *Journal of Official Statistics* 6 (3): 251–73.

Balk, Bert M. 1980. "A Method for Constructing Price Indices for Seasonal Commodities." *The Journal of the Royal Statistical Society Series A* 143: 68–75.

Balk, Bert M. 1995. "Axiomatic Price Index Theory: A Survey." *International Statistical Review* 63: 69–93.

Balk, Bert M. 1996. "Consistency in Aggregation and Stuvel Indices." *The Review of Income and Wealth* 42: 353–63.

Balk, Bert M. 2008. *Price and Quantity Index Numbers*. New York: Cambridge University Press.

Bean, Louis H., and Oscar C. Stine. 1924. "Four Types of Index Numbers of Farm Prices." *Journal of the American Statistical Association* 19: 30–35.

Blackorby, Charles, and Daniel Primont. 1980. "Index Numbers and Consistency in Aggregation." *Journal of Economic Theory* 22: 97–98.

Bowley, Arthur L. 1899. "Wages, Nominal and Real." In *Dictionary of Political Economy*, Volume 3, edited by R.H. Inglis Palgrave (pp. 640–651). London: Macmillan.

Bowley, Arthur L. 1901. *Elements of Statistics*. Westminster: Orchard House.

Bowley, Arthur L. 1919. "The Measurement of Changes in the Cost of Living." *Journal of the Royal Statistical Society* 82: 343–61.

[96] In order for this index to be well defined, we require all period 1 prices to be positive. If a product is present in just one of the two periods under consideration, then it is necessary to exclude that product from the index or, alternatively, to generate an imputed price for the product for the period where it is missing.

[97] If $k = 1$, (A5.29) shows that the two-stage Young index is equal to its single-stage counterpart.

Carruthers, A.G., D.J. Sellwood, and P.W. Ward. 1980. "Recent Developments in the Retail Prices Index." *The Statistician* 29: 1–32.

Cobb, Charles, and Paul H. Douglas. 1928. "A Theory of Production." *American Economic Review* 18: 139–65.

Dalén, Jörgen 1992. "Computing Elementary Aggregates in the Swedish Consumer Price Index." *Journal of Official Statistics* 8: 129–47.

Diewert, W. Erwin. 1978. "Superlative Index Numbers and Consistency in Aggregation." *Econometrica* 46: 883–900.

Diewert, W. Erwin. 1980. "Aggregation Problems in the Measurement of Capital." In *The Measurement of Capital*, edited by Dan Usher, Studies in Income and Wealth, Vol. 45, National Bureau of Economics Research (pp. 433–528). Chicago: University of Chicago Press.

Diewert, W. Erwin. 1992. "Fisher Ideal Output, Input and Productivity Indexes Revisited." *Journal of Productivity Analysis* 3: 211–48.

Diewert, W. Erwin. 1993a. "The Early History of Price Index Research." In *Essays in Index Number Theory*, Volume 1, edited by W. Erwin Diewert and Alice O. Nakamura (pp. 33–65). Amsterdam: North-Holland.

Diewert, W. Erwin. 1993b. "Symmetric Means and Choice under Uncertainty." In *Essays in Index Number Theory*, Volume 1, edited by W. Erwin Diewert and Alice O. Nakamura (pp. 355–433). Amsterdam: North-Holland.

Diewert, W. Erwin. 1995. "Axiomatic and Economic Approaches to Elementary Price Indexes." Discussion Paper No. 95–01, Department of Economics, University of British Columbia, Vancouver, Canada.

Diewert, W. Erwin. 1997. "Commentary on Mathew D. Shapiro and David W. Wilcox: Alternative Strategies for Aggregating Price in the CPI." *The Federal Reserve Bank of St. Louis Review* 79 (3): 127–37.

Diewert, W. Erwin. 2001. "The Consumer Price Index and Index Number Purpose." *Journal of Economic and Social Measurement* 27: 167–248.

Diewert, W. Erwin. 2009. "Similarity Indexes and Criteria for Spatial Linking." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao (pp. 183–216). Cheltenham, UK: Edward Elgar.

Drobisch, Moritz W. 1871a. "Über die Berechnung der Veränderungen der Waarenpreise und des Geldwerths." *Jahrbücher für Nationalökonomie und Statistik* 16, 143–56.

Drobisch, Moritz W. 1871b. "Über einige Einwürfe gegen die in diesen Jahrbüchern veröffentlichte neue Methode, die Veränderungen der Waarenpreise und des Geldwerths zu berechten." *Jahrbücher für Nationalökonomie und Statistik* 16: 416–27.

Edgeworth, Francis Y. 1888. "Some New Methods of Measuring variation in General Prices." *Journal of the Royal Statistical Society* 51: 346–68.

Edgeworth, Francis Y. 1925. *Papers Relating to Political Economy*, Volume 1. New York: Burt Franklin.

Eichhorn, Wolfgang 1978. *Functional Equations in Economics*. Reading, MA: Addison-Wesley Publishing Company.

Eurostat. 2018. *Harmonized Index of Consumer Prices (HICP) Methodological Manual*. Luxembourg: Publications Office of the European Union.

Fisher, Irving. 1897. "The Role of Capital in Economic Theory." *Economic Journal* 7: 511–37.

Fisher, Irving. 1911. *The Purchasing Power of Money*. London: Macmillan.

Fisher, Irving. 1921. "The Best Form of Index Number." *Journal of the American Statistical Association* 17: 533–37.

Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton-Mifflin.

Frisch, Ragnar. 1930. "Necessary and Sufficient Conditions Regarding the Form of an Index Number which shall Meet Certain of Fisher's Tests." *Journal of the American Statistical Association* 25: 397–406.

Funke, H., G. Hacker, and Joachim Voeller. 1979. "Fisher's Circular Test Reconsidered." *Schweizerische Zeitschrift für Volkswirtshaft und Statistik* 115: 677–87.

Hardy, Godfrey H., John E. Littlewood, and George Pólya. 1934. *Inequalities*. Cambridge: Cambridge University Press.

Hill, Robert J. 1995. *Purchasing Power Methods of Making International Comparisons*, Ph. D. dissertation, Vancouver: The University of British Columbia.

Hill, Robert J. 1999a. "Comparing Price Levels across Countries Using Minimum Spanning Trees." *The Review of Economics and Statistics* 81: 135–42.

Hill, Robert J. 1999b. "International Comparisons using Spanning Trees." In *International and Interarea Comparisons of Income, Output and Prices*, edited by Alan Heston and Robert E. Lipsey (pp. 109–120), Studies in Income and Wealth Volume 61, NBER, Chicago: The University of Chicago Press.

Hill, Robert J. 2001. "Measuring Inflation and Growth Using Spanning Trees." *International Economic Review* 42: 167–85.

Hill, Robert J. 2009. "Comparing Per Capita Income Levels Across Countries Using Spanning Trees: Robustness, Prior Restrictions, Hybrids and Hierarchies." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao (pp. 217–244). Cheltenham UK: Edward Elgar.

Hill, T. Peter 1988. "Recent Developments in Index Number Theory and Practice." *OECD Economic Studies* 10: 123–48.

Hill, T. Peter 1993. "Price and Volume Measures." In *System of National Accounts 1993* (pp. 379–406). Eurostat, IMF, OECD, UN and World Bank, Luxembourg, Washington, DC, Paris, New York, and Washington, D.C.

Hill, T. Peter 1996. *Inflation Accounting: A Manual on National Accounting under Conditions of High Inflation*. Paris: OECD.

Jevons, William S. 1865. "The variation of Prices and the Value of the Currency since 1782." *Journal of the Statistical Society of London* 28: 294–320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119–50.

Konüs, Alexander A., and Sergei S. Byushgens. 1926. "K probleme pokupatelnoi cili deneg." *Voprosi Konyunkturi* 2: 151–72.

Knibbs, Sir George H. 1924. "The Nature of an Unequivocal Price Index and Quantity Index." *Journal of the American Statistical Association* 19: 42–60 and 196–205.

Laspeyres, Etienne 1871. "Die Berechnung einer mittleren Waarenpreissteigerung." *Jahrbücher für Nationalökonomie und Statistik* 16: 296–314.

Lehr, Julius 1885. *Beitrage zur Statistik der Preise*. Frankfurt: J.D. Sauerlander.

Lowe, Joseph 1823. *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition. London: Longman, Hurst, Rees, Orme and Brown.

Marshall, Alfred 1887. "Remedies for Fluctuations of General Prices." *Contemporary Review* 51, 355–75.

Paasche, Hermann 1874. "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen." *Jahrbücher fur Nationalökonomie und Statistik* 12, 168–78.

Pollak, Robert A. 1983. "The Theory of the Cost-of-Living Index." In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette (pp. 87–161). Ottawa: Statistics Canada; reprinted as pp. 3–52 in R.A. Pollak, *The Theory of the Cost-of-Living Index*. Oxford: Oxford University Press, 1989; also reprinted as pp. 5–77 in *Price Level Measurement*, edited by W. Erwin Diewert. Amsterdam: North-Holland, 1990.

Schlömilch, Oskar. 1858. "Über Mïttelgrössen verschiedener Ordnungen." *Zeitschrift für Mathematik und Physik* 3: 308–10.

Schultz, Bohdan. 1999. "Effects of Using various Macro-Index Formulae in Longitudinal Price and Comparisons: Empirical

Studies." In *Proceedings of the Fourth Meeting of the International Working Group on Price Indices*, edited by W. Lane (pp. 236–249). Washington, DC: Bureau of Labor Statistics, U.S. Department of Labor.

Szulc, Bohdan J. 1983. "Linking Price Index Numbers." *in Price Level Measurement*, W. Erwin Diewert and Claude Montmarquette (pp. 537–66). Ottawa: Statistics Canada.

Sidgwick, Henry. 1883. *The Principles of Political Economy*. London: Macmillan.

Törnqvist, Leo. 1936. "The Bank of Finland's Consumption Price Index." *Bank of Finland Monthly Bulletin* 10: 1–8.

Törnqvist, Leo, and Egil Törnqvist. 1937. "Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1–39 reprinted as pp. 121–60 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

Triplett, Jack E. 1981. "Reconciling the CPI and the PCE Deflator." *Monthly Labor Review* (September): 3–15.

Vogt, A. 1980. "Der Zeit und der Faktorumkehrtest als 'Finders of Tests'." *Statistische Hefte* 21: 66–71.

Vogt, Arthur, and János Barta. 1997. *The Making of Tests for Index Numbers*. Heidelberg: Physica-Verlag.

von Bortkiewicz, Ladislaus. 1923. "Zweck und Struktur einer Preisindexzahl." *Nordisk Statistisk Tidsskrift* 2, 369–408.

von der Lippe, Peter. 2001. *Chain Indices: A Study in Price Index Theory*, Volume 16 of the Publication Series Spectrum of Federal Statistics, Wiesbaden: Statistisches Bundesamt.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Walsh, C. Moylan. 1921a. *The Problem of Estimation*. London: P.S. King & Son.

Walsh, C. Moylan. 1921b. "Discussion." *Journal of the American Statistical Association* 17: 537–44.

Walsh, C. Moylan. 1932. "Index Numbers." In *Encyclopedia of the Social Sciences*, Volume 7, edited by Edwin R. A. Seligman (editor in chief) (pp. 652–58). New York: The Macmillan Co.

Westergaard, Harald. 1890. *Die Grundzüge der Theorie der Statistik*. Jena: Fischer.

Young, Arthur. 1812. *An Inquiry into the Progressive Value of Money in England as Marked by the Price of Agricultural Products*. London: McMillan.

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THE AXIOMATIC OR TEST APPROACH TO INDEX NUMBER THEORY* 3

1. Introduction

As was seen in Chapter 2, it was useful to be able to evaluate various index number formulae that have been proposed in terms of their properties. If a formula turns out to have rather undesirable properties, then doubt is cast on its suitability as an index that could be used by a statistical agency as a target index. Looking at the mathematical properties of index number formulae leads to the *test* or *axiomatic approach to index number theory*. In this approach, desirable properties for an index number formula are proposed, and then it is attempted to determine whether any formula is consistent with these properties or tests. An ideal outcome is the situation where the proposed tests are both desirable and completely determine the functional form for the formula.

The axiomatic approach to index number theory is not completely straightforward since choices have to be made in two dimensions:

- The index number framework must be determined.
- Once the framework has been decided upon, it must be decided what tests or properties should be imposed on the index number.

The second point is straightforward: different price statisticians may have different ideas about what tests are important, and alternative sets of axioms can lead to alternative "best" index number functional forms. This point must be kept in mind while reading this chapter, since there is no universal agreement on the "best" set of "reasonable" axioms. Hence, the axiomatic approach can lead to more than one "best" index number formula.

The first point about choices listed here requires further discussion. In the previous chapter, for the most part, the focus was on *bilateral index number theory*; that is, it was assumed that prices and quantities for the same *N* commodities were given for two periods and the object of the index number formula was to compare the overall level of prices in one period with that in the other period. In this framework, both sets of price and quantity vectors were regarded as variables that could be *independently varied* so that, for example, variations in the prices of one period did not affect the prices of the other period or the quantities in either period. The emphasis was on comparing the overall cost of a fixed basket of quantities in the two periods or taking averages of such fixed basket indices. This is an example of an index number framework.

However, other index number frameworks are possible. For example, instead of decomposing a value ratio into a term that represents price change between the two periods times another term that represents quantity change, one could attempt to decompose a value aggregate for one period into a single number that represents the price level in the period times another number that represents the quantity level in the period. In this approach, the price level is supposed to be a function of the *N* commodity prices pertaining to that aggregate in the period under consideration, and the quantity level is supposed to be a function of the *N* commodity quantities pertaining to the aggregate in the period. The resulting price level function was called an *absolute index number* by Frisch (1930; 397), a *price level* by Eichhorn (1978; 141), and a *unilateral price index* by Anderson, Jones, and Nesmith (1997; 75). This approach to index number theory (in the context of the axiomatic approach) will be considered in Section 2.[1]

The remaining approaches in this chapter are largely bilateral approaches; that is, the prices and quantities in an aggregate are compared for two periods. In Sections 3 and 4, the value ratio decomposition approach is taken. In Sections 3–6, the bilateral price and quantity indices, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, are regarded as functions of the price vectors pertaining to the two periods, p^0 and p^1, and the two quantity vectors, q^0 and q^1. The axioms or tests that are placed on the price index function $P(p^0,p^1,q^0,q^1)$ not only reflect "reasonable" price index properties but some tests have their origin as "reasonable" tests on the companion quantity index $Q(p^0,p^1,q^0,q^1)$. The approach taken in Sections 3 and 4 simultaneously determines the "best" price and quantity indices. Section 5 looks at the test performance of various bilateral index number formulae.

The implications of the circularity test are examined in Section 6. It turns out that only a few index number formulae satisfy this test. Recall that in Chapter 2, the Lowe index was introduced. This index does not fit precisely into the bilateral framework since the quantity weights used in this index do not necessarily correspond to the quantities that pertain to either of the periods, which are characterized by the price vectors p^0 and p^1. In Section 6, the axiomatic properties of the class of indices of the form $P(p^0,p^1,q)$ will be discussed.

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[1] In Chapters 7 and 8, the levels approach will be considered again but from the viewpoint of the economic approach to index number theory rather than the test approach. Using the economic approach, quantities cannot be varied independently of prices.

One of Walsh's (1921a) approaches to index number theory[2] was an attempt to determine the "best" weighted average of the price relatives, r_n, where $r_n \equiv p_n^1/p_n^0$ for $n = 1,\ldots,N$. This is equivalent to using an axiomatic approach to try and determine the "best" index of the form $P^*(r,v^0,v^1)$, where $r \equiv [r_1,\ldots,r_N]$ and v^t is a vector of expenditures on the N commodities during period t for $t = 0,1$. This approach will be considered in Sections 7 and 8.[3]

In Section 9, the index number framework explained in Sections 7 and 8 is used to develop a methodological approach to the problem of decomposing overall price change into additive components that are functions of the percentage changes in individual commodity prices.

An annex provides proofs of some complex results.

2. The Test Approach to Index Number Theory Using Price Levels

Denote the price and quantity of commodity i in period t by p_i^t and q_i^t, respectively for $i = 1,2,\ldots,N$ and $t = 0,1,\ldots,T$. The variable q_i^t is interpreted as the total amount of commodity i transacted within period t. In order to conserve the value of transactions, it is necessary that p_i^t be defined as a unit value; that is, p_i^t must be equal to the value of transactions in commodity i for period t divided by the total quantity transacted, q_i^t. In principle, the period of time should be chosen so that variations in commodity prices within a period are very small compared to their variations between periods.[4] For $t = 0,1,\ldots,T$ and $i = 1,\ldots,N$, define the value of transactions in commodity i as $v_i^t \equiv p_i^t q_i^t$ and define the *total value of transactions in period t* as

$$V^t \equiv \Sigma_{i=1}^N v_i^t = \Sigma_{i=1}^N p_i^t q_i^t;\ t = 0,1,\ldots,T. \tag{1}$$

Using this notation, the following *levels version of the index number problem* is defined as follows: For $t = 0,1,\ldots,T$, find scalar numbers P^t and Q^t such that

$$V^t = P^t Q^t;\ t = 0,1,\ldots,T. \tag{2}$$

The number P^t is interpreted as an aggregate period t price level, while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t, while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector q^t; hence,

$$P^t = c(p^t);\ Q^t = f(q^t);\ t = 0,1,\ldots,T. \tag{3}$$

The functions c and f are to be determined somehow. Note that (3) requires that the functional forms for the price aggregation function c and for the quantity aggregation function f be independent of time. This is a reasonable requirement, since there is no reason to change the method of aggregation as time changes.

Substituting (3) and (2) into (1) and dropping the superscripts t means that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

$$c(p)f(q) = \Sigma_{i=1}^N p_i q_i \text{ for all } p_i > 0 \text{ and all } q_i > 0. \tag{4}$$

It is natural to assume that the functions $c(p)$ and $f(q)$ be positive if all prices and quantities are positive:

$$c(p_1,\ldots,p_N) > 0;\ f(q_1,\ldots,q_N) > 0 \text{ if all } p_i > 0 \text{ and all } q_i > 0. \tag{5}$$

Let 1_N denote an n dimensional vector of ones. Then (5) implies that when $p = 1_N$, $c(1_N)$ is a positive number, $a > 0$, and when $q = 1_N$, then $f(1_N)$ is also a positive number, $b > 0$; that is, (5) implies that c and f satisfy

$$c(1_N) = a > 0;\ f(1_N) = b > 0. \tag{6}$$

Let $p = 1_N$, and substitute the first equation in (6) into (4) in order to obtain the following equation:

$$f(q) = \Sigma_{i=1}^N q_i/a \text{ for all } q_i > 0. \tag{7}$$

Now let $q = 1_N$, and substitute the second equation in (6) into (4) in order to obtain the following equation:

$$c(p) = \Sigma_{i=1}^N p_i/b \text{ for all } p_i > 0. \tag{8}$$

Finally, substitute (7) and (8) into the left-hand side of (4), and the following equation is obtained:

$$[\Sigma_{i=1}^N q_i/a][\Sigma_{i=1}^N p_i/b] = \Sigma_{i=1}^N p_i q_i \text{ for all } p_i > 0 \text{ and all } q_i > 0. \tag{9}$$

If N is greater than one, it is obvious that equation (9) cannot be satisfied for all strictly positive p and q vectors. Thus,

[2] Walsh (1901) also considered basket-type approaches to index number theory, as was seen in Chapter 2.

[3] In Section 7, rather than starting with indices of the form $P(r,v^0,v^1)$, indices of the form $P(p^0,p^1,v^0,v^1)$ are considered. However, if the invariance to changes in the units of measurement test is imposed on this index, it is equivalent to studying indices of the form $P(r,v^0,v^1)$. vartia (1976) also used a variation of this approach to index number theory.

[4] This treatment of prices as unit values over time follows Walsh (1901; 96) (1921a; 88) and Fisher (1922; 318). Fisher and Hicks both had the idea that the length of the period should be short enough so that variations in price for a single commodity within the period could be ignored as the following quotations indicate: "Throughout this book 'the price' of any commodity or 'the quantity' of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered throughout the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since, ordinarily, the variation during a year, so far, at least, as prices are concerned, is too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point. The quantities sold will, of course, vary widely. What is needed is their sum for the year (which, of course, is the same thing as the simple arithmetic average of the per annum rates for the separate months or other subdivisions). In short, the simple arithmetic average, both of prices and of quantities, may be used. Or, if it is worthwhile to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold" (Irving Fisher (1922; 318)). "*I* shall define a week as that period of time during which variations in prices can be neglected. For theoretical purposes this means that prices will be supposed to change, not continuously, but at short intervals. The calendar length of the week is of course quite arbitrary; by taking it to be very short, our theoretical scheme can be fitted as closely as we like to that ceaseless oscillation which is a characteristic of prices in certain markets" (John Hicks (1946; 122)).

if the number of commodities N exceeds one, then there do not exist any functions c and f that satisfy (4) and (5).[5]

Thus, this *levels test approach* to index number theory comes to an abrupt halt; it is fruitless to look for price and quantity level functions, $P^t = c(p^t)$ and $Q^t = f(q^t)$, that satisfy (2) or (4) and also satisfy the very reasonable positivity requirements (5).[6] Thus, in the following sections of this chapter, the levels approach to price measurement will be replaced by the bilateral comparisons approach that was used in Chapter 2.

3. Tests for Bilateral Price Indices

In this section and the following section, the strategy will be to assume that the bilateral price index formula, $P(p^0,p^1,q^0,q^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[7] The word "bilateral"[8] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; that is, P is regarded as a function of the two sets of price and quantity vectors, p^0,p^1,q^0,q^1, that are to be aggregated into a single number that summarizes the overall change in the N price ratios, $p_1^1/p_1^0,\ldots,p_N^1/p_N^0$.

The bilateral price index function, $P(p^0,p^1,q^0,q^1)$, is assumed to be well defined if all prices and quantities are positive for the two periods under consideration. If a commodity is missing in both periods, then it can simply be ignored. However, if it is missing in one period but not in the other period, this can create problems; that is, $P(p^0,p^1,q^0,q^1)$ may not be well defined if one or more prices or quantities are equal to 0.[9] In this case, where some prices and quantities may be equal to 0, we assume that the following conditions hold:[10]

$$p^0 > 0_N;\ p^1 > 0_N;\ q^0 > 0_N;\ q^1 > 0_N;\ p^0\cdot q^0 > 0;\quad p^1\cdot q^1 > 0;\ p^0\cdot q^1 > 0;\ p^1\cdot q^0 > 0. \tag{10}$$

In the remainder of this chapter, we assume that p^0,p^1,q^0,q^1 satisfy either the strict positivity conditions, $p^0 >> 0_N$; $p^1 >> 0_N$; $q^0 >> 0_N$; $q^1 >> 0_N$, or the weaker conditions (10).[11]

In this section, the value ratio decomposition approach to index number theory will be taken; that is, along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indices equals the value ratio between the two periods.[12] Thus, throughout this chapter, it is assumed that P and Q satisfy the following *product test*:

$$V^1/V^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1). \tag{11}$$

The period t values, V^t, for $t = 0,1$ are defined by (1). Equation (11) means that as soon as the functional form for the price index P is determined, (11) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that if a reasonable test is imposed on the quantity index Q, then (11) can be used to translate this test on the quantity index into a corresponding test on the price index P.[13]

If $N = 1$, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0, the single price ratio, and a natural candidate for Q is q_1^1/q_1^0, the single quantity ratio. If the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multidimensional analogues to the one good price index formula, p_1^1/p_1^0. Here, some 20 tests are listed that turn out to characterize the Fisher ideal price index. If it is desired to set $q^0 = q^1$, the common quantity vector is denoted by q; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

The first two tests are not very controversial, and so they will not be discussed in detail.

T1: *Positivity*[14]: $P(p^0,p^1,q^0,q^1) > 0$.
T2: *Continuity*[15]: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

The next two tests are somewhat more controversial.

T3: *Identity or Constant Prices Test*[16]: $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.

T4: *Fixed Basket or Constant Quantities Test*[17]: $P(p^0,p^1,q,q) = p^1\cdot q/p^0\cdot q$.

[5] Eichhorn (1978; 144) established this result.
[6] It is important to keep in mind that this result follows under the assumption that prices and quantities can vary *independently* from each other. When taking the economic approach to index number theory in Chapters 5 and 8, prices can vary independently, but quantities will depend on prices, so *quantities cannot vary independently from prices.* Thus, when taking the economic approach, it is quite possible to find functions $c(p)$ and $f(q)$ such that $c(p)f(q) = \Sigma_{n=1}^N p_nq_n$.
[7] Much of the material in this section is drawn from Sections 2 and 3 of Diewert (1992). For subsequent surveys of the axiomatic approach, see Balk (1995) (2008).
[8] Multilateral index number theory refers to the case where there are more than two situations whose prices and quantities need to be aggregated.
[9] The problems caused by missing prices and quantities will be addressed in Chapters 7 and 8.
[10] Notation: $p >> 0_N$ means each component of p is positive, $p \geq 0_N$ means each component of p is nonnegative, $p > 0_N$ means $p \geq 0_N$ and $p \,\Sigma\, 0_N$ and $p\cdot q \equiv \Sigma_{n=1}^N p_nq_n$, where $p \equiv [p_1,\ldots,p_N]$ and $q \equiv [q_1,\ldots,q_N]$.
[11] Test T14 requires the additional assumption of strict positivity of the base period prices; that is, T14 requires that $p^0 >> 0_N$.
[12] See Section 2 of Chapter 2 for more on this approach, which was initially due to Fisher (1911; 403) (1922).
[13] This observation was first made by Fisher (1911; 400–406). Vogt (1980) and Diewert (1992) also pursued this idea.
[14] Eichhorn and Voeller (1976, 23) suggested this test.
[15] Fisher (1922; 207–215) informally suggested the essence of this test.
[16] Laspeyres (1871; 308), Walsh (1901; 308), and Eichhorn and Voeller (1976; 24) have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871), which does not satisfy this test. This test is also a special case of Fisher's (1911; 409–410) price proportionality test. This test could be called the *strong identity test*. The corresponding *weak identity test* is $P(p,p,q,q) = 1$; that is, if *both* prices and quantities are equal for the two periods under consideration, then the price index should equal unity. This version of the identity test is not controversial.
[17] The origins of this test go back at least 200 years to the Massachusetts legislature, which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution;

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i \equiv p^1 \cdot q$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i \equiv p^0 \cdot q$.

If the price index P satisfies test T4 and P and Q jointly satisfy the product test, (11), then it is easy to show[18] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors p^0,p^1,q. This *constant quantities test* for Q is also somewhat controversial since p^0 and p^1 are allowed to be different.[19]

The following four tests restrict the behavior of the price index P as the *scale* of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in Current Prices*[20]: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number l, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree 1 in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related *proportionality test* $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact, Walsh (1901, 385) noted that this last test implies the identity test T3.

In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T6: *Inverse Proportionality in Base Period Prices*[21]: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1)$ if $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number l, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus 1 in the components of the period 0 price vector p^0.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*:

$$P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1) \text{ for all } \lambda > 0.$$

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree 0 in the components of the period 1 quantity vector q^1. Vogt (1980, 70) was the first to propose this test,[22] and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; that is, suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then, using the product test (11), it can be seen that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities*[23]:

$$P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1) \text{ for all } \lambda > 0.$$

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree 0 in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (11) the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

The next five tests are *invariance* or *symmetry tests*. Fisher (1922; 62–63, 458–460) and Walsh (1901; 105) (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62–63) spoke of fairness, but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as will be done in Section 4. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1),$$

where p^{t*} denotes a permutation of the components of the vector p^t, and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test was developed by Irving Fisher (1922; 63),[24] and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test, which will be considered subsequently.

see Willard Fisher (1913). Other researchers who have suggested the test over the years include Lowe (1823, Annex, 95), Scrope (1833, 406), Jevons (1865), Sidgwick (1883, 67–68), Edgeworth (1925, 215) originally published in 1887, Marshall (1887, 363), Pierson (1895, 332), Walsh (1901, 540) (1921b; 543–544), and Bowley (1901, 227). Vogt and Barta (1997; 49) correctly observed that this test is a special case of Fisher's (1911; 411) proportionality test for quantity indices that Fisher (1911; 405) translated into a test for the price index using the product test (11).

[18] See Vogt (1980; 70).

[19] A weaker version of Tests 3 and 4 is the following test: T3*: $P(p,p,q,q) = 1$ for all $p > 0_N$ and $q > 0_N$. Obviously, if prices and quantities are identical in the two periods under consideration, it is very reasonable that the bilateral price index (and the companion quantity index) equals unity. However, if prices are identical across periods but the two quantity vectors are different, then it is not clear that a bilateral index number formula that uses quantity or value weights should equal 1. An example of a weighted bilateral index number formula that satisfies T3* but not T3 is *the unit value price index* defined by $P_{UV}(p^0,p^1,q^0,q^1) \equiv [p^1 \cdot q^1/1_N \cdot q^1]/[p^0 \cdot q^0/1_N \cdot q^0]$. The properties of unit value price indices will be studied in Chapter 7.

[20] This test was proposed by Walsh (1901, 385), Eichhorn and Voeller (1976, 24), and Vogt (1980, 68).

[21] Eichhorn and Voeller (1976; 28) suggested this test.

[22] Fisher (1911; 405) proposed the related test $P_L(p^0,p^1,q^0,\lambda q^0) = P_L(p^0,p^1,q^0,q^0) \equiv p^1 \cdot q^0/p^0 \cdot q^0$.

[23] This test was proposed by Diewert (1992; 216).

[24] "This [test] is so simple as never to have been formulated. It is merely taken for granted and observed instinctively. Any rule for averaging the commodities must be so general as to apply interchangeably to all of the terms averaged" (Irving Fisher (1922; 63)).

The next test asks that the index be invariant to changes in the units of measurement.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ \alpha_1^{-1} q_1^0,\ldots,\alpha_N^{-1} q_N^0;\ \alpha_1^{-1} q_1^1,\ldots,\alpha_N^{-1} q_N^1) =$
$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ q_1^0,\ldots,q_N^0;\ q_1^1,\ldots,q_N^1)$ for all $\alpha_1 > 0,\ldots,\alpha_N > 0$.

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1863; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test*, and later, Fisher (1922; 420) called it the *commensurability test*.

The next test asks that the formula be invariant to the period chosen as the base period.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test will be satisfied (as are all of the other tests listed in this section). When the number of goods is greater than 1, many commonly used price indices fail this test; for example, the Laspeyres (1871) price index, P_L, defined by $P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0 / p^0 \cdot q^0$, and the Paasche (1874) price index, P_P, defined by $P_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / p^0 \cdot q^1$, both fail this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test; he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

The next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory.[25] However, these tests are quite consistent with the weighted stochastic approach to index number theory, which is discussed later in this chapter.

T12: *Quantity Reversal Test* (quantity weights symmetry test): $P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even-handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

[25]The economic approach to index number theory assumes that given prices (and income), households choose quantity vectors that maximize their welfare or utility. Thus, when prices change, in general household consumption vectors will change. Thus, tests 12 and 13 are not consistent with the economic approach to bilateral index number theory.

The next test is the analogue to T12 applied to quantity indices:

T13: *Price Reversal Test* (price weights symmetry test)[26]:

$$[p^1 \cdot q^1 / p^0 \cdot q^0]/P(p^0,p^1,q^0,q^1) = [p^0 \cdot q^1 / p^1 \cdot q^0]/P(p^1,p^0,q^0,q^1).$$

Thus, if we use (11) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

$$Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1). \tag{12}$$

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus, if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests.

T14: *Mean Value Test for Prices*[27]:

$$min_n\,\{p_n^1/p_n^0;\ n = 1,\ldots,N\} \le P(p^0,p^1,q^0,q^1) \le max_n\,\{p_n^1/p_n^0;\ n = 1,\ldots,N\}. \tag{13}$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be interpreted as some sort of an average of the N price ratios, p_n^1/p_n^0, it seems essential that the price index P satisfy this test.

The next test is the analogue to T14 applied to quantity indices:

T15: *Mean Value Test for Quantities*[28]:

$$min_n\,\{q_n^1/q_n^0;\ n = 1,\ldots,N\} \le [V^1/V^0]/P(p^0,p^1,q^0,q^1) \le max_n\,\{q_n^1/q_n^0;\ n = 1,\ldots,N\}, \tag{14}$$

where $V^t \equiv p^t \cdot q^t$ is the period t value for the aggregate defined by (1). Using the product test (11) to define the quantity index Q in terms of the price index P, it can be seen that T15 is equivalent to the following property for the associated quantity index Q:

$$min_n\,\{q_n^1/q_n^0;\ n = 1,\ldots,N\} \le Q(p^0,p^1,q^0,q^1) \le max_n\,\{q_n^1/q_n^0;\ n = 1,\ldots,N\}. \tag{15}$$

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_n^1/q_n^0 of the individual quantities.

In Section 4 of Chapter 2, it was argued that it was very reasonable to take an average of the Laspeyres and Paasche

[26]This test was proposed by Diewert (1992; 218).

[27]In the present context, this test seems to have been first proposed by Eichhorn and Voeller (1976; 10). Samuelson (1947) and Pollak (1971) showed that this test was satisfied by the Konüs true cost of living index, which will be considered in Chapter 5. Note that this test requires that $p^0 \gg 0_N$ so that all of the price ratios p_n^1/p_n^0 are well defined.

[28]This test was proposed by Diewert (1992; 219). Note that this test requires that $q^0 \gg 0_N$ so that the quantity ratios q_n^1/q_n^0 are well defined.

price indices as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test*[29]: The price index P lies between the Laspeyres and Paasche indices, $P_L \equiv p^1 \cdot q^0 / p^0 \cdot q^0$ and $P_P \equiv p^1 \cdot q^1 / p^0 \cdot q^1$.

A test could be proposed where the implicit quantity index Q that corresponds to P via (11) is to lie between the Laspeyres and Paasche quantity indices, Q_P and Q_L, defined as follows:

$$Q_L(p^0,p^1,q^0,q^1) \equiv p^0 \cdot q^1 / p^0 \cdot q^0;\ Q_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / p^1 \cdot q^0. \qquad (16)$$

However, the resulting test turns out to be equivalent to test T16.

The final four tests are monotonicity tests; that is, how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^{1*},q^0,q^1)$ if $p^1 < p^{1*}$.

That is, if some period 1 price increases, then the price index must increase so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976; 23), and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^{0*},p^1,q^0,q^1)$ if $p^0 < p^{0*}$.

That is, if any period 0 price increases, then the price index must decrease so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0. This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^{1*}$, then

$$[p^1 \cdot q^1 / p^0 \cdot q^0] / P(p^0,p^1,q^0,q^1) < [p^1 \cdot q^{1*} / p^0 \cdot q^0] / P(p^0,p^1,q^0,q^{1*}). \qquad (17)$$

T20: *Monotonicity in Base Quantities*: if $q^0 < q^{0*}$, then

$$[p^1 \cdot q^1 / p^0 \cdot q^0] / P(p^0,p^1,q^0,q^1) > [p^1 \cdot q^1 / p^0 \cdot q^{0*}] / P(p^0,p^1,q^{0*},q^1). \qquad (18)$$

Let Q be the implicit quantity index that corresponds to P using (11). Then it is found that T19 translates into the following inequality involving Q:

$$Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^{1*}) \text{ if } q^1 < q^{1*}. \qquad (19)$$

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into

$$Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^{0*},q^1) \text{ if } q^0 < q^{0*}. \qquad (20)$$

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 were developed by Vogt (1980, 70).

[29] Both Bowley (1901; 227) and Fisher (1922; 403) endorsed this property for a price index.

This concludes the listing of tests. In the next section, we ask whether any index number formula $P(p^0,p^1,q^0,q^1)$ exists that can satisfy all 20 tests.

4. The Fisher Ideal Index and the Test Approach

It can be shown that the only index number formula $P(p^0,p^1,q^0,q^1)$ that satisfies tests T1–T20 is the Fisher ideal price index P_F defined as the geometric mean of the Laspeyres and Paasche indices:[30]

$$P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1) P_P(p^0,p^1,q^0,q^1)]^{1/2}, \qquad (21)$$

where $P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0 / p^0 \cdot q^0$ and $P_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / p^0 \cdot q^1$.

To prove this assertion, it is relatively straightforward to show that the Fisher index satisfies all 20 tests. The more difficult part of the proof is to show that it is the *only* index number formula that satisfies these tests. This part of the proof follows from the fact that if P satisfies the positivity test T1 and the three reversal tests, T11–T13, then P must equal P_F. To see this, rearrange the terms in the statement of test T13 into the following equation:

$$\begin{aligned} [p^1 \cdot q^1 / p^0 \cdot q^0] / [p^0 \cdot q^1 / p^1 \cdot q^0] &= P(p^0,p^1,q^0,q^1) / P(p^1,p^0,q^0,q^1) \qquad (22) \\ &= P(p^0,p^1,q^0,q^1) / P(p^1,p^0,q^1,q^0) \text{ using } T12, \\ &\quad \text{the quantity reversal test} \\ &= P(p^0,p^1,q^0,q^1) P(p^0,p^1,q^0,q^1) \text{ using } T11, \\ &\quad \text{the time reversal test.} \end{aligned}$$

Now take positive square roots on both sides of (22), and it can be seen that the left-hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by (21) and the right-hand side is $P(p^0,p^1,q^0,q^1)$. Thus, if P satisfies T1, T11, T12, and T13, it must equal the Fisher ideal index P_F.

The quantity index that corresponds to the Fisher price index using the product test (11) is Q_F, the Fisher quantity index, defined as follows:

$$\begin{aligned} Q_F(p^0,p^1,q^0,q^1) &\equiv [V^1/V^0] / P_F(p^0,p^1,q^0,q^1) \\ &= [Q_L(p^0,p^1,q^0,q^1) Q_P(p^0,p^1,q^0,q^1)]^{1/2}, \qquad (23) \end{aligned}$$

where the Laspeyres and Paasche quantity indices, Q_L and Q_P, are defined by (16). Thus, the Fisher quantity index that corresponds via the product test (11) to the Fisher price index is also equal to the geometric mean of the Laspeyres and Paasche quantity indices.

It turns out that P_F satisfies yet another test, T21, which was Irving Fisher's (1921; 534) (1922; 72–81) *third reversal test* (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):

$$P(p^0,p^1,q^0,q^1) P(q^0,q^1,p^0,p^1) = V^1/V^0. \qquad (24)$$

[30] See Diewert (1992; 221).

A justification for this test is the following one: If $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if the roles of prices and quantities are reversed, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument), and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1/V^0. The second part of this argument does not seem to be valid, and thus many researchers over the years have objected to the factor reversal test. However, if one is willing to embrace T21 as a basic test, Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ that satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test), and T21 (factor reversal test) is the Fisher ideal index P_F defined by (21). Thus, the price reversal test T13 can be replaced by the factor reversal test in order to obtain a minimal set of four tests that lead to the Fisher price index.[31]

5. The Test Performance of Other Indices

The Fisher price index P_F satisfies all 20 of the tests listed in Section 3. Which tests do other commonly used price indices satisfy? The Laspeyres, Paasche, and Fisher price indices, P_L, P_P, and P_F, have been defined earlier. Two other indices that played a prominent role in Chapter 2 were the Walsh and Törnqvist indices defined as follows:

$$P_W(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} [q_n^0 q_n^1]^{1/2} p_n^1 / \Sigma_{n=1}^{N} [q_n^0 q_n^1]^{1/2} p_n^0, \quad (25)$$

$$P_T(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0], \quad (26)$$

where $s_n^t \equiv p_n^t q_n^t / p^t \cdot q^t$ for $n = 1,\ldots,N$ and $t = 0,1$. Note that in order that $P_T(p^0,p^1,q^0,q^1)$ be well defined, it is required that $p^0 >> 0_N$; that is, each base period price p_n^0 must be positive.

Straightforward computations show that the Paasche and Laspeyres price indices, P_L and P_P, fail only the three reversal tests, T11, T12, and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of P_L and P_P seem at first sight to be quite good. However, the failure of the time reversal test, T11, is a severe limitation associated with the use of these indices.

The Walsh price index, P_W, fails four tests: T13, the price reversal test; T16, the Paasche and Laspeyres bounding test; T19, the monotonicity in current quantities test; and T20, the monotonicity in base quantities test.

Finally, the Törnqvist price index P_T fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test), and the four monotonicity tests T17 to T20. Thus, the Törnqvist index is subject to a rather high failure rate from the viewpoint of this axiomatic approach to index number theory.[32]

The tentative conclusion that can be drawn from these results is that from the viewpoint of this particular bilateral test approach to index numbers, the Fisher ideal price index P_F appears to be "best" since it satisfies all 20 tests.[33] The Paasche and Laspeyres indices are next best if we treat each test as being equally important. However, both of these indices fail the very important time reversal test. The remaining two indices, the Walsh and Törnqvist price indices, both satisfy the time reversal test, but the Walsh index emerges as being "better" since it passes 16 of the 20 tests, whereas the Törnqvist only satisfies 11 tests. However, in Section 7, we will change the axiomatic framework, and in this new framework, the Törnqvist price index will emerge as "best" in this alternative framework. Before this new framework is considered, one important additional test in the present axiomatic framework will be discussed in the following section.

6. The Circularity Test

If the identity test T3 is true, then the time reversal test T11 can be rewritten as follows:

$$1 = P(p^0,p^0,q^0,q^0) = P(p^0,p^1,q^0,q^1)P(p^1,p^0,q^1,q^0). \quad (27)$$

Thus, if one starts out with the prices p^0 in period 0 and go to the prices p^1 in period 1 but then returns to the prices of period 0 in period 2, and if the tests T3 and T11 are satisfied, then the product of the price movement from period 0 to 1, $P(p^0,p^1,q^0,q^1)$, and the price movement from period 1 to 2, $P(p^1,p^0,q^1,q^0)$, turns out to equal 1, indicating that the *chained price index* in period 2 has returned to its period 0 level of 1. An obvious generalization of (27) would be to replace the assumption that the period 2 price and quantity vectors in this formula are the same as the period 0 price and quantity vectors, p^0 and q^0, and allow for arbitrary period 2 price and quantity vectors, p^2 and q^2. With this replacement, (27) becomes

$$P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2). \quad (28)$$

If an index number formula P satisfies (28), then we say that P satisfies the *circularity test*.[34]

What is the meaning of (28)? The index number on the left-hand side of (28) compares prices in period 2 directly with prices in period 0, and $P(p^0,p^2,q^0,q^2)$ is called the *fixed-base price index* for period 2. The *chained price index* for period 2, $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$, on the right-hand side of (28) compares prices in period 2 with those in period 0 by first comparing prices in period 1 with those in period 0 (this is the *chain link index* $P(p^0,p^1,q^0,q^1)$) and multiplies that index by the chain link index that compares prices in period

[31] Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985) (1995).

[32] However, it will be shown later in Chapter 5 that the Törnqvist index approximates the Fisher index quite closely using "normal" time series data that are subject to relatively smooth trends. Hence, under these circumstances, the Törnqvist index can be regarded as passing the 20 tests to a reasonably high degree of approximation.

[33] This assertion needs to be qualified: There are many other tests that we have not discussed, and price statisticians could differ on the importance of satisfying various sets of tests. Some references that discuss other tests are Auer (2002), Eichhorn and Voeller (1976), Balk (1995) (2008), and Vogt and Barta (1997). In Section 7, it is shown that the Törnqvist index is "best" for a different set of axioms.

[34] The test was named after Fisher (1922; 413), and the concept was originally proposed by Westergaard (1890; 218–219).

2 to those of period 1, $P(p^1,p^2,q^1,q^2)$. If the index number formula P satisfies the circularity test (28), then it does not matter whether we use the *chained index* (the right-hand side of (28)) to compare prices in period 2 with those of the base period 0 or if we use the *fixed-base index* (the left-hand side of (28)): *We get the same answer either way*. Obviously, it would be desirable if one could find an index number formula that satisfied the circularity test and had satisfactory axiomatic properties with respect to the other tests that have been considered.

Unfortunately, it turns out that index number formulae that satisfy the circularity test have other properties that make it unsatisfactory. Consider the following result:

Proposition 1

Suppose that the index number formula P satisfies the following tests: T1 (positivity), T2 (continuity), T3 (identity), T5 (proportionality in current prices), T10 (commensurability), and T17 (monotonicity in current prices) in addition to the circularity test above. Then, P must have the following functional form due originally to Konüs and Byushgens[35] (1926; 163–166):[36]

$$P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0]^{\alpha_n}, \qquad (29)$$

where the N constants α_n satisfy the conditions $\Sigma_{n=1}^{N} \alpha_n = 1$ and $\alpha_n > 0$ for $n = 1,\ldots,N$.

A proof of this result is in the annex. This result says that under fairly weak regularity conditions, *the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios*, the weights being constant through time. The α_n weights could be chosen to be the average expenditure shares on the N commodities over the time period when the index number formula is being used. If expenditure shares are close to being constant over the sample period, the resulting weighted geometric mean index defined by (29) will be a perfectly good index. However, if there are strong (divergent) trends in expenditure shares and strong (divergent) trends in the prices of the N commodities, then the index will not have representative weights over the entire sample period and thus will not be able to adequately represent price movements over the entire sample period.[37]

An interesting special case of the family of indices defined by (29) occurs when the weights α_n are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

$$P_J(p^0,p^1,q^0,q^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0]^{1/N}. \qquad (30)$$

The problem with the indices defined by Konüs and Byushgens and Jevons is that the individual price ratios, p_n^1/p_n^0, have weights (either α_n or $1/N$) that are *independent* of the economic importance of commodity n in the two periods under consideration. Put another way, these price weights for commodity n are independent of the quantities of commodity n consumed or the expenditures on commodity n during the two periods. Hence, these indices are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.[38]

Proposition 1 is a result that applies to bilateral index number functions of the form $P(p^0,p^1,q^0,q^1)$, where it is assumed that all prices and quantities can vary independently. The Lowe index defined in Chapter 2 does not fit into this framework; however, the Lowe index is widely used by national statistical offices NSOs. Recall that the *Lowe index* is defined as follows:

$$P_{Lo}(p^0,p^1,q) \equiv \Sigma_{n=1}^{N} p_n^1 q_n / \Sigma_{n=1}^{N} p_n^0 q_n = p^1 \cdot q / p^0 \cdot q, \qquad (31)$$

where $p^0 \equiv [p_1^0,\ldots,p_N^0]$ and $p^1 \equiv [p_1^1,\ldots,p_N^1]$ are the price vectors for periods 0 and 1 and $q \equiv [q_1,\ldots,q_N]$ is a representative quantity vector. It can be seen that the Lowe index does not fit into the axiomatic framework that was developed for index number formulae of the form $P(p^0,p^1,q^0,q^1)$.

It is possible to adapt many of the tests listed in Section 3 to a new index number framework that looks at the axiomatic properties of indices of the form $P(p^0,p^1,q)$. Thus, the Section 3 tests T1–T20 that can be adapted to this new framework have the following counterpart tests:

T1: *Positivity*: $P(p^0,p^1,q) > 0$.[39]

T2: *Continuity*: $P(p^0,p^1,q)$ is a continuous function of its arguments.

T3: *Identity or Constant Prices Test*: $P(p,p,q) = 1$.

T5: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,q) = \lambda P(p^0,p^1,q)$ for $\lambda > 0$.

T6: *Inverse Proportionality in Base Period Prices*: $P(1p^0,p^1,q) = \lambda^{-1} P(p^0,p^1,q)$ if $\lambda > 0$.

T7: *Invariance to Proportional Changes in Quantities*:

$$P(p^0,p^1,\lambda q) = P(p^0,p^1,q) \text{ for all } \lambda > 0.$$

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^*) = P(p^0,p^1,q),$$

[35] Konüs and Byushgens (1926) showed that the index defined by (29) is exact for Cobb–Douglas (1928) preferences; see also Pollak (1971). The concept of an exact index number formula will be explained in Chapter 5 when the economic approach to index number theory is studied.

[36] See also Eichhorn (1978; 167–168) and Vogt and Barta (1997; 47). Proofs of related results can be found in Funke, Hacker, and Voeller (1979) and Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition, who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*. . . ." Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. . . . Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

[37] This lack of representative weights problem will be particularly acute if there are disappearing and newly appearing commodities. However, note that Proposition 1 does not deal adequately with this problem since it is assumed that all prices and quantities are positive for the two periods under consideration. The problems associated with missing prices and quantities will be addressed in Chapters 5, 7, and 8.

[38] However, as mentioned earlier, if the expenditure shares are not changing much from period to period (or better yet, are constant), then by choosing α_n to be these constant expenditure shares, the Konüs and Byushgens price index will reduce to the Törnqvist price index P_T, defined by (26), which has good statistical properties.

[39] Unless otherwise specified, the domain of definition for $P(p^0,p^1,q)$ is $p^0 > 0_N$, $p^1 > 0_N$, $q > 0_N$, $p^0 \cdot q > 0$, and $p^1 \cdot q > 0$.

where p^{t*} denotes a permutation of the components of the vector p^t for $t = 0,1$ and q^* denotes the same permutation of the components of q.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ \alpha_1^{-1} q_1,\ldots,\alpha_N^{-1} q_N) = P(p^0,p^1,q).$$

T11: *Time Reversal Test*: $P(p^0,p^1,q) = 1/P(p^1,p^0,q)$.

T14: *Mean Value Test for Prices*:

$$\min_n \{p_n^1/p_n^0;\ n = 1,\ldots,N\} \le P(p^0,p^1,q) \le \max_n \{p_n^1/p_n^0;\ n = 1,\ldots,N\}.$$

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q) < P(p^0,p^{1*},q)$ if $0_N < p^1 < p^{1*}$ and $q >> 0_N$.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q) > P(p^{0*},p^1,q)$ if $0_N < p^0 < p^{0*}$ and $q >> 0_N$.

Thus, 12 of the 20 tests listed in Section 3 have counterparts that can be applied to indices of the form $P(p^0,p^1,q)$. The counterpart to the circularity test in the present framework is the following test:

T22: *Circularity*: $P(p^0,p^2,q) = P(p^0,p^1,q)P(p^1,p^2,q)$.

A question of interest is: Are the above tests sufficient to determine the functional form for $P(p^1,p^2,q)$? Using the positivity test T1, rewrite the circularity test T22 in the following form:

$$P(p^1,p^2,q) = P(p^0,p^2,q)/P(p^0,p^1,q). \quad (32)$$

Now hold p^0 constant at some fixed value, say $p^* >> 0_N$, and define the function $f(p,q)$ as follows:

$$f(p,q) \equiv P(p^*,p,q) > 0 \text{ for all } p >> 0_N \text{ and } q >> 0_N, \quad (33)$$

where the positivity of $f(p,q)$ follows from T1. Substituting definition (33) back into (32) gives us the following representation for $P(p^1,p^2,q)$:

$$P(p^1,p^2,q) = f(p^2,q)/f(p^1,q). \quad (34)$$

Thus, in this axiomatic framework, the price index $P(p^1,p^2,q)$ is equal to the price level for period 2, $f(p^2,q)$, divided by the price level for period 1, $f(p^1,q)$. The function $f(p,q)$ determines the functional form for the price index. various properties on f can be imposed so that the above tests are satisfied by the price index function, $P(p^1,p^2,q)$. Imposing continuity on $f(p,q)$ will ensure that test T2 is satisfied. The identity test T3 will automatically be satisfied by a $P(p^1,p^2,q)$ defined as $f(p^2,q)/f(p^1,q)$. Imposing the linear homogeneity property $f(\lambda p,q) = \lambda f(p,q)$ for all $\lambda > 0$ will ensure that $P(p^1,p^2,q)$ will satisfy tests T5 and T6. Imposing the linear homogeneity property $f(p,\lambda q) = \lambda f(p,q)$ for all $\lambda > 0$ will ensure that $P(p^1,p^2,q)$ will satisfy test T7. Imposing the commodity reversal and commensurability tests on $f(p,q)$ will ensure that $P(p^1,p^2,q)$ will satisfy tests T9 and T10. The time reversal test T11 will automatically be satisfied by a $P(p^1,p^2,q)$, defined as $f(p^2,q)/f(p^1,q)$. Simple conditions on $f(p,q)$ that will ensure that $P(p^1,p^2,q)$ will satisfy the mean value test T14 are difficult to determine. If $f(p,q)$ is monotonically increasing in the components of p, then $P(p^1,p^2,q)$ will satisfy the monotonicity tests T17 and T18. In general, it appears that the tests listed here are not sufficient to determine the functional form for $f(p,q)$, and hence the listed tests (including the circularity test) do not determine a unique functional form for $P(p^1,p^2,q)$.

Recall the test T4 from Section 3, the *Fixed Basket or Constant Quantities Test*, which was the following test:

$$P(p^0,p^1,q,q) = p^1 \cdot q / p^0 \cdot q. \quad (35)$$

This test is relevant in the present context. where we have only a single reference quantity vector q. Test T4 from Section 3 suggests that if the quantities purchased in periods 0 and 1 were identical, then the price index should equal $p^1 \cdot q/p^0 \cdot q$, where q is the common quantity vector; that is, $q = q^0 = q^1$. Thus, if $q^0 = q^1$, then the representative quantity vector q is obviously this common quantity vector, so in this case, $P(p^0,p^1,q)$ should equal $p^1 \cdot q/p^0 \cdot q$.[40] This suggests that even if q^0 is not equal to q^1, the functional form for $P(p^0,p^1,q)$ should be set equal to $p^1 \cdot q/p^0 \cdot q$, since this functional form will register the correct result for a price index when the quantity vectors for periods 0 and 1 are identical (or proportional). Thus, the application of test T4 to the present context pins down the functional form for $P(p^0,p^1,q)$:

$$P(p^0,p^1,q) \equiv p^1 \cdot q / p^0 \cdot q \equiv P_{Lo}(p^0,p^1,q). \quad (36)$$

Thus, the new axiomatic framework leads to the Lowe index, $P_{Lo}(p^0,p^1,q)$, as being "best" in this framework. It is straightforward to show that the Lowe index satisfies tests T1, T2, T3, T5, T6, T7, T9, T10, T11, T14, T17, T18, and T22; that is, it satisfies all of the modified tests that were listed in this section.

The Lowe index works well if the quantities demanded grow in a proportional manner (or approximately proportional manner) over time. But if prices and quantities have divergent trends over time, the Lowe index will be subject to *substitution bias*; that is, it will tend to register higher rates of inflation than the economic indices to be considered in Chapter 5, which deals more adequately with substitution bias.

As was seen earlier, it is possible to find index number formulae (see (29) and (36)) that satisfy the circularity test, but the resulting indices are not entirely satisfactory.

In the following section, another axiomatic framework for bilateral index number formulae will be discussed.

7. An Alternative Axiomatic Approach to Bilateral Index Number Theory

One of Walsh's approaches to index number theory was an attempt to determine the "best" weighted average of the price relatives, $r_n \equiv p_n^1/p_n^0$.[41] This is equivalent to using an

[40] Thus, $f(p,q) = p \cdot q$.

[41] Fisher also took this point of view when describing his approach to index number theory: "An index number of the prices of a number of commodities is an average of their price relatives. This definition has,

axiomatic approach to try to determine the "best" index of the form $P(r,v^0,v^1)$, where v^0 and v^1 are the vectors of expenditures on the N commodities during periods 0 and 1.[42] However, initially, rather than starting with indices of the form $P(r,v^0,v^1)$, indices of the form $P(p^0,p^1,v^0,v^1)$ will be considered, since this framework is more comparable to the first bilateral axiomatic framework taken in Sections 3 and 4. As will be seen later, if the invariance to changes in the units of measurement test is imposed on an index of the form $P(p^0,p^1,v^0,v^1)$, then $P(p^0,p^1,v^0,v^1)$ can be written in the form $P^*(r,v^0,v^1)$.

Recall that the product test (11) was used in order to define the quantity index, $Q(p^0,p^1,q^0,q^1) \equiv V^1/V^0 \mathrm{P}(p^0,p^1,q^0,q^1)$, that corresponded to the bilateral price index $P(p^0,p^1,q^0,q^1)$. A similar product test holds in the present framework; that is, given that the functional form for the price index $P(p^0,p^1,v^0,v^1)$ has been determined, the corresponding *implicit quantity index* can be defined in terms of P as follows:

$$Q(p^0,p^1,v^0,v^1) \equiv [\textstyle\sum_{n=1}^N v_n^1]/[(\textstyle\sum_{n=1}^N v_n^0)\ P(p^0,p^1,v^0,v^1)]. \qquad (37)$$

In Sections 3 and 4, the price and quantity indices $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ were determined *jointly*; that is, not only were axioms imposed on $P(p^0,p^1,q^0,q^1)$, but they were also imposed on $Q(p^0,p^1,q^0,q^1)$, and the product test (11) was used to translate these tests on Q into tests on P. In what follows, only tests on $P(p^0,p^1,v^0,v^1)$ will be used in order to determine the "best" price index of this form. Thus, there is a parallel theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$, where it is attempted to find the "best" value-weighted average of the quantity relatives, q_n^1/q_n^0.[43]

For the most part, the tests that will be imposed on the price index $P(p^0,p^1,v^0,v^1)$ in this section are counterparts to the tests that were imposed on the price index $P(p^0,p^1,q^0,q^1)$ in Section 3. It will be assumed that every component of each price and value vector is positive; that is, $p^t >> 0_N$ and $v^t >> 0_N$ for $t = 0,1$. If it is desired to set $v^0 = v^1$, the common expenditure vector is denoted by v; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

for concreteness, been expressed in terms of prices. But in like manner, an index number can be calculated for wages, for quantities of goods imported or exported, and, in fact, for any subject matter involving divergent changes of a group of magnitudes. Again, this definition has been expressed in terms of time. But an index number can be applied with equal propriety to comparisons between two places or, in fact, to comparisons between the magnitudes of a group of elements under any one set of circumstances and their magnitudes under another set of circumstances" (Irving Fisher (1922; 3)). However, in setting up his axiomatic approach, Fisher imposed axioms on the price and quantity indices written as functions of the two price vectors, p^0 and p^1, and the two quantity vectors, q^0 and q^1; that is, he did not write his price index in the form $P(r,v^0,v^1)$ and impose axioms on indices of this type. Of course, in the end, his ideal price index turned out to be the geometric mean of the Laspeyres and Paasche price indices, and, as was seen in Chapter 2, each of these indices can be written as expenditure share-weighted averages of the N price relatives, $r_n \equiv p_n^1/p_n^0$.

[42] Chapter 3 in vartia (1976) considered a variant of this axiomatic approach.

[43] It turns out that the price index that corresponds to this "best" quantity index, defined as $P^*(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1/[\sum_{n=1}^N v_n^0\, Q(q^0,q^1,v^0,v^1)]$, will not equal the "best" price index, $P(p^0,p^1,v^0,v^1)$. Thus, the axiomatic approach to be developed in this section generates separate "best" price and quantity indices whose product does not equal the value ratio in general. This is a disadvantage of this third axiomatic approach to bilateral indices compared to the first approach studied in Sections 3 and 4.

The first two tests are straightforward counterparts to the corresponding tests in Section 3.

T1: *Positivity*: $P(p^0,p^1,v^0,v^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,v^0,v^1)$ is a continuous function of its arguments.

T3: *Identity or Constant Prices Test*: $P(p,p,v^0,v^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the value vectors are. Note that the two value vectors are allowed to be different in the above test.

The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,v^0,v^1 changes.

T4: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,v^0,v^1) = \lambda P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number l, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree 1 in the components of the period 1 price vector p^1. This test is the counterpart to test T5 in Section 3.

In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T5: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1,v^0,v^1) = \lambda^{-1}\mathrm{P}(p^0,p^1,v^0,v^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number l, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree minus 1 in the components of the period 0 price vector p^0. This test is the counterpart to test T6 in Section 3.

The following two homogeneity tests can also be regarded as invariance tests.

T6: *Invariance to Proportional Changes in Current Period Values*:

$$P(p^0,p^1,v^0,\lambda v^1) = P(p^0,p^1,v^0,v^1) \text{ for all } \lambda > 0.$$

That is, if current period values are all multiplied by the number l, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree 0 in the components of the period 1 value vector v^1.

T7: *Invariance to Proportional Changes in Base Period Values*:

$$P(p^0,p^1,\lambda v^0,v^1) = P(p^0,p^1,v^0,v^1) \text{ for all } \lambda > 0.$$

That is, if base period values are all multiplied by the number l, then the price index remains unchanged. Hence, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree 0 in the components of the period 0 value vector v^0.

T6 and T7 together impose the property that the price index P does not depend on the *absolute* magnitudes of the value vectors v^0 and v^1. Using test T6 with $\lambda = 1/\sum_{i=1}^N v_i^1$ and using test T7 with $\lambda = 1/\sum_{i=1}^{N} v_i^0$, it can be seen that P has the following property:

$$P(p^0,p^1,v^0,v^1) = P(p^0,p^1,s^0,s^1), \tag{38}$$

where s^0 and s^1 are the vectors of expenditure shares for periods 0 and 1; that is, the ith component of s^t is $s_i^t \equiv v_i^t/\sum_{k=1}^{N} v_k^t$ for $t = 0,1$ and $i = 1,\ldots,N$. Thus, the tests T6 and T7 imply that the price index function P is a function of the two price vectors p^0 and p^1 and the two vectors of expenditure shares, s^0 and s^1.

Walsh suggested the spirit of tests T6 and T7, as the following quotation indicates:

> What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [that is, the price relatives] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered.
>
> Correa Moylan Walsh (1901; 104)

Walsh also realized that weighting the ith price relative r_i by the arithmetic mean of the value weights in the two periods under consideration, $(1/2)[v_i^0 + v_i^1]$ would give too much weight to the expenditures of the period that had the highest level of prices:

> At first sight it might be thought sufficient to add up the weights of every class at the two periods and to divide by two. This would give the (arithmetic) mean size of every class over the two periods together. But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the first period. Or in a comparison between two countries, greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even*.
>
> Correa Moylan Walsh (1901; 104–105)

As a solution to the above weighting problem, Walsh (1901; 202) (1921a; 97) proposed the following *geometric Walsh price index*:

$$P_{GW}(p^0,p^1,v^0,v^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0]^{w_n}, \tag{39}$$

where the nth weight in the above formula was defined as

$$w_n \equiv (v_n^0 v_n^1)^{1/2}/\sum_{i=1}^{N} (v_i^0 v_i^1)^{1/2} = (s_n^0 s_n^1)^{1/2}/\sum_{i=1}^{N} (s_i^0 s_i^1)^{1/2};\ n = 1,\ldots,N. \tag{40}$$

The second equation in (40) shows that Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ can also be written as a function of the expenditure share vectors, s^0 and s^1; that is, $P_{GW}(p^0,p^1,v^0,v^1)$ is homogeneous of degree 0 in the components of the value vectors v^0 and v^1 and so $P_{GW}(p^0,p^1,v^0,v^1) = P_{GW}(p^0,p^1,s^0,s^1)$. Thus, Walsh came very close to deriving the Törnqvist index defined earlier by (26).[44]

The next five tests are *invariance* or *symmetry tests*, and four of them are direct counterparts to similar tests in Section 3. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed.

T8: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*}, p^{1*},v^{0*},v^{1*}) = P(p^0,p^1,v^0,v^1),$$

where p^{t*} denotes a permutation of the components of the vector p^t, and v^{t*} denotes the same permutation of the components of v^t for $t = 0,1$.

The next test asks that the index be invariant to changes in the units of measurement.

T9: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1) = P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1)$$
$$\text{for all } \alpha_1 > 0, \ldots, \alpha_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. Note that the expenditure on commodity i during period t, v_i^t, does not change if the unit by which commodity i is measured changes.

The last test has a very important implication. Let $\alpha_1 = 1/p_1^0, \ldots, \alpha_N = 1/p_N^0$ and substitute these values for the a_i into the definition of the test. The following equation is obtained:

$$P(p^0,p^1,v^0,v^1) = P(1_N,r,v^0,v^1) \equiv P^*(r,v^0,v^1), \tag{41}$$

where 1_N is a vector of ones of dimension N and r is a vector of the price relatives; that is, the ith component of r is $r_i \equiv p_i^1/p_i^0$. Thus, if the commensurability test T9 is satisfied, then the price index $P(p^0,p^1,v^0,v^1)$, which is a function of 4N variables, can be written as a function of 3N variables, $P^*(r,v^0,v^1)$, where r is the vector of price relatives and $P^*(r,v^0,v^1)$ is defined as $P(1_N, r,v^0,v^1)$.

The next test asks that the formula be invariant to the period chosen as the base period.

T10: *Time Reversal Test*: $P(p^0,p^1,v^0,v^1) = 1/P(p^1,p^0,v^1,v^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when

[44] It is evident that Walsh's geometric price index will closely approximate the Törnqvist index using normal time series data. More formally, regarding both indices as functions of p^0,p^1,v^0,v^1, it can be shown that $P_{GW}(p^0,p^1,v^0,v^1)$ approximates $P_T(p^0,p^1,v^0,v^1)$ to the second order around an equal price (that is, $p^0 = p^1$) and expenditure (that is, $v^0 = v^1$) point.

the price index is simply the single price ratio, this test will be satisfied (as are all of the other tests listed in this section).

The next test is a variant of the *circularity test*, which was introduced in Section 6.

T11: *Transitivity in Prices for Fixed Value Weights*: $P(p^0,p^1,v^r,v^s)P(p^1,p^2,v^r,v^s) = P(p^0,p^2,v^r,v^s)$.

In this test, the expenditure weighting vectors, v^r and v^s, are held constant while making all price comparisons. However, given that these weights are held constant, the test asks that the product of the index going from period 0 to 1, $P(p^0,p^1,v^r,v^s)$, times the index going from period 1 to 2, $P(p^1,p^2,v^r,v^s)$, should equal the direct index that compares the prices of period 2 with those of period 0, $P(p^0,p^2,v^r,v^s)$. Obviously, this test is a many commodity counterpart to a property that holds for a single price relative.

The final test in this section captures the idea that the value weights should enter the index number formula in a symmetric manner.

T12: *Quantity Weights Symmetry Test*: $P(p^0,p^1,v^0,v^1) = P(p^0,p^1,v^1,v^0)$.

That is, if the expenditure vectors for the two periods are interchanged, then the price index remains invariant. This property means that if values are used to weight the prices in the index number formula, then the period 0 values v^0 and the period 1 values v^1 must enter the formula in a symmetric or even-handed manner.

The next test is a *mean value test*.

T13: *Mean Value Test*: $\min_i (p_i^1/p_i^0 : i = 1,\ldots,N) \le P(p^0,p^1,v^0,v^1) \le \max_i (p_i^1/p_i^0 : i = 1,\ldots,N)$.

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is to be interpreted as an average of the N price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfies this test.

The next two tests in this section are *monotonicity tests*; that is, how should the price index $P(p^0,p^1,v^0,v^1)$ change as any component of the two price vectors p^0 and p^1 increases.

T14: *Monotonicity in Current Prices*: $P(p^0,p^1,v^0,v^1) < P(p^0,p^2,v^0,v^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase (holding the value vectors fixed) so that $P(p^0,p^1,v^0,v^1)$ is increasing in the components of p^1 for fixed p^0, v^0 and v^1.

T15: *Monotonicity in Base Prices*: $P(p^0,p^1,v^0,v^1) > P(p^2,p^1,v^0,v^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease so that $P(p^0,p^1,v^0,v^1)$ is decreasing in the components of p^0 for fixed p^1, v^0 and v^1.

These tests are not sufficient to determine the functional form of the price index; for example, it can be shown that both Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (39) and the Törnqvist index $P_T(p^0,p^1,v^0,v^1)$ defined by (26)[45] satisfy all of the above axioms. Thus, at least one more test will be required in order to determine the functional form for the price index $P(p^0,p^1,v^0,v^1)$.

The tests proposed thus far do not specify exactly how the expenditure share vectors s^0 and s^1 are to be used in order to weight, for example, the first price relative, p_1^1/p_1^0. The next test says that only the expenditure shares s_1^0 and s_1^1 pertaining to the first commodity are to be used in order to weight the prices that correspond to commodity 1, p_1^1 and p_1^0.

T16: *Own Share Price Weighting*:

$$P(p_1^0,1,\ldots,1;p_1^1,1,\ldots,1;v^0;v^1) = f(p_1^0,p_1^1,v_1^0/\textstyle\sum_{n=1}^N v_n^0,\ v_1^1/\sum_{n=1}^N v_n^1). \qquad (42)$$

Note that $v_1^t/\sum_{k=1}^N v_k^t$ equals s_1^t, the expenditure share for commodity 1 in period t. This test says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity 1 and the two expenditure shares for commodity 1. The axiom says that a function of 2 + 2N variables is actually only a function of four variables.[46]

If test T16 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property:

$$P(1,\ldots,1,p_i^0,1,\ldots,1;1,\ldots,1,p_i^1,1,\ldots,1;v^0;v^1) = f(p_i^0,p_i^1,v_i^0/\textstyle\sum_{n=1}^N v_n^0,\ v_i^1/\sum_{n=1}^N v_n^1);\ i = 1,\ldots,N. \qquad (43)$$

Equation (43) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity i and the two expenditure shares for commodity i.

The final test that also involves the weighting of prices is the following one:

T17: *Irrelevance of Price Change with Tiny Value Weights*:

$$P(p_1^0,1,\ldots,1;\ p_1^1,1,\ldots,1;\ 0,v_2^0,\ldots,v_N^0;\ 0,v_2^1,\ldots,v_N^1) = 1. \qquad (44)$$

The test T17 says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, and the expenditures on commodity 1 are 0 in the two periods but the expenditures on the other commodities are arbitrarily given, then the index is equal to 1.[47] Thus, roughly speaking, if the value weights for commodity 1 are tiny, then it does not matter what the price of commodity 1 is during the two periods.

Of course, if test T17 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property: for $i = 1,\ldots,N$:

$$P(1,\ldots,1,p_i^0,1,\ldots,1;\ 1,\ldots,1,p_i^1,1,\ldots,1;\ v_1^0,\ldots,v_{i-1}^0,0,v_{i+1}^0,\ldots,v_N^0;\ v_1^1,\ldots,v_{i-1}^1,0,v_{i+1}^1,\ldots,v_N^1) = 1. \qquad (45)$$

[45] The share weights s_n^t in definition (26) can be rewritten as $v_n^t/v^t\cdot 1_N$ for $n = 1,\ldots,N$ and $t = 0,1$. Thus, $P_T(p^0,p^1,q^0,q^1)$ can be rewritten in the form $P_T(p^0,p^1,v^0,v^1)$.

[46] In the economics literature, axioms of this type are known as separability axioms.

[47] Strictly speaking, since all prices and values are required to be positive, the left-hand side of (44) should be replaced by the limit as the commodity 1 values, v_1^0 and v_1^1, approach 0.

Equation (45) says that if all of the prices are set equal to 1 except the prices for commodity *i* in the two periods, and the expenditures on commodity *i* are 0 during the two periods but the other expenditures in the two periods are arbitrarily given, then the index is equal to 1.

This completes the listing of tests for the weighted average of price relatives approach to bilateral index number theory. It turns out that these tests are sufficient to imply a specific functional form for the price index, as will be seen in the next section.

8. The Törnqvist Price Index and the Alternative Approach to Bilateral Indices

It turns out that the Törnqvist price index is the only index that satisfies the axioms in the previous section. In Section 6, the framework/tests were adapted to the context of a Lowe index. Similarly, it would be possible to adapt the tests in Section 7 to a new index number framework that looks at the axiomatic properties of indices of the form $P(p0,p1,v)$. The index that would come out of such a framework is the geometric Young Index.

Proposition 2

If the number of commodities N exceeds two and the bilateral price index function $P(p^0,p^1,v^0,v^1)$ satisfies the 17 axioms listed in Section 7, then P must be the Törnqvist price index $P_T(p^0,p^1,v^0,v^1)$ defined by (26).[48]

Thus, the 17 properties or tests listed in Section 7 provide an axiomatic characterization of the Törnqvist price index, just as the 20 tests listed in Section 3 provided an axiomatic characterization for the Fisher ideal price index. For a proof of Proposition 2, see the annex.

Obviously, there is a parallel axiomatic theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$ that depend on the two quantity vectors for periods 0 and 1, q^0 and q^1, as well as on the corresponding two expenditure vectors, v^0 and v^1. Thus, if $Q(q^0,q^1,v^0,v^1)$ satisfies the quantity counterparts to tests T1 to T17, then Q must be equal to the Törnqvist quantity index $Q_T(q^0,q^1,v^0,v^1)$, whose logarithm is defined as follows:

$$ln Q_T(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) ln(q_n^1/q_n^0), \qquad (46)$$

where as usual the period t expenditure share on commodity i, s_i^t, is defined as $v_i^t/\sum_{k=1}^{N} v_k^t$ for $i = 1,\ldots,N$ and $t = 0,1$.

Unfortunately, the implicit Törnqvist–Theil price index, $P_{IT}(q^0,q^1,v^0,v^1)$ that corresponds to the Törnqvist quantity index Q_T defined by (46) using the product test is *not* equal to the direct Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined earlier by (26). The product test equation that defines P_{IT} in the present context is given by the following definition:

$$P_{IT}(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^{N} v_n^1/[\sum_{n=1}^{N} v_n^0 Q_T(q^0,q^1,v^0,v^1)]$$

$$= v^1\cdot 1_N/[v^0\cdot 1_N Q_T(q^0,q^1,v^0,v^1)]. \qquad (47)$$

The fact that the direct Törnqvist price index P_T is not in general equal to the implicit Törnqvist–Theil price index P_{IT} defined by (47) can be slightly disadvantageous compared to the axiomatic approach outlined in Sections 3 and 4, which led to the Fisher ideal price and quantity indices as being "best". Using the Fisher approach meant that it was not necessary to decide whether one wanted a "best" price index or a "best" quantity index: The theory outlined in Sections 3 and 4 determined both indices simultaneously. However, in the Törnqvist approach outlined in this section, it is necessary to *choose* whether one wants a "best" price index or a "best" quantity index.[49]

Other tests are of course possible. A counterpart to test T16 in Section 3, the Paasche and Laspeyres bounding test, is the following *geometric Paasche and Laspeyres bounding test*:

$$P_{GL}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GP}(p^0,p^1,v^0,v^1) \text{ or}$$

$$P_{GP}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GL}(p^0,p^1,v^0,v^1), \qquad (48)$$

where the logarithms of the *geometric Laspeyres* and *geometric Paasche* price indices, P_{GL} and P_{GP}, are defined as follows:

$$ln P_{GL}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^{N} s_n^0 ln(p_n^1/p_n^0), \qquad (49)$$

$$ln P_{GP}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^{N} s_n^1 ln(p_n^1/p_n^0). \qquad (50)$$

It can be shown that the Törnqvist price index $P_T(p^0,p^1,v^0,v^1)$ defined by (26) satisfies the geometric Laspeyres and Paasche bounding test, but the geometric Walsh price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (39) does not satisfy it.

The geometric Paasche and Laspeyres bounding test was not included as a primary test in Section 7 because, a priori, it was not known what form of averaging of the price relatives (for example, geometric or arithmetic or harmonic) would turn out to be appropriate in this test framework. The test (48) is an appropriate one if it has been decided that geometric averaging of the price relatives is the appropriate framework, since the geometric Paasche and Laspeyres indices correspond to "extreme" forms of value weighting in the context of geometric averaging, and it is natural to require that the "best" price index lie between these extreme indices.

Walsh (1901; 408) pointed out a problem with his geometric price index P_{GW} defined by (39), which also applies to the Törnqvist price index $P_T(p^0,p^1,v^0,v^1)$: These geometric-type indices do not give the "right" answer when the quantity vectors are constant (or proportional) over the two periods. In this case, Walsh thought that the "right" answer must be

[48] The Törnqvist price index satisfies all 17 tests, but the proof in the annex did not use all of these tests to establish the result in the opposite direction: Tests 5, 13, 15, and one of 10 or 12 were not required in order to show that an index satisfying the remaining tests must be the Törnqvist price index. For alternative characterizations of the Törnqvist–Theil price index, see Balk and Diewert (2001) and Hillinger (2002).

[49] Hillinger (2002) suggested taking the geometric mean of the direct and implicit Törnqvist price indices in order to resolve this conflict. Unfortunately, the resulting index is not "best" for either set of axioms that were suggested in this section. For more on Hillinger's approach to index number theory, see Hillinger (2002).

the *Lowe* (1823) *index*, which is the ratio of the costs of purchasing the constant basket during the two periods.[50] Put another way, the geometric indices P_{GW} and P_T do not satisfy the fixed basket test, T4, in Section 3.

There is one additional test that should be mentioned. Fisher (1911; 401) introduced this test in his first book that dealt with the test approach to index number theory. He called it the *test of determinateness as to prices* and described it as follows:

> A price index should not be rendered zero, infinity, or indeterminate by an individual price becoming zero. Thus, if any commodity should in 1910 be a glut on the market, becoming a 'free good', that fact ought not to render the index number for 1910 zero.
>
> Irving Fisher (1911; 401)

In the present context, this test could be interpreted as the following one: If any single price p_i^0 or p_i^1 tends to 0, then the price index $P(p^0,p,v^0,v^1)$ should not tend to 0 or plus infinity. However, with this interpretation of the test, which regards the values v_i^t as remaining constant as the p_i^0 or p_i^1 tends to 0, none of the commonly used index number formulae would satisfy this test. Hence, this test should be interpreted as a test that applies to price indices $P(p^0,p^1,q^0,q^1)$ of the type that were studied in Sections 3 and 4, which is how Fisher intended the test to apply. Thus, Fisher's price determinateness test should be interpreted as follows: If any single price p_i^0 or p_i^1 tends to 0, then the price index $P(p^0,p,q^0,q^1)$ should not tend to 0 or plus infinity. With this interpretation of the test, it can be verified that Laspeyres, Paasche, and Fisher indices satisfy this test but the Törnqvist price index will not satisfy this test. Thus, when using the Törnqvist price index, *care must be taken to bound the prices away from 0 in order to avoid a meaningless index number value*.

Walsh was aware that geometric average-type indices like the Törnqvist–Theil price index P_T or Walsh's geometric price index P_{GW} defined by (39) become somewhat unstable[51] as individual price relatives become very large or small:

> Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes [i. e., expenditures] are very unequal and the price variations are very great, this average may deflect considerably.
>
> Correa Moylan Walsh (1901; 373)

> In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together.
>
> Correa Moylan Walsh (1901; 404)

Weighing all of the arguments and tests presented in this chapter, there is a preference for the use of the Fisher ideal price index as a suitable target index for a statistical agency that wishes to use the axiomatic approach, but of course, opinions can differ on which set of axioms is the most appropriate to use in practice.

9. Defining Contributions to Overall Percentage Change for a Bilateral Index

Business analysts often want statistical agencies to provide decompositions of overall price change into explanatory components that reflect individual commodity price change. Chapter 9 of the *CPI Manual* discussed contributions to change. This decomposition problem can be defined more precisely as follows. A bilateral price index of the form $P(p^0, p^1,q^0,q^1)$ can be interpreted as the ratio of a period 1 price level, P^1, to a period 0 price level, P^0. Thus, the *percentage change in the overall price level* is

$$P(p^0,p^1,q^0,q^1) - 1 = [P^1/P^0] - 1 = [P^1 - P^0]/P^0. \qquad (51)$$

The *percentage change in commodity price n* is $(p_n^1/p_n^0) - 1$ for $n = 1,\ldots,N$.[52] The desired decomposition has the following form:

$$P(p^0,p^1,q^0,q^1) - 1 = \Sigma_{n=1}^{N} w_n[(p_n^1/p_n^0) - 1], \qquad (52)$$

where w_n are the *weighting factors* to be determined. The overall *contribution factor for commodity n* is defined as

$$C_n \equiv w_n[(p_n^1/p_n^0) - 1];\ n = 1,\ldots,N. \qquad (53)$$

The problem is: How exactly are the weighting factors w_n to be determined? At the outset, it should be recognized that there need not be a unique determination for these weighting factors since the weighting factors are allowed to be functions of the 4N variables, p^0,p^1,q^0,q^1. The price index $P(p^0,p^1,q^0,q^1)$ will typically be a rather complicated function of the variables, p^0,p^1,q^0,q^1, and thus there can be many ways of decomposing $P(p^0,p^1,q^0,q^1) - 1$ into the form given by the right-hand side of (52).[53]

The approach taken in this section is to use a simple first-order Taylor series approximation to the index number formula to give us an approximate decomposition of the form (52). However, the suggested approximation requires an extra assumption—namely, that the given index number formula, $P(p^0,p^1,q^0,q^1)$, can be rewritten in the form $P^*(r,s^0,s^1)$, where $r \equiv [r_1,\ldots,r_N] = [p_1^1/p_1^0,\ldots,p_N^1/p_N^0]$ and s^t is the usual vector of expenditure shares on the N commodities for period t for $t = 0,1$. Thus, it is assumed that $P(p^0,p^1,q^0,q^1)$ can be expressed in the price ratio and expenditure share framework for bilateral indices that was explained in Section 7.[54] Definitions (54)–(58) show how the Laspeyres, Paasche, Fisher, Törnqvist, and Walsh indices can be expressed in the form $P^*(r,s^0,s^1)$:

[50] Of course, the Fisher ideal index does have this property and gives the "right" answer when q^1 is equal or proportional to q^0.

[51] That is, the index may approach 0 or plus infinity.

[52] In this section, it is assumed that all period 0 prices are positive so that the ratios p_n^1/p_n^0 are well defined.

[53] For example, see the alternative decompositions of the form (52) for the Fisher ideal index $P_F(p^0,p^1,q^0,q^1)$ that were obtained by Van IJzeren (1987), Ehemann, Katz and Moulton (2002), Diewert (2002), and Reinsdorf, Diewert and Ehemann (2002).

[54] See equation (41).

$$P_L^*(r,s^0,s^1) \equiv \Sigma_{n=1}^N s_n^0 r_n; \quad (54)$$
$$P_P^*(r,s^0,s^1) \equiv [\Sigma_{n=1}^N s_n^1 (r_n)^{-1}]^{-1}; \quad (55)$$
$$P_F^*(r,s^0,s^1) \equiv [P_L^*(r,s^0,s^1) P_P^*(r,s^0,s^1)]^{1/2}; \quad (56)$$
$$P_T^*(r,s^0,s^1) \equiv \prod_{n=1}^N r_n^{(1/2)(s_n^0 + s_n^1)}; \quad (57)$$
$$P_W^*(r,s^0,s^1) \equiv \Sigma_{n=1}^N (s_n^0 s_n^1)^{1/2} (r_n)^{1/2} / \Sigma_{n=1}^N (s_n^0 s_n^1)^{1/2} (r_n)^{-1/2}. \quad (58)$$

Using the new notation, the desired decomposition of overall percentage price change for $P(p^0,p^1,q^0,q^1)$ given by (52) can be rewritten as follows:

$$P^*(r,s^0,s^1) - 1 = \Sigma_{n=1}^N w_n (r_n - 1). \quad (59)$$

Assume that $P(p^0,p^1,q^0,q^1)$ satisfies the identity test, $P(p,p,q^0,q^1) = 1$, for all $p >> 0_N$, $q^0 >> 0_N$ and $q^1 >> 0_N$. Then the companion $P^*(r,s^0,s^1)$ will satisfy

$$P^*(1_N,s^0,s^1) = 1 \textit{ for all } s^0 >> 0_N \textit{ and } s^1 >> 0_N, \quad (60)$$

where 1_N is a vector of ones of dimension N.[55] Assuming that $P^*(r,s^0,s^1)$ is differentiable with respect to the components of r at $r = 1_N$, the *first-order Taylor series approximation* to $P^*(r, s^0,s^1)$ around the point $r = 1_N$ is

$$P^*(r,s^0,s^1) \approx P^*(1_N,s^0,s^1) + \Sigma_{n=1}^N [\partial P^*(1_N,s^0,s^1)/\partial r_n][r_n - 1]$$
$$= 1 + \Sigma_{n=1}^N [\partial P^*(1_N,s^0,s^1)/\partial r_n][r_n - 1], \quad (61)$$

where the second line was derived from (60). Rearranging (61) leads to the following approximate equality:

$$P^*(r,s^0,s^1) - 1 \approx \Sigma_{n=1}^N [\partial P^*(1_N,s^0,s^1)/\partial r_n][r_n - 1]. \quad (62)$$

Thus, (62) has the same structure as (59) except that (62) is only an approximate equality. The w_n weighting factors from (62) are the partial derivatives of $P^*(r,s^0,s^1)$ with respect to the components of r, evaluated at $r = 1_N$. The interpretation of these weighting factors is fairly straightforward.

Denote the nth weighting factor for the Laspeyres, Paasche, Fisher, Törnqvist, and Walsh indices as w_{Ln}, w_{Pn}, w_{Fn}, w_{Tn}, and w_{Wn} respectively. Straightforward calculations of the respective partial derivatives show that these weighting factors are equal to the following expressions for $n = 1,\ldots,N$:

$$w_{Ln} = s_n^0; \quad (63)$$
$$w_{Pn} = s_n^1; \quad (64)$$
$$w_{Fn} = ½(s_n^0 + s_n^1); \quad (65)$$
$$w_{Tn} = ½(s_n^0 + s_n^1); \quad (66)$$
$$w_{Wn} = (s_n^0 s_n^1)^{1/2} / \Sigma_{i=1}^N (s_i^0 s_i^1)^{1/2}.\text{[56]} \quad (67)$$

These weighting factors can be substituted into the approximate equations $P_X^*(r,s^0,s^1) - 1 \approx \Sigma_{n=1}^N w_{Xn}(r_n - 1)$, where X = L, P, F, T, or W. The resulting equation will be exact for the Laspeyres index, but, in general, it will not be exact for the remaining indices. For the remaining indices, the difference between $P_X^*(r,s^0,s^1) - 1$ and $\Sigma_{n=1}^N w_{Xn}(r_n - 1)$ can be labeled as a statistical discrepancy, or the discrepancy could be distributed across the N contribution factors.

The aforementioned methodology for defining contribution factors is only one of many possible approaches. However, it does generate results that analysts will probably find suitable for their purposes.

We conclude this section by discussing the problems associated with deriving a decomposition of the rate of change of the Lowe index into explanatory factors involving rates of change for individual commodity prices.[57] This topic is of some interest since the European Union makes a great deal of use of the Lowe formula when computing its HICP.[58] However, their Lowe index uses a combination of fixed-base and chained indices, as will be seen subsequently.

We consider the case where a statistical agency uses a fixed-base Lowe index for 13 consecutive months in a year that starts in December; that is, for the first 13 months of the index, December of, say, 2018, is used as the base month. For these months, the following fixed-base Lowe index is used at higher levels of aggregation:[59]

$$P_{Lo}(p^0,p^t,q^{b0}) \equiv p^t \cdot q^{b0} / p^0 \cdot q^{b0};\ t = 0,1,\ldots,12$$
$$= \Sigma_{n=1}^N (p_n^t/p_n^0) p_n^0 q_n^{b0} / p^0 \cdot q^{b0}$$
$$= \Sigma_{n=1}^N s_n^0 b^0 r_n, \quad (68)$$

where p^0 is the monthly price vector for December 2018, $p^1,\ldots,p^{12}$ are the relevant month price vectors for January–December of 2019, q^{b0} is a base year quantity vector for a prior year, and the *price ratios* r_n^t and *hybrid shares* $s_n^0 b^0$ are defined as follows:

$$r_n^t \equiv p_n^t/p_n^0;\ s_n^0 b^0 \equiv p_n^0 q_n^{b0} / p^0 \cdot q^{b0} = (p_n^0/p_n^{b0})$$
$$s_n^{b0} / \Sigma_{i=1}^N (p_i^0/p_i^{b0}) s_i^{b0};\ n = 1,\ldots,N, \quad (69)$$

where $s_n^{b0} \equiv p_n^{b0} q_n^{b0} / p^0 \cdot q^{b0}$ is the base year expenditure share for commodity n and p_n^{b0} and q_n^{b0} are the base year price and quantity for commodity n for $n = 1,\ldots,N$. The second equation for the hybrid share $s_n^0 b^0$ shows that it can be written using only the price ratios p_n^0/p_n^{b0} and the annual expenditure shares s_n^{b0} for base year 0.[60]

[55] Note that P_L^*, P_P^*, P_F^*, P_T^*, and P_W^* all satisfy the identity test (60).

[56] If we approximate the geometric mean $(s_n^0 s_n^1)^{1/2}$ by the corresponding arithmetic mean, $(½)(s_n^0 + s_n^1)$, then it can be seen that $(½)(s_n^0 + s_n^1)/\Sigma_{i=1}^N (½)(s_i^0 + s_i^1) = (½)(s_n^0 + s_n^1)$ so that $w_{Wn} \approx (½)(s_n^0 + s_n^1) = w_{Fn} = w_{Tn}$. Thus, the contribution factors for the Fisher, Törnqvist, and Walsh indices will all be approximately equal (and intuitively sensible).

[57] Our analysis is based on the work of Balk (2017), de Haan and Akem (2017), and Eurostat (2018; 180–183).

[58] See Chapter 8 of Eurostat (2018) for details. This chapter was written by Bert Balk and Jens Mehrhoff. Other countries such as Australia and the United Kingdom use a similar annually chained Lowe index methodology. It should be noted that some member states of the European Union do not use an annually chained Lowe index at higher levels of aggregation; they use an annually chained Young index when constructing their HICP.

[59] All prices are assumed to be positive.

[60] It is also possible to show that the Lowe index $P_{Lo}(p^0,p^t,q^{b0}) \equiv p^t \cdot q^{b0} / p^0 \cdot q^{b0}$ defined in terms of the annual basket vector q^{b0} can also be written as a

Since the shares $s_n^0 b^0$ sum to 1, it can be seen that the Lowe index has the following exact decomposition into *commodity price change contribution factors*:

$$P_{Lo}(p^0,p^t,q^{b0}) - 1 = \Sigma_{n=1}^{N} s_n^0 b^0 (r_n^t - 1);\ t = 1,...,12. \quad (70)$$

However, typically analysts do not want to measure contributions to general inflation from December of the previous year to a particular month t in the subsequent year; they will be interested in month-to-month inflation in the subsequent year. It is not a problem to measure month-to-month inflation during the year subsequent to the base month. Since the Lowe index satisfies the circularity test for months $t = 0, 1,...,12$, the Lowe index going from month t to month $t + 1$ is the following index:

$$\begin{aligned} P_{Lo}(p^t,p^{t+1},q^{b0}) &\equiv p^{t+1}\cdot q^{b0}/p^t\cdot q^{b0};\ t = 0,1,...,11 \\ &= \Sigma_{n=1}^{N} (p_n^{t+1}/p_n^t)p_n^t q_n^{b0}/p^t\cdot q^{b0} \\ &= \Sigma_{n=1}^{N} s_n^{tb0} r_n^{t*}, \end{aligned} \quad (71)$$

where the *short-term price ratios* r_n^* and *short-term hybrid shares* s_n^{tb0} are defined as follows:

$$r_n^{t*} \equiv p_n^{t+1}/p_n^t;\ s_n^{tb0} \equiv p_n^t q_n^{b0}/p^t\cdot q^{b0};\ n = 1,...,N;\ t = 1,...,11. \quad (72)$$

Since the shares s_n^{tb0} sum to 1, it can be seen that the short-term Lowe index has the following exact decomposition into *monthly commodity price change contribution factors*:

$$P_{Lo}(p^t,p^{t+1},q^{b0}) - 1 = \Sigma_{n=1}^{N} s_n^{tb0} (r_n^{t*} - 1);\ t = 1,...,11. \quad (73)$$

Thus, the problem of defining monthly contribution factors for the Lowe index for a single year is solved using this framework. However, now suppose that the statistical agency does not use the base year quantity weights q^{b0} beyond one year; the annual weights are changed each year. The above algebra describes how the index is constructed for the months of say December 2018 through to December 2019. In December 2019, a new annual quantity vector is introduced, say q^{b1}. The Lowe index for December 2018 is $P^{12} \equiv P_{Lo}(p^0,p^{12},q^{b0})$. The Lowe index using December 2018 as the reference price base for the next 12 months uses the new annual quantity vector q^{b1}. This new Lowe index is multiplied by the index level for December 2019 to give the overall index level P^t relative to December 2018 defined as follows:[61]

$$\begin{aligned} P^t &\equiv P_{Lo}(p^0,p^{12},q^{b0})P_{Lo}(p^{12},p^t,q^{b1})\ t = 13,...,24 \\ &= [p^{12}\cdot q^{b0}/p^0\cdot q^{b0}][p^t\cdot q^{b1}/p^{12}\cdot q^{b1}] \\ &= [p^t\cdot q^{b1}/p^0\cdot q^{b0}]/[p^{12}\cdot q^{b1}/p^{12}\cdot q^{b0}]. \end{aligned} \quad (74)$$

The first term in the last equality in (74) shows that the prices of month t (which is a month in 2020) are compared to the prices in month 0 (December of 2018) by the index $p^t\cdot q^{b1}/p^0\cdot q^{b0}$. However, the annual baskets, q^{b0} and q^{b1}, are not held constant in this comparison, so this index is divided by a quantity index that compares the annual quantity vector q^{b1} to the prior annual quantity vector q^{b0}, using the price weights p^{12}, which are the monthly price weights for December 2019.[62]

Since the right-hand side of (74) is a rather complicated function of p^0, p^1, p^{12}, q^{b0} and q^{b1}, it is difficult to develop a straightforward contribution to percentage change decomposition for this index. Many analysts will be interested in year-over-year contributions to overall percentage change. In this case, the price level in month $12 + t$, P^{12+t}, is compared to the price level in month t of the base year, which is P^t, for $t = 1,...,12$. Using (68) and (74), this ratio is equal to[63]

$$\begin{aligned} P^{12+t}/P^t &= P_{Lo}(p^0,p^{12},q^{b0})P_{Lo}(p^{12},p^{12+t},q^{b1})/P_{Lo}(p^0,p^t,q^{b0});\ t = 1,...,12 \\ &= [p^{12}\cdot q^{b0}/p^0\cdot q^{b0}][p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}]/[p^t\cdot q^{b0}/p^0\cdot q^{b0}] \\ &= [p^{12+t}\cdot q^{b1}/p^t\cdot q^{b0}][p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}]. \end{aligned} \quad (75)$$

Again, it is not straightforward to rewrite (75) as a function of the year-over-year price ratios, p_n^{12+t}/p_n^t, for $t = 1,...,12$ and expenditure shares. However, it is possible to rewrite (75) in the following two alternative forms for $t = 1,...,12$:

$$\begin{aligned} P^{12+t}/P^t &= \kappa_{b0}[p^{12+t}\cdot q^{b0}/p^t\cdot q^{b0}] \\ &= \kappa_{b0}[\Sigma_{n=1}^{N} s_n^{b0} (p_n^{12+t}/p_n^t)]; \end{aligned} \quad (76)$$

$$\begin{aligned} P^{12+t}/P^t &= \kappa_{b1}[p^{12+t}\cdot q^{b1}/p^t\cdot q^{b1}] \\ &= \kappa_{b1}[\Sigma_{n=1}^{N} s_n^{b1} (p_n^{12+t}/p_n^t)], \end{aligned} \quad (77)$$

where κ_{b0} and κ_{b1} are defined by (78) and the hybrid shares s_n^{b0} and s_n^{b1} are defined by (79):

$$\begin{aligned} \kappa_{b0} &\equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^{12+t}\cdot q^{b1}/p^{12+t}\cdot q^{b0}]; \\ \kappa_{b1} &\equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^t\cdot q^{b1}/p^t\cdot q^{b0}]; \end{aligned} \quad (78)$$

$$s_n^{b0} \equiv p_n^t q_n^{b0}/p^t\cdot q^{b0};\ s_n^{b1} \equiv p_n^t q_n^{b1}/p^t\cdot q^{b1};\ n = 1,...,N. \quad (79)$$

Note that indices of the form $p\cdot q^{b1}/p\cdot q^{b0} \equiv Q_{Lo}(q^{b0},q^{b1},p)$, where p is a vector of reference prices, are *Lowe-type quantity indices* that indicate the effects of the change in annual

function of *relative prices* and the *base year share vector*; that is, we have $P_{Lo}(p^0,p^t,q^{b0}) = \Sigma_{n=1}^{N} (p_n^1/p_n^{b0})s_n^{b0}/\Sigma_{i=1}^{N} (p_i^1/p_i^{b0})s_i^{b0}$. It turns out that all of the formulae exhibited in this section can replace the use of q^{b0} by using an equivalent formula which uses the vector of base year expenditure shares. We will use the quantity vector q^{b0} in place of the base year share vector s^{b0} in order to simplify the formulae.

[61] Note that the months have been numbered in a consecutive manner. Thus month 0 is December 2018, month 1 is January 2019,...,month 12 is December 2019, month 13 is January 2020,...., and month 24 is December 2020.

[62] Note that the index defined by (74) does not satisfy the identity test; that is, it is not necessarily the case that $P_{Lo}(p^0,p^{12},q^{b0})P_{Lo}(p^t,p^{12},q^{b1}) = 1$ if $p^t = p^0$; see Balk (2017; 9). Of course, if $q^{b1} = q^{b0}$, then the identity test will be satisfied.

[63] This analysis follows that of Balk (2017; 8). Balk rearranged the terms in the last equality of (75) to give the following decomposition: $P^{12+t}/P^t = [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}][p^{12}\cdot q^{b0}/p^t\cdot q^{b0}]$. Thus, Balk noted that the year over year annually chained Lowe index is a product of two Lowe indices where the first index uses quantity weights q^{b1} and the second uses the quantity weights q^{b0}.

quantities going from q^{b0} to q^{b1}, holding prices fixed at p. It can be seen that using the definition of the Lowe quantity index, κ_{b0} and κ_{b1} can be written as follows:

$$\kappa_{b0} \equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^{12+t}\cdot q^{b1}/p^{12+t}\cdot q^{b0}] = Q_{Lo}(q^{b0},q^{b1},p^{12+t})/Q_{Lo}(q^{b0},q^{b1},p^{12}); \quad (80)$$

$$\kappa_{b1} \equiv [p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}][p^{t}\cdot q^{b1}/p^{t}\cdot q^{b0}] = Q_{Lo}(q^{b0},q^{b1},p^{t})/Q_{Lo}(q^{b0},q^{b1},p^{12}). \quad (81)$$

Looking at (80) and (81), it can be seen that κ_{b0} and κ_{b1} represent the effects of changes in the annual quantity weights, and it is likely that κ_{b0} and κ_{b1} will be close to one.

Using (76), we have the following exact decomposition of the year-over-year percentage change in the overall annually chained Lowe index:[64]

$$\begin{aligned}[P^{12+t}/P^{t}] - 1 &= \kappa_{b0}[\Sigma_{n=1}^{N} s_n^{b0}(p_n^{12+t}/p_n^{t})] - 1 \quad t = 1,\ldots,12\\ &= \kappa_{b0}[\Sigma_{n=1}^{N} s_n^{b0}(p_n^{12+t}/p_n^{t})] - \kappa_{b0} + \kappa_{b0} - 1\\ &= \Sigma_{n=1}^{N} \kappa_{b0}s_n^{b0}[(p_n^{12+t}/p_n^{t}) - 1] + [\kappa_{b0} - 1]. \quad (82)\end{aligned}$$

Thus, the year-over-year percentage change in the annually chained Lowe index for month t is no longer a *share-weighted average* of the commodity price annual rates of change, $(p_n^{12+t}/p_n^{t}) - 1$; it is expressed as a *weighted sum* of the price changes $(p_n^{12+t}/p_n^{t}) - 1$ (with weight $\kappa_{b0}s_n^{b0}$ for commodity n) plus a term that reflects the changes in the annual quantity weights, which is $[\kappa_{b0} - 1]$. Of course, if κ_{b0} is equal to 1, then the quantity weights change term vanishes and (82) becomes the usual share-weighted decomposition.

Using (77) instead of (76) leads to the following alternative exact decomposition of the year-over-year percentage change in the overall annually chained Lowe index:

$$\begin{aligned}[P^{12+t}/P^{t}] - 1 &= \kappa_{b1}[\Sigma_{n=1}^{N} s_n^{b1}(p_n^{12+t}/p_n^{t})] - 1 \quad t = 1,\ldots,12\\ &= \kappa_{b1}[\Sigma_{n=1}^{N} s_n^{b1}(p_n^{12+t}/p_n^{t})] - \kappa_{b1} + \kappa_{b1} - 1\\ &= \Sigma_{n=1}^{N} \kappa_{b1}s_n^{b1}[(p_n^{12+t}/p_n^{t}) - 1] + [\kappa_{b1} - 1]. \quad (83)\end{aligned}$$

If κ_{b1} equals 1, then (83) collapses down to a traditional share-weighted decomposition.

Since the decompositions defined by (82) and (83) are equally plausible, it is best to combine them into the following exact decomposition for $t = 1,\ldots,12$:

$$[P^{12+t}/P^{t}] - 1 = \Sigma_{n=1}^{N} (½)(\kappa_{b0}s_n^{b0} + \kappa_{b1}s_n^{b1})[(p_n^{12+t}/p_n^{t}) - 1] + [(½)(\kappa_{b0} + \kappa_{b1}) - 1]. \quad (84)$$

The approach taken by Eurostat and the OECD to provide a decomposition of $[P^{12+t}/P^{t}] - 1$ was developed by Ribe (1999), and it can be explained as follows (see also paragraphs 9.107–9.110 of the *CPI Manual*). Using equation (75), we have

$$\begin{aligned}P^{12+t}/P^{t} - 1 &= [p^{12+t}\cdot q^{b1}/p^{t}\cdot q^{b0}][p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}] - 1 \quad t = 1,\ldots,12\\ &= [p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}][p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}] - 1\\ &= [p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}]\{[p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}] - 1\} + [p^{12}\cdot q^{b0}/p^{t}\cdot q^{b0}] - 1\\ &= P_{Lo}(p^{t},p^{12},q^{b0})\{\Sigma_{n=1}^{N} s_n^{b1*}[(p_n^{12+t}/p_n^{12}) - 1]\} + \{\Sigma_{n=1}^{N} s_n^{b0}[(p_n^{12}/p_n^{t}) - 1]\}, \quad (85)\end{aligned}$$

where the hybrid shares s_n^{b0} were defined earlier in definitions (79) and the new hybrid shares s_n^{b1*} are defined as follows:

$$s_n^{b1*} \equiv p_n^{12}q_n^{b1}/p^{12}\cdot q^{b1};\ n = 1,\ldots,N. \quad (86)$$

The first term on the right-hand side of the last equation in (85) is regarded as a "this year" term that looks at the contribution of price change from month 12 to month 12 + t, while the second term, $\Sigma_{n=1}^{N} s_n^{b0}[(p_n^{12}/p_n^{t}) - 1]$, is regarded as a "last year" contribution term that looks at the price change from month t in the first year to month 12 in the first year.

Balk commented on the decomposition defined by (85) as follows:

> However, by looking at the structure of the right-hand side [of (85)] it becomes clear that this decomposition is not completely satisfactory. Though the second factor between brackets can be interpreted as previous year's contribution, and the first factor between brackets likewise as current year's contribution (and both factors can be decomposed commodity-wise), this first factor is multiplied by previous year's price change. Thus there seems to be a whiff of double-counting here.
>
> Bert M. Balk (2017; 9)

Thus, Balk noted that the Lowe index $P_{Lo}(p^{t},p^{12},q^{b0})$ precedes the first price decomposition term, $\Sigma_{n=1}^{N} s_n^{b1*}[(p_n^{12+t}/p_n^{12}) - 1]$, and this Lowe index involves the overall amount of inflation going from month t in the first year to December of the first year, and this amount of overall inflation estimate augments the commodity-specific contributions going from December of the first year to month t of the second year. The decomposition defined by (84) seems to be conceptually cleaner with year-over-year contributions to overall year-over-year inflation listed in single terms (rather than as a sum of two terms) plus a final term that measures the overall contribution made by the changing annual baskets.[65]

It is possible to obtain an alternative decomposition to (85) that is equally plausible.[66] Again using equation (75), we have

[64] This type of decomposition where there is a *separate term* for the effects of weight changes follows the methodological approach explained by de Haan and Akem (2017). Their approach to year-over-year contributions analysis was implemented by the Australian Bureau of Statistics (2017; 7).

[65] De Haan and Akem's (2017) decomposition is similar in structure to (84) except that their decomposition of overall inflation is not an exact one.

[66] The analysis that follows was performed by Jens Mehrhoff.

$$[P^{12+t}/P^t] - 1 = [p^{12+t}\cdot q^{b1}/p^t \times q^{b0}][p^{12}\cdot q^{b0}/p^{12}\cdot q^{b1}] - 1 \quad t = 1,\ldots,12$$
$$= [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}][p^{12}\cdot q^{b0}/p^t\cdot q^{b0}] - 1$$
$$= [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}]\{[p^{12}\cdot q^{b0}/p^t\cdot q^{b0}] - 1\}$$
$$+ [p^{12+t}\cdot q^{b1}/p^{12}\cdot q^{b1}] - 1$$
$$= P_{Lo}(p^{12},p^{12+t},q^{b1})\{\Sigma_{n=1}^{N} s_n^{b0}[(p_n^{12}/p_n^t) - 1]\}$$
$$+ \{\Sigma_{n=1}^{N} s_n^{b1*}[(p_n^{12+t}/p_n^{12}) - 1]\}, \quad (87)$$

where the hybrid shares s_n^{b0} are defined by (79) and the hybrid shares s_n^{b1*} are defined by (86).

When two equally plausible estimates for the same thing are available and a single estimate is required, it is best to take an evenly weighted average of the two estimates to form a final estimate. Thus, taking the arithmetic mean of the two estimates defined by (85) and (87) leads to the following *Mehrhoff decomposition* of year-over-year price change into explanatory components:

$$[P^{12+t}/P^t] - 1 = \Sigma_{n=1}^{N} (1/2)s_n^{b0}[1 + P_{Lo}(p^{12},p^{12+t},q^{b1})][(p_n^{12}/p_n^t) - 1]$$
$$t = 1,\ldots,12$$
$$+ \Sigma_{n=1}^{N} (1/2)s_n^{b1*}[1 + P_{Lo}(p^t,p^{12},q^{b0})]$$
$$[(p_n^{12+t}/p_n^{12}) - 1]. \quad (88)$$

Of course, there are many other decompositions that have been suggested in the literature.[67] As was indicated earlier, there is no unambiguous “best” solution to this decomposition problem.

[67]See Walshots (2016), Balk (2017), de Haan and Akem (2017), OECD (2018), and Chapter 8 in Eurostat (2018).

Annex: Proof of Propositions

Proof of Proposition 1

Using the positivity test T1, rewrite the circularity test (28) in the following form:

$$P(p^1,p^2,q^1,q^2) = P(p^0,p^2,q^0,q^2)/P(p^0,p^1,q^0,q^1). \qquad (A1)$$

Now hold p^0 and q^0 constant at some fixed values, say $p^* >> 0_N$ and $q^* >> 0_N$, and define the function $f(p,q)$ as follows:

$$f(p,q) \equiv P(p^*,p,q^*,q) > 0 \text{ for all } p>>0_N \text{ and } q >> 0_N, \qquad (A2)$$

where the positivity of $f(p,q)$ follows from T1. Substituting definition (A2) back into (A1) gives us the following representation for $P(p^1,p^2,q^1,q^2)$:

$$P(p^1,p^2,q^1,q^2) = f(p^2,q^2)/f(p^1,q^1). \qquad (A3)$$

Now let $p^1 = p^2 = p$ in (A3), and apply the identity test T3 to the resulting equation. We obtain:

$$1 = P(p,p,q^1,q^2) = f(p,q^2)/f(p,q^1); \quad p >> 0_N;\ q^1 >> 0_N;\ q^2 >> 0_N. \qquad (A4)$$

Equation (A4) implies that $f(p,q^1) = f(p,q^2)$ for all $p >> 0_N$; $q^1 >> 0_N$; $q^2 >> 0_N$, which in turn implies that $f(p,q)$ does not depend on q; that is, we have

$$f(p,q) = f(p,q^*) \text{ for all } p >> 0_N;\ q >> 0_N. \qquad (A5)$$

Define the function $c(p)$ for all $p >> 0_N$ as

$$c(p) \equiv f(p,q^*) = P(p^*,p,q^*,q^*). \qquad (A6)$$

Substitute (A5) and (A6) back into (A3), and we obtain the following representation for the index number formula, $P(p^1,p^2,q^1,q^2)$:

$$P(p^1,p^2,q^1,q^2) = c(p^2)/c(p^1). \qquad (A7)$$

Now apply the commensurability test, T10, to the P that is defined by (A7), where we set $\alpha_n = (p_n^0)^{-1}$ for $n = 1,\ldots,N$. Using the representation for P given by (61), we find that c must satisfy the following functional equation:

$$c(p^1)/c(p^0) = c(p_1^1/p_1^0,p_2^1/p_2^0,\ldots,p_N^1/p_N^0)/c(1_N); \quad p^0 >> 0_N;\ p^1 >> 0_N. \qquad (A8)$$

Define $h(p)$ as follows:

$$h(p) \equiv c(p)/c(1_N) > 0,\ p >> 0_N, \qquad (A9)$$

where the positivity of h follows from the positivity of c. Using definition (A9), we have

$$h(p_1^1/p_1^0,p_2^1/p_2^0,\ldots,p_N^1/p_N^0) = c(p_1^1/p_1^0,p_2^1/p_2^0,\ldots,p_N^1/p_N^0)/c(1_N)\ p^0 >> 0_N;\ p^1 >> 0_N \qquad (A10)$$

$$= c(p^1)/c(p^0) \quad \text{using (A8)}$$
$$= [c(p^1)/c(1_N)]/[c(p^0)/c(1_N)] \quad \text{using T1}$$
$$= h(p^1)/h(p^0) \quad \text{using (A9) twice.}$$

Thus, h must satisfy the following functional equation:

$$h(p^0)h(p_1^1/p_1^0,p_2^1/p_2^0,\ldots,p_1^1/p_1^0) = h(p^1); \quad p^0 >> 0_N;\ p^1 >> 0_N. \qquad (A11)$$

Define the vector x as the vector p^0 and the vector y as p_1^1/p_1^0, $p_2^1/p_2^0,\ldots,p_N^1/p_N^0$. Hence, the product of the nth components of x and y is equal to the nth component of the vector p^1, and it can be seen that the functional equation (A11) is equivalent to the following functional equation:

$$h(x_1y_1,x_2y_2,\ldots,x_Ny_N) = h(x_1,x_2,\ldots,x_N)\ h(y_1,y_2,\ldots,y_N);\ x >> 0_N;\ y >> 0_N. \qquad (A12)$$

Equation (A12) becomes the following equation if we allow x_1 and y_1 to vary freely but fix all x_i and y_i at 1 for $i = 2,3,\ldots,N$:

$$h(x_1y_1,1,\ldots,1) = h(x_1,1,\ldots,1)\ h(y_1,1,\ldots,1);\ x_1 > 0;\ y_1 > 0. \qquad (A13)$$

But (A13) is an example of *Cauchy*'s (1821) *fourth functional equation*. Using the T1 (positivity) and T2 (continuity) properties of P, which carry over to h, we see that the solution to (A13) is

$$h(x_1,1,\ldots,1) = x_1^{c(1)}, \qquad (A14)$$

where $c(1)$ is an arbitrary constant. In a similar fashion, (A12) becomes the following equation if we allow x_2 and y_2 to vary freely but fix all other x_i and y_i at 1:

$$h(1,x_2y_2,1,\ldots,1) = h(1,x_2,1,\ldots,1)\ h(1,y_2,1,\ldots,1);\ x_2 > 0;\ y_2 > 0. \qquad (A15)$$

The solution to (A15) is

$$h(1,x_2,1,\ldots,1) = x_2^{c(2)}, \qquad (A16)$$

where $c(2)$ is an arbitrary constant. In a similar fashion, we find that

$$h(1,1,x_3,1,\ldots,1) = x_3^{c(3)};\ \ldots;\ h(1,1,\ldots,1,x_N) = x_N^{c(N)}, \qquad (A17)$$

where $c(i)$ are the arbitrary constants. Using (66) repeatedly, we can show

$$\begin{aligned} h(x_1,x_2,...,x_N) &= h(x_1,1,...,1)h(1,x_2,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)h(1,1,1,x_4,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)...h(1,1,1,...,1,x_N) \\ &= \prod_{i=1}^{N} x_i^{c(i)} \text{ using (A14), (A16), and (17).} \end{aligned} \quad \text{(A18)}$$

Thus, we have determined the functional form for the function h. Now use (A9) to determine the function $c(p)$ in terms of $h(p)$:

$$\begin{aligned} c(p) &= c(1_N)h(p) \\ &= c(1_N)\prod_{i=1}^{N} p_i^{c(i)}. \end{aligned} \quad \text{(A19)}$$

Using (A7), we can express P in terms of c as follows:

$$\begin{aligned} P(p^0,p^1,q^0,q^1) &= c(p^1)/c(p^0) \\ &= c(1_N)\prod_{i=1}^{N}(p_i^1)^{c(i)}/c(1_N)\prod_{i=1}^{N}(p_i^0)^{c(i)} \text{ using (A19)} \\ &= \prod_{i=1}^{N}(p_i^1/p_i^0)^{c(i)}. \end{aligned} \quad \text{(A20)}$$

Now apply test T5, proportionality in current prices, to the P defined by (A20). It is easy to see that this test implies that the constants $c(i)$ must sum to 1.

Finally, apply test T17, monotonicity in current prices, to conclude that the constants $c(i)$ must be positive. Hence, we can set the $c(i)$ equal to the α_i and we have proved the proposition.

It should be noted that Konüs and Byushgens (1926) and Frisch (1930) provided alternative proofs for this result, assuming differentiability of the price index function. They used solutions to partial differential equations in place of Cauchy's fourth fundamental functional equation.

Proof of Proposition 2

Define $r_i \equiv p_i^1/p_i^0$ for $i = 1,...,N$. Using T1, T9, and (41), $P(p^0,p^1,v^0,v^1) = P^*(r, v^0,v^1)$. Using T6, T7, and (41):

$$P(p^0,p^1,v^0,v^1) = P^*(r,s^0,s^1), \quad \text{(A21)}$$

where s^t is the period t expenditure share vector for $t = 0,1$.

Let $x \equiv (x_1,...,x_N)$ and $y \equiv (y_1,...,y_N)$ be strictly positive vectors. The transitivity test T11 and (A21) imply that the function P^* has the following property:

$$P^*(x,s^0,s^1)P^*(y,s^0,s^1) = P^*(x_1y_1,...,x_Ny_N,s^0,s^1). \quad \text{(A22)}$$

Using T1, $P^*(r,s^0,s^1) > 0$, and using T14, $P^*(r, s^0,s^1)$ is strictly increasing in the components of r. The identity test T3 implies that

$$P^*(1_N,s^0,s^1) = 1, \quad \text{(A23)}$$

where 1_N is a vector of ones of dimension N. Using a result due to Eichhorn (1978; 66), it can be seen that these properties of P^* are sufficient to imply that there exist positive functions $\alpha_i(s^0,s^1)$ for $i = 1,...,N$ such that P^* has the following representation:

$$\ln P^*(r,s^0,s^1) = \sum_{i=1}^{N}\alpha_i(s^0,s^1)\ln r_i. \quad \text{(A24)}$$

The continuity test T2 implies that the positive functions $\alpha_i(s^0,s^1)$ are continuous. For $\lambda > 0$, the linear homogeneity test T4 implies that

$$\begin{aligned} \ln P^*(\lambda r,s^0,s^1) &= \ln\lambda + \ln P^*(r,s^0,s^1) \\ &= \sum_{i=1}^{N}\alpha_i(s^0,s^1)\ln\lambda r_i \text{ using (A24)} \\ &= \sum_{i=1}^{N}\alpha_i(s^0,s^1)\ln\lambda + \sum_{i=1}^{N}\alpha_i(s^0,s^1)\ln r_i \\ &= \sum_{i=1}^{N}\alpha_i(s^0,s^1)\ln\lambda + \ln P^*(r,s^0,s^1) \text{ using (A24) again.} \end{aligned} \quad \text{(A25)}$$

Equating the right-hand sides of the first and last lines in (A25) shows that the functions $\alpha_i(s^0,s^1)$ must satisfy the following restriction

$$\sum_{i=1}^{N}\alpha_i(s^0,s^1) = 1 \quad \text{(A26)}$$

for all strictly positive vectors s^0 and s^1.

Using the weighting test T16 and the commodity reversal test T8, equation (43) holds. Equation (43) combined with the commensurability test T9 implies that P^* satisfies the following equations

$$P^*(1,...,1,r_i,1,...,1;s^0;s^1) = f(1,s_i^0,s_i^1),\ i = 1,...,N, \quad \text{(A27)}$$

for all $r_i > 0$, where f is the function defined in test T16.

Substitute equation (A27) into equation (A24) in order to obtain the following system of equations:

$$\begin{aligned} \ln P^*(1,...,1,r_i,1,...,1;s^0;s^1) &= \ln f(1,s_i^0,s_i^1) \\ &= \alpha_i(s^0,s^1)\ln r_i;\ i = 1,...,N. \end{aligned} \quad \text{(A28)}$$

But equation i in (A28) implies that the positive continuous function of $2N$ variables $\alpha_i(s^0,s^1)$ is constant with respect to all of its arguments except s_i^0 and s_i^1, and this property holds for each i. Thus, each $\alpha_i(s^0,s^1)$ can be replaced by the positive continuous function of two variables $\beta_i(s_i^0,s_i^1)$ for $i = 1,...,N$.[68] Now replace the $\alpha_i(s^0,s^1)$ in equation (A24) by the $\beta_i(s_i^0,s_i^1)$ for $i = 1,...,N$, and the following representation for P^* is obtained:

$$\ln P^*(r,s^0,s^1) = \sum_{i=1}^{N}\beta_i(s_i^0,s_i^1)\ln r_i. \quad \text{(A29)}$$

Equation (A26) imply that the functions $\beta_i(s_i^0,s_i^1)$ also satisfy the following restrictions:

[68] More explicitly, $\beta_1(s_1^0,s_1^1) \equiv \alpha_1(s_1^0,1,...,1;s_1^1,1,...,1)$ and so on. That is, in defining $\beta_1(s_1^0,s_1^1)$, the function $\alpha_1(s_1^0,1,...,1;s_1^1,1,...,1)$ is used where all components of the vectors s^0 and s^1 except the first are set equal to an arbitrary positive number like 1.

$$\sum_{n=1}^{N} s_n^{\,0} = 1; \sum_{n=1}^{N} s_n^{\,1} = 1 \text{ implies } \sum_{i=1}^{N} \beta_i(s_i^{\,0},s_i^{1}) = 1. \qquad \text{(A30)}$$

Assume that the weighting test T17 holds and substitute equations (43) into (A29) in order to obtain the following equations:

$$\beta_i(0,0) \ln [p_i^{1}/p_i^{0}] = 0;\ i = 1,\ldots,N. \qquad \text{(A31)}$$

Since p_i^1 and p_i^0 can be arbitrary positive numbers, it can be seen that (A31) implies

$$\beta_i(0,0) = 0;\ i = 1,\ldots,N. \qquad \text{(A32)}$$

Assume that the number of commodities N is equal to or greater than 3. Using (A10) and (A12), Theorem 2 in Aczél (1987; 8) can be applied, and the following functional form for each of the $\beta_i(s_i^0,s_i^1)$ is obtained:

$$\beta_i(s_i^{\,0},s_i^{1}) = \mathrm{c}s_i^{\,0} + (1-\gamma)s_i^{1},\ i = 1,\ldots,N, \qquad \text{(A33)}$$

where γ is a positive number satisfying $0 < \gamma < 1$.

Finally, the time reversal test T10 *or* the quantity weights symmetry test T12 can be used to show that γ must equal ½. Substituting this value for γ back into (A33) and then substituting those equations back into (A29), the functional form for P^* and hence P is determined as

$$\ln \mathrm{P}(p^0,p^1,v^0,v^1) = \ln \mathrm{P}^*(\,r,s^0,s^1) = \sum_{n=1}^{N} (1/2)[\mathrm{s}_n^{\,0} + \mathrm{s}_n^{\,1}] \ln (p_n^{\,1}/p_n^{\,0}). \qquad \text{(A34)}$$

References

Aczél, János. 1987. *A Short Course on Functional Equations*. Dordrecht: Reidel Publishing Co.

Anderson, Richard G., Barry E. Jones and Travis Nesmith. 1997. "Building New Monetary Services Indices: Concepts, Data and Methods." *Federal Reserve Bank of St. Louis Review* 79(1): 53–83.

Auer, Ludwig von. 2002. "Spurious Inflation: The Legacy of Laspeyres and Others." *The Quarterly Review of Economics and Finance* 42: 529–42.

Australian Bureau of Statistics. 2017. "An Implementation Plan to Annually Re-Weight the Australian CPI." ABS Catalogue No. 6401.0.60.005, Canberra: Australian Bureau of Statistics.

Balk, Bert M. 1985. "A Simple Characterization of Fisher's Price Index." *Statistische Hefte* 26: 59–63.

Balk, Bert M. 1995. "Axiomatic Price Index Theory: A Survey." *International Statistical Review* 63: 69–93.

Balk, Bert M. 2008. *Price and Quantity Index Numbers*. New York: Cambridge University Press.

Balk, Bert M. 2017. "Mixed-Form Indices: A Study of Their Properties." Paper presented at the 15th meeting of the Ottawa Group, 10–12 May 2017, Altville am Rhein, Germany.

Balk, Bert M., and W.E. Diewert. 2001. "A Characterization of the Törnqvist Price Index." *Economics Letters* 73: 279–81.

Bowley, Arthur L. 1901. *Elements of Statistics*. Westminster: Orchard House.

Cauchy, Augustin L. 1821. *Cours d'analyse de l'École Polytechnique*, *Volume 1*, *Analyse algébrique*. Paris.

Cobb, Charles and Paul H. Douglas. 1928. "A Theory of Production." *American Economic Review* 18: 139–65.

de Haan, Jan, and A. Akem. 2017. "Calculating Contributions of Product Categories to Percentage Changes in the Annually Chained Australian CPI." Paper presented at the EMG Workshop 2017, Centre for Applied Economic Research, UNSW Sydney, December 1. https://tinyurl.com/tlwlheg

Diewert, W. Erwin. 1992. "Fisher Ideal Output, Input and Productivity Indices Revisited." *Journal of Productivity Analysis* 3: 211–48.

Diewert, W. Erwin. 2002. "The Quadratic Approximation Lemma and Decompositions of Superlative Indices." *Journal of Economic and Social Measurement* 28: 63–88.

Drobisch, Moritz W. 1871. "Über die Berechnung der Veränderungen der Waarenpreise und des Geldwerths." *Jahrbücher für Nationalökonomie und Statistik* 16: 143–56.

Edgeworth, Francis Y. 1925. *Papers Relating to Political Economy*, Volume 1. New York: Burt Franklin.

Ehemann, Christian, Arnold J. Katz, and Brent R. Moulton. 2002. "The Chain-Additivity Issue and the U.S. National Accounts." *Journal of Economic and Social Measurement* 28: 37–49.

Eichhorn, Wolfgang. 1978. *Functional Equations in Economics*. Reading, MA: Addison-Wesley Publishing Company.

Eichhorn, Wolfgang and Joachim Voeller. 1976. *Theory of the Price Index*. Lecture Notes in Economics and Mathematical Systems, Vol. 140, Berlin: Springer-Verlag.

Eurostat. 2018. *Harmonized Index of Consumer Prices (HICP) Methodological Manual*. Luxembourg: Publications Office of the European Union.

Fisher, Irving. 1911. *The Purchasing Power of Money*. London: Macmillan.

Fisher, Irving. 1921. "The Best Form of Index Number." *Journal of the American Statistical Association* 17: 533–37.

Fisher, Irving. 1922. *The Making of Index Numbers*. Houghton-Mifflin, Boston.

Fisher, Willard C. 1913. "The Tabular Standard in Massachusetts History." *Quarterly Journal of Economics* 27: 417–51.

Frisch, Ragnar. 1930. "Necessary and Sufficient Conditions Regarding the Form of an Index Number which shall Meet Certain of Fisher's Tests." *Journal of the American Statistical Association* 25: 397–406.

Funke, H., G. Hacker, and Joachim Voeller. 1979. "Fisher's Circular Test Reconsidered." *Schweizerische Zeitschrift für Volkswirtshaft und Statistik* 115: 677–87.

Funke, H. and Joachim Voeller. 1978. "A Note on the Characterization of Fisher's Ideal Index." In *Theory and Applications of Economic Indices*, edited by Wolfgang Eichhorn, Rudolf Henn, Otto Opitz, and Ronald W. Shephard. Würzburg: PhysicaVerlag, pp. 177–81.

Hicks, John R. 1946. *Value and Capital*, Second Edition. Oxford: Clarendon Press.

Hillinger, Claude. 2002. "A General Theory of Price and Quantity Aggregation and Welfare Measurement." SEMECON, University of Munich, Ludwigstrasse 33/IV, D-80539 Munich, Germany.

Jevons, William S. 1863. "A Serious Fall in the Value of Gold Ascertained and its Social Effects Set Forth." In *Investigations in Currency and Finance* (1884), London: Macmillan and Co, pp. 13–118.

Jevons, William S. 1865. "The variation of Prices and the Value of the Currency since 1782." *Journal of the Statistical Society of London* 28: 294–320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119–50.

Konüs, Alexander A. and Sergei S. Byushgens. 1926. "K probleme pokupatelnoi cili deneg." *Voprosi Konyunkturi* 2: 151–72.

Laspeyres, Etienne. 1871. "Die Berechnung einer mittleren Waarenpreissteigerung." *Jahrbücher für Nationalökonomie und Statistik* 16: 296–314.

Lowe, Joseph. 1823. *The Present State of England in Regard to Agriculture, Trade and Finance*, Second edition. London: Longman, Hurst, Rees, Orme and Brown.

Marshall, Alfred. 1887. "Remedies for Fluctuations of General Prices." *Contemporary Review* 51: 355–75.

OECD. 2018. "OECD Calculation of Contributions to Overall Annual Inflation." Release May 2, 2018, Paris: OECD.

http://www.oecd.org/sdd/prices-ppp/OECD-calculation-contributions-annual-inflation.pdf

Paasche, Hermann. 1874. "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen." *Jahrbücher fur Nationalökonomie und Statistik* 12: 168–78.

Pierson, Nicolaas G. 1895. "Index Numbers and Appreciation of Gold." *Economic Journal* 5: 329–35.

Pierson, Nicolaas G. 1896. "Further Considerations on Index-Numbers." *Economic Journal* 6: 127–31.

Pollak, Robert A. 1971. "The Theory of the Cost of Living Index." Research Discussion Paper 11, Bureau of Labor Statistics, Washington, DC. Published in *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette. Ottawa: Statistics Canada, 1983, pp. 87–161.

Reinsdorf, Marshall B., W. Erwin Diewert and Christian Ehemann. 2002. "Additive Decompositions for the Fisher, Törnqvist and geometric Mean Indices." *Journal of Economic and Social Measurement* 28: 51–61.

Ribe, Martin. 1999. "Effects of Subcomponents on Chained Price Indices Like the HICPs and the MUICP." Unpublished paper, Statistics Sweden.

Samuelson, Paul A. 1947. *Foundations of Economic Analysis*. Cambridge, MA: Harvard University Press.

Scrope, George P. 1833. *Principles of Political Economy*. London: Longman, Rees, Orme, Brown, Green and Longman.

Sidgwick, Henry. 1883. *The Principles of Political Economy*. London: Macmillan.

Van Ijzeren. 1987. *Bias in International Index Numbers: A Mathematical Elucidation*. Dissertation for the Hungarian Academy of Sciences, Den Haag: Koninklijke Bibliotheek.

vartia, Yrjö O. 1976. *Relative Changes and Index Numbers*. Helsinki: The Research Institute of the Finnish Economy.

Vogt, Arthur. 1980. "Der Zeit und der Faktorumkehrtest als 'Finders of Tests'." *Statistische Hefte* 21: 66–71.

Vogt, Arthur and Janos Barta. 1997. *The Making of Tests for Index Numbers*. Heidelberg: Physica-Verlag.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Walsh, C. Moylan. 1921a. *The Problem of Estimation*. London: P.S. King & Son.

Walsh, C. Moylan. 1921b. "Discussion." *Journal of the American Statistical Association* 17: 537–44.

Westergaard, Harald. 1890. *Die Grundzüge der Theorie der Statistik*. Jena: Fischer.

Walshots, Jan. 2016. "Contributions to and Impacts on Inflation." The Hague: Central Bureau of Statistics.

STOCHASTIC APPROACHES TO INDEX NUMBER THEORY* 4

Introduction

"In drawing our averages the independent fluctuations will more or less destroy each other; the one required variation of gold will remain undiminished."

W. Stanley Jevons (1884, 26).

The stochastic approach to the determination of the price index can be traced back to the work of Jevons and Edgeworth over a hundred years ago.[1] In "Early Unweighted Stochastic Approaches to Bilateral Index Number Theory" section, the work of these early pioneers will be explained. Basically, their approach was to take an *average of the price ratios* pertaining to the two periods as their index number. However, Keynes (1930) was critical of this approach to index number theory because it did not take into account the *economic importance* of each commodity in the index. Thus in "The Weighted Stochastic Approach of Theil" section, the weighted stochastic approach of Theil (1967) will be explained. This approach does take into account the economic importance of each commodity.

In "The Time Product Dummy Approach to Bilateral Index Number Theory" section, an introduction to the *time product dummy stochastic approach* to index number theory will be presented. Using this approach, the focus is on providing representative *price levels* for two periods.[2] Weighted versions of this approach are described in "The Weighted Time Product Dummy Approach to Bilateral Index Number Theory" section.

A weakness of the material presented in this chapter is that it is assumed that all prices are positive. In Chapters 7 and 8, this assumption will be relaxed. The reason for postponing a discussion of index number theory when there are missing prices is that it is useful to develop the economic approach to index number theory before discussing the problem of missing prices. The missing price problem and the treatment of new and disappearing products will be studied in some detail in Chapters 7 and 8. The economic approach to index number theory will be discussed in Chapters 5 and 8.

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[1] For references to the literature, see Diewert (1993, 37–38) (2010).

[2] The extension of the price levels approach to many periods will be discussed in Chapter 7.

Early Unweighted Stochastic Approaches to Bilateral Index Number Theory

The basic idea behind the early stochastic approaches to index number theory is that each price relative, p_n^1/p_n^0 for $n = 1,2, . . .,N$, can be regarded as an estimate of a common inflation rate a between periods 0 and 1; that is, it is assumed that

$$p_n^1/p_n^0 = \alpha + \varepsilon_n, \quad n = 1,2, . . .,N \qquad (1)$$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the *Carli* (1764) *price index* P_C, defined as

$$P_C(p^0,p^1) \equiv \sum_{n=1}^N (1/N)\, p_n^1/p_n^0. \qquad (2)$$

Unfortunately, P_C does not satisfy the time reversal test,[3] that is, $P_C(p^1,p^0) \neq 1/ P_C(p^0,p^1)$.

Now suppose that the stochastic specification of the error terms is changed; that is, assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus, we have

$$\ln(p_n^1/p_n^0) = \beta + \varepsilon_n, \quad n = 1,2, . . .,N \qquad (3)$$

where $\beta \equiv \ln\alpha$ and ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence, the corresponding estimate for the common inflation rate α[4] is the *Jevons* (1865) *price index* P_J, which is defined as

[3] In fact, Fisher (1922, 66) noted that $P_C(p^0,p^1)p_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; that is, Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula. The upward bias of the Carli index will be illustrated in Chapter 6.

[4] Greenlees (1999) pointed out that although $(1/N)\Sigma_{n=1}^N \ln(p_n^1/p_n^0)$ is an unbiased estimator for β, the corresponding exponential of this estimator, P_J defined by (4), will generally *not* be an unbiased estimator for α under our stochastic assumptions. To see this, let $x_n = \ln(p_n^1/p_n^0)$. Taking expectations, we have $Ex_n = \beta = \ln(\alpha)$. Thus, each x_n is an unbiased estimator of overall log price change. If we wish to measure overall price change α instead of log price change β, then use $y_n \equiv \exp[x_n]$ as an estimator for α. Define the positive convex function f of one variable x by $f(x) \equiv e^x$. By Jensen's (1906) inequality, we have $Ef(x) \geq f(Ex)$. Letting x equal the random variable x_n, this inequality becomes $E(p_n^1/p_n^0) = Ef(x_n) \geq f(Ex_n) = f(\beta) = e^\beta = e^{\ln\alpha} = \alpha$. Thus, for each n, we have $E(p_n^1/p_n^0) \geq \alpha$, and it can be seen that the Jevons price index defined by (4) will generally have an upward bias from a statistical point of view. However,

$$P_J(p^0,p^1) \equiv \prod_{n=1}^{N} (p_n^1/p_n^0)^{1/N}. \quad (4)$$

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. However, both the Jevons and Carli price indices suffer from a fatal flaw: each price relative p_n^1/p_n^0 is regarded as being *equally important* and is given an equal weight in the index number formulae (2) and (4). Keynes was particularly critical of this *unweighted stochastic approach* to index number theory. He directed the following criticism toward this approach, which was vigorously advocated by Edgeworth (1923):

> Nevertheless I venture to maintain that such ideas, which I have endeavoured to expound above as fairly and as plausibly as I can, are root-and-branch erroneous. The "errors of observation", the "faulty shots aimed at a single bull's eye" conception of the index number of prices, Edgeworth's "objective mean variation of general prices", is the result of confusion of thought. There is no bull's eye. There is no moving but unique centre, to be called the general price level or the objective mean variation of general prices, round which are scattered the moving price levels of individual things. There are all the various, quite definite, conceptions of price levels of composite commodities appropriate for various purposes and inquiries which have been scheduled above, and many others too. There is nothing else. Jevons was pursuing a mirage.
>
> What is the flaw in the argument? In the first place it assumed that the fluctuations of individual prices round the "mean" are "random" in the sense required by the theory of the combination of independent observations. In this theory the divergence of one "observation" from the true position is assumed to have no influence on the divergences of other "observations". But in the case of prices, a movement in the price of one commodity necessarily influences the movement in the prices of other commodities, whilst the magnitudes of these compensatory movements depend on the magnitude of the change in expenditure on the first commodity as compared with the importance of the expenditure on the commodities secondarily affected. Thus, instead of "independence", there is between the "errors" in the successive "observations" what some writers on probability have called "connexity", or, as Lexis expressed it, there is "sub-normal dispersion".
>
> We cannot, therefore, proceed further until we have enunciated the appropriate law of connexity. But the law of connexity cannot be enunciated without reference to the relative importance of the commodities affected—which brings us back to the problem that we have been trying to avoid, of weighting the items of a composite commodity.
>
> John Maynard Keynes (1930, 76–77)

One of the points Keynes makes in this quotation is that prices in the economy are not independently distributed from each other and from quantities. In current macroeconomic terminology, we can interpret Keynes as saying that a macroeconomic shock will be distributed across all prices and quantities in the economy through the normal interaction between supply and demand; that is, through the workings of the general equilibrium system. Thus, Keynes seemed to be leaning toward the economic approach to index number theory (even before it was developed to any great extent), where quantity movements are functionally related to price movements. A second point that Keynes made in this quotation is that there is no such thing as the inflation rate; there are only price changes that pertain to well-specified sets of commodities or transactions; that is, the domain of definition of the price index must be carefully specified. A final point that Keynes made is that price movements must be weighted by their *economic importance*; that is, by quantities or expenditures.[5]

In addition to these theoretical criticisms, Keynes also made the following strong empirical attack on Edgeworth's unweighted stochastic approach:

> The Jevons—Edgeworth "objective mean variation of general prices", or "indefinite" standard, has generally been identified, by those who were not as alive as Edgeworth himself was to the subtleties of the case, with the purchasing power of money—if only for the excellent reason that it was difficult to visualise it as anything else. And since any respectable index number, however weighted, which covered a fairly large number of commodities could, in accordance with the argument, be regarded as a fair approximation to the indefinite standard, it seemed natural to regard any such index as a fair approximation to the purchasing power of money also.
>
> Finally, the conclusion that all the standards "come to much the same thing in the end" has been reinforced "inductively" by the fact that rival index numbers (all of them, however, of the wholesale type) have shown a considerable measure of agreement with one another in spite of their different compositions.
>
> . . . On the contrary, the tables given above (pp. 53,55) supply strong presumptive evidence that over long periods as well as over short periods the movements of the wholesale and of the consumption standards respectively are capable of being widely divergent.
>
> John Maynard Keynes (1930, 80–81)

In this quotation, Keynes noted that the proponents of the unweighted stochastic approach to price change measurement were comforted by the fact that all of the then existing (unweighted) indices of wholesale prices showed broadly similar movements. However, Keynes showed empirically that these wholesale price indices moved quite differently than his CPIs.[6]

if we make the measurement of *average log price change* our estimation target, then the Jevons index is no longer biased for this alternative target index.

[5]An empirical example in the annex to Chapter 6 will illustrate the importance of weighting. This example also illustrates that there can be substantial differences between the Jevons and Carli indices.

[6]Using the OECD national accounts data for the last five decades, some broad trends in the rates of increase in prices for the various components of GDP can be observed: Rates of increase for the prices of internationally traded goods have been the lowest, followed by the prices of reproducible capital goods, followed by consumer prices, followed by wage rates. From other sources, land prices have shown the highest rate of

In order to overcome the Keynesian criticisms of the unweighted stochastic approach to index numbers, it is necessary to

- have a definite domain of definition for the index number and
- weight the price relatives by their economic importance.

On the second dot point, it should be noted that the issue of weighting price ratios came up early in the history of index number theory. Young (1812) advocated some form of rough weighting of the price relatives according to their relative value over the period being considered, but the precise form of the required value weighting was not indicated.[7] However, it was Walsh (1901, 83–121) (1921, 81–90) who stressed the importance of weighting the individual price ratios, where the weights are functions of the associated values for the commodities in each period, and each period is to be treated symmetrically in the resulting formula:

> What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [price ratios] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered.
>
> Correa Moylan Walsh (1901, 104)

> Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period and there is a second period which is compared with it. Price variations[8] have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods.
>
> Correa Moylan Walsh (1921, 90)

In the following section, Theil's solution to the weighting problem will be described.

The Weighted Stochastic Approach of Theil

> It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. . . . A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth.
>
> Correa Moylan Walsh (1921, 82–83)

Theil (1967, 136–37) proposed a solution to the lack of weighting in the Jevons index defined by (4). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0 q_n^0 / p^0 \cdot q^0$, the period 0 expenditure share for commodity n. The resulting overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^{N} s_n^0 \ln(p_n^1/p_n^0)$. Now repeat the aforementioned mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^{N} s_n^1 \ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change is equally valid, so it is best to take a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change.[9] Theil[10] argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

price increase over this period. Of course, if a country adjusts the price of computer-related equipment for quality improvements, then the aggregate price of capital machinery and equipment tends to move *downward* in recent years. Another source of long-run differential rates of price increase is due to the fact that service prices tend to increase more rapidly than product prices. Thus, there are long-term systematic differences in price movements over different domains of definition.

[7] Walsh (1901, 84) refers to Young's contributions as follows: "Still, although few of the practical investigators have actually employed anything but even weighting, they have almost always recognized the theoretical need of allowing for the relative importance of the different classes ever since this need was first pointed out, near the commencement of the century just ended, by Arthur Young. . . . Arthur Young advised simply that the classes should be weighted according to their importance."

[8] A price variation is a price ratio or price relative in Walsh's terminology.

[9] "The [asymmetric] price index (1.6) has certain merits. It is, for example, independent of the units in which we measure the quantities of the various commodities (tons, gallons, etc.). It has the disadvantage, however, of being one sided in the sense that it is based on the distribution of expenditure in the *a*th region. We could equally well apply our random selection procedure to the *b*th region, in which case, w_{ia} is replaced by w_{ib} in (1.5) and (1.6). We must conclude that (1.6) is an asymmetric index number, which is a disadvantage because the question asked is symmetric: If the price level of the *b*th region exceeds that of the *a*th by a factor 1.2, say, we should expect that the price level of the latter region exceed that of the former by a factor 1/1.2" (Henri Theil (1967, 137)).

[10] "The price index number defined in (1.8) and (1.9) uses the n individual logarithmic price differences as the basic ingredients. They are combined linearly by means of a two-stage random selection procedure: First, we give each region the same chance ½ of being selected, and second, we give each dollar spent in the selected region the same chance ($1/m_a$ or $1/m_b$) of being drawn" (Henri Theil (1967, 138)).

$$\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} (½)(s_n^{\,0} + s_n^{\,1})\ln(p_n^{\,1}/p_n^{\,0}). \quad (5)$$

It is possible to give the following statistical interpretation of the right-hand side of (5). Define the *n*th logarithmic price ratio r_n by

$$r_n \equiv \ln(p_n^{\,1}/p_n^{\,0}). \qquad n = 1, \ldots, N. \quad (6)$$

Now define the discrete random variable, say *R*, as the random variable that can take on the values r_n with probabilities $\rho_n \equiv (1/2)(s_n^{\,0} + s_n^{\,1})$ for $n = 1, \ldots, N$. Note that since each set of expenditure shares, $s_n^{\,0}$ and $s_n^{\,1}$, sum to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable *R* is

$$\mathrm{E}[R] \equiv \sum_{n=1}^{N} \rho_n r_n = \sum_{n=1}^{N} (½)(s_n^{\,0} + s_n^{\,1}) \ln(p_n^{\,1}/p_n^{\,0}) = \ln P_T(p^0,p^1,q^0,q^1) \quad (7)$$

using (5) and (6). Thus, the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the *N* discrete price ratios in this domain of definition are weighted according to Theil's probability weights, $\rho_n \equiv (½)(s_n^{\,0} + s_n^{\,1})$ for $n = 1, \ldots, N$.

Taking antilogs of both sides of (7), we obtain the Törnqvist (1936) (1937) Theil price index, P_T. This index number formula has a number of good properties.[11] In particular, P_T satisfies the *time reversal test*:

$$P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1). \quad (8)$$

The price index P_T also satisfies the following *linear homogeneity test in current period prices*:

$$P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1), \quad (9)$$

for all positive numbers λ and strictly positive vectors p^0,p^1,q^0,q^1. Thus, if all period one prices increase by the same positive number λ and if the price index *P* satisfies the test (9), then the price index increases by this same scalar factor λ.

The time reversal test and the linearly homogeneous test can be used to justify Theil's (arithmetic) method of forming an average of the two sets of expenditure shares in order to obtain his probability weights, $\rho_n \equiv (½)[s_n^{\,0} + s_n^{\,1}]$ for $n = 1, \ldots, N$. Consider the following *symmetric mean* class of Theil-type *logarithmic index number formulae*:

$$\ln P_{ml}(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} m(s_n^{\,0},s_n^{\,1})\ln(p_n^{\,1}/p_n^{\,0}), \quad (10)$$

where $m(s_n^{\,0},s_n^{\,1})$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares, $s_n^{\,0}$ and $s_n^{\,1}$, respectively. In order for P_{ml} to satisfy the time reversal test, it is necessary that the mean function m be symmetric. In order for the weights in (10) to sum to one so that the linear homogeneity test is satisfied and the weights can be interpreted as probability weights, it can be shown that the homogeneous symmetric mean function $m(a,b)$ that appears in (10) *must* be the arithmetic mean.

The stochastic approach of Theil has another nice symmetry property. Instead of considering the distribution of the price ratios $r_n = \ln(p_n^{\,1}/p_n^{\,0})$, we could also consider the distribution of the *reciprocals* of these price ratios, say:

$$\begin{aligned} t_n &\equiv \ln(p_n^{\,0}/p_n^{\,1}); \qquad n = 1, \ldots, N \\ &= \ln(p_n^{\,1}/p_n^{\,0})^{-1} \\ &= -\ln(p_n^{\,1}/p_n^{\,0}) \\ &= -r_n, \end{aligned} \quad (11)$$

where the last equality follows from definitions (6). We can still associate the symmetric probability, $\rho_n \equiv (1/2)[s_n^{\,0} + s_n^{\,1}]$, with the *n*th reciprocal logarithmic price ratio t_n for $n = 1, \ldots, N$. Now define the discrete random variable, say *T*, as the random variable that can take on the values t_n with probabilities $\rho_n \equiv (1/2)(s_n^{\,0} + s_n^{\,1})$ for $n = 1, \ldots, N$. It can be seen that the expected value of the discrete random variable *T* is

$$\begin{aligned} E[T] &\equiv \sum_{n=1}^{N} \rho_n t_n \\ &= -\sum_{n=1}^{N} \rho_n r_n \quad \text{using (11)} \\ &= -E[R] \quad \text{using (7)} \\ &= -\ln P_T(p^0,p^1,q^0,q^1). \end{aligned} \quad (12)$$

Thus, it can be seen that the distribution of the random variable *T* is equal to minus the distribution of the random variable *R*. Hence, it does not matter whether we consider the distribution of the original logarithmic price ratios, $r_n \equiv \ln(p_n^{\,1}/p_n^{\,0})$, or the distribution of their reciprocals, $t_n \equiv \ln(p_n^{\,0}/p_n^{\,1})$: we obtain essentially the same stochastic theory.

It is possible to consider weighted stochastic approaches to index number theory where we look at the distribution of price ratios, $p_n^{\,1}/p_n^{\,0}$, rather than the distribution of the logarithmic price ratios, $\ln(p_n^{\,1}/p_n^{\,0})$. Thus, again following in the footsteps of Theil, suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then, the probability that we will draw the *n*th price relative is equal to $s_n^{\,0} \equiv p_n^{\,0}q_n^{\,0}/p^0\cdot q^0$, the period 0 expenditure share for commodity *n*. Now the overall mean (period 0 weighted) price change is

$$P_L(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} s_n^{\,0}(p_n^{\,1}/p_n^{\,0}), \quad (13)$$

which turns out to be the Laspeyres price index P_L defined in Chapter 2.

Now repeat this mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) price change equal to

$$P_{Pal}(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} s_n^{\,1}(p_n^{\,1}/p_n^{\,0}). \quad (14)$$

The right-hand side of (14) is known as the Palgrave (1886) index number formula, P_{Pal}.[12]

[11] See Section 5 of Chapter 3 for a listing of the test properties of the Törnqvist–Theil index.

[12] It is formula number 9 in Fisher's (1922, 466) listing of index number formulae.

It can be verified that neither the Laspeyres nor Palgrave price indices satisfy the time reversal test (8). Again following in the footsteps of Theil, we might try to obtain a formula that satisfied the time reversal test by taking a symmetric average of the two sets of shares. Consider the following class of *symmetric mean index number formulae*:

$$P_{SM}(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} m(s_n^0,s_n^1)(p_n^1/p_n^0), \quad (15)$$

where $m(s_n^0,s_n^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares, s_n^0 and s_n^1, respectively. However, in order to interpret the right-hand side of (15) as an expected value of the price ratios p_n^1/p_n^0, it is necessary that

$$\Sigma_{n=1}^{N} m(s_n^0,s_n^1) = 1. \quad (16)$$

However, in order to satisfy (16), *m* cannot be a symmetric geometric or harmonic mean or any of the commonly used homogeneous symmetric mean. In fact, the only simple homogeneous symmetric mean that satisfies (16) is the arithmetic mean.[13] With this choice of *m*, (15) becomes the following (unnamed) index number formula, P_U:

$$P_U(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N} (½)(s_n^0 + s_n^1)(p_n^1/p_n^0). \quad (17)$$

Unfortunately, the unnamed index P_U does not satisfy the time reversal test either.

These considerations explain why Theil's stochastic index number formula P_T seems to be the preferred member of this class of index number formula.

In the following two sections, stochastic approaches to index number theory that focus on the estimation of *price levels* rather than *bilateral price indices* will be considered. In Section 4, the price level approach will be applied to the case of two time periods, while in Section 5 the price level approach will be applied to many periods.

The Time Product Dummy Approach to Bilateral Index Number Theory

In this section, a stochastic model that estimates the average price levels for two periods will be derived using an adaptation of the country product dummy model.[14] The adaptation is to move from the context of comparing prices across two countries to the time series context where the comparison of prices is made between two time periods.

Consider the following model of price behavior for the value aggregate under consideration:

$$p_n^t = \pi_t \alpha_n e_{tn}. \qquad t = 0,1; n = 1, \ldots, N. \quad (18)$$

The parameter π_t can be interpreted as the TPD *price level* for period *t*, α_n can be interpreted as a commodity *n quality adjustment factor*,[15] and e_{tn} is a positive stochastic error term with a mean that is assumed to be 1. Define the logarithms of p_n^t and e_{tn} as $y_{tn} \equiv \ln p_n^t$ and $\varepsilon_{tn} \equiv \ln e_{tn}$ for $t = 0,1$; $n = 1, \ldots, N$; define the logarithm of π_t as $\rho_t \equiv \ln \pi_t$ for $t = 0,1$; and define the logarithm of α_n as $\beta_n \equiv \ln \alpha_n$ for $n = 1, \ldots, N$. Then taking logarithms of both sides of (18) leads to the following *linear regression model*:

$$y_{tn} = \rho_t + \beta_n + \varepsilon_{tn}. \qquad t = 0,1; n = 1, \ldots, N. \quad (19)$$

It can be seen that the parameters in the linear regression model defined by (19), ρ_t and β_n, are not uniquely determined. If any number λ is added to each ρ_t and the same number λ is subtracted from each β_n, the right-hand side of each equation in (19) will not change. Thus, in order to obtain unique estimates for ρ_t and β_n on the right-hand side of equations (19), we need to impose a normalization on these parameters. Impose the following normalization:

$$\rho_0 \equiv 0. \quad (20)$$

This normalization corresponds to setting the period 0 price level, $\pi_0 \equiv \exp[\rho_0]$, equal to 1. Thus, $\pi_1/\pi_0 = \pi_1$ and thus the estimated $\pi_1^* \equiv \exp[\rho_1^*]$ can be interpreted as a *bilateral index number*, where ρ_1^* and $\beta_1^*, \ldots, \beta_N^*$ solve the following *least squares minimization problem*:

$$\min_{\rho 1,\beta 1,\ldots \beta N} \Sigma_{n=1}^{N} (y_0 n - 0 - \beta_n)^2 + \Sigma_{n=1}^{N} (y_1 n - \rho_1 - \beta_n)^2. \quad (21)$$

The first-order necessary (and sufficient) conditions for solving (21) are equation (22) and the *N* equations (23) are listed here:

$$N\rho_1 + \Sigma_{n=1}^{N} \beta_n = \Sigma_{n=1}^{N} y_1 n_{,} \quad (22)$$

$$\rho_1 + 2\beta_n = y_0 n + y_1 n \qquad n = 1, \ldots, N. \quad (23)$$

The solution to equations (22) and (23) is given by the following estimators:

$$\rho_1^* = (1/N)\Sigma_{n=1}^{N} [y_{1n} - y_{0n}], \quad (24)$$

$$\beta_n^* = (½)y_{0n} + (½)[y_{1n} - \rho_1^*]. \qquad n = 1, \ldots, N. \quad (25)$$

Exponentiating the estimators defined by (24) and (25) leads to the following estimators for the period 1 price level (and price index) $\pi_1^* \equiv \exp[\rho_1^*]$ and the quality adjustment factors $\alpha_n^* \equiv \exp[\beta_n^*]$:

$$\pi_1^* \equiv \prod_{n=1}^{N} (p_n^1/p_n^0)^{1/N} = P_J(p^0,p^1), \quad (26)$$

$$\alpha_n^* \equiv (p_n^0)^{1/2}(p_n^1/\pi_1^*)^{1/2} \qquad n = 1, \ldots, N \quad (27)$$

[13] For a proof of this assertion, see Diewert (2000).

[14] See Summers (1973), who introduced the CPD model. Balk (1980) was the first to adapt the CPD method to the time series context.

[15] In the context of commodities that are close substitutes, the interpretation of the α_n as quality adjustment factors is intuitively plausible. In the context of commodities that are not close substitutes, the α_n can be interpreted as *relative utility valuation factors*; i.e., α_n represents the marginal utility value to purchasers of the product of an extra unit of q_n. This interpretation relies on the economic approach to index number theory and is pursued in more depth in Chapter 8.

where $P_J(p^0,p^1)$ is the Jevons price index defined earlier by (4). This is Summer's (1973) country product dummy multilateral method adapted to the time series context for the case of two time periods with no missing observations.

The model defined by (18) or (19) can be interpreted as a highly simplified *hedonic regression model*,[16] where α_n are interpreted as the quality adjustment factors for each product n. The only characteristic of each commodity is the commodity itself. As noted earlier, this model is also a special case of the country product dummy method for making international comparisons between the prices of different countries. A possible advantage of this regression method for constructing a price index is that a standard error for the period 1 log price level ρ_1 (and hence for π_1) can be obtained. This advantage of the stochastic approach to index number theory was stressed by Selvanathan and Rao (1994). However, suppose that the standard error (or variance) for the estimated π_1^* were 0. Then all of the error terms e_{tn} in (18) must be equal to 1, and under these conditions, with $\pi_0 \equiv 1$, equations (18) imply that $p^1 = \pi_1^* p^0$ so that prices move in a *proportional manner* going from period 0 to period 1. Thus, a nonzero standard error simply means that prices did *not* move in a proportional manner going from period 0 to 1. This fact does not imply that a larger standard error for π_1^* means that the overall inflation rate for the commodity group is more uncertain. For example, if the quantity vector q for periods 0 and 1 were constant, then most economists would agree that the appropriate measure of overall purchaser inflation is exactly measured by the Lowe index, $p^1{\cdot}q/p^0{\cdot}q$. Prices need not move in a proportional manner under these conditions so the standard error for π_1^* could be large but yet a very precise exact measure of overall inflation is available. Thus, it must be kept in mind that standard errors for price levels or price indices that are generated by a stochastic approach to index number theory are measures of parameter dispersion that are *conditional on the underlying model of price formation*. If the underlying model is faulty and the error variance is high, then the parameter standard errors that are generated by the model should be viewed with some degree of caution.

The Weighted Time Product Dummy Approach to Bilateral Index Number Theory

There is a problem with the unweighted least squares model defined by (21), namely, that the logarithm of each price quote is given exactly the *same weight* in the model no matter what the expenditure on that item was in each period. This is obviously unsatisfactory since a price that has very little economic importance (that is, a low expenditure share in each period) is given the same weight in the regression model compared to a very important item. As was mentioned earlier, Walsh was the first serious index number economist to stress the importance of weighting. Keynes was quick to follow up on the importance of weighting[17] and Fisher emphatically endorsed weighting.[18] Griliches also endorsed weighting in the hedonic regression context.[19] Thus, it is useful to consider how to introduce weights into the TPD model that reflect the economic importance of the various commodities into the model.

In order to take economic importance into account, replace (21) by the following *weighted least squares minimization problem*:[20]

$$\min_{\rho_1,\beta_1,\ldots\beta_N} \Sigma_{n=1}^N q_n^0[\ln p_n^0 - \beta_n]^2 + \Sigma_{n=1}^N q_n^1 [\ln p_n^1 - \rho_1 - \beta_n]^2, \qquad (28)$$

where we have set $\rho_0 = 0$. The squared error for product n in period t is repeated q_n^t times to reflect the sales of product n in period t. Thus, the new problem (28) takes into account the popularity of each product.[21]

The first-order necessary conditions for the minimization problem defined by (28) are the following $N + 1$ equations:

$$(q_n^0 + q_n^1)\beta_n = q_n^0 \ln p_n^0 + q_n^1(\ln p_n^1 - \rho_1), \quad n = 1, \ldots, N \qquad (29)$$

$$(\Sigma_{n=1}^N q_n^1)\rho_1 = \Sigma_{n=1}^N q_n^1(\ln p_n^1 - \beta_n). \qquad (30)$$

The solution to (29) and (30) is given as follows:[22]

$$\rho_1^* \equiv \Sigma_{n=1}^N q_n^0 q_n^1 (q_n^0 + q_n^1)^{-1} \ln(p_n^1/p_n^0)/\Sigma_{i=1}^N q_i^0 q_i^1 (q_i^0 + q_i^1)^{-1}, \qquad (31)$$

$$\beta_n^* \equiv q_n^0 (q_n^0 + q_n^1)^{-1} \ln(p_n^0) + q_n^1 (q_n^0 + q_n^1)^{-1} \ln(p_n^1/\pi_1^*), \quad n = 1, \ldots, N \qquad (32)$$

[16] For an introduction to hedonic regression models, see Griliches (1971). Hedonic regression models will be studied in great detail in Chapter 8.

[17] "It is also clear that the so-called unweighted index numbers, usually employed by practical statisticians, are the worst of all and are liable to large errors which could have been easily avoided" J. M. Keynes (1909, 79). This paper won the Cambridge University Adam Smith Prize for that year. Keynes (1930, 76–77) again stressed the importance of weighting in his later 1930 paper which drew heavily on his 1909 paper.

[18] "It has already been observed that the purpose of any index number is to strike a fair average of the price movements or movements of other groups of magnitudes. At first a simple average seemed fair, just because it treated all terms alike. And, in the absence of any knowledge of the relative importance of the various commodities included in the average, the simple average is fair. But it was early recognized that there are enormous differences in importance. Everyone knows that pork is more important than coffee and wheat than quinine. Thus the quest for fairness led to the introduction of weighting" Irving Fisher (1922, 43).

[19] "But even here, we should use a weighted regression approach, since we are interested in an estimate of a weighted average of the pure price change, rather than just an unweighted average over all possible models, no matter how peculiar or rare" (Zvi Griliches (1971, 8)).

[20] Balk (1980, 70) was the first to both apply the country product dummy model to the time series context and he was the first to introduce some form of weighting to the basic model. However, the specific forms of weighting used in this section were introduced by Diewert (2005) for the models defined by (28), (35) and (42). Rao (1995) (2005) introduced the form of weighting for the model defined by (38).

[21] One can think of repeating the term $[\ln p_n^0 - \beta_n]^2$ for each unit of product n sold in period 0. The result is the term $q_n^0[\ln p_n^0 - \beta_n]^2$. A similar justification based on repeating the price according to its sales can also be made. This repetition methodology makes the stochastic specification of the error terms somewhat complicated but the least squares minimization problem is simple enough.

[22] This solution was derived by Diewert (2005).

where $\pi_1^* \equiv \exp[\rho_1^*]$. Note that the weight for the term $\ln(p_n^1/p_n^0)$ in (31) can be written as follows:

$$q_n^* \equiv \Sigma_{n=1}^N q_n^0 q_n^1 (q_n^0 + q_n^1)^{-1} / \Sigma_{i=1}^N q_i^0 q_i^1 (q_i^0 + q_i^1)^{-1}, \qquad n = 1,\ldots,N$$
$$= h(q_n^0, q_n^1) / \Sigma_{i=1}^N h(q_i^0, q_i^1), \tag{33}$$

where $h(a,b) \equiv 2ab/(a+b) = [\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}]^{-1}$ is the *harmonic mean* of a and b.[23]

Note that the q_n^* sum to 1, and thus ρ_1^* is a weighted average of the logarithmic price ratios $\ln(p_n^1/p_n^0)$. Using $\pi_1^* = \exp[\rho_1^*]$ and $\pi_0^* = \exp[\rho_0^*] = \exp[0] = 1$, the bilateral price index that is generated by the solution to (28) is

$$\pi_1^*/\pi_0^* = \exp[\rho_1^*] = \exp[\Sigma_{n=1}^N q_n^* \ln(p_n^1/p_n^0)]. \tag{34}$$

Thus, π_1^*/π_0^* is a weighted geometric mean of the price ratios p_n^1/p_n^0 with weights q_n^* defined by (33). Although this seems to be a reasonable bilateral index number formula, it must be rejected for practical use on the grounds that *the index is not invariant to changes in the units of measurement*.

Since values are invariant to changes in the units of measurement, the lack of invariance problem could be solved if we replace the quantity weights in (28) with expenditure or sales weights.[24] This leads to the following *weighted least squares minimization problem* where the weights v_n^t are defined as $p_n^t q_n^t$ for $t = 0,1$ and $n = 1,\ldots,N$:

$$\min_{\rho 1,\beta 1,\ldots\beta N} \Sigma_{n=1}^N v_n^0 [\ln p_n^0 - \beta_n]^2 + \Sigma_{n=1}^N v_n^1 [\ln p_n^1 - \rho_1 - \beta_n]^2. \tag{35}$$

It can be seen that problem (35) has exactly the same mathematical form as problem (28) except that v_n^t has replaced q_n^t and so the solutions (31) and (32) will be valid in the present context if v_n^t replaces q_n^t in these formulae. Thus, the solution to (35) is

$$\rho_1^* \equiv \Sigma_{n=1}^N v_n^0 v_n^1 (v_n^0 + v_n^1)^{-1} \ln(p_n^1/p_n^0) / \Sigma_{i=1}^N v_i^0 v_i^1 (v_i^0 + v_i^1)^{-1}, \tag{36}$$

$$\beta_n^* \equiv v_n^0 (v_n^0 + v_n^1)^{-1} \ln(p_n^0) + v_n^1 (v_n^0 + v_n^1)^{-1} \ln(p_n^1/\pi_1^*), \qquad n = 1,\ldots,N \tag{37}$$

where $\pi_1^* \equiv \exp[\rho_1^*]$.

The resulting price index, $\pi_1^*/\pi_0^* = \pi_1^* = \exp[\rho_1^*]$, is indeed invariant to changes in the units of measurement. However, if we regard π_1^* as a function of the price and quantity vectors for the two periods, say $P(p^0,p^1,q^0,q^1)$, then another problem emerges for the price index defined by the solution to (35): $P(p^0,p^1,q^0,q^1)$ is not homogeneous of degree 0 in the components of q^0 or in the components of q^1. These properties are important because it is desirable that the companion implicit quantity index defined as $Q(p^0,p^1,q^0,q^1) \equiv [p^1 \cdot q^1 / p^0 \cdot q^0] / P(p^0,p^1,q^0,q^1)$ be homogeneous of degree 1 in the components of q^1 and homogeneous of degree minus 1 in the components of q^0.[25] We also want $P(p^0,p^1,q^0,q^1)$ to be homogeneous of degree 1 in the components of p^1 and homogeneous of degree minus 1 in the components of p^0, and these properties are also not satisfied. Thus, we conclude that the solution to the weighted least squares problem defined by (35) does not generate a satisfactory price index formula.

The aforementioned deficiencies can be remedied if the *expenditure amounts* v_n^t in (35) are replaced by *expenditure shares*, s_n^t, where $v^t \equiv \Sigma_{n=1}^N v_n^t$ for $t = 0,1$ and $s_n^t \equiv v_n^t/v^t$ for $t = 0,1$ and $n = 1,\ldots,N$. This replacement leads to the following *weighted least squares minimization problem*:[26]

$$\min_{\rho 1,\beta 1,\ldots\beta N} \Sigma_{n=1}^N s_n^0 [\ln p_n^0 - \beta_n]^2 + \Sigma_{n=1}^N s_n^1 [\ln p_n^1 - \rho_1 - \beta_n]^2. \tag{38}$$

Again, it can be seen that problem (38) has exactly the same mathematical form as problem (28) except that s_n^t has replaced q_n^t, and so the solutions (31) and (32) will be valid in the present context if s_n^t replaces q_n^t in these formulae. Thus, the solution to (38) is

$$\rho_1^* \equiv \Sigma_{n=1}^N s_n^0 s_n^1 (s_n^0 + s_n^1)^{-1} \ln(p_n^1/p_n^0) / \Sigma_{i=1}^N s_i^0 s_i^1 (s_i^0 + s_i^1)^{-1}, \tag{39}$$

$$\beta_n^* \equiv s_n^0 (s_n^0 + s_n^1)^{-1} \ln(p_n^0) + s_n^1 (s_n^0 + s_n^1)^{-1} \ln(p_n^1/\pi_1^*), \qquad n = 1,\ldots,N \tag{40}$$

where $\pi_1^* \equiv \exp[\rho_1^*]$. Define the *normalized harmonic mean share weights* as $s_n^* \equiv h(s_n^0, s_n^1)/\Sigma_{i=1}^N h(s_i^0, s_i^1)$ for $n = 1,\ldots,N$. Then, the *weighted* TPD *bilateral price index*, $P_{\text{WTPD}}(p^0,p^1,q^0,q^1) \equiv \pi_1^*/\pi_0^* = \pi_1^*$, has the following logarithm:

$$\ln P_{\text{WTPD}}(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N s_n^* \ln(p_n^1/p_n^0). \tag{41}$$

[23] $h(a,b)$ is well defined by $2ab/(a+b)$ if a and b are nonnegative and at least one of these numbers is positive. In order to write $h(a,b)$ as $[\frac{1}{2} a^{-1} + \frac{1}{2} b^{-1}]^{-1}$, we require that $a > 0$ and $b > 0$.

[24] "But on what principle shall we weight the terms? Arthur Young's guess and other guesses at weighting represent, consciously or unconsciously, the idea that relative money values of the various commodities should determine their weights. A value is, of course, the product of a price per unit, multiplied by the number of units taken. Such values afford the only common measure for comparing the streams of commodities produced, exchanged, or consumed, and afford almost the only basis of weighting which has ever been seriously proposed" (Irving Fisher (1922, 45)).

[25] Thus, we want Q to have the following properties: $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1} Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$; see Chapter 3 for a list of desirable properties or tests for bilateral price indices of the form $P(p^0,p^1,q^0,q^1)$.

[26] Note that the minimization problem defined by (38) is equivalent to the problem of minimizing $\Sigma_{n=1}^N e_0 n^2 + \Sigma_{n=1}^N e_1 n^2$ with respect to $\rho_1, \beta_1, \ldots, \beta_N$ where the error terms e_{tn} are defined by the equations $(s_n^0)^{1/2} \ln p_n^0 = (s_n^0)^{1/2} \beta_n + e_1 n$ for $n = 1,\ldots,N$ and $(s_n^1)^{1/2} \ln p_n^1 = (s_n^1)^{1/2} \rho_1 + (s_n^1)^{1/2} \beta_n + e_{2n}$ for $n = 1,\ldots,N$. Thus the solution to (38) can be found by running a linear regression using the previous two sets of estimating equations. The numerical equivalence of the least squares estimates obtained by repeating multiple observations or by using the square root of the weight transformation was noticed long ago as the following quotation indicates: "It is evident that an observation of weight w enters into the equations exactly as if it were w separate observations each of weight unity. The best practical method of accounting for the weight is, however, to prepare the equations of condition by multiplying each equation throughout by the square root of its weight" (E. T. Whittaker and G. Robinson (1940, 224)).

Thus, $P_{WTPD}(p^0,p^1,q^0,q^1)$ is equal to a share-weighted geometric mean of the price ratios, p_n^1/p_n^0.[27] This index is satisfactory from the viewpoint of the test approach to index number theory. Of the first 16 tests listed in Chapter 3, it can be shown that $P_{WTPD}(p^0,p^1,q^0,q^1)$ satisfies all of these tests (assuming strictly positive prices and quantities) except for Test 4 (the basket test, $P(p^0,p^1,q,q) = p^1{\cdot}q/p^0{\cdot}q$), Test 12 (the quantity reversal test), Test 13 (the price reversal test), Test 15 (the mean value test for quantities), and Test 16 (the Paasche and Laspeyres bounding test). It is likely that $P_{WTPD}(p^0,p^1,q^0,q^1)$ passes the monotonicity in prices tests, T17 and T18, and it is not likely that $P_{WTPD}(p^0,p^1,q^0,q^1)$ passes the monotonicity in quantity tests, T19 and T20.[28] Moreover, Diewert (2005, 564) showed that $P_{WTPD}(p^1,p^2,q^1,q^2)$ approximated the Fisher, Walsh, and Törnqvist–Theil indices to the second order around an equal price and quantity point where $p^0 = p^1$ and $q^0 = q^1$. Thus, if changes in prices and quantities going from one period to the next are not too large and there are no missing products, P_{WTPD} should be close to these indices that have emerged as being "best" in several contexts.[29]

Recall the unweighted least squares minimization problem defined by (21) that was introduced at the beginning of Section 4. The solution to this unweighted bilateral TPD regression model generated the Jevons index as its solution. But appropriate weighting of the squared errors has changed the solution index dramatically: The index defined by (41) weights products by their economic importance and has good test properties, whereas the Jevons index can generate very problematic results because of its lack of weighting according to economic importance. Note that both models have the same underlying structure; that is, they assume that p_{tn} is approximately equal to $\pi_t\alpha_n$ for $t = 0,1$ and $n = 1, \ldots, N$. Thus, *weighting by economic importance has converted a least squares minimization problem that generates a rather poor price index into a problem that generates a rather good index*.

There is one more weighting scheme that generates an even better index in the bilateral context where we are running a TPD hedonic regression using the price and quantity data for only two periods. Consider the following weighted least squares minimization problem:

$$\min_{\rho_1,\beta_1,\ldots\beta_N} \Sigma_{n=1}^N (½)(s_n^0 + s_n^1)[\ln p_n^0 - \beta_n]^2 + \Sigma_{n=1}^N (½)(s_n^0 + s_n^1)[\ln p_n^1 - \rho_1 - \beta_n]^2. \qquad (42)$$

As usual, it can be seen that problem (42) has exactly the same mathematical form as problem (28) except that $(½)(s_n^0 + s_n^1)$ has replaced q_n^0 and q_n^1 and so the solutions (31) and (32) will be valid in the present context if $(½)(s_{1n} + s_{2n})$ replaces q_n^t in these formulae. Thus, the solutions to (42) simplify to the following solutions:

$$\rho_1^* \equiv \Sigma_{n=1}^N (½)(s_n^0 + s_n^1)\ln(p_n^1/p_n^0), \qquad (43)$$

$$\beta_n^* \equiv (½)\ln(p_n^0) + (½)\ln(p_n^1/\pi_1^*), \qquad n = 1, \ldots, N \qquad (44)$$

where $\pi_1^* \equiv \exp[\rho_1^*]$ and $\pi_0 \equiv \exp[\rho_0] = \exp[0] = 1$ since we have set $\rho_0 = 0$. Thus, the bilateral index number formula that emerges from the solution to (42) is $\pi_1^*/\pi_0 = \exp[\Sigma_{n=1}^N (½)(s_n^0 + s_n^1)\ln(p_n^1/p_n^0)] \equiv P_T(p^0,p^1,q^0,q^1)$, which is the Törnqvist (1936), Theil (1967, 137–38) bilateral index number formula. Thus, the use of the weights in (42) has generated an even better bilateral index number formula than the formula that resulted from the use of the weights in (38).[30] This result reinforces the case for using appropriately weighted versions of the basic TPD hedonic regression model.[31] However, if the implied residuals in the minimization problem (42) are small (or, equivalently, if the fit in the linear regression model that can be associated with (42) is high, so that predicted values for log prices are close to actual log prices), then *weighting will not matter very much* and the unweighted version of (42), which was (21) in the previous section, will give results that are similar to (42). This comment applies to all of the weighted hedonic regression models that are considered in this section.[32] But in most practical applications of index number theory, *prices will not move in a proportional manner* over time. Under these conditions, weighting according to the economic importance of the individual commodities will lead to more representative estimates of overall price change; that is, the measures of price change generated by the models defined by the minimization problems (38) and (42) are to be preferred over the unweighted minimization problem defined by (21) in the previous section.

In Chapter 7, generalizations of the bilateral weighted TPD model defined by the weighted least squares minimization problem (38) will be generalized from 2 periods to T periods. The problems arising from missing prices will also be addressed in this chapter.

At this point, it is perhaps useful to contrast stochastic approaches to index number theory to the approaches explained in Chapter 2 (basket approaches) and in Chapter 3 (axiomatic or test approaches). These earlier approaches led to a small number of preferred indices such as the

[27] See Diewert (2005) for this formula. Note that Rao (1995) (2005) considered the extension of the model defined by (38) to T periods and so he pioneered this class of models.

[28] See Diewert (2006) who initiated an investigation of the test properties of hedonic regressions.

[29] However, with large changes in price and quantities going from period 0 to 1, P_{WTPD} will tend to lie below these alternative indices. Consider a case with only two commodities. Let the price vectors be $p^0 \equiv [1,1]$ and $p^1 \equiv [1,0.5]$ and let the share vectors be $s^0 \equiv [0.5,0.5]$ and $s^1 \equiv [0.1,0.9]$. Thus the two products are highly substitutable and when the price of product 2 goes on sale at half price, its market share jumps from 0.5 to 0.9. The Törnqvist Theil index for this example is 0.6156 which is considerably above the weighted TPD index value which is 0.5767. This example is based on an example due to Diewert and Fox (2017). Missing prices can also cause substantial differences between these indices.

[30] Diewert (1992, 223) and Balk (2008) listed the commonly used tests that $P_T(p^0,p^1,q^0,q^1)$ satisfies; see also Chapter 3.

[31] Note that the bilateral regression model defined by the minimization problem (38) is readily generalized to the case of T periods whereas the bilateral regression model defined by the minimization problem (42) cannot be generalized to the case of T periods. These facts were noted by de Haan and Krsinich (2014).

[32] If the residuals are small for (42), then prices will vary almost proportionally over time and all reasonable index number formulae will register price levels that are close to the estimated π_1^*; that is, we will have $p^1 \approx \pi_1^* p^0$ and hence all reasonable bilateral index number formulae will generate an estimate that is close to π_1^*.

Fisher and Walsh indices. The *stochastic approach* or the *descriptive statistics approach* to index number theory attempts to find a single summary measure of a distribution of price changes. The practical problem associated with this method is that there are many ways of summarizing relative price distributions as was seen at the end of Section 3. We chose Theil's summary measure of price change because it satisfied some key tests; that is, we had to draw on the test approach to index number theory in order to pin down our final specific estimator of price change. Similarly, in this section, we again drew on the test approach to index number theory to determine "best" measures of price change. This is the problem with the stochastic approach to index number theory: By itself, it does not narrow down the range of possible estimators of price change. Nevertheless, in subsequent chapters, we will utilize the stochastic approach to index number theory in order to address the problems associated with measuring quality change. However, as was done in this chapter, we will draw on the other approaches to index number theory to help us to narrow down the range of possible stochastic specifications that could be used to measure quality change.

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), de Haan (2004), Diewert (2004) (2010), Rao (2004), Clements, Izan, and Selvanathan (2006), de Haan and Krsinich (2014), and Rao and Hajargasht (2016).

References

Balk, Bert M. 1980. "A Method for Constructing Price Indexes for Seasonal Commodities." *Journal of the Royal Statistical Society A* 143: 68–75.

Balk, Bert M. 2008. *Price and Quantity Index Numbers*. New York: Cambridge University Press.

Carli, Gian-Rinaldo. 1804. "Del valore e della proporzione de' metalli monetati." In *Scrittori classici italiani di economia politica*, vol. 13, 297–366. Milano: G. G. Destefanis (originally published in 1764).

Clements, Kenneth W., H. Y. Izan, and E. Antony Selvanathan. 2006. "Stochastic Index Numbers: A Review." *International Statistical Review* 74: 235–70.

de Haan, Jan. 2004. "Hedonic Regressions: The Time Dummy Index as a Special Case of the Törnqvist Index." Paper presented at the Eighth Meeting of the International Working Group on Price Indexes, August 23–25, Statistics Finland, Helsinki.

de Haan, Jan, and Frances Krsinich. 2014. "Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes." *Journal of Business and Economic Statistics* 32 (3): 341–58.

Diewert, W. Erwin. 1992. "Fisher Ideal Output, Input and Productivity Indexes Revisited." *Journal of Productivity Analysis* 3: 211–48.

Diewert, W. Erwin. 1993. "The Early History of Price Index Research." In *Essays in Index Number Theory*, edited by W. Erwin Diewert and Alice O. Nakamura, vol. 1, 33–65. Amsterdam: North-Holland.

Diewert, W. Erwin. 2000. "Notes on Producing an Annual Superlative Index Using Monthly Price Data." Discussion Paper 00–08, Department of Economics, University of British Columbia, Vancouver, Canada, V6T 1Z1, 30 pp.

Diewert, W. Erwin. 2004. "On the Stochastic Approach to Linking the Regions in the ICP." Discussion Paper no. 04–16, Department of Economics, The University of British Columbia, Vancouver, Canada.

Diewert, W. Erwin. 2005. "Weighted Country Product Dummy variable Regressions and Index Number Formulae." *The Review of Income and Wealth* 51 (4): 561–71.

Diewert, W. Erwin. 2006. "Adjacent Period Dummy variable Hedonic Regressions and Bilateral Index Number Theory." *Annales d'économie et de statistique* (79/80): 1–28.

Diewert, W. Erwin. 2010. "On the Stochastic Approach to Index Numbers." In *Price and Productivity Measurement, Volume 6—Index Number Theory*, Chapter 11, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 235–62. Trafford Press.

Diewert, W. Erwin, and Kevin J. Fox. 2017. "Substitution Bias in Multilateral Methods for CPI Construction Using Scanner Data." Discussion Paper 17–02, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.

Edgeworth, Francis Ysidro. 1888. "Some New Methods of Measuring variation in General Prices." *Journal of the Royal Statistical Society* 51: 346–68.

Edgeworth, Francis Ysidro. 1923. "The Doctrine of Index Numbers According to Mr. Correa Walsh." *The Economic Journal* 11: 343–51.

Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton-Mifflin.

Greenlees, John. 1999. "Random Errors and Superlative Indexes." Paper presented at the Annual Conference of the Western Economic Association in San Diego, CA, July 8, 1999.

Griliches, Zvi. 1971. "Introduction: Hedonic Price Indexes Revisited." In *Price Indexes and Quality Change*, edited by Zvi Griliches, 3–15. Cambridge, MA: Harvard University Press.

Jensen, Johan Ludwig W. V. 1906. "Sur les fonctions convexes et les inégalités entre les valeurs moyennes." *Acta Mathematica* 30: 175–93.

Jevons, William Stanley. 1865. "The variation of Prices and the Value of the Currency Since 1782." *Journal of the Statistical Society of London* 28: 294–320.

Jevons, William Stanley. 1884. "A Serious Fall in the Value of Gold Ascertained and Its Social Effects Set Forth (1863)." In *Investigations in Currency and Finance*, 13–118. London: Macmillan and Co.

Keynes, John M. 1909. "The Method of Index Numbers with Special Reference to the Measurement of General Exchange Value." Reprinted as in *The Collected Writings of John Maynard Keynes* (1983), vol. 11, edited by Don Moggridge, 49–156. Cambridge: Cambridge University Press.

Keynes, John M. 1930. *Treatise on Money*, vol. 1. London: Macmillan.

Palgrave, R. H. Inglis. 1886. "Currency and Standard of Value in England, France and India and the Rates of Exchange Between these Countries." Memorandum Submitted to the Royal Commission on Depression of Trade and Industry, Third Report, Annex B, pp. 312–90.

Rao, D. S. Prasada. 1995. "On the Equivalence of the Generalized Country-Product-Dummy (CPD) Method and the Rao System for Multilateral Comparisons." Working Paper No. 5, Centre for International Comparisons, University of Pennsylvania, Philadelphia.

Rao, D. S. Prasada. 2004. "The Country-Product-Dummy Method: A Stochastic Approach to the Computation of Purchasing Power parities in the ICP." Paper presented at the SSHRC Conference on Index Numbers and Productivity Measurement, June 30–July 3, 2004, Vancouver, Canada.

Rao, D. S. Prasada. 2005. "On the Equivalence of Weighted Country-Product-Dummy(CPD) Method and The Rao-System for Multilateral Price Comparisons." *The Review of Income and Wealth* 51: 571–80.

Rao, D. S. Prasada, and Gholamreza Hajargasht. 2016. "Stochastic Approach to Computation of Purchasing Power Parities in the International Comparison Program (ICP)." *Journal of Econometrics* 191 (2): 414–25.

Selvanathan, Eliyathamby A., and D. S. Prasada Rao. 1994. *Index Numbers: A Stochastic Approach*. Ann Arbor: The University of Michigan Press.

Summers, Robert. 1973. "International Comparisons with Incomplete Data." *Review of Income and Wealth* 29 (1): 1–16.

Theil, Herni. 1967. *Economics and Information Theory*. Amsterdam: North-Holland Publishing.

Törnqvist, Leo. 1936. "The Bank of Finland's Consumption Price Index." *Bank of Finland Monthly Bulletin* 10: 1–8.

Törnqvist, Leo, and Egil Törnqvist. 1937. "Vilket är förhällandet mellan finska markens och svenska kronans köpkraft?" *Ekonomiska Samfundets Tidskrift* 39: 1–39 reprinted as in *Collected Scientific Papers of Leo Törnqvist*, 121–60. Helsinki: The Research Institute of the Finnish Economy, 1981.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Walsh, C. Moylan. 1921. *The Problem of Estimation*. London: P. S. King & Son.

Whittaker, Edmund T., and George Robinson. 1940. *The Calculus of Observations*. 3rd ed. London: Blackie & Sons.

Wynne, Mark A. 1999. "Core Inflation: A Review of Some Conceptual Issues." Research Department Working Paper 99–03, Federal Reserve Bank of Dallas, 2200 N. Pearl Street, Dallas, TX 75201–2272.

Young, Arthur. 1812. *An Inquiry into the Progressive Value of Money in England as Marked by the Price of Agricultural Products*. London: Macmillan.

THE ECONOMIC APPROACH TO INDEX NUMBER THEORY*

5

1. Introduction

Economics is the study of choice under constraints. Thus, the economic approach to index number theory applied to households generally involves the assumption of cost-minimizing or utility-maximizing behavior on the part of consumers subject to one or more constraints. It is unlikely that actual consumer behavior is completely described by the optimization models that will be considered in this chapter, but it seems that the economic approach to index number theory allows us to address some difficult measurement problems that other approaches to index number theory cannot address.

Some of the material in this chapter relies on advanced microeconomic theory. Some attempt is made to explain the various theories, but if the explanations are not adequate, references to the underlying literature are given.

In Section 2, the Konüs Cost of Living Index (COLI) for a single household is explained. This section is a fundamental one. It allows us to conceptualize the role of substitution as a response to changes in relative prices. In this section, the underlying utility or preference function is a general one. In Section 3, the theory described in Section 2 is specialized for the case of homothetic preferences. Preferences are homothetic if they can be represented by a linearly homogeneous utility function. It turns out that the assumption of homothetic preferences enables the price statistician to deal with product substitution in a very straightforward way. In Section 4, two results from microeconomic theory are discussed: Wold's Identity and Shephard's Lemma. These two results will be used in Sections 5–7, where certain formulae or functional forms for price and quantity indices are introduced and their connection to the economic approach to index number theory is established. Section 5 introduces the concept of a flexible functional form for a utility function. A flexible functional form can approximate an arbitrary twice continuously differentiable, linearly homogeneous functional form to the second order around any given point. Thus, it is useful to have index number formulae that are exactly consistent with preferences that can be represented by a flexible functional form since these functions can accommodate a wide variety of substitution responses on the part of consumers to changes in prices. Sections 5, 6, and 7 show that there are flexible functional forms for consumer utility functions that are exactly consistent with three well-known index number formulae: the Fisher, Walsh, and Törnqvist Theil indices. An index number formula that is exactly consistent with a flexible functional form is called a superlative index. In Section 8, it is shown that the superlative indices studied in Sections 5–7 all approximate each other to the second order around an equal price and quantity point, so in general, it will not matter too much which one of these three formulae is chosen in practice. The first eight sections of this chapter are the most important ones. The remaining sections deal with specific measurement topics that extend the basic theory in various directions.

In Sections 9 and 10, index number formulae, which are exact for two functional forms for the consumer's utility function that are not flexible, are given. These two functional forms are the Cobb–Douglas and Constant Elasticity of Substitution (CES) functions. Since they are widely used by economists and statisticians, it is useful to study these two functional forms and their corresponding exact index number formulae.

In Section 11, the Allen quantity index is introduced. In the previous sections, quantity indices that were exact for homothetic preferences were defined. The Allen quantity index is well defined even if preferences are not homothetic. It turns out that various Allen indices match up with various Konüs cost of living indices. The Törnqvist Theil price and quantity indices turn out to be very useful in this context. They are also very useful in the following two sections, which show how changes in tastes can be accommodated (Section 12) and how price indices that are conditional on environmental factors can be defined (Section 13).

In Section 14, the concept of a Hicksian reservation price is introduced. A reservation price is an imputed price that is just high enough to induce consumers not to purchase a product. It turns out that this concept is useful in the context of dealing with the problems that arise when new products are introduced and old, obsolete products disappear.

In Section 15, it is noted that consumers face not only a budget constraint but also a time constraint. The consumer's allocation of time interacts with his or her budget constraint, and this interaction leads to difficult measurement problems when constructing CPIs. An introduction to some of these problems is provided in this section.

Sections 16 and 17 generalize the single-household Konüs price index and Allen quantity index concepts to

*Much of the material in Sections 2–8 of this chapter is drawn from Chapter 17 of *The Consumer Price Index Manual: Theory and Practice* and from Diewert (2009). The author thanks Paul Armknecht, Bert Balk, Peter Hill, Alice Nakamura, Mick Silver, and Kim Zieschang for their helpful comments on the material in this chapter that was taken from the previous *Manual*. The author thanks Charles Andrew Barclay, Kevin Fox, Ismail Hossain, Ronald Johnson, Daniel Melser, Chihiro Shimizu, Paul Schreyer, Rui Xiao, and Clément Yélou for their helpful comments on the current draft.

many households. Fisher indices play a large role in these sections.

There are demands on statistical agencies to produce price and volume indices that take into account changes in the distribution of income among households. Section 18 provides the reader with an introduction to this topic.

Finally, Section 19 discusses the matching problem. If we attempt to construct a cost of living index for a single household, then due to the fact that many household purchases are made infrequently, it proves to be difficult to match the prices and quantities of purchased products over consecutive periods. For example, a seasonal product may be purchased only during certain seasons. Or a big discounted price may induce a household to stock up on a product this month and not purchase the product again for several months. This leads to a lack of matching of products problem that makes the construction of price indices difficult. Section 19 offers some possible solutions to this problem.

The Annex provides proofs of various theoretical results that are stated in the main text.

2. The Konüs Cost of Living Index for a Single Consumer

In this section, we outline the theory of the cost of living index for a single consumer (or household)[1] that was first developed by the Russian economist Konüs (1924). This theory relies on the assumption of *optimizing behavior* on the part of households. Given an observed vector of commodity or input prices p^t that the household faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is a solution to a cost (or expenditure) minimization problem that involves the consumer's preference or utility function $f(q)$. Thus, in contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors q^0 and q^1 discussed in previous chapters are independent of the two price vectors p^0 and p^1 that the household faces in periods 0 and 1. In the economic approach, the period t quantity vector q^t is determined by the consumer's preference function f and the period t vector of prices p^t that the consumer faces in period t.[2]

We assume that the consumer (or household) has well-defined *preferences* over different combinations of the N consumer commodities or items.[3] Each combination of items can be represented by a nonnegative vector $q \equiv [q_1, \ldots, q_N]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, increasing[4] and concave[5] utility function f.[6] Thus, if $f(q^1) > f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. It is further assumed that *the consumer minimizes the cost of achieving the observed period t utility level* $u^t \equiv f(q^t)$ for periods $t = 0,1$. Thus, the economic approach to index number theory assumes that the observed period t consumption vector $q^t >> 0_N$ solves the following *period t cost minimization problem*:[7]

$$C(u^t,p^t) \equiv \min_q \{p^t \cdot q : f(q) \geq u^t; q \geq 0_N\} = p^t \cdot q^t; \; t = 0,1. \qquad (1)$$

The consumer's cost minimization problem for period 0 is to choose a consumption vector $q \equiv [q_1, \ldots, q_N]$, which will minimize the cost $p^0 \cdot q \equiv \Sigma_{n=1}^N p_n^0 q_n$ of achieving at least the given utility level u^0, given that the consumer's preferences can be represented by the function $f(q)$. The period 0 observed consumption vector for the consumer is $q^0 \equiv [q_1^0, \ldots, q_N^0]$, where it is assumed that each q_n^0 is positive. An assumption that is imbedded in this definition for the period 0 cost minimization problem is that the period 0 reference utility level is u^0 defined as $f(q^0)$. The final assumption that is imbedded in the period 0 cost minimization problem defined by (1) previously is that the consumer's observed period 0 quantity vector is a solution to the period 0 cost minimization problem. A similar interpretation applies to the period 1 cost minimization problem. We also assume that the period t price vector for the N commodities under consideration that the consumer faces in each period is strictly positive; that is, we assume that $p^t >> 0_N$ for $t = 0,1$. Thus, there is a fair amount of complexity hidden behind the cost minimization problems (and their solutions) defined by (1).

Note that the solution to the cost or expenditure minimization problem (1) for a general utility level u and general vector of commodity prices p defines the *consumer's cost function* $C(u,p)$. This cost function will be used in order to define the consumer's cost of living price index. It can be shown that $C(u,p)$ has the following mathematical properties under our regularity conditions on $f(q)$: (*i*) $C(u,p)$ is nonnegative for all $u \geq 0$ and $p >> 0_N$; (ii) for each $p >> 0_N$, $C(u,p)$ is an increasing continuous function of u; and (iii) for each $u \geq 0$, $C(u,p)$ is a continuous, concave, and linearly

[1] A household may consist of more than one individual. Our exposition ignores the complications that can arise in multi-person households; that is, we assume that the household has consistent preferences of the type explained subsequently.

[2] In principle, the price p_n^t is a *period t unit value price for product n* for the household under consideration. The corresponding q_n^t is equal to the total purchases of product n by the household in period t. Thus, the product $p_n^t q_n^t$ is the total expenditure of the household on product n during period t.

[3] In this section, these preferences are assumed to be invariant over time. Changing preferences and the complications that arise when the number of available products changes over time will be postponed to Sections 12 and 14 and subsequent chapters.

[4] $f(q)$ is *increasing* in q if $q^2 >> q^1 \geq 0_N$ implies $f(q^2) > f(q^1)$.

[5] f is *concave* over the set of nonnegative q if $f(\lambda q^1 + (1\text{-l})q^2) \geq \lambda f(q^1) + (1 - \lambda)f(q^2)$ for all $0 \leq \lambda \leq 1$ and all $q^1 \geq 0_N$ and $q^2 \geq 0_N$. Note that $q \geq 0_N$ means that each component of the N-dimensional vector q is nonnegative, $q >> 0_N$ means that each component of q is positive, and $q > 0_N$ means that $q \geq 0_N$ but $q \neq 0_N$; that is, q is nonnegative, but at least one component is positive.

[6] For convenience, we assume that $f(0_N) = 0$ and $f(q)$ tends to plus infinity as all components of q tend to plus infinity.

[7] Notation: $p^t \equiv [p_1^t, \ldots, p_N^t]$, $q^t \equiv [q_1^t, \ldots, q_N^t]$ and $p^t \cdot q^t \equiv \Sigma_{n=1}^N p_n^t q_n^t$ for $t = 0,1$. Note that we are assuming that all prices and quantities are positive. Thus, $C(f(q^t),p^t) > 0$ for $t = 0,1$.

homogeneous function of p[8] that increases[9] if all components of p increase.[10]

The Konüs (1924) family of *true cost of living indices* pertaining to the two periods, $P_K(p^0,p^1,q)$, where the consumer faces the strictly positive price vectors $p^0 \equiv (p_1^0, \ldots, p_N^0)$ and $p^1 \equiv (p_1^1, \ldots, p_N^1)$ in periods 0 and 1, respectively, is defined as *the ratio of the minimum costs of achieving the same utility level* $u \equiv f(q)$, where $q \equiv (q_1, \ldots, q_N) >> 0_N$ is a positive reference quantity vector:

$$P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0]. \quad (2)$$

Definition (2) defines a *family* of price indices because there is one such index for each reference quantity vector q chosen.

It is natural to choose two specific reference quantity vectors q in definition (2): the observed base period quantity vector q^0 and the current period quantity vector q^1. The first of these two choices leads to the *following Laspeyres–Konüs true cost of living index*:

$$\begin{aligned} P_K(p^0,p^1,q^0) &\equiv C[f(q^0),p^1]/C[f(q^0),p^0] \quad (3)\\ &= C[f(q^0),p^1]/p^0\cdot q^0 \quad \text{using (1) for } t = 0\\ &= \min_q \{p^1\cdot q : f(q) \geq f(q^0);\ q \geq 0_N\}/p^0\cdot q^0\\ &\quad \text{using the definition of } C[f(q^0),p^1]\\ &\leq p^1\cdot q^0/p^0\cdot q^0 \quad \text{since } q^0 \text{ is feasible for the minimization problem}\\ &= P_L(p^0,p^1,q^0,q^1) \end{aligned}$$

where P_L is the Laspeyres price index defined in earlier chapters. *Thus, the (unobservable) Laspeyres–Konüs true cost of living index is bounded from above by the observable Laspeyres price index*.[11]

The second of the two natural choices for a reference quantity vector q in definition (2) leads to the *following Paasche–Konüs true cost of living index*:

$$\begin{aligned} P_K(p^0,p^1,q^1) &\equiv C[f(q^1),p^1]/C[f(q^1),p^0] \quad (4)\\ &= p^1\cdot q^1/C[f(q^1),p^0] \text{ using (1) for } t = 1\\ &= p^1\cdot q^1/\min_q\{p^0\cdot q : f(q) \geq f(q^1);\ q \geq 0_N\}\\ &\quad \text{using the definition of } C[f(q^1),p^0]\\ &\geq p^1\cdot q^1/p^0\cdot q^1 \text{ since } q^1 \text{ is feasible for the minimization problem and thus}\\ &\min_q\{p^0\cdot q : f(q) \geq f(q^1)\} \leq p^0\cdot q^1 \text{ and hence}\\ &1/C[f(q^1),p^0] \geq 1/p^0\cdot q^1\\ &= P_P(p^0,p^1,q^0,q^1), \end{aligned}$$

where P_P is the Paasche price index defined in earlier chapters. *Thus, the (unobservable) Paasche–Konüs true cost of living index is bounded from below by the observable Paasche price index*.[12]

Figure 5.1 illustrates the bounds given by (3) and (4) for the case of two commodities.

The solution to the period 0 cost minimization problem is the vector q^0 and the straight line through C represents the consumer's period 0 budget constraint, the set of quantity points q_1,q_2 such that $p_1^0q_1 + p_2^0q_2 = p_1^0q_1^0 + p_2^0q_2^0$. The curved line through q^0 is the consumer's period 0 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^0,q_2^0)$; that is, it is the set of consumption vectors that give the same utility as the observed period 0 consumption vector q^0. The solution to the period 1 cost minimization problem is the vector q^1, and the straight line through D represents the consumer's period 1 budget constraint, the set of quantity points q_1,q_2 such that $p_1^1q_1 + p_2^1q_2 = p_1^1q_1^1 + p_2^1q_2^1$. The curved line through q^1 is the consumer's period 1 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^1,q_2^1)$; that is, it is the set of consumption vectors that give the same utility as the observed period 1 consumption vector q^1. The point q^{0*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the base period utility level $u^0 \equiv f(q^0)$ when facing the period 1 price vector $p^1 = (p_1^1,p_2^1)$. Thus, we have $C[u^0,p^1] = p_1^1q_1^{0*} + p_2^1q_2^{0*}$ and the dashed line through A is the corresponding isocost line $p_1^1q_1 + p_2^1q_2 = C[u^0,p^1]$.

Note that the hypothetical cost line through A is parallel to the actual period 1 cost line through D. From (3), the Laspeyres–Konüs true index is $C[u^0,p^1]/[p_1^0q_1^0 + p_2^0q_2^0]$, while the ordinary Laspeyres index is $[p_1^1q_1^0 + p_2^1q_2^0]/[p_1^0q_1^0 + p_2^0q_2^0]$. Since the denominators for these two indices are the same, the difference between the indices is due to the differences in their numerators. In Figure 5.1, this difference in the numerators is expressed by the fact that the cost line through A lies *below* the parallel cost line through B.

If the consumer's indifference curve through the observed period 0 consumption vector q^0 were L shaped with vertex at q^0, then the consumer would not change his or her consumption pattern in response to a change in the relative prices of the two commodities while keeping a fixed standard of living. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dashed line through A would coincide with the dashed line through B and the true Laspeyres–Konüs index would *coincide* with the ordinary Laspeyres index. However, L-shaped indifference curves are not generally consistent with consumer behavior; that is, when the price of a commodity decreases, consumers generally demand more of it.

[8] This property is the following one: Let $u \geq 0$, $p >> 0_N$, and $\lambda \geq 0$; then $C(u,\lambda p) = \lambda C(u,p)$.

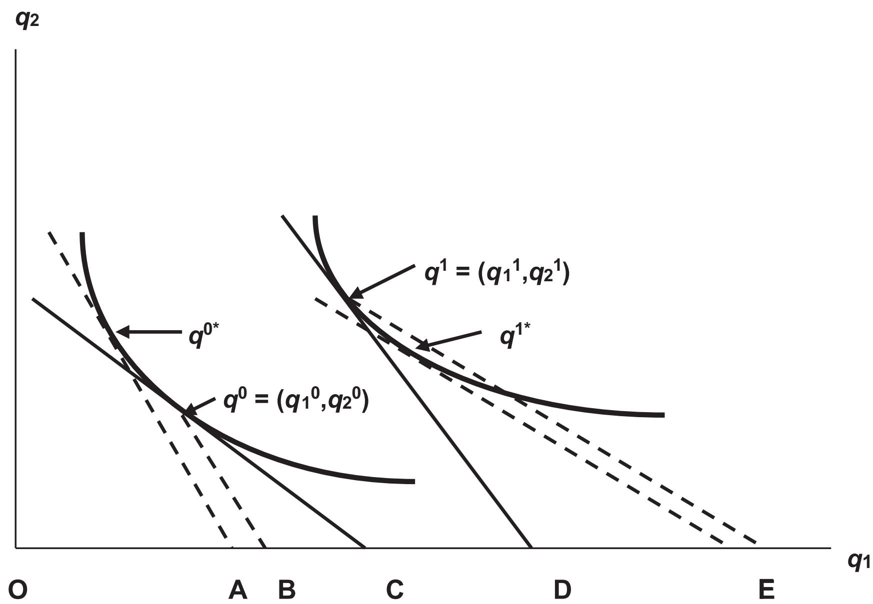
[9] This property is the following one: Let $u > 0$ and $0_N << p^1 << p^2$; then $C(u,p^1) < C(u,p^2)$.

[10] For additional materials on the properties of cost functions and references to the literature, see Diewert (1993a). The restriction that $f(q)$ be a concave function is not the usual assumption in the economics literature, but drawing on the work of Afriat (1967) and Diewert (1973), it can be shown that this assumption is not restrictive in practice.

[11] This inequality was first obtained by Konüs (1924) (1939, 17). See also Pollak (1983).

[12] This inequality was obtained by Konüs (1924) (1939, 19).

Figure 5.1 The Laspeyres and Paasche Bounds to the True Cost of Living q_2



Thus, in the general case, there will be a gap between the points *A* and *B*. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; that is, the Laspeyres index will generally be *greater* than the corresponding true cost of living index, $P_K(p^0,p^1,q^0)$.

Figure 5.1 can also be used to illustrate the inequality (4). First note that the dashed lines through *E* and *F* are parallel to the period 0 isocost line through *C*. The point q^{1*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the current period utility level $u^1 \equiv f(q^1)$ when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. Thus, we have $C[u^1,p^0] = p_1^0q_1^{1*} + p_2^0q_2^{1*}$ and the dashed line through *E* is the corresponding isocost line $p_1^1q_1 + p_2^1q_2 = C[u^0,p^1]$. From (4), the Paasche–Konüs true index is $[p_1^1q_1^1 + p_2^1q_2^1]/C[u^1,p^0]$, while the ordinary Paasche index is $[p_1^1q_1^1 + p_2^1q_2^1]/[p_1^0q_1^1 + p_2^0q_2^1]$. Since the numerators for these two indices are the same, the difference between the indices is due to the differences in their denominators. In Figure 5.1, this difference in the denominators is expressed by the fact that the cost line through *E* lies *below* the parallel cost line through *F*. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; that is, the Paasche index will generally be *less* than the corresponding true cost of living index, $P_K(p^0,p^1,q^1)$. Note that this inequality goes in the opposite direction to the previous inequality between the two Laspeyres indices. The reason for this change in direction is due to the fact that one set of differences between the two indices takes place in the numerators of the two indices (the Laspeyres inequalities), while the other set takes place in the denominators of the two indices (the Paasche inequalities).

The bound (3) on the Laspeyres–Konüs true cost of living $P_K(p^0,p^1,q^0)$ using the base period level of utility as the living standard is *one sided* as is the bound (4) on the Paasche–Konüs true cost of living $P_K(p^0,p^1,q^1)$ using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1924, 20) showed that there exists an intermediate consumption vector q^* that is on the straight line joining the base period consumption vector q^0 and the current period consumption vector q^1 such that the corresponding (unobservable) true cost of living index $P_K(p^0,p^1,q^*)$ is between the observable Laspeyres and Paasche indices, P_L and P_P.[13] The Konüs result is the following Proposition:

Proposition 1: There exists a number λ^* between 0 and 1 such that

$$P_L \le P_K(p^0,p^1, \lambda^*q^0 + (1 - \lambda^*)q^1) \le P_P$$
$$\text{or } P_P \le P_K(p^0,p^1, \lambda^*q^0 + (1 - \lambda^*)q^1) \le P_L. \quad (5)$$

The first set of inequalities holds when $P_L \le P_P$, and the second holds when $P_P \le P_L$. For a proof of this result, see the Annex.

The aforementioned inequalities are of some practical importance. If the observable (in principle) Paasche and Laspeyres indices are not too far apart, then taking a symmetric average of these indices should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. To determine the precise symmetric average of the Paasche and Laspeyres indices, we can appeal to the results in Chapter 2 and take the geometric mean, which is the Fisher price index. Thus, the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical cost of living index.

[13] See Diewert (1983, 191).

The bounds (3)–(5) are the best bounds that we can obtain on the true cost of living indices without making further assumptions. In subsequent sections, we will make further assumptions on the class of utility functions that describe the consumer's tastes for the N commodities under consideration. By making specific functional form assumptions about the utility function $f(q)$ or about the corresponding cost function $C(u,p)$, it will be possible to determine the functional form for the consumer's true cost of living index.

Before proceeding further, it may be useful to discuss some problems with the economic approach to index number theory. A major objection to this approach is the assumption of cost-minimizing (or equivalently of utility-maximizing) behavior on the part of households. Do households even have consistent preferences over alternative combinations of goods and services, let alone minimize the cost of achieving a given level of utility or welfare? Even if households do not have perfectly consistent preferences, experience has shown that when the price of a product is significantly decreased, households will buy more of it, and conversely, if the price of a product rises significantly, households will tend to purchase less of it. The economic approach to index number theory simply formalizes this behavior, and at the same time, it is able to generate measures of possible changes in consumer welfare along with measures of changes in the cost of living. These measures are imperfect, but they are valued by economists and policy makers. Thus, it is useful to take an economic approach to index number theory. Moreover, government statisticians are obliged to produce *economic statistics*. It seems sensible for official statisticians to be at least aware of economic approaches to index number theory while producing economic statistics. Finally, as will be seen in later sections, the economic approach to index number theory provides useful insights into difficult measurement problems that other approaches to index number theory are unable to address.

Some of the limitations of the present framework will be relaxed in subsequent sections; that is, the assumption that all prices and quantities are positive will be relaxed, the assumption of constant preferences will also be relaxed, and the problems associated with the appearance of new products and the disappearance of existing products will be addressed. However, one problem that will not be addressed is *the stock piling problem*; that is, when storable products go on sale, households may purchase large amounts of the products so that the period of consumption of these products does not coincide with the period of purchase. These left-over stocks will affect demand for the products in subsequent periods, and the model of economic behavior used in this section does not take this possibility into account.[14] The problem of storable goods not being consumed in the period of purchase suggests that the Konüs true cost of living index should not be implemented if the length of the period is very short. Thus, *daily economic price indices* for *individual households* may be more or less meaningless from the viewpoint of the economic approach to index numbers. The length of the accounting period for individual households should be a longer period, such as a month or a quarter.

[14]The treatment of purchases of durable goods will be addressed in Chapter 10. A *durable good* (for example, an automobile or a house) is able to provide a stream of services over its useful lifetime; a *storable good* (for example, a can of beans) can only be used once but its consumption can be postponed from its period of purchase to a later period of consumption.

3. The Cost of Living Index When Preferences Are Homothetic

Up to now, the consumer's preference function f did not have to satisfy any particular homogeneity assumption. In this section, we assume that f is (positively*) linearly homogeneous*;[15] that is, we assume that the consumer's utility function has the following property:

$$f(\lambda q) = \lambda f(q) \text{ for all } \lambda > 0 \text{ and all } q \geq 0_N. \qquad (6)$$

Given the continuity of f, it can be seen that property (6) implies that $f(0_N) = 0$. Furthermore, f also satisfies $f(q) > 0$ if $q >> 0_N$.

In the economics literature, assumption (6) is known as the assumption of *homothetic preferences*.[16] This assumption is not strictly justified from the viewpoint of actual economic behavior, but, as will be seen later, it leads to economic price indices that do not depend on the consumer's standard of living; that is, the resulting aggregate prices do not depend on quantities.[17] Under this assumption, the consumer's expenditure or cost function, $C(u,p)$ defined by (1), decomposes into the product of two terms. For positive commodity prices $p >> 0_N$ and a positive utility level u, we have the following decomposition of the cost function:

$$
\begin{aligned}
C(u,p) &\equiv \min_q\{p \cdot q : f(q) \geq u;\ q\ {}^3\ 0_N\} \qquad (7)\\
&= \min_q\{p \cdot q : (1/u)f(q) \geq 1;\ q \geq 0_N\} \quad \text{dividing both sides of the constraint by } u > 0\\
&= \min_q\{p \cdot q : f(q/u) \geq 1;\ q \geq 0_N\} \quad \text{using the linear homogeneity of } f\\
&= u \min_q\{p \cdot q/u : f(q/u) \geq 1;\ q\ {}^3\ 0_N\} \quad \text{using the assumption that } u \text{ is positive}\\
&= u \min_z\{p \cdot z : f(z) \geq 1;\ z \geq 0_N\} \quad \text{defining } z \equiv q/u\\
&= uC(1,p) \quad \text{using definition (1) with } u = 1\\
&= uc(p),
\end{aligned}
$$

[15]This assumption is fairly restrictive in the consumer context. It implies that each indifference curve or surface is a radial projection of the unit utility indifference curve or surface. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence. However, at lower levels of aggregation, the homotheticity assumption for the relevant subutility function is probably an acceptable approximation to reality.

[16]More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

[17]This particular branch of the economic approach to index number theory was developed by Shephard (1953) (1970) and Samuelson and Swamy (1974). Shephard, in particular, realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindices of the overall cost of living index.

where $c(p) \equiv C(1,p)$ is the *unit cost function* that corresponds to f.[18] It can be shown that the unit cost function $c(p)$ satisfies the same regularity conditions that f satisfies; that is, $c(p)$ is positive, concave, and (positively) linearly homogeneous for positive price vectors.[19] Substituting (7) into (1) and using $u^t = f(q^t)$ leads to the following equation:

$$p^t \cdot q^t = c(p^t)f(q^t) \text{ for } t = 0,1. \qquad (8)$$

Thus, under the linear homogeneity assumption on the utility function f, observed period t expenditure on the N commodities (the left-hand side of (8)) is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$ (the right-hand side of (8)). Obviously, we can identify the period t unit cost, $c(p^t)$, as the *period t price level* P^t and the period t level of utility, $f(q^t)$, as the *period t quantity level* Q^t. Note that P^t does not depend on q^t and Q^t does not depend on p^t. This is the main advantage of assuming homothetic preferences when we use the economic approach to index number theory: We can decompose period t aggregate value, $p^t \cdot q^t$, into the product of an aggregate period t price level, $P^t \equiv c(p^t)$, which just depends on the vector of period t commodity prices p^t, times an aggregate period t quantity level, $Q^t \equiv f(q^t)$, which just depends on the period t quantity vector q^t.

The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indices, $P_K(p^0,p^1,q)$, defined by (2). Using definition (2) for an arbitrary reference quantity vector q, we have[20]

$$\begin{aligned} P_K(p^0,p^1,q) &\equiv C[f(q),p^1]/C[f(q),p^0] \qquad (9)\\ &= c(p^1)f(q)/c(p^0)f(q) \text{ using (8) twice}\\ &= c(p^1)/c(p^0). \end{aligned}$$

Thus, under the homothetic preferences assumption, the entire family of Konüs true cost of living indices collapses to a single index, $c(p^1)/c(p^0)$, the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices, respectively. Put another way, *under the homothetic preferences assumption, $P_K(p^0,p^1,q)$ does not depend on the reference quantity vector q*.

Substitute (9) into the inequalities (3) and (4), which, of course, are still valid under the homothetic preferences assumption. The resulting two inequalities simplify into the following two inequalities:

$$\begin{aligned} p^1 \cdot q^1/p^0 \cdot q^1 &\equiv P_P(p^0,p^1,q^0,q^1) \le c(p^1)/c(p^0)\\ &= P_K(p^0,p^1,q) \le P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0/p^0 \cdot q^0. \qquad (10) \end{aligned}$$

Thus, under the homothetic preferences assumption, every Konüs true cost of living index $P_K(p^0,p^1,q)$ is bounded from above by the ordinary Laspeyres price index and bounded from below by the ordinary Paasche price index. Moreover, if we can observe the quantity vectors for periods 0 and 1 that are generated by a cost-minimizing consumer that has homothetic preferences, then we can calculate the Laspeyres and Paasche indices for this consumer, and it must be the case that not only will the consumer's true cost of living index be bounded by these two indices but also the Paasche index is equal to or less than the corresponding Laspeyres index.[21]

If we use the Konüs true cost of living index defined by the right-hand side of (9) as our price index concept, then the corresponding *implicit quantity index* defined by deflating the value ratio by this price index is the following index:[22]

$$\begin{aligned} Q(p^0,p^1,q^0,q^1,q) &\equiv p^1 \cdot q^1/\{p^0 \cdot q^0 P_K(p^0,p^1,q)\} \qquad (11)\\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)P_K(p^0,p^1,q)\} \text{ using (8) twice}\\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} \text{ using (9)}\\ &= f(q^1)/f(q^0). \end{aligned}$$

Thus, under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is a natural definition for a quantity index.

The bounds (3), (4), and (10) are the best nonparametric bounds that we can obtain on the Konüs true cost of living index $P_K(p^0,p^1,q)$. In subsequent sections, we will assume specific functional forms for $f(q)$ or $c(p)$ and find price indices that are consistent with the chosen functional forms. Before this is done, we will require two additional results from microeconomic theory: Wold's Identity and Shephard's Lemma.

4. Wold's Identity and Shephard's Lemma

Instead of using the assumption that a household minimizes the cost of achieving a given utility level, one can use the assumption that the household maximizes utility subject to a budget constraint. Thus, let $p^t >> 0_N$ and $q^t >> 0_N$ be the household's observed period t price and quantity vectors for $t = 0,1$. Define the household's *period t observed expenditure* e^t as

$$e^t \equiv p^t \cdot q^t;\ t = 0,1. \qquad (12)$$

[18] Economists will recognize the producer theory counterpart to the result $C(u,p) = uc(p)$: If a producer's production function f is subject to constant returns to scale, then the corresponding total cost function $C(u,p)$ (where $u > 0$ is output and p is a vector of input prices) is equal to the product of the output level u times the unit cost $c(p)$.

[19] Obviously, the utility function f determines the consumer's cost function $C(u,p)$ as the solution to the cost minimization problem defined by (1). Then the unit cost function $c(p)$ is defined as $C(1,p)$. Thus, f determines c. But we can also use c to determine f under appropriate regularity conditions. In the economics literature, this is known as *duality theory*. For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953), McFadden (1966) (1978), and Diewert (1974a, 110–13) (1993a, 107–23).

[20] Konüs and Byushgens (1926, 168) were the first to establish this result. Pollak (1971) (1983) independently established this result later.

[21] This result was established by Konüs and Byushgens (1926, 168).

[22] The Product Test from Chapter 2 is used to define the implicit quantity index that corresponds to the price index defined by (9).

The household's *period t utility maximization problem* is defined as the following constrained maximization problem:

$$\max_q \{f(q) : p^t \cdot q \leq e^t;\ q \geq 0_N\} \equiv g(e^t, p^t);\ t = 0,1. \quad (13)$$

Instead of assuming that the household's observed consumption vector q^t is a solution to the period t cost minimization problem defined earlier by (1), an equivalent assumption (under Section 2 regularity conditions on f) is that the observed q^t solves the period t utility maximization problem defined by (13). The period t optimized objective function in (13) is defined as the consumer's *indirect utility function*, $g(e^t,p^t)$.[23] This function is the maximum utility that the consumer can achieve given that they face the period t price vector p^t and has "income" e^t to spend on the N commodities under consideration.

If we assume that the observed period t consumption vector q^t is a solution to (13) for $t = 0,1$ and, in addition, $f(q)$ has partial derivatives at q^0 and q^1, then it is possible to establish the following connection of these partial derivatives to the observed period 0 and 1 price vectors, p^0 and p^1.

Proposition 2 (Wold's [1944, 69–71] [1953, 145] Identity): Suppose that (i) $p^0 >> 0_N$, $p^1 >> 0_N$; (ii) the consumer's utility function $f(q)$ is increasing, continuous, and concave for all $q \geq 0_N$; (iii) $f(q)$ has first-order partial derivatives at the points q^0 and q^1; and (iv) $q^t >> 0_N$ is a solution to the household's period t utility maximization problem (13) for $t = 0,1$. Then the following equations hold:

$$p_i^t / p^t \cdot q^t = [\partial f(q^t)/\partial q_i] / \Sigma_{k=1}^N q_k^t \partial f(q^t)/\partial q_k;$$
$$t = 0,1;\ i = 1,\ldots,N, \quad (14)$$

where $\partial f(q^t)/\partial q_i$ denotes the partial derivative of the utility function f with respect to the ith quantity q_i evaluated at the period t quantity vector q^t.

A proof of Proposition 2 may be found in the Annex.

It is useful to express equations (14) using some alternative notation. Denote the N dimensional vector of first-order partial derivatives of $f(q^t)$ as $\nabla f(q^t) \equiv [\partial f(q^t)/\partial q_1, \ldots, \partial f(q^t)/\partial q_N]$ for $t = 0,1$. Using this notation, equations (14) can be rewritten more succinctly as follows:

$$p^t / p^t \cdot q^t = \nabla f(q^t) / q^t \cdot \nabla f(q^t);\ t = 0,1. \quad (15)$$

If in addition to the assumptions made for Proposition 2, the utility function $f(q)$ is linearly homogeneous, then it turns out that the terms $q^t \cdot \nabla f(q^t) = \Sigma_{n=1}^N q_n^t \partial f(q^t)/\partial q_n$ are equal to $f(q^t)$ for $t = 0,1$; that is, if $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$, then we have the following identities:[24]

$$f(q^t) = q^t \cdot \nabla f(q^t);\ t = 0,1. \quad (16)$$

Substituting (16) into (15) leads to the following very useful equations:

$$p^t / p^t \cdot q^t = \nabla f(q^t) / f(q^t);\ t = 0,1. \quad (17)$$

We turn now to the implications of differentiability of the consumer's cost function, $C(u,p)$, with respect to components of the commodity price vector p. If $C(f(q^t),p^t)$ has first-order partial derivatives $\partial C(u^t,p^t)/\partial p_n$ for $n = 1,\ldots,N$ and $t = 0,1$ where $u^t = f(q^t)$, then we have the following Proposition:

Proposition 3 (Shephard's [1953, 11] Lemma): Suppose (i) the utility function $f(q)$ is increasing, continuous, and concave in q; (ii) $p^t >> 0_N$ for $t = 0,1$; (iii) $q^t \equiv [q_1^t,\ldots,q_N^t] > 0_N$ is a solution to the cost minimization problem defined by (1) for $t = 0,1$; and (iv) for $u^t \equiv f(q^t)$, the first-order partial derivatives of $C(u^t,p^t)$ with respect to the components of p existing for $t = 0,1$, then

$$q_n^t = \partial C(u^t,p^t)/\partial p_n;\ n = 1,\ldots,N;\ t = 0,1. \quad (18)$$

Moreover, q^t is the unique solution to the cost minimization problem defined by (1) for $t = 0,1$.

A proof of Proposition 3 can be found in the Annex.

Let the vector of first-order partial derivatives of $C(u^t,p^t)$ with respect to the components of the price vector p be denoted as $\nabla_p C(u^t,p^t) \equiv [\partial C(u^t,p^t)/\partial p_1,\ldots,\partial C(u^t,p^t)/\partial p_N]$ for $t = 0,1$. Using this notation, equations (18) can be written more succinctly as follows:

$$q^t = \nabla_p C(u^t,p^t);\ t = 0,1. \quad (19)$$

The result obtained here has the following implication: postulate a differentiable functional form for the cost function $C(u,p)$ that satisfies the appropriate regularity conditions on the cost function listed after definitions (1). Then differentiating $C(u,p)$ with respect to the components of the product price vector p generates the consumer's system of Hicksian cost-minimizing input demand functions,[25] $x(u,p) \equiv \nabla_p C(u,p)$.

If we make the *homothetic preferences assumption* and assume that the utility function is linearly homogeneous, then using (7), we have $C(u^t,p^t) = u^t c(p^t) = p^t \cdot q^t$, where $u^t \equiv f(q^t)$ for $t = 0,1$. Shephard's Lemma (19) becomes $q^t = \nabla_p C(u^t,p^t) = u^t \nabla c(p^t)$ for $t = 0,1$. Using these equations, we find that

$$q^t / p^t \cdot q^t = \nabla c(p^t) / c(p^t);\ t = 0,1. \quad (20)$$

These equations will be very useful in subsequent sections of this chapter. Note the nice symmetry between the Shephard's Lemma equations (20) and the Wold's Identity equations (17). In the following sections, specific functional forms for a linearly homogeneous utility function $f(q)$ or for a unit cost function $c(p)$ will be made and index number formulae that are exactly correct for these specific functional

[23] When the consumer's utility function $f(q)$ is linearly homogeneous, concave, and increasing in q, then the corresponding indirect utility function defined by (13) is equal to $u^t \equiv g(e^t,p^t) = e^t/c(p^t)$ since $e^t = u^t c(p^t)$. Thus, if we set $e^t = 1$ in (13), we obtain the following explicit formula for calculating the unit cost function from a knowledge of f: $c(p^t) = 1/\max_q \{f(q) : p^t \cdot q \leq 1;\ q \geq 0_N\}$. Alternatively, we can define $c(p^t)$ in the usual way as $c(p^t) \equiv \min_q \{p^t \cdot q;\ f(q) \geq 1;\ q \geq 0_N\}$.

[24] Proof: Partially differentiate both sides of $f(\lambda q) = \lambda f(q)$ with respect to λ and evaluate the resulting partial derivatives at $\lambda = 1$ and $q = q^t$. This is Euler's theorem on linearly homogeneous functions.

[25] Hicks (1946, 311–31) introduced this type of demand function into the economics literature.

forms will be derived. Thus, in the next two sections, we assume that the consumer's preference function is linearly homogeneous.

5. Superlative Indices: The Fisher Ideal Index

Suppose the consumer has the following utility function:[26]

$$f(q_1, \ldots, q_N) \equiv [\Sigma_{i=1}^{N} \Sigma_{k=1}^{N} a_{ik} q_i q_k]^{1/2}, \quad (21)$$

where the N^2 parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all indices i and k. Thus, there are only $N(N + 1)/2$ independent parameters in this functional form. Note that $f(q)$ defined by (21) is linearly homogeneous.

Differentiating $f(q)$ defined by (21) with respect to q_i yields the following equations:

$$\begin{aligned} \partial f(q)/\partial q_i &= (1/2)[\Sigma_{j=1}^{N} \Sigma_{k=1}^{N} a_{jk} q_j q_k]^{-1/2} \, 2\Sigma_{k=1}^{N} a_{ik} q_k; \quad i = 1, \ldots, N \\ &= \Sigma_{k=1}^{N} a_{ik} q_k / f(q), \end{aligned} \quad (22)$$

where it is necessary to use the symmetry conditions, $a_{ik} = a_{ki}$ for $1 \leq i,k$ £ N, in order to derive the first set of equations in (22) and the second set of equations follows from definition (21). Now evaluate the second set of equations in (22) at the observed period t quantity vector $q^t \equiv (q_1^t, \ldots, q_N^t)$ and divide both sides of the resulting equations by $f(q^t)$. We obtain the following equations:

$$[\partial f(q^t)/\partial q_i]/f(q^t) = \Sigma_{k=1}^{N} a_{ik} q_k^t / [f(q^t)]^2 \quad t = 0,1; \quad i = 1, \ldots, N. \quad (23)$$

At this point, it is convenient to rewrite equations (23) using matrix notation. Thus, in what follows, we interpret the vectors p^t and q^t for $t = 0,1$ as column vectors. Denote the transpose of a column vector x by x^T, which is the row vector $[x_1, \ldots, x_N]$. Define $A \equiv [a_{ik}]$ as the N by N matrix that has component a_{ik} in row i and column k of A. We assume that A is a symmetric matrix so that its transpose A^T is equal to the original matrix A. Thus, using matrix notation, $f(q) \equiv [q^T A q]^{1/2}$, where $A = A^T$.

Using this matrix notation, equations (22) can be written as the following vector equation:

$$\begin{aligned} \nabla f(q) &= Aq/[q^T A q]^{1/2} \\ &= Aq/f(q) \end{aligned} \quad (24)$$

because $f(q) \equiv [q^T A q]^{1/2}$. Using matrix notation, equations (23) can be denoted as follows:

$$\nabla f(q^t)/f(q^t) = Aq^t/[f(q^t)]^2; \; t = 0,1. \quad (25)$$

$f(q)$ defined by (21) is obviously linearly homogeneous. But we also need it to be positive (if $q > 0_N$), nondecreasing, and concave in q over at least a subset of the nonnegative orthant. Suppose the symmetric matrix A has one positive eigenvalue with a corresponding strictly positive eigenvector and the remaining $N - 1$ eigenvalues of A are either 0 or negative.[27] Then $f(q)$ defined by (21) will be positive, nondecreasing, and concave over the region of regularity S defined as follows:[28]

$$S \equiv \{q : q \geq 0_N; \; Aq \geq 0_N; \; q^T A q > 0\}. \quad (26)$$

Now assume utility-maximizing behavior for the consumer in periods 0 and 1; that is, assume that $q^t >> 0_N$ is a solution to the period t utility maximization problem defined by (13), where $p^t >> 0_N$ and $e^t \equiv p^t \cdot q^t$ for $t = 0,1$ and the utility function $f(q)$ is defined by (21), where matrix A satisfies the aforementioned regularity conditions. Assume that q^0 and q^1 are both in the regularity region defined by (26). Since the utility function f defined by (21) is linearly homogeneous and differentiable over S, equations (17) (Wold's Identity) will hold for periods 0 and 1. Thus, using (17), we have

$$\begin{aligned} p^t/p^t \cdot q^t &= \nabla f(q^t)/f(q^t); \; t = 0,1 \\ &= Aq^t/[f(q^t)]^2, \end{aligned} \quad (27)$$

where the second set of equations follows from equations (25).

As usual, the *Fisher* (1922) *ideal quantity index*, Q_F, is defined as $Q_F(p^0,p^1,q^0,q^1) \equiv [p^0 \cdot q^1 p^1 \cdot q^1 / p^0 \cdot q^0 p^1 \cdot q^0]^{1/2}$.

Thus, the square of the Fisher quantity index is equal to

$$\begin{aligned} Q_F(p^0,p^1,q^0,q^1)^2 &= p^0 \cdot q^1 p^1 \cdot q^1 / p^0 \cdot q^0 p^1 \cdot q^0 \\ &= [p^0/p^0 \cdot q^0]^T q^1 / [p^1/p^1 \cdot q^1]^T q^0 \\ &= \{q^0 T A^T q^1 / f(q^0)^2\} / \{q^1 T A^T q^0 / f(q^1)^2\} \text{ using (27)} \\ &= \{1/f(q^0)^2\} / \{1/f(q^1)^2\} \text{ since } q^0 T A^T q^1 = q^1 T A^T q^0 \text{ using } A = A^T \\ &= [f(q^1)/f(q^0)]^2. \end{aligned} \quad (28)$$

Taking positive square roots of both sides of (28) shows that, according to this hypotheses, the Fisher quantity index is *exactly* equal to the utility ratio, which is the *consumer's true volume index*; that is, we have

$$Q_F(p^0,p^1,q^0,q^1) = f(q^1)/f(q^0). \quad (29)$$

Finally, use the following *Product Test* to define the price index that corresponds to the Fisher volume index:

$$\begin{aligned} P_F(p^0,p^1,q^0,q^1) &\equiv p^1 \cdot q^1 / \{p^0 \cdot q^0 Q_F(p^0,p^1,q^0,q^1)\} \\ &= [p^1 \cdot q^0 p^1 \cdot q^1 / p^0 \cdot q^0 p^0 \cdot q^1]^{1/2} \text{ using definition (27).} \end{aligned} \quad (30)$$

[26]This functional form was indirectly introduced into the economics literature by Konüs and Byushgens (1926, 171) and Diewert (1974b, 123) (1976, 116). Pollak (1971), Afriat (1972, 45), and others also considered this functional form but did not work out the region where the utility function was well-behaved.

[27]These conditions were imposed on A by Diewert (1976, 116).

[28]See Diewert and Hill (2010, 272–74) for a proof of this result. It turns out that $f(q) \equiv (q^T Aq)^{1/2}$ is a concave function over the regularity region $S \equiv \{q: Aq \geq 0_N; \; q \geq 0_N$ and $q^T Aq > 0\}$ if A has a positive eigenvalue with a corresponding strictly positive eigenvector and the other eigenvalues of A are negative or 0.

Let $c(p)$ be the unit cost function that corresponds to the utility function $f(q)$ defined by (21).[29] Then, for this $c(p)$, equations (8) will hold; that is, we will have $p^t \cdot q^t = f(q^t)c(p^t)$ for $t = 0,1$. Substituting these equations into the first line of (30), we obtain the following equation:

$$\begin{aligned} P_F(p^0,p^1,q^0,q^1) &\equiv c(p^1)f(q^1)/\{c(p^0)f(q^0)Q_F(p^0,p^1,q^0,q^1)\} \\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)[f(q^1)/f(q^0)]\} \text{ using (29)} \\ &= c(p^1)/c(p^0), \end{aligned} \tag{31}$$

which is the Konüs true cost of living index defined by (9) when preferences are homothetic. Thus, under the assumption that the consumer engages in cost-minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (21), the *Fisher ideal price index* P_F *is exactly equal to the true cost of living index*, $c(p^1)/c(p^0)$.

What is useful about the results obtained here is that *it is not necessary to estimate econometrically the N(N + 1)/2 parameters in the A matrix in order to find an estimator for the consumer's true cost of living index and the corresponding true volume index*.

There is another useful property of the utility function $f(q)$ that is defined by (21): This function is a flexible functional form. Diewert (1974a, 113) defined a twice continuously differentiable linearly homogeneous function of N variables, $f(q)$, to be a *flexible functional form* if it could approximate an arbitrary twice continuously differentiable linearly homogeneous function of N variables, say $f^*(q)$, to the second order around an arbitrary positive vector $q^* >> 0_N$. Thus, if $f^*(q)$ is an arbitrary linearly homogeneous function that is twice continuously differentiable at the given arbitrary point $q^* >> 0_N$, then the linearly homogeneous twice continuously differentiable function $f(q)$ is a *flexible functional form* if it has a sufficient number of free parameters so that the following $1 + N + N^2$ equations can be satisfied:

$$f(q^*) = f^*(q^*); \tag{32}$$

$$\nabla f(q^*) = \nabla f^*(q^*); \tag{33}$$

$$\nabla^2 f(q^*) = \nabla^2 f^*(q^*), \tag{34}$$

where $\nabla f(q^*) \equiv [\partial f(q^*)/\partial q_1, \ldots, \partial f(q^*)/\partial q_N]^T$ is the vector of first-order partial derivatives of $f(q)$ evaluated at the point q^* and $\nabla^2 f(q^*) \equiv [\partial^2 f(q^*)/\partial q_i \partial q_k]$ is the N by N matrix of second-order partial derivatives of $f(q)$ evaluated at the point q^*, where the element in row i and column k is $\partial^2 f(q^*)/\partial q_i \partial q_k$ for $i,k = 1, \ldots, N$.

If $f(q)$ is a flexible functional form, then it can approximate an arbitrary twice differentiable linearly homogeneous utility function very closely in a neighborhood of any arbitrarily chosen point q^*. Thus, if q^0 and q^1, the consumer's observed quantity choices for periods 0 and 1, are fairly close to each other, then a flexible utility function $f(q)$ can approximate the consumer's true utility function $f^*(q)$ reasonably closely, and so index numbers based on the assumption that the consumer maximizes utility using the utility function $f(q)$ instead of the true one $f^*(q)$ will be able to provide a good approximation to the consumer's behavior.[30]

Proposition 4: The utility function defined as $f(q) \equiv (q^T A q)^{1/2}$ over the region S defined by (27) where $A = A^T$ is a flexible functional form.

For a proof, see the Annex.

Diewert (1976, 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[31] Equation (29) plus the fact that the homogeneous quadratic function $f(q)$ defined by (21) is a flexible functional form shows that the Fisher ideal quantity index Q_F defined (27) is a superlative index number formula. Since the corresponding implicit Fisher ideal price index P_F satisfies (31) where $c(p)$ is the unit cost function that is generated by the homogeneous quadratic utility function, we also call P_F a superlative index number formula.

There is a special case of the homogeneous quadratic preferences that will play an important role in later chapters and that is the case of *linear preferences*. Thus, suppose that the consumer has the following linear utility function:

$$f(q) = S_{n=1}^{N} a_n q_n, \tag{35}$$

where the parameters a_n are positive. If $N = 2$, the indifference curves for a consumer with linear preferences are a family of parallel straight lines. The parameters a_n are *quality adjustment parameters*; that is, a_n is the marginal increment to the consumer's welfare due to the consumption of an extra unit of the nth commodity. The absolute magnitudes of the a_n are not meaningful (since the units of measurement for utility are not observable), but the relative valuations a_n/a_k are meaningful. If a consumer has linear preferences, then we say that the N products are *perfect substitutes*.

To see that the preferences defined by (35) are a special case of the preferences defined by $f(q) = (q^T A q)^{1/2}$, let matrix A be defined as the following rank 1 matrix:

$$A \equiv aa^T, \tag{36}$$

where the row vector a^T is defined as $a^T \equiv [a_1, \ldots, a_N]$. Thus, if $f(q)$ is defined as $(q^T A q)^{1/2}$, then using the A defined by (36), we have $f(q) = (q^T A q)^{1/2} = (q^T a a^T q)^{1/2} = ([a^T q]^2)^{1/2} = a^T q = \Sigma_{n=1}^{N} a_n q_n$. With linear preferences, the consumer's utility maximization problem (13) becomes the following linear programming problem:

$$\max_q \{a^T q : p^t \cdot q \leq e^t; q \geq 0_N\} = \max_n \{e^t a_n / p_n^t; \; n = 1, \ldots, N\}. \tag{37}$$

[29] It may not be easy to find an explicit formula for c(p) in terms of matrix A. If matrix A has an inverse, then it can be shown that the unit cost function that corresponds to the utility function $f(q)$ defined by (21) is $c(p) \equiv (p^T A^{-1} p)^{1/2}$ for price vectors p belonging to the region of prices defined by $S^* \equiv \{p$: $A^{-1}p$ ³ 0_N ; $p \geq 0_N$ and $p^T A^{-1} p > 0_N\}$.

[30] A first-order approximation to a consumer's utility function will not be able to provide a first-order approximation to the consumer's system of consumer demand functions. A first-order approximation will not be able to adequately describe a consumer's reactions to changes in relative prices.

[31] Fisher (1922, 247) used the term "superlative" to describe the Fisher ideal price index. Thus, Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

Thus, if a consumer has linear preferences, then they will usually end up at a corner solution where one or more commodities are not consumed at all. However, if a utility-maximizing consumer with linear preferences ends up choosing a positive amount of each commodity for period t, then it must be the case that $a_n/p_n{}^t = \lambda_t$ for $n = 1, \ldots, N$. Thus, if a utility-maximizing consumer with linear preferences consumes positive amounts of all N products in periods 0 and 1, then it must be the case that prices are varying in a proportional manner over periods 0 and 1; that is, the period t price vector p^t must be equal to $1_t a$, where $\lambda_t > 0$ for $t = 0,1$.[32] It is not realistic to assume that prices vary in strict proportion over time, but if the variation in prices is approximately proportional, then it is not unrealistic to assume that a utility-maximizing consumer's preferences can be adequately approximated by a linear utility function. The assumption of linear preferences will play a large role in our treatment of quality change (Chapter 8). The important point to take away from this discussion of utility-maximizing behavior where the consumer has a linear utility function is that the use of the Fisher quantity index to measure quantity change (and hence to measure welfare change) is perfectly consistent with the assumption of linear preferences.

It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (21), we can start with the assumption that the consumer's unit cost function is a homogeneous quadratic. Thus, suppose that the consumer has the following unit cost function:

$$c(p_1, \ldots, p_N) \equiv [\Sigma_{i=1}{}^N \Sigma_{k=1}{}^N b_{ik} p_i p_k]^{1/2}, \qquad (38)$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all $1 \leq i, k \leq N$. Thus, there are $N(N + 1)/2$ independent parameters in the functional form for $c(p)$ defined by (38).[33] Let $B \equiv [b_{ik}]$ be the N by N matrix that has b_{ik} in row i and column k of B. Then $c(p_1, \ldots, p_N) = c(p)$ can be defined as follows:

$$c(p) = (p^T B p)^{1/2};\ B = B^T. \qquad (39)$$

Using this matrix notation, the vector of first-order partial derivatives of the unit cost function defined by (39) is equal to the following expression:

$$\begin{aligned} \nabla c(p) &= Bp/[p^T Bp]^{1/2} \\ &= Bp/c(p) \end{aligned} \qquad (40)$$

where the second equation in (40) follows because $c(p) \equiv [p^T Bp]^{1/2}$. Now evaluate (40) when $p = p^t$ for $t = 0,1$, where $p^t >> 0_N$ is the positive period t price vector facing the consumer. Divide the resulting equation t by $c(p^t)$ for $t = 0,1$ and we obtain the following equations:

$$\nabla c(p^t)/c(p^t) = Bp^t/[c(p^t)]^2;\ t = 0,1. \qquad (41)$$

The $c(p)$ defined by (39) is obviously linearly homogeneous. But we also need it to be positive (if $p > 0_N$), nondecreasing, and concave in p over at least a subset of the nonnegative orthant. Suppose the symmetric matrix B has one positive eigenvalue with a corresponding strictly positive eigenvector and the remaining $N - 1$ eigenvalues of B are either 0 or negative.[34] Then $c(p)$ defined by (39) will be positive, nondecreasing, and concave over the region of regularity S^* defined as follows:[35]

$$S^* \equiv \{p : p \geq 0_N;\ Bp \geq 0_N;\ p^T Bp > 0\}. \qquad (42)$$

Now assume cost-minimizing behavior for the consumer in periods 0 and 1; that is, assume that $q^t >> 0_N$ is a solution to the consumer's period t cost minimization problem when the consumer faces the price vector $p^t >> 0_N$ for $t = 0,1$. Assume that the consumer has homothetic preferences and the consumer's unit cost function is $c(p)$ defined by (39). Finally assume that p^t belongs to the regularity region for prices S^* defined by (42) for $t = 0,1$. Shephard's Lemma (20) applied to the $c(p)$ defined by (39) gives us the following equations:

$$\begin{aligned} q^t/p^t \cdot q^t &= \nabla c(p^t)/c(p^t)\ t = 0,1 \\ &= Bp^t/[c(p^t)]^2 \text{ using (41).} \end{aligned} \qquad (43)$$

Recall that the Fisher (1922) ideal price index was defined earlier by (30); that is, $P_F(p^0,p^1,q^0,q^1)$ was defined as $[p^1 \cdot q^0 p^1 \cdot q^1 / p^0 \cdot q^0 p^0 \cdot q^1]^{1/2}$. Thus, the square of the Fisher price index is equal to

$$\begin{aligned} [P_F(p^0,p^1,q^0,q^1)]^2 &= p^1 \cdot q^0 p^1 \cdot q^1 / p^0 \cdot q^0 p^0 \cdot q^1 \\ &= p^1 T[q^0/p^0 \cdot q^0]/p^0 T[q^1/p^1 \cdot q^1] \\ &= p^1 T\{Bp^0/[c(p^0)]^2\}/p^0 T\{Bp^1/[c(p^1)]^2\} \\ &= \{1/c(p^0)^2\}/\{1/c(p^1)^2\} \text{ since } p^1 TBp^0 = p^0 TBp^1 \text{ using } B = B^T \\ &= [c(p^1)/c(p^0)]^2. \end{aligned} \qquad (44)$$

Taking positive square roots of both sides of (44) shows that, under the aforementioned hypotheses, the Fisher price index is *exactly* equal to the unit cost ratio, which is the *consumer's true cost of living index* in the case of homothetic preferences; that is, we have

$$P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0). \qquad (45)$$

[32] In this case, the solution set to the period t utility maximization problem defined by (37) is the set $\{q : p^t \cdot q = e^t;\ q \geq 0_N\}$. This analysis for the case of a linear utility function follows that of Pollak (1971) (1983).

[33] This functional form for a unit cost function was essentially developed by Konüs and Byushgens (1926, 168), and they showed the relationship of this functional form to the Fisher ideal price index. See also Diewert (1976) and Diewert and Hill (2010).

[34] These regularity conditions on B are counterparts to our earlier regularity conditions that were imposed on A.

[35] Again see Diewert and Hill (2010, 272–74) for a proof of this result.

Finally, use the *Product Test* to define a quantity index Q_F^* that corresponds to the Fisher price index defined by (30):

$$Q_F^*(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/\{p^0\cdot q^0 P_F(p^0,p^1,q^0,q^1)\} \quad (46)$$
$$= [p^0\cdot q^1 p^1\cdot q^1/p^0\cdot q^0 p^1\cdot q^0]^{1/2} \text{ using definition (30)}$$
$$= Q_F(p^0,p^1,q^0,q^1),$$

where $Q_F(p^0,p^1,q^0,q^1)$ was defined earlier by (27). Thus, the implicit quantity index that corresponds to the Fisher price index defined by (30) is the Fisher quantity index defined by (27), and the implicit price index that corresponds to the Fisher quantity index defined by (27) is the Fisher price index.

Since preferences are homothetic, equations (8) will hold; that is, we have $p^t\cdot q^t = c(p^t)f(q^t)$ for $t = 0,1$. From (46), we have

$$Q_F(p^0,p^1,q^0,q^1) = p^1\cdot q^1/\{p^0\cdot q^0 P_F(p^0,p^1,q^0,q^1)\} \quad (47)$$
$$= c(p^1)f(q^1)/\{c(p^0)f(q^0)P_F(p^0,p^1,q^0,q^1)\} \text{ using (8)}$$
$$= c(p^1)f(q^1)/\{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} \text{ using (45)}$$
$$= f(q^1)/f(q^0).$$

Again, the Fisher quantity index is equal to the utility ratio under our assumptions on consumer behavior.

The proof of Proposition 4 can be adapted to show that $c(p) \equiv (p^T Bp)^{1/2}$ is a flexible functional form. Thus, we have again shown that the Fisher ideal price index is a *superlative index*; that is, it is exact for a flexible functional form for the unit cost function.

An important special case of this functional form is the case where matrix B is equal to a rank 1 matrix; that is, suppose B is given by

$$B = bb^T, \quad (48)$$

where $b^T \equiv [b_1, \ldots, b_N]$ and $b_n > 0$ for $n = 1, \ldots, N$. Using Shephard's Lemma (19) for the cost function $C(u^t,p^t) \equiv u^t c(p^t) = u^t(p^T bb^T p)^{1/2} = u^t b^T p$ for periods $t = 0,1$ leads to the following equations to describe the period t demand vectors, q^t:

$$q^t = u^t \nabla c(p^t) = u^t b;\ t = 0,1. \quad (49)$$

Thus, $q_n^1/q_n^0 = u^1/u^0$ for $n = 1, \ldots, N$ and the demand for each commodity moves in a proportional manner over the two periods. Note also that *changes in commodity prices do not change the demands*. Thus, preferences are such that the consumer will not substitute cheaper products for more expensive products as prices change over time. For $N = 2$, the consumer's family of indifference curves are L shaped. The preferences that are represented by the cost function $uc(p) = ub\cdot p = u\,\Sigma_{n=1}^N b_n p_n$ are called *no substitution* or *Leontief preferences* in the economic literature.[36] These preferences are completely opposite to linear preferences where products were perfect substitutes. What is interesting is that the Fisher ideal price and quantity indices are completely consistent with utility-maximizing behavior for both types of preferences.

We conclude this section by showing how a linearly homogeneous utility function $f(q)$ can be derived from its dual unit cost function $c(p)$. Suppose that the unit cost function $c(p)$ is given and it is nonnegative, increasing, linearly homogeneous, concave, and continuous for $q \geq 0_N$. Let $q^* >> 0_N$. The utility level $u \equiv f(q)$ that corresponds to $c(p)$ must satisfy the inequality $c(p)u \leq p\cdot q^*$ for all $p > 0_N$. Since $c(p)$ and $p\cdot q^*$ are linearly homogeneous in p, we can replace the set of p such that $p > 0_N$ by the set $\{p : p \geq 0_N;\ p\cdot q^* = 1\}$. Thus, the inequalities $c(p)u \leq p\cdot q^*$ for all $p > 0_N$ are equivalent to the inequalities $c(p)u \leq 1$ for all $p \geq 0_N;\ p\cdot q^* = 1$. Since $c(p)$ will be positive for all such p vectors, this last set of inequalities can be replaced by $u \leq 1/c(p)$ for all $p \geq 0_N;\ p\cdot q^* = 1$. The biggest such $u = f(q^*)$ that will satisfy all of the inequalities is given by $1/c(p^*)$, where p^* solves the concave programming problem: $\max_p \{c(p) : p\cdot q^* = 1;\ p \geq 0_N\}$. Thus, we have the following representation for $f(q^*)$ in terms of $c(p)$:[37]

$$f(q^*) = 1/\max_p \{c(p) : p\cdot q^* = 1;\ p \geq 0_N\}. \quad (50)$$

We can use this formula in order to calculate the utility function that corresponds to the no-substitution unit cost function defined as $c(p) \equiv b\cdot p$. The constrained maximization problem that appears in (50) for this unit cost function is

$$\max_p \{\Sigma_{n=1}^N b_n p_n : \Sigma_{n=1}^N q_n^* p_n = 1;$$
$$p \geq 0_N\} = \max_n \{b_n/q_n^* : n = 1, \ldots, N\}. \quad (51)$$

Since all of the numbers b_n and q_n^* are assumed to be positive, $1/\max_n \{b_n/q_n^* : n = 1, \ldots, N\}$ will equal $\min_n \{q_n^*/b_n : n = 1, \ldots, N\}$. Using this equality and (51), (50) becomes the following explicit representation for the *no substitution preference function*:

$$f(q^*) = \min_n \{q_n^*/b_n : n = 1, \ldots, N\}. \quad (52)$$

Another special case of the homogeneous quadratic unit cost function defined by (39) is the case where matrix B has an inverse.[38] Let $c(p) = (p^T Bp)^{1/2}$, where $B = B^T$ and B has one positive eigenvalue with a strictly positive eigenvector and the remaining $N - 1$ eigenvalues of B are negative. In this case, B has full rank and so B^{-1} exists. We show in the annex how a modification of formula (50) can be used to calculate $f(q^*)$ for some $q^* >> 0_N$.

Proposition 5: Let $c(p) = (p^T Bp)^{1/2}$, where $B = B^T$ and B has one positive eigenvalue with a strictly positive eigenvector and the remaining $N - 1$ eigenvalues of B are negative. Let $q^* >> 0_N$ and suppose also that $B^{-1}q^* >> 0_N$. Let $f(q)$ be the utility function that is dual to $c(p)$. Then $f(q^*) = (q^{*T}B^{-1}q^*)^{1/2}$.

[36] See Diewert (1971).

[37] This formula may be found in the work of Diewert (1974a, 112).

[38] This is the model of consumer behavior considered by Konüs and Byushgens (1926, 168).

In the following sections, we will exhibit some additional exact index number formulae.

6. Quadratic Means of Order *r* and the Walsh Index

It turns out that there are many other superlative index number formulae; that is, there exist many quantity indices $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indices $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define two families of superlative indices in this section.

Suppose the consumer has the *following quadratic mean of order r utility function*:[39]

$$f^r(q_1,\ldots,q_N) \equiv [\Sigma_{i=1}^N \Sigma_{k=1}^N a_{ik} q_i^{r/2} q_k^{r/2}]^{1/r}, \quad (53)$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. It turns out that $f^r(q)$ is a flexible functional form.

Proposition 6: For each $r \neq 0$, $f^r(q)$ defined by (53) is a flexible functional form.

See the Annex for a proof. From the proof of Proposition 6, it can be seen that the quadratic mean of order r utility function defined by (53) can adequately represent the preferences for a utility-maximizing consumer for quantity vectors q in a neighborhood around any strictly positive q^* since there will be a neighborhood around q^*, where $f^r(q)$ will be concave and increasing. Hence, for this region, $f^r(q)$ can provide an adequate approximation to arbitrary differentiable homothetic preferences. However, this neighborhood may not be very large and this point should be kept in mind.[40]

Let $r \neq 0$ and define the *quadratic mean of order r quantity index* Q^r by

$$Q^r(p^0,p^1,q^0,q^1) \equiv \{\Sigma_{i=1}^N s_i^0 (q_i^1/q_i^0)^{r/2}\}^{1/r} \{\Sigma_{i=1}^N s_i^1 (q_i^1/q_i^0)^{-r/2}\}^{-1/r}, \quad (54)$$

where $s_i^t \equiv p_i^t q_i^t / \Sigma_{k=1}^N p_k^t q_k^t$ is the period t expenditure share for commodity i for $i = 1,\ldots,N$ and $t = 0,1$. It can be verified that when $r = 2$, Q^r simplifies into Q_F, the Fisher ideal quantity index.

Proposition 7: Let $r \neq 0$ and define $f^r(q)$ by (53) over an open convex set S of positive quantity vectors q. We assume that $f^r(q)$ defined by (53) is positive, increasing, and concave over S.[41] Finally assume that q^t solves the following period t local utility maximization problem where $p^t >> 0_N$ and $e^t > 0$ for $t = 0,1$:

$$\max_q \{f^r(q) : p^t \cdot q \leq e^t;\ q \in S\}. \quad (55)$$

Then $Q^r(p^0,p^1,q^0,q^1)$ defined by (54) is exact for $f^r(q)$ defined by (53); that is, we have

$$Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0). \quad (56)$$

See the Annex for a proof of Proposition 7.

Thus, under the assumption that the consumer engages in utility-maximizing behavior during periods 0 and 1 and has local preferences over the N commodities that correspond to the utility function defined by (53) for a region that includes q^0 and q^1, then the quadratic mean of order r quantity index Q^r is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[42] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus, there are an infinite number of superlative quantity indices.[43]

For each quantity index Q^r, we can use the product test in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

$$P^{r*}(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/\{p^0\cdot q^0 Q^r(p^0,p^1,q^0,q^1)\} = c^{r*}(p^1)/c^{r*}(p^0), \quad (57)$$

where c^{r*} is the unit cost function that corresponds to the aggregator function f^r defined by (53). For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

When $r = 2$, as noted earlier, Q^r defined by (54) simplifies to Q_F, the Fisher ideal quantity index, and P^{r*} defined by (57) simplifies to P_F, the Fisher ideal price index. When $r = 1$, Q^r defined by (54) simplifies to

$$\begin{aligned} Q^1(p^0,p^1,q^0,q^1) &\equiv \{\Sigma_{i=1}^N s_i^0 (q_i^1/q_i^0)^{1/2}\}/\{\Sigma_{i=1}^N s_i^1 (q_i^1/q_i^0)^{-1/2}\} \quad (58)\\ &= \{[\Sigma_{i=1}^N p_i^0 q_i^0/\Sigma_{i=1}^N p_i^0 q_i^0](q_i^1/q_i^0)^{1/2}\}/\{[\Sigma_{i=1}^N p_i^1 q_i^1/\Sigma_{i=1}^N p_i^1 q_i^1](q_i^1/q_i^0)^{-1/2}\}\\ &= \{\Sigma_{i=1}^N p_i^0 (q_i^0 q_i^1)^{1/2}/p^0\cdot q^0\}/\{\Sigma_{i=1}^N p_i^1 (q_i^0 q_i^1)^{1/2}/p^1\cdot q^1\}\\ &= [p^1\cdot q^1/p^0\cdot q^0]/P_W(p^0,p^1,q^0,q^1), \end{aligned}$$

where P_W is the *Walsh* (1901) (1921) *price index* defined in Chapter 2. Thus, P^{1*} is equal to P_W, the *Walsh price index*, and hence it is also a superlative price index.[44]

Suppose the consumer has the *following quadratic mean of order r unit cost function*:[45]

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[39] This terminology was adopted by Diewert (1976, 129). When $r = 1$, $f^r(q)$ simplifies into the Generalized Linear Utility Function; see Diewert (1971).

[40] This index number formula was derived by Diewert (1976, 130).

[41] Using the techniques described in Blackorby and Diewert (1979), the utility function $f^r(q)$ that satisfies the appropriate regularity conditions over the set S can be extended to preferences that are defined over $q \geq 0_N$.

[42] See Diewert (1976, 130).

[43] However, as r becomes large in magnitude, the region where $f^r(q)$ can approximate a well-behaved utility function will tend to shrink. In the limiting cases where r tends to plus or minus infinity, Hill (2006) showed that $f^r(q)$ loses its flexibility property. Thus, it is recommended that $Q^r(p^0,p^1,q^0,q^1)$ only be used for r small in magnitude.

[44] For $r = 1$, the utility function defined by (53) turns out to be the Generalized Linear function that was introduced to the economics literature by Diewert (1971).

[45] This terminology was adopted by Diewert (1976, 130). This unit cost function was first defined by Denny (1974). We restrict p to belong to a set of prices S^* that is defined in Proposition 8.

$$c^r(p_1, \ldots, p_N) \equiv [\Sigma_{i=1}^{N} \Sigma_{k=1}^{N} b_{ik} p_i^{r/2} p_k^{r/2}]^{1/r}, \qquad (59)$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Note that when $r = 2$, c^r equals the homogeneous quadratic unit cost function defined by (39).[46]

Proposition 8: For each $r \neq 0$, $c^r(p)$ defined by (59) is a flexible functional form.[47]

The proof of this proposition is analogous to the proof of Proposition 6: just replace q by p and replace $f^r(q)$ by $c^r(p)$.

Since $c^r(p)$ is unlikely to be a well-behaved unit cost function over the entire set of positive price vectors, we need a method for recovering preferences defined by a unit cost function defined over a smaller set of prices where $c^r(p)$ satisfies the necessary conditions for unit cost function; that is, where it is increasing and concave.[48] Thus, let S^* be a set of prices that satisfies the following conditions:[49]

(60) S^* is a set of N dimensional vectors that has the following properties: (i) if $p \in S^*$, then $p >> 0_N$; (ii) S^* is an open set;[50] (iii) S^* is a convex set;[51] (iv) S^* is a cone;[52] (v) if p belongs to S^*, then $\nabla c^r(p) >> 0_N$; and (vi) $c^r(p)$ is a concave function over S^*.

We need to find the utility function $f^{r*}(q)$ that is consistent with the unit cost function $c^r(p)$ defined by (59) over S^*. We can find this corresponding utility function but it will not be defined over all nonnegative quantity vectors, $q \geq 0_N$. It will be defined over the set S defined as follows:

$$S \equiv \{q: q = \lambda \nabla c^r(p); \lambda > 0; p \in S^*\}. \qquad (61)$$

It can be seen using property (v) in (60) that S will also be a cone and moreover, if $q \in S$, then $q >> 0_N$.

If S^* turns out to be the interior of the nonnegative orthant, then $f^{r*}(q^*)$ that is generated by the unit cost function $c^r(p)$ for $q^* >> 0_N$ can be defined as follows:

$$f^{r*}(q^*) \equiv \max_{u>0,p} \{u: c^r(p)u \leq p \cdot q^*; p > 0_N\} \qquad (62)$$

$= \max_{u>0,p} \{u$: $c^r(p)u$ £ e; e $= p \cdot q^*; p > 0_N\}$, where e > 0 is an arbitrary positive number[53]

$= \max_{u>0,p} \{u: u \leq$ e$/c^r(p)$; e $= p \cdot q^*$; p ³ $0_N\}$[54]

$= $ e$/\max_p \{c^r(p)$; e $= p \cdot q^*$; p ³ $0_N\}$.

However, in general, $c^r(p)$ will not be a well-behaved unit cost function for all $p > 0_N$. Thus, in the following definition for $f^{r*}(q^*)$, we restrict p to belong to the set S^* that has the properties listed in (60), and we restrict q^* to belong to S, where S is defined by (61). Thus, let q^* belong to S and define $f^{r*}(q^*)$ as follows:[55]

$f^{r*}(q^*) \equiv \max_{u>0,p} \{u: c^r(p)u \leq p \cdot q^*$; pÎ$S^*\}$ (63)

$= \max_{u>0,p} \{u$: $c^r(p)u$ £ e; e $= p \cdot q^*; p \in S^*\}$, where e > 0 is an arbitrary positive number

$= \max_{u>0,p} \{u: u \leq$ e$/c^r(p)$; e $= p \cdot q^*$; pÎ$S^*\}$

$=$ e$/\max_p \{c^r(p)$; e $= p \cdot q^*$; pÎ$S^*\}$.

The previous representation for $f^{r*}(q^*)$ will be used in the proof of the following Proposition:

Proposition 9: Let $c^r(p)$ be defined by (59) for $p \in S^*$, where S^* is defined by (60). Let $e^t > 0$ and $p^t \in S^*$. Define q^t as

$$q^t \equiv e^t \nabla c^r(p^t)/c^r(p^t). \qquad (64)$$

Then p^t is a solution to $\max_p \{c^r(p)$; e $= p \cdot q^t$; pÎ$S^*\}$. Define $f^{r*}(q)$ by (63) (with e $= e^t$) for $q \in S$, where S is defined by (61). Then $f^{r*}(q^t)$ is equal to the following expression:

$$f^{r*}(q^t) = e^t/c^r(p^t). \qquad (65)$$

Finally, the q^t defined by (64) is a solution to the consumer's local utility maximization problem defined as follows:

$$\max_q \{f^{r*}(q) : p^t \cdot q = e^t; q \in S\}. \qquad (66)$$

For a proof of Proposition 9, see the Annex.

Note that using (64), we have

$$p^t \cdot q^t = p^t \cdot e^t \nabla c^r(p^t)/c^r(p^t) \qquad (67)$$

$= e^t c^r(p^t)/c^r(p^t)$ since $c^r(p^t) = p^t \cdot e^t \nabla c^r(p^t)$

$= e^t$

$= f^{r*}(q^t) c^r(p^t)$ using (65).

Using (64) and $p^t \cdot q^t = e^t$, we also have the Shephard's Lemma equality:[56]

$$q^t/p^t \cdot q^t = \nabla c^r(p^t)/c^r(p^t). \qquad (68)$$

[46] When $r = 1$, $c^r(p)$ defined by (59) becomes the Generalized Leontief functional form for a cost function; see Diewert (1971).
[47] See Diewert (1976, 130).
[48] The $c^r(p)$ defined by (59) is automatically linearly homogeneous over the set of prices where it is positive, increasing, and concave since linear homogeneity is imposed on the functional form by its definition.
[49] Using the techniques described in Blackorby and Diewert (1979), if $c^r(p)$ is linearly homogeneous, increasing, and concave over S^*, then the domain of definition of $c^r(p)$ can be extended to all $p \geq 0_N$.
[50] This means if $p \in S^*$, then there exists a $\delta > 0$ such that the open ball of radius δ, $B_\delta(p)$, also belongs to S^*, where $B_\delta(p) \equiv \{x: (x - p) \cdot (x - p) < d^2\}$.
[51] This means if p^1 and p^2 belong to S^* and $0 < \lambda < 1$, then $\lambda p^1 + (1 - \lambda) p^2$ also belongs to S^*.
[52] If p belongs to S^*, then lp also belongs to S^* for all $\lambda > 0$.
[53] The number e is a fixed positive number. In order to justify moving from the first equality in (62) to the second equality, we need to use the fact that $c^r(p)$ is linearly homogeneous.
[54] Since e > 0, $p \geq 0_N$, $q^* >> 0_N$, and $p \cdot q^* =$ e, we can replace the constraints $p > 0_N$ by $p \geq 0_N$.
[55] Again, using the methods described in Blackorby and Diewert (1979), the domain of definition for $f^{r*}(q)$ can be extended to $q \geq 0_N$. However, for $q > 0_N$ but $q \notin S$, the extended $f^{r*}(q)$ may not represent the true preferences of the consumer.
[56] Recall (20).

We will relate the preferences defined by $c^r(p)$ to the following price index formula. Let $r \neq 0$ and define the *quadratic mean of order r price index* P^r by

$$P^r(p^0,p^1,q^0,q^1) \equiv \{\Sigma_{i=1}^N s_i^0(p_i^1/p_i^0)^{r/2}\}^{1/r} \{\Sigma_{i=1}^N s_i^1(p_i^1/p_i^0)^{-r/2}\}^{-1/r}, \quad (69)$$

where $s_i^t \equiv p_i^t q_i^t / p^t \cdot q^t$ is the period t expenditure share for commodity i for $i = 1, \ldots, N$ and $t = 0,1$. It can be verified that when $r = 2$, P^r simplifies into P_F, the Fisher ideal price index.

Proposition 10: Let $r \neq 0$ and assume that $c^r(p)$ given by (59) is defined over a set S^* that satisfies conditions (60). Define the set S by (61) and define the locally dual utility function $f^{r*}(q^*)$ for $q^* \in S$ by (63) for any e > 0. Let e^t equal the consumer's "income" in period t that is allocated to spending on the N commodities for $t = 0,1$. Let p^0 and p^1 belong to S^* and q^0 and q^1 are defined by:

$$q^t \equiv e^t \nabla c^r(p^t)/c^r(p^t);\ t = 0,1. \quad (70)$$

Then, q^t solves the local utility maximization problem, $\max_q \{f^{r*}(q);\ p^t \cdot q = e^t;\ q \in S\}$, for $t = 0,1$. Moreover, P^r defined by (69) is exact for the preferences defined by $f^{r*}(q)$ over the set S; that is, we have

$$P^r(p^0,p^1,q^0,q^1) = c^r(p^1)/c^r(p^0). \quad (71)$$

See the Annex for a proof.

Thus, under the assumption that the consumer engages in cost-minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the unit cost function defined by (59), the quadratic mean of order r price index P^r is *exactly* equal to the true price index, $c^r(p^1)/c^r(p^0)$.[57] Since P^r is exact for c^r and c^r is a flexible functional form, we see that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus, there are an infinite number of superlative price indices.

For each price index P^r, we can use the product test in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

$$Q^{r*}(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / \{p^1 \cdot q^1 P^r(p^0,p^1,q^0,q^1)\} = f^{r*}(q^1)/f^{r*}(q^0), \quad (72)$$

where f^{r*} is the utility function that corresponds to the unit cost function c^r defined by (53). For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

When $r = 2$, P^r defined by (69) simplifies to P_F, the Fisher ideal price index and Q^{r*} defined by (72) simplifies to Q_F, the Fisher ideal quantity index. When $r = 1$, P^r simplifies to

$$P^1(p^0,p^1,q^0,q^1) \equiv \{\Sigma_{i=1}^N s_i^0 (p_i^1/p_i^0)^{1/2}\} / \{\Sigma_{i=1}^N s_i^1 (p_i^1/p_i^0)^{-1/2}\} \quad (73)$$
$$= \{[\Sigma_{i=1}^N p_i^0 q_i^0 / p^0 \cdot q^0](p_i^1/p_i^0)^{1/2}\} / \{[p^1 \cdot q^1 / \Sigma_{i=1}^N p_i^1 q_i^1] (p_i^1/p_i^0)^{-1/2}\}$$
$$= \{\Sigma_{i=1}^N q_i^0 (p_i^0 p_i^1)^{1/2} / p^0 \cdot q^0\} / \{\Sigma_{i=1}^N q_i^1 (p_i^0 p_i^1)^{1/2} / p^1 \cdot q^1\}$$
$$= [p^1 \cdot q^1 / p^0 \cdot q^0] / Q_W(p^0,p^1,q^0,q^1),$$

where Q_W is the *Walsh quantity index*. Thus, Q^{1*} is equal to Q_W, the Walsh quantity index, and hence it is also a superlative quantity index.[58]

The results in this section can be summed up as follows:

- Superlative indices are nice in theory since they enable statisticians to compute price and volume indices that are consistent with the economic approach to index number theory where the underlying preference functions and their corresponding unit cost functions can approximate arbitrary differentiable preferences to the second order around an arbitrary point. These superlative indices do not require econometric estimation in order to be implemented.
- These indices are consistent with a wide range of substitution responses on the part of consumers to changes in prices.
- However, superlative indices have the disadvantage that the quantity and price regions where the underlying preferences are well behaved is generally not known to the statistician. If there are large fluctuations in prices and quantities across periods, then the various exact indices may no longer be exact![59]
- It is of some comfort that the Fisher and Walsh indices that have been recommended as "best" from the approaches to index number theory that were described in previous chapters emerge as being "best" from the economic approach as well.

We turn our attention to yet another superlative index number formula.

7. Superlative Indices: The Törnqvist-Theil Index

In this section, we will revert to the assumptions made on the consumer in Section 2. In particular, we do not assume that the consumer's utility function f is necessarily linearly homogeneous as in Sections 3–6.

Before we derive our main result, we require a preliminary result. Suppose the function of N variables, $f(z_1, \ldots, z_N) \equiv f(z)$, is quadratic; that is,

$$f(z_1, \ldots, z_N) \equiv a_0 + \Sigma_{i=1}^N a_i z_i + (1/2) \Sigma_{i=1}^N \Sigma_{k=1}^N a_{ik} z_i z_k;\ a_{ik} = a_{ki} \text{ for all } i \text{ and } k, \quad (74)$$

where a_i and a_{ik} are constants. Let $f_i(z)$ denote the first-order partial derivative of f evaluated at z with respect to the ith component of z and z_i. Let $f_{ik}(z)$ denote the second-order

[57] See Diewert (1976, 133–34).

[58] The Walsh quantity index is a useful one for national income accountants since it is a superlative index, but it is also an index that defines real output for periods 0 and 1 as $Q^t \equiv \Sigma_{n=1}^N (p_n^0 p_n^1)^{1/2} q_n^t$ for $t = 0,1$. Thus, the price weights are *constant* over the two periods, and the quantity aggregate Q^t for period t is *linear* in the period t quantities, q_n^t. See Diewert (1996).

[59] This warning is particularly relevant for the use of the quadratic mean of order r functional forms where r is large in magnitude. The regularity regions for these functions will tend to shrink as r approaches plus or minus infinity.

partial derivative of f with respect to z_i and z_k. Then it is well known that the second-order Taylor series approximation to a quadratic function is *exact*; that is, if f is defined by (74), then for any two points, z^0 and z^1, we have

$$\begin{aligned} f(z^1) - f(z^0) &= \Sigma_{i=1}^N f_i(z^0)[z_i^1 - z_i^0] + (1/2) \\ &\quad \Sigma_{i=1}^N \Sigma_{k=1}^N f_{ik}(z^0)[z_i^1 - z_i^0][z_k^1 - z_k^0] \\ &= \nabla f(z^0)\cdot[z^1 - z^0] + (1/2)[z^1 - z^0]^T \nabla^2 f(z^0)^T [z^1 - z^0]. \end{aligned} \tag{75}$$

It is less well known that *an average of two first-order Taylor series approximations* to a quadratic function is also *exact*; that is, if f is defined by (74), then for any two points, z^0 and z^1, we have[60]

$$\begin{aligned} f(z^1) - f(z^0) &= (1/2)\Sigma_{i=1}^N [f_i(z^0) + f_i(z^1)][z_i^1 - z_i^0] \\ &= (1/2)[\nabla f(z^0) + \nabla f(z^1)]^T[z^1 - z^0]. \end{aligned} \tag{76}$$

Diewert (1976, 118) and Lau (1979) showed that equation (76) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will refer to (76) as the *quadratic identity*.

We now suppose that the consumer's *cost function*,[61] $C(u,p)$, has the following *translog functional form*:[62]

$$\begin{aligned} lnC(u,p) &\equiv a_0 + \Sigma_{i=1}^N a_i lnp_i + (1/2)\, \Sigma_{i=1}^N \\ &\quad \Sigma_{k=1}^N a_{ik} lnp_i lnp_k \\ &\quad + b_0 lnu + \Sigma_{i=1}^N b_i lnp_i lnu + (1/2)\, b_{00}[lnu]^2, \end{aligned} \tag{77}$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

$$a_{ik} = a_{ki};\ i,k = 1, \ldots, N; \tag{78}$$

$$\Sigma_{i=1}^N a_i = 1; \tag{79}$$

$$\Sigma_{i=1}^N b_i = 0; \tag{80}$$

$$\Sigma_{k=1}^N a_{ik} = 0;\ i = 1, \ldots, N. \tag{81}$$

The parameter restrictions (78)–(81) ensure that $C(u,p)$ defined by (77) is linearly homogeneous in p, a property that a cost function must have. It can be shown that the translog cost function defined by (77)–(81) can provide a second-order Taylor series approximation to an arbitrary cost function.[63]

We assume that the consumer has preferences that correspond to the translog cost function and that the consumer engages in cost-minimizing behavior during periods 0 and 1. Let p^0 and p^1 be the period 0 and 1 observed price vectors,[64] and let q^0 and q^1 be the period 0 and 1 observed quantity vectors. Using the assumption of cost-minimizing behavior, we have

$$C(u^0,p^0) = p^0\cdot q^0 \text{ and } C(u^1,p^1) = p^1\cdot q^1, \tag{82}$$

where C is the translog cost function defined earlier. We can also apply Shephard's Lemma[65] to $C(u^t,p^t)$ defined by (77):

$$\begin{aligned} q_i^t &= \partial C(u^t,p^t)/\partial p_i;\ i = 1, \ldots, N;\ t = 0,1 \\ &= [C(u^t,p^t)/p_i^t]\partial lnC(u^t,p^t)/\partial lnp_i. \end{aligned} \tag{83}$$

Now use (82) to replace $C(u^t,p^t)$ in (83). After some cross multiplication, equations (83) become the following system of equations:

$$p_i^t q_i^t / \Sigma_{k=1}^N p_k^t q_k^t \equiv s_i^t = \partial lnC(u^t,p^t)/\partial lnp_i;\quad i = 1, \ldots, N;\ t = 0,1 \text{ or} \tag{84}$$

$$s_i^t = a_i + \Sigma_{k=1}^N a_{ik} lnp_k^t + b_i lnu^t;\quad i = 1, \ldots, N;\ t = 0,1, \tag{85}$$

where s_i^t is the period t expenditure share on commodity i and (85) follows from (84) by differentiating (77) with respect to lnp_i for $t = 0,1$ and $i = 1, \ldots, N$.

Define the geometric average of the period 0 and 1 utility levels as u^*; that is, define

$$u^* \equiv [u^0 u^1]^{1/2}\ . \tag{86}$$

Now observe that the right-hand side of the equation that defines the natural logarithm of the translog cost function, equation (77), is a quadratic function of the variables $z_i \equiv lnp_i$ if we hold utility constant at the level u^*. Hence, we can apply the quadratic identity, (76), and get the following equation:

$$\begin{aligned} & lnC(u^*,p^1) - lnC(u^*,p^0) \\ &= (1/2)\Sigma_{i=1}^N [\partial lnC(u^*,p^0)/\partial lnp_i + \partial lnC(u^*,p^1)/\partial lnp_i][lnp_i^1 - lnp_i^0] \\ &= (1/2)\Sigma_{i=1}^N [a_i + \Sigma_{k=1}^N a_{ik} lnp_k^0 + b_i lnu^* + a_i + \Sigma_{k=1}^N a_{ik} lnp_k^1 + b_i lnu^*][lnp_i^1 - lnp_i^0] \text{ differentiating (77) at the points } (u^*,p^0) \text{ and } (u^*,p^1) \\ &= (1/2)\Sigma_{i=1}^N [a_i + \Sigma_{k=1}^N a_{ik} lnp_k^0 + b_i ln[u^0u^1]^{1/2} + a_i + \Sigma_{k=1}^N a_{ik} lnp_k^1 + b_i ln[u^0u^1]^{1/2}][lnp_i^1 - lnp_i^0] \text{ using definition (86) for } u^* \\ &= (1/2)\Sigma_{i=1}^N [a_i + \Sigma_{k=1}^N a_{ik} lnp_k^0 + b_i\, lnu^0 + a_i + \mathring{a}_{k=1}^N a_{ik}\, lnp_k^1 + b_i lnu^1][lnp_i^1 - lnp_i^0] \text{ rearranging terms} \end{aligned} \tag{87}$$

[60] To prove that (75) and (76) are true, use definition (74) and substitute into the left-hand sides of (75) and (76). Then calculate the partial derivatives of the quadratic function defined by (74) and substitute these derivatives into the right-hand side of (75) and (76).

[61] The consumer's cost function was defined by (1).

[62] Christensen, Jorgenson, and Lau (1971) (1975) introduced this function into the economics literature.

[63] It can also be shown that if $b_0 = 1$ and $b_i = 0$ for $i = 1, \ldots, N$ and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; that is, with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences. Note that we also assume that utility u is scaled so that u is always positive.

[64] We need to assume that (u^0,p^0) and (u^1,p^1) belong to the region of prices S^* where the translog $C(u,p)$ satisfies the regularity conditions that a cost function must satisfy. If we think of $C(u,p)$ as an approximation to an arbitrary differentiable cost function, then because of the flexibility property of the translog cost function, it is not a problem to assume that (u^0, p^0) belongs to S^*, but if the vector (u^1,p^1) is not close to (u^0, p^0), then (u^1,p^1) may not belong to the regularity region so that equation (83) for $t = 1$ may not hold and hence equation (87) may not be valid.

[65] See (18).

$= (1/2)\Sigma_{i=1}^{N} [\partial lnC(u^0,p^0)/\partial lnp_i + \partial lnC(u^1,p^1)/\partial lnp_i][lnp_i^1 - lnp_i^0]$
differentiating (77) at the points (u^0,p^0) and (u^1,p^1)
$= (1/2)\Sigma_{i=1}^{N} [s_i^0 + s_i^1][lnp_i^1 - lnp_i^0]$ using equations (85).

The last equation in (87) can be recognized as the logarithm of the Törnqvist[66] Theil (1967) index number formula P_T defined in Chapter 4. Hence, exponentiating both sides of (87) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level u^* and the observable Törnqvist Theil index P_T:[67]

$$C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1). \quad (88)$$

Since the translog cost function which appears on the left-hand side of (88) is a flexible functional form, the Törnqvist Theil price index P_T is also a *superlative index*. Note that it is not necessary to assume homothetic preferences to derive this result.

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left-hand side of equation (88) can be *exactly* estimated by an *observable* index number formula, but the key to this mystery is the assumption of cost-minimizing behavior and the quadratic identity (76) along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's Lemma, (18). In fact, all of the exact index number results derived in this section and the previous section can be derived using transformations of the quadratic identity along with Shephard's Lemma (or Wold's identity (15)).[68] Fortunately, for most empirical applications, assuming that the consumer has (transformed) quadratic preferences will be an adequate assumption, so the results presented in this section and the previous section are quite useful to index number practitioners who are willing to adopt the economic approach to index number theory. Essentially, the economic approach to index number theory provides a strong justification for the use of the Fisher price index P_F, the Törnqvist Theil price index P_T, the implicit quadratic mean of order r price indices P^{r*} defined by (57) (when $r = 1$, this index is the Walsh price index P_W), and the quadratic mean of order r price indices P^r defined by (69), provided that r is a number that is small in magnitude.

8. The Numerical Approximation Properties of Superlative Indices

In the previous section, we have exhibited two families of superlative price and quantity indices, Q^r and P^{r*} defined by (54) and (57), and P^r and Q^{r*} defined by (69) and (72) for each $r \neq 0$. The Fisher index P_F was a special case of P^r with $r = 2$ and the Walsh index P_W was a special case of P^{r*} with $r = 1$. Another superlative index was the Törnqvist Theil index P_T. A natural question to ask at this point is: how different will these indices be? It is possible to show that all of the price indices P^r approximate each other to the second order around any point where the price vectors p^0 and p^1 are equal and where the quantity vectors q^0 and q^1 are equal; that is, we have the following equalities if the first- and second-order partial derivatives are evaluated at $p^0 = p^1 = p >> 0_N$ and $q^0 = q^1 = q >> 0_N$ for any $r \neq 0$:[69]

$$P_F(p^0,p^1,q^0,q^1) = P_T(p^0,p^1,q^0,q^1) = P_W(p^0,p^1,q^0,q^1)$$
$$= P^r(p^0,p^1,q^0,q^1) = P^{r*}(p^0,p^1,q^0,q^1); \quad (89)$$
$$\nabla P_F(p^0,p^1,q^0,q^1) = \nabla P_T(p^0,p^1,q^0,q^1)$$
$$= \nabla P_W(p^0,p^1,q^0,q^1) = \nabla P^r(p^0,p^1,q^0,q^1)$$
$$= \nabla P^{r*}(p^0,p^1,q^0,q^1); \quad (90)$$
$$\nabla^2 P_F(p^0,p^1,q^0,q^1) = \nabla^2 P_T(p^0,p^1,q^0,q^1)$$
$$= \nabla^2 P_W(p^0,p^1,q^0,q^1) = \nabla^2 P^r(p^0,p^1,q^0,q^1)$$
$$= \nabla^2 P^{r*}(p^0,p^1,q^0,q^1) . \quad (91)$$

The vector of first-order partial derivatives of the function of 4N variables $P_F(p^0,p^1,q^0,q^1)$ is the vector of dimension 4N denoted by $\nabla P_F(p^0,p^1,q^0,q^1)$ and the matrix of second-order partial derivatives of $P_F(p^0,p^1,q^0,q^1)$ is a 4N by 4N matrix denoted by $\nabla^2 P_F(p^0,p^1,q^0,q^1)$, and so on. A similar set of equalities holds for the companion quantity indices that match up to P_F, P_T, P_W, P^r, and P^{r*} using the product test, $Q(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/p^0 \cdot q^0 P(p^0,p^1,q^0,q^1)$. The implication of these equalities is that if prices and quantities do not change much over the two periods being compared, then all of the previous price indices will give much the same answer.

For empirical comparisons of some of the aforementioned indices, see Diewert (1978, 894–95) and Hill (2006). Hill (2006) showed that the second-order approximation property of the mean of order r indices breaks down as r approaches plus or minus infinity. However, in most empirical applications, we generally choose r equal to 2 (the Fisher case) or 1 (the Walsh case) or 0 (the Törnqvist Theil case). For these cases, the resulting indices generally approximate each other very closely.[70]

It turns out that the Laspeyres and Paasche price indices approximate each other (and superlative indices like the Fisher index) to the first order around an equal price and quantity point *but not to the second order*; that is, we have the following equalities if the first-order partial derivatives are evaluated at $p^0 = p^1 = p >> 0_N$ and $q^0 = q^1 = q >> 0_N$:

$$P_F(p^0,p^1,q^0,q^1) = P_L(p^0,p^1,q^0,q^1)$$
$$= P_P(p^0,p^1,q^0,q^1); \quad (92)$$
$$\nabla P_F(p^0,p^1,q^0,q^1) = \nabla P_L(p^0,p^1,q^0,q^1)$$
$$= \nabla P_P(p^0,p^1,q^0,q^1). \quad (93)$$

[66] See Törnqvist and Törnqvist (1937).
[67] This result was obtained by Diewert (1976, 122).
[68] See Diewert (2002). However, when applying Wold's Identity or Shephard's Lemma to observed price and quantity data, we need the assumption of optimizing behavior on the part of the consumer, and we need the observed data to be in the regions of regularity for the utility function or cost function that we are working with.

[69] The proof is a straightforward differentiation exercise; see Diewert (1978, 889). In fact, the equalities in (89)–(91) are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and m > 0.
[70] The approximations will be close if we are using annual time series data where price and quantity changes are generally smooth. However, if we are making international comparisons or using panel data or using sub-annual time series data, then the approximations may not be close.

Up to this point, we have considered four different approaches to index number theory:

- Fixed basket approaches and averages of baskets;
- Test approaches to index number theory;
- Stochastic or descriptive statistics approaches to index number theory; and
- Economic approaches.

The first approach led to the Fisher and Walsh indices as being "best," the second approach led to the Fisher and Törnqvist Theil indices as being "best," the third approach led to the Törnqvist Theil index as being "best," and the economic approach led to the Fisher, Walsh, and Törnqvist Theil indices as being the "best" indices. Thus, P_F, P_W, and P_T keep emerging as "best' indices. The results in this section tell us that if prices and quantities do not change that much going from the first period to the second period, then all three of these indices will give us more or less the same answer.

9. The Cobb–Douglas Price Index

Suppose that the consumer's utility function for all $q \geq 0_N$ is defined as follows:

$$f(q) \equiv \alpha_0 \prod_{n=1}^{N} q_n^{\alpha_n} \tag{94}$$

where $\alpha_n > 0$ for $n = 0,1, \ldots, N$ and in addition satisfy the following constraint:

$$\Sigma_{n=1}^{N} \alpha_n = 1. \tag{95}$$

This is the Cobb–Douglas functional form.[71] It can be seen that $f(q)$ defined by (94) is linearly homogeneous. It is also positive, concave, and increasing over the set of strictly positive quantity vectors.

Let the consumer's preferences be represented by $f(q)$ and suppose that the commodity price vector $p >> 0_N$ is given. The *consumer's unit cost minimization problem* is defined as follows:

$$\min_q \{p \cdot q : f(q) \geq 1; q \geq 0_N\} \equiv c(p). \tag{96}$$

Proposition 11: The solution to the unit cost minimization problem defined by (96) when $f(q)$ is the Cobb–Douglas utility function defined by (94) and (95) is the *Cobb–Douglas unit cost function defined* as follows for $p >> 0_N$:

$$c(p) \equiv \kappa \prod_{n=1}^{N} (p_n)^{\alpha_n}; \ \kappa \equiv [\alpha_0 \prod_{n=1}^{N} \alpha_n^{\alpha_n}]^{-1}. \tag{97}$$

See the Annex for a proof.

It can be seen that the Cobb–Douglas unit cost function has more or less the same functional form as the Cobb–Douglas utility function: P replaces q when we move from the utility function to the unit cost function.

Let $p^t >> 0_N$ for $t = 0,1$. Suppose the consumer has Cobb–Douglas preferences and faces the prices p^t in period t for $t = 0,1$. The observed period t quantity vector is $q^t >> 0_N$. Assume that the consumer minimizes the cost of achieving the utility level $u^t \equiv f(q^t)$ for each period. Then the components of $q^t \equiv [q_1^t, \ldots, q_N^t]$ must satisfy the following equations obtained using (97) and Shephard's Lemma:

$$\begin{aligned} q_n^t &= [\partial c(p^t)/\partial p_n] f(q^t); \ n = 1, \ldots, N; \ t = 0,1 \\ &= \alpha_n c(p^t)[p_n^t]^{-1} f(q^t). \end{aligned} \tag{98}$$

By multiplying both sides of equation n in period t by p_n^t, we obtain the following equations:

$$p_n^t q_n^t = \alpha_n c(p^t) f(q^t); \ n = 1, \ldots, N; \ t = 0,1. \tag{99}$$

Summing equations (99) for each period t gives us the following equations, making use of $\Sigma_{n=1}^{N} \alpha_n = 1$:

$$p^t \cdot q^t = c(p^t) f(q^t); \ t = 0,1. \tag{100}$$

Using equations (99) and (100), we see that the following equations hold:

$$\begin{aligned} s_n^t \equiv p_n^t q_n^t / p^t \cdot q^t &= \alpha_n c(p^t) f(q^t) / c(p^t) f(q^t) \\ &= \alpha_n; \ n = 1, \ldots, N; \ t = 0,1. \end{aligned} \tag{101}$$

Equations (101) are important: They tell us that a utility-maximizing consumer who has Cobb–Douglas preferences will have expenditure shares on each commodity that will remain constant across all time periods. This assumption is unlikely to be satisfied in practice. Nevertheless, equations (101) lead to an exact Konüs true cost of living index, as will be seen later.

Since Cobb–Douglas preferences are homothetic, the true cost of living index going from period 0 to 1 is $c(p^1)/c(p^0)$, where $c(p)$ is defined by (97). Thus, we have the following *exact index number formula* for a Cobb–Douglas consumer:

$$\begin{aligned} c(p^1)/c(p^0) &= \kappa \prod_{n=1}^{N} (p_n^1)^{\alpha_n} / \kappa \prod_{n=1}^{N} (p_n^0)^{\alpha_n} \\ &= \prod_{n=1}^{N} (p_n^1/p_n^0)^{\alpha_n} \\ &= \prod_{n=1}^{N} (p_n^1/p_n^0)^{s_n^0} \text{ using (101) for } t = 0 \\ &\equiv P_{KB}(p^0,p^1,q^0,q^1), \end{aligned} \tag{102}$$

where $P_{KB}(p^0,p^1,q^0,q^1)$ is the Konus–Byushgens or Cobb–Douglas price index. This formula is useful for price statisticians: The price index for a current period can be evaluated using only the prices p_n^0 and expenditure shares s_n^0 for a past period 0 and prices p_n^1 for the current period 1.

We turn now to a functional form for the utility function that is more flexible than the Cobb–Douglas utility function but is still not completely flexible.

[71] This functional form was used as a production function for the case $N = 2$ by Cobb and Douglas (1928). It was also used by Knut Wicksell as a production function much earlier in 1916; see Olsson (1971). This functional form was first used as a utility function for the N commodity case in Section 8 of Konüs and Byushgens (1926). Our algebra in this section was more or less worked out by Konüs and Byushgens. In particular, these authors realized that the assumption of Cobb–Douglas preferences implied that commodity expenditure shares must be constant over time. See also Pollak (1971) (1983) for his analysis of Cobb–Douglas preferences, which is followed in the present section.

10. Constant Elasticity of Substitution (CES) Preferences

It is useful to introduce a family of functions that calculate an *average* of N positive numbers, $x \equiv [x_1, \ldots, x_N]$. Assume that the number r is not equal to 0 and the positive weights α_n sum to 1 so that $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$ satisfies conditions (95). Define the *weighted mean of order* r of the N components of the x vector as follows:[72]

$$M_r(x) \equiv [\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{1/r}. \qquad (103)$$

The functional form defined by (103) occurs frequently in the economics literature. If $r = 1$, then $M_r(x)$ equals $\alpha \cdot x$, a linear function of x. As r tends to plus infinity, $M_r(x)$ tends to $\max_n \{x_n: n = 1, \ldots, N\}$. As r tends to minus infinity, $M_r(x)$ tends to $\min_n \{x_n: n = 1, \ldots, N\}$. As r tends to 0, $M_r(x)$ tends to the Cobb–Douglas functional form, which is the weighted geometric mean, $\prod_{n=1}^N (x_n)^{\alpha_n}$. It is readily verified that $M_r(\lambda x) = \lambda M_r(x)$ for all $\lambda > 0$ and $x >> 0_N$. If we multiply $M_r(x)$ by a constant, then we obtain the CES (*Constant Elasticity of Substitution*) functional form popularized by Arrow et al. (1961) in the context of production theory (where x is an input vector and $\alpha_0 M_r(x)$ is the output produced by the input vector x). This functional form is also widely used as a utility function, and it is also used extensively when measures of income inequality are constructed.[73] We note that the function $M_r(x)$ is flexible if $r \neq 0$ and $N = 2$. It is not flexible if $N > 2$.

For future reference, the first- and second-order partial derivatives of $M_r(x)$ for $x >> 0_N$ are as follows:

$$\begin{aligned}\partial M_r(x)/\partial x_i &= (1/r)[\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-1}\alpha_i\, r\, x_i^{\,r-1} \\ &= [\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-1}\alpha_i\, x_i^{\,r-1};\ i = 1, \ldots, N.\end{aligned} \qquad (104)$$

Differentiating (104) again with respect to x_i yields the following second-order partial derivatives for $i = 1, \ldots, N$:

$$\begin{aligned}\partial^2 M_r(x)/\partial x_i^2 &= [(1/r) - 1][\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-2}\alpha_i\, r x_i^{\,r-1}\alpha_i\, x_i^{\,r-1} \\ &\quad + [\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-1}\alpha_i\,(r-1)x_i^{\,r-2};\ i = 1, \ldots, N \\ &= [r-1][\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-2}\{[\Sigma_{n=1}^N \alpha_n x_n^{\,r}]\alpha_i\, x_i^{\,r-2} - \alpha_i^2\, x_i^{\,2r-2}\}.\end{aligned} \qquad (105)$$

Differentiating (104) with respect to x_k for $k \neq i$ yields the following:

$$\begin{aligned}\partial^2 M_r(x)/\partial x_i \partial x_k &= [(1/r) - 1][\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-2}\alpha_k\, r x_k^{\,r-1}\alpha_i\, x_i^{\,r-1};\ k \neq i \\ &= (1-r)[\Sigma_{n=1}^N \alpha_n x_n^{\,r}]^{(1/r)-2}\,\alpha_i\alpha_k x_i^{\,r-1} x_k^{\,r-1}.\end{aligned} \qquad (106)$$

It can be shown if $r \leq 1$, then the matrix of second-order partial derivatives of $M_r(x)$, $\nabla^2 M_r(x)$, is a negative semidefinite matrix for all $x >> 0_N$, and this property in turn implies that $M_r(x)$ is a concave function over the set of positive x vectors.[74] Hence, $M_r(q)$ is a suitable functional form for a utility function, and $M_r(p)$ is a suitable functional form for a unit cost function if $r \leq 1$. These functions satisfy the required regularity conditions over the entire positive orthant. For future reference, the derivatives defined by (104)–(106) can be used in order to establish the following equalities:

$$\begin{aligned}&M_r(x)[\partial^2 M_r(x)/\partial x_i \partial x_k]/[\partial M_r(x)/\partial x_i][\partial M_r(x)/\partial x_k] \\ &= (1-r);\ x >> 0_N;\ 1 \leq i \neq k \leq N.\end{aligned} \qquad (107)$$

Suppose that the unit cost function has the following CES functional form for $r \leq 1$:

$$\begin{aligned}c(p) &\equiv \alpha_0 [\Sigma_{n=1}^N \alpha_n (p_n)^r]^{1/r}\ \textit{if}\ r \neq 0; \\ &\equiv \alpha_0 \prod_{n=1}^N p_n^{\alpha_n}\ \textit{if}\ r = 0,\end{aligned} \qquad (108)$$

where $\alpha_0 > 0$ and $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$ satisfies conditions (95).

Under the assumption of cost-minimizing behavior on the part of the consumer in period t, Shephard's Lemma, (18), tells us that the observed period t consumption of commodity i, q_i^t will be equal to $u^t \partial c(p^t)/\partial p_i$, where $\partial c(p^t)/\partial p_i$ is the first-order partial derivative of the unit cost function with respect to the ith commodity price evaluated at the period t prices and $u^t = f(q^t)$ is the aggregate (unobservable) level of period t utility. Using the CES unit cost function defined by (108) and assuming that $r \neq 0$, the following equations are obtained that express the components of the consumer's observed consumption vector q^t in terms of the period t prices p^t facing the consumer and either the period t utility level for the consumer u^t or the observed period t expenditure for the consumer, $e^t \equiv p^t \cdot q^t$:

$$\begin{aligned}q_i^{\,t} &= u^t\alpha_0 [\Sigma_{n=1}^N \alpha_n (p_n^{\,t})^{\,r}]^{(1/r)-1}\alpha_i\,(p_i^{\,t})^{r-1}; \\ &\quad t = 0,1;\ i = 1, \ldots, N \\ &= u^t c(p^t)\,\alpha_i (p_i^{\,t})^{r-1}/\Sigma_{n=1}^N \alpha_n (p_n^{\,t})^r \\ &= e^t\alpha_i (p_i^{\,t})^{r-1}/\Sigma_{n=1}^N a_n (p_n^{\,t})^r\end{aligned} \qquad (109)$$

where the last equation follows from the fact that the observed period t expenditure, e^t, is equal to $p^t \cdot q^t$, which in turn is equal to $u^t c(p^t)$. The last set of equations in (109) could be used to estimate the unknown parameters r and α that appear in definition (108).[75]

Equations (109) can be rewritten as

$$\begin{aligned}s_i^{\,t} &\equiv p_i^t q_i^t/p^t \cdot q^t = p_i^t q_i^t/u^t c(p^t) = \alpha_i (p_i^{\,t})^r/\Sigma_{n=1}^N \alpha_n (p_n^{\,t})^{\,r}; \\ &\quad t = 0,1;\ i = 1, \ldots, N.\end{aligned} \qquad (110)$$

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[72] Hardy, Littlewood, and Polyá (1934, 12–14) refer to this family of means or averages as elementary weighted mean values and study their properties in great detail. $M_r(x)$ has the following properties, where $x >> 0_N$: (i) $M_r(\lambda 1_N) = \lambda$ for any $\lambda > 0$; (ii) $\nabla M_r(x) >> 0_N$ so that $M_r(x)$ is increasing in x; (iii) $\min \{x_n; n = 1, \ldots, N\} \leq M_r(x) \leq \max \{x_n; n = 1, \ldots, N\}$; and (iv) $M_r(\lambda x) = \lambda M_r(x)$. Thus, $M_r(x)$ is a homogeneous mean. See Diewert (1993b) for materials on mean functions and their application to economics.

[73] See Diewert (1993b).

[74] The definition of $M_r(x)$ can be extended to the set $x \geq 0_N$; see Hardy, Littlewood, and Polyá (1934).

[75] Note that the parameter α_0 cannot be identified using observable data. This makes sense since the scale of utility cannot be observed, and so some arbitrary decision will have to be made in order to determine the utility scale. Usually, we normalize period 0 utility u^0 (which is equal to the period 0 volume level Q^0) to equal period 0 observed expenditure $e^0 = p^0 \cdot q^0$. This normalization determines the units of measurement for utility.

Equations (110) give observed expenditure shares s^t as functions of consumer prices p^t and the unknown parameters r and $\alpha_1, \ldots, \alpha_N$. These equations could also be used as estimating equations for the unknown parameters in an econometric model.[76]

Recall the definition of the consumer's cost function (1), which we repeat here for convenience for some positive level of utility u, given that the consumer is facing the positive vector of consumer prices $p >> 0_N$:

$$C(u,p) \equiv \min_q \{p \cdot q : f(q) \geq u; q^3\, 0_N\}. \quad (111)$$

If the cost function $C(u,p)$ is differentiable with respect to the components of the commodity price vector p, then Shephard's Lemma (18) applies and the consumer's system of commodity demand functions as functions of the chosen utility level u and the commodity price vector p, $q(u,p)$, is equal to the vector of first-order partial derivatives of the cost or expenditure function with respect to the components of p:

$$q(u,p) = \nabla_p C(u,p), \quad (112)$$

where $q(u,p) \equiv [q_1(u,p), \ldots, q_N(u,p)]$. The demand functions $q_n(u,p) \equiv \partial C(u,p)/\partial p_n$ are known as *Hicksian*[77] *demand functions*. We expect that the demand for commodity i will increase if the price of commodity k (not equal to i) increases if i and k are substitutes in consumption; that is, we expect $\partial q_i(u,p)/\partial p_k > 0$ if i and k are substitutes. Note that $q_i(u,p) = \partial C(u,p)/\partial p_i$ so that $\partial q_i(u,p)/\partial p_k = \partial^2 C(u,p)/\partial p_i \partial p_k$. A unit-free measure of the magnitude of the response of the demand for product i due to an increase in the price of product k is the *elasticity function* $e_{ik}(u,p)$, which is defined as

$$\begin{aligned}\varepsilon_{ik}(u,p) &\equiv [\partial q_i(u,p)/\partial p_k][p_k/q_i(u,p)]\\ &= p_k[\partial^2 C(u,p)/\partial p_i \partial p_k]/\partial C(u,p)/\partial p_i\\ &= p_k[\partial^2 C(u,p)/\partial p_i \partial p_k]/q_i(u,p). \end{aligned} \quad (113)$$

Allen (1938, 504) and Uzawa (1962)[78] suggested the following measure of the response of product i to a change in the price of product k:

$$\sigma_{ik}(u,p) \equiv C(u,p)[\partial^2 C(u,p)/\partial p_i \partial p_k]/[\partial C(u,p)/\partial p_i][\partial C(u,p)/\partial p_k]; \; i \neq k. \quad (114)$$

The Allen–Uzawa measure is also independent of the units of measurement, but their measure converted the response into a measure that applied to both i and k. The bigger are $\varepsilon_{ik}(u,p)$ and $\sigma_{ik}(u,p)$, the more *substitutable* are the products.[79] Thus, $\sigma_{ik}(u,p)$ defined by (114) is called the *elasticity of substitution* between products i and k. Note that $\sigma_{ik}(u,p) = \sigma_{ki}(u,p)$.

Define the cost function to be $C(u,p) = uc(p)$, where $c(p)$ is defined by (108). Using equations (104)–(106), which apply to the CES functional form, it can be verified that $\sigma_{ik}(u,p)$ defined by (114) simplify to the following equations:

$$\sigma_{ik}(u,p) = 1 - r \equiv \sigma \geq 0; \; i \neq k, \quad (115)$$

where we have defined $\sigma \equiv 1 - r$. Thus, if the consumer has CES preferences, which are dual to the unit cost function defined by (108), then *the elasticity of substitution between every pair of products is equal to the same number*, $1 - r \equiv \sigma$, which is equal to or greater than 0, since in order for $c(p)$ to be a concave function, we require $r \leq 1$. Thus, the CES functional form rules out complementary commodities and is far from being able to model arbitrary preferences if $N \geq 3$. However, the CES functional form is still a useful one, since it can model both Leontief and Cobb–Douglas preferences: Simply set $r = 1$ or $r = 0$ to get these two special cases.[80]

We turn now to the problem of finding exact index number formulae for preferences that are defined by the CES unit cost function. Our first exact index number formula requires an estimate for the elasticity of substitution, $\sigma \equiv 1 - r$. For $\sigma \neq 1$, define the *Lloyd* (1975) *Moulton* (1996) *price index* $P_{LM}(p^0,p^1,q^0,q^1)$ for $p^t >> 0_N$ and $q^t >> 0_N$, $t = 0,1$ as follows:

$$P_{LM}(p^0,p^1,q^0,q^1) \equiv [\Sigma_{i=1}^N s_i^0 (p_i^1/p_i^0)^{1-\sigma}]^{1/(1-\sigma)}; \; \sigma \neq 1, \quad (116)$$

where s_i^0 is the period 0 expenditure share of commodity i as usual. Substitute equations (110) for s_i^0 into the right-hand side of (116) and we obtain the following equation:

$$\begin{aligned} P_{LM}(p^0,p^1,q^0,q^1) &\equiv [\Sigma_{i=1}^N s_i^0 (p_i^1/p_i^0)^r]^{1/r} \text{ letting } r = 1 - \sigma \quad (117)\\ &= [\Sigma_{i=1}^N \{\alpha_i (p_i^0)^r / \Sigma_{n=1}^N \alpha_n (p_n^0)^r\} (p_i^1/p_i^0)^r]^{1/r} \text{ using (110)}\\ &= [\Sigma_{i=1}^N \alpha_i (p_i^1)^r / \Sigma_{n=1}^N \alpha_n (p_n^0)^r]^{1/r}\\ &= [\Sigma_{i=1}^N \alpha_i (p_i^1)^r]^{1/r} / [\Sigma_{n=1}^N \alpha_n (p_n^0)^r]^{1/r}\\ &= \alpha_0[\Sigma_{i=1}^N \alpha_i (p_i^1)^r]^{1/r} / \alpha_0[\Sigma_{n=1}^N \alpha_n (p_n^0)^r]^{1/r}\\ &= c(p^1)/c(p^0) \text{ using definition (108) for } p = p^0 \text{ and } p = p^1. \end{aligned}$$

Equation (117) shows that the Lloyd–Moulton index number formula P_{LM} is *exact* for CES preferences. Lloyd (1975) and Moulton (1996) independently derived this result but it was

[76] Note that the right-hand sides of equations (110) are homogeneous of degree 0 in the α_n parameters. However, the normalization $\Sigma_{n=1}^N \alpha_n = 1$ can be used to solve for say $\alpha_N = 1 - \Sigma_{n=1}^{N-1} \alpha_n$, which will allow all of the parameters to be identified. Because $\Sigma_{n=1}^N s_n^t = 1$ for $t = 0,1$, the N share equations for period t are not statistically independent, and hence one of these estimating equations should be dropped from the estimation procedure. Similar adjustments need to be made to the system of estimating equations defined by (109) since the equations $p^t \cdot q^t = e^t$ hold without error for $t = 0,1$.

[77] See Hicks (1946, 311–31).

[78] They suggested their measure in the context of production theory, but it carries over to Hicksian demand functions.

[79] Hicks (1946) showed that if $N = 2$, then $\varepsilon_{12}(u,p)$ and $\sigma_{12}(u,p)$ must be nonnegative. However, if $N \geq 3$, then $\varepsilon_{12}(u,p)$ and $\sigma_{12}(u,p)$ could be negative. In this case, products 1 and 2 are called *complements*.

[80] It can also model linear preferences by letting r tend to plus infinity.

Moulton who appreciated the significance of the formula (117) for statistical agency purposes. Note that in order to evaluate (116) numerically, it is necessary to have information on

- Base period expenditure shares s_i^0;
- The price relatives p_i^1/p_i^0 between the base period and the current period; and
- An estimate of the elasticity of substitution between the commodities in the aggregate, s.

The first two pieces of information are the standard information sets that statistical agencies use to evaluate the Laspeyres price index P_L (note that P_{LM} reduces to P_L if $\sigma = 0$ or $r = 1$). Hence, if the statistical agency is able to estimate the elasticity of substitution σ based on past experience,[81] *then the Lloyd–Moulton price index can be evaluated using essentially the same information set that is used in order to evaluate the traditional Laspeyres index*. Moreover, the resulting consumer price index may be free of *substitution bias* to a reasonable degree of approximation.[82] Of course, the practical problem with implementing this methodology is that estimates of the elasticity of substitution parameter *s* are bound to be somewhat uncertain and hence the resulting Lloyd–Moulton index may be subject to charges that it is not *objective* or *reproducible*. The statistical agency will have to balance the benefits of reducing substitution bias with these possible costs.

Our second index number formula that is exact for CES preferences does not require an estimate for the elasticity of substitution. Suppose that $p^t >> 0_N$ and $q^t >> 0_N$ for $t = 0,1$. The logarithm of the *Sato* (1976)–*vartia* (1976) price index $P_{SV}(p^0,p^1,q^0,q^1)$ is defined by the following equation:[83]

$$lnP_{SV}(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N w_n ln(p_n^1/p_n^0), \qquad (118)$$

where the weights w_n are calculated in two stages. The first stage set of weights is defined as $w_n^* \equiv (s_n^1 - s_n^0)/(lns_n^1 - lns_n^0)$ for $n = 1,\ldots,N$ provided that $s_n^1 \neq s_n^0$. If $s_n^1 = s_n^0$, then define $w_n^* \equiv s_n^1 = s_n^0$. The second stage weights are defined as $w_n \equiv w_n^*/\Sigma_{i=1}^N w_i^*$ for $n = 1,\ldots,N$. Note that in order for the logarithm of $P_{SV}(p^0,p^1,q^0,q^1)$ to be well defined, we require that $s_n^0 > 0$, $s_n^1 > 0$, $p_n^0 > 0$ and $p_n^1 > 0$ for all $n = 1,\ldots,N$; that is, all prices and quantities must be positive for all products in both periods.

Proposition 12: The Sato vartia price index is exact for CES preferences; that is, if the consumer faces the positive prices $p^0 >> 0_N$ and $p^1 >> 0_N$ in periods 0 and 1, has CES preferences dual to the unit cost function defined by (108) and maximizes utility in periods 0 and 1 with solution vectors $q^0 >> 0_N$ and $q^1 >> 0_N$ for periods 0 and 1, then we have

$$\begin{aligned} P_{SV}(p^0,p^1,q^0,q^1) &= c(p^1)/c(p^0) \qquad (119)\\ &= [\Sigma_{n=1}^N \alpha_n (p_n^1)^r]^{1/r}/[\Sigma_{n=1}^N \alpha_n (p_n^0)^r]^{1/r} \text{ if } r \neq 0\\ &= \Pi_{n=1}^N (p_n^1)^{\alpha_n}/\Pi_{n=1}^N (p_n^0)^{\alpha_n} \text{ if } r = 0. \end{aligned}$$

For a proof of this proposition, see the Annex.

We noted earlier that equations (109) and (110) could be used to estimate the unknown parameters $r = 1 - \sigma$ and the a_n that characterize the CES unit cost function defined by (108). However, if our focus is on obtaining an estimate for r (or equivalently for the elasticity of substitution $\sigma \equiv 1 - r$), then much simpler systems of estimating equations can be derived, as will be indicated later.

Recall the system of share equations defined by (110) that express cost-minimizing expenditure shares as functions of prices. Extend this system of equations to period T, take logarithms of both sides of the resulting equations and add error terms η_i^t.[84] The following system of estimating equations is obtained:

$$lns_n^t = ln\alpha_n + rln(p_n^t) - ln[\Sigma_{i=1}^N a_i(p_i^t)^r] + \eta_n^t;\quad t = 0,1,\ldots,T;\ n = 1,\ldots,N. \qquad (120)$$

Now difference the *logarithms* of the s_n^t with respect to time; that is, define Δs_n^t as follows:

$$\Delta s_n^t \equiv ln(s_n^t) - ln(s_n^{t-1});\ n = 1,\ldots,N;\quad t = 1,\ldots,T. \qquad (121)$$

Now pick product N as the numeraire product[85] and the difference Δs_n^t with respect to product N, giving rise to the following *double differenced log variable*, ds_n^t:

$$\begin{aligned} ds_n^t &\equiv \Delta s_n^t - \Delta s_N^t;\ n = 1,\ldots,N-1;\quad t = 1,\ldots,T \qquad (122)\\ &= lns_n^t - lns_n^{t-1} - [lns_N^t - lns_N^{t-1}]. \end{aligned}$$

Define the *double differenced log price variables* in a similar manner:

$$\begin{aligned} dp_n^t &\equiv \Delta p_n^t - \Delta p_N^t;\ n = 1,\ldots,N-1;\quad t = 1,\ldots,T. \qquad (123)\\ &= lnp_n^t - lnp_n^{t-1} - [lnp_N^t - lnp_N^{t-1}]. \end{aligned}$$

[81] For the first application of this methodology (in the context of the consumer price index), see Shapiro and Wilcox (1997, 121–23). They calculated superlative Törnqvist indices for the United States for the years 1986–1995 and then calculated the Lloyd Moulton CES index for the same period using various values of σ. They then chose the value of σ (which was 0.7) that caused the CES index to most closely approximate the Törnqvist index. Essentially, the same methodology was used by Alterman, Diewert, and Feenstra (1999) in their study of US import and export price indices. Alternative methods for estimating σ will be considered later.

[82] What is a "reasonable" degree of approximation depends on the context. Assuming that consumers have CES preferences may not be a reasonable assumption in the context of forming index numbers over an aggregate that contains heterogeneous products where elasticities of demand for the various products are very different. However, if the aggregate consists of fairly similar products, then it may be adequate to assume a CES approximation to preferences over these relatively homogeneous products, which are presumably highly substitutable with each other.

[83] Sato and vartia both defined P_{SV} independently. Sato (1976, 225) showed that P_{SV} was exact for CES preferences; that is, Sato provided a (somewhat sketchy) proof of a dual version of Proposition 12.

[84] A standard specification for the error terms η_n^t is that they have 0 means, a constant variance-covariance matrix for the error terms belonging to the same period t and zero covariances across time periods.

[85] In practice, the numeraire commodity should be chosen to be a commodity that has a small predicted variance and a large expenditure share. However, it is not straightforward to find such a commodity. Later, an alternative method of estimation will be suggested that avoids the need to choose a numeraire commodity.

Finally, define the *double differenced error variables* $d\eta_n{}^t$ as follows:

$$d\eta_n{}^t \equiv \eta_n{}^t - \eta_n{}^{t-1} - \eta_N{}^t + \eta_N{}^{t-1} \equiv \varepsilon_n{}^t;\ n = 1, \ldots, N-1;\ t = 1, \ldots, T. \quad (124)$$

Using definitions (121)–(124) and equations (120), it can be verified that the double differenced log shares $ds_n{}^t$ satisfy the following system of $(N-1)T$ estimating equations under our assumptions:

$$ds_n{}^t = rdp_n{}^t + \varepsilon_n{}^t;\ n = 1, \ldots, N-1;\ t = 1, \ldots, T; \quad (125)$$

where the new residuals $\varepsilon_n{}^t$ have means 0 and a constant covariance matrix with 0 covariances for observations that are separated by two or more time periods. Thus, we have a system of linear estimating equations with only one unknown parameter across all equations, namely the parameter r. This is almost the simplest possible system of estimating equations that one could imagine. This *double differencing method* for estimating the elasticity of substitution when consumers have CES preferences was suggested by Feenstra (1994).[86]

Instead of starting with the share equations (110), one could start with the demand functions defined by equations (109). Extend this system of equations to period T, take logarithms of both sides of the resulting equations and add error terms $\eta_i{}^t$. The following system of estimating equations is obtained:[87]

$$lnq_n{}^t = lne^t + ln\alpha_n + (r-1)lnp_n{}^t - ln[\Sigma_{i=1}{}^N \alpha_i(p_i{}^t)^r] + \eta_n{}^t;\ t = 0,1, \ldots, T;\ n = 1, \ldots, N. \quad (126)$$

Define $\Delta q_n{}^t$ as the *time difference for the logarithms of quantities* as follows:

$$\Delta q_n{}^t \equiv lnq_n{}^t - lnq_n{}^{t-1};\ n = 1, \ldots, N;\ t = 1,, \ldots, T. \quad (127)$$

Again, pick product N as the numeraire product and the difference $\Delta q_n{}^t$ with respect to product N, giving rise to the following *double differenced log variable*, $dq_n{}^t$:

$$dq_n{}^t \equiv \Delta q_n{}^t - \Delta q_N{}^t;\ n = 1, \ldots, N-1;\ t = 1, \ldots, T = lnq_n{}^t - lnq_n{}^{t-1} - (lnq_N{}^t - lnq_N{}^{t-1}). \quad (128)$$

Define the double differenced price and error variables, $dp_n{}^t$ and $d\eta_n{}^t$ by (123) and (124). Using these definitions and (126)–(128), it is straightforward to show that the following equations will hold:

$$dq_n{}^t = (r-1)dp_n{}^t + d\eta_n{}^t;\ n = 1, \ldots, N-1;\ t = 1, \ldots, T = -\sigma dp_n{}^t + e_n{}^t \quad (129)$$

since the elasticity of substitution σ is equal to $1 - r$. Again, this is an extremely simple system of estimating equations.

The double differenced share equation specification given by (125) and the double difference quantity demanded specification given by (129) both depend on the choice of the numeraire commodity. This dependence could be a problem for statistical agencies in that the estimation procedure is not completely reproducible: Different statisticians could pick different commodities as the numeraire commodity and get different estimates for the elasticity of substitution. It is possible to modify the double difference method so that it is not dependent on the choice of a numeraire commodity.

For each time period t, define the geometric average of the $s_n{}^t$ and $p_n{}^t$ as $s_\bullet{}^t$ and $p_\bullet{}^t$ respectively for $t = 0,1, \ldots, T$. For each time period t, define the arithmetic average of the $\eta_n{}^t$ as $\eta_\bullet{}^t$ for $t = 0,1, \ldots, T$. Finally define the geometric average of α_n as $\alpha_\bullet$. Recall equations (120). For each time period t, take the arithmetic average of both sides of equations (120) for all N observations in period t. The following equations are the result of these operations:

$$lns_\bullet{}^t = ln\alpha_\bullet + rlnp_\bullet{}^t - ln[\Sigma_{i=1}{}^N \alpha_i(p_i{}^t)^r] + \eta_\bullet{}^t;\ t = 0,1, \ldots, T. \quad (130)$$

Now the difference $lns_n{}^t$ defined by equations (120) with the $lns_\bullet{}^t$ defined by (130); that is, essentially we are choosing the *average* (over commodities n) *log shares* in place of the log shares of a numeraire commodity. The following equations are obtained:

$$lns_n{}^t - lns_\bullet{}^t = ln\alpha_n - ln\alpha_\bullet + rlnp_n{}^t - rlnp_\bullet{}^t + \eta_n{}^t - \eta_\bullet{}^t;\ t = 0,1, \ldots, T;\ n = 1, \ldots, N. \quad (131)$$

Now taking the difference between the variables $lns_n{}^t$ and $lns_\bullet{}^t$ with respect to time, we obtain the following estimating equations:[88]

$$lns_n{}^t - lns_n{}^{t-1} - lns_\bullet{}^t + lns_\bullet{}^{t-1} = r[lnp_n{}^t - lnp_n{}^{t-1} - lnp_\bullet{}^t + lnp_\bullet{}^{t-1}] + \varepsilon_n{}^t;\ t = 1, \ldots, T;\ n = 1, \ldots, N, \quad (132)$$

where $\varepsilon_n{}^t \equiv \eta_n{}^t - \eta_n{}^{t-1} - \eta_\bullet{}^t + \eta_\bullet{}^{t-1}$. Again, we have a system of estimating equations that is linear in the single parameter r.

Instead of starting with the share equations (110), one could start with the demand functions defined by equations (109). Extend this system of equations to period T, take logarithms of both sides of the resulting equations, and add the error terms $\eta_n{}^t$. The system of estimating equations defined by (126) is obtained. Now define the geometric average of $q_n{}^t$ for period t as $q_\bullet{}^t$ for $t = 0,1, \ldots, T$. Apply the same definitions

[86] For an empirical application of the method, see Diewert and Feenstra (2019). The variance-covariance structure is not quite classical due to the correlation of residuals between adjacent time periods. Another problem with the method is that the estimates for r will generally depend on the choice of the numeraire commodity.

[87] The error terms in (126) are different from the error terms in (120). For convenience, we did not introduce a new notation for the error terms in (126).

[88] Note that for each t, we have the following equalities: $0 = \Sigma_{n=1}{}^N [lns_n{}^t - lns_\bullet{}^t] = \Sigma_{n=1}{}^N [ln\alpha_n - ln\alpha_\bullet] = S_{n=1}{}^N [lnp_n{}^t - lnp_\bullet{}^t] = \Sigma_{n=1}{}^N [\eta_n{}^t - \eta_\bullet{}^t]$. Thus, for each t, the N equations for $lns_n{}^t - lns_\bullet{}^t$ for $n = 1, \ldots, N$ are linearly dependent, and hence any one of these N equations can be dropped. If the commodity N equations are dropped, then we use equations (132) as estimating equations only for $t = 1, \ldots, T$ and $n = 1, \ldots, N-1$. Under an appropriate stochastic specification, the estimate for r will not depend on which equation is dropped.

and techniques that led to equations (130)–(132) and we obtain the following system of estimating equations:

$$\begin{aligned} &lnq_n^{\,t} - lnq_n^{\,t-1} - lnq_\cdot^{\,t} + lnq_\cdot^{\,t-1} = (r-1)[lnp_n^{\,t} - lnp_n^{\,t-1} \\ &\quad - lnp_\cdot^{\,t} + lnp_\cdot^{\,t-1}] + \varepsilon_n^{\,t};\ t = 1,\ldots,T;\ n = 1,\ldots,N \\ &\quad = -\sigma[lnp_n^{\,t} - lnp_n^{\,t-1} - lnp_\cdot^{\,t} + lnp_\cdot^{\,t-1}] + \varepsilon_n^{\,t}, \end{aligned} \tag{133}$$

where $\varepsilon_n^{\,t} \equiv \eta_n^{\,t} - \eta_n^{\,t-1} - \eta_\cdot^{\,t} + \eta_\cdot^{\,t-1}$. Equations (133) are a system of estimating equations that is linear in the single parameter σ, which is the elasticity of substitution between all pairs of commodities.

It turns out that estimating the consumer's utility function directly (rather than estimating the dual unit cost function) is advantageous when estimates of reservation prices[89] for products that are not available are required. In the case of CES preferences, this advantage is not immediately apparent since the CES reservation prices are automatically set equal to infinity. But it turns out that there may be advantages in estimating the CES utility function directly because of econometric considerations as we shall see later. Thus, we will conclude this section by deriving the consumer demand functions that are consistent with the maximization of a CES utility function.

We now assume that the utility function $f(q)$ is defined directly as the following *CES utility function*:

$$f(q_1,\ldots,q_N) \equiv [\Sigma_{n=1}^{N} \beta_n q_n^{\,s}]^{1/s}, \tag{134}$$

where the parameters β_n are positive and sum to 1 and s is a parameter that satisfies $s \le 1$ (so that $f(q)$ will be a concave function of q and $s \ne 0$ (in which case $f(q)$ is a Cobb–Douglas utility function). Thus, $f(q)$ is a mean of order s.

Suppose $s = 1$ and let $p >> 0_N$. In this case, the utility function is the *linear function* $f(q) \equiv \beta\cdot q = \Sigma_{n=1}^{N} \beta_n q_n$. The cost minimization problem that defines the dual unit cost function for this case is the following linear programming problem:

$$\min_q \{p\cdot q: \beta\cdot q \ge 1; q \ge 0_N\} = \min_n \{p_n/\beta_n: n = 1,\ldots,N\} \equiv c(p). \tag{135}$$

The unit cost function $c(p)$ defined by the solution to (135) is not differentiable but it is a well-defined continuous, increasing, linearly homogeneous, and concave function of p. If the minimum over n is unique and attained for say product 1, then the solution q^* to (135) is unique and is given by $q_1^* = 1/\beta_1$ with $q_i^* = 0$ for $i = 2,3,\ldots,N$. If p happens to equal $\lambda\beta$ for some $\lambda > 0$, then the solution set of q vectors that solve (135) is the set $\{q: \beta\cdot q = 1/\lambda; q \ge 0_N\}$.

We turn our attention to the case where s satisfies $s < 1$ and $s \ne 0$. Suppose $p \equiv (p_1,\ldots,p_N) >> 0_N$. Ignoring the constraints $q \ge 0_N$ for the moment, the first-order necessary (and sufficient) conditions that can be used to solve the unit cost minimization problem defined by (96) when $f(q)$ is defined by (134) are the following conditions:

$$p_n = \lambda^*\beta_n q_n^{\,s-1};\ n = 1,\ldots,N; \tag{136}$$

$$1 = [\Sigma_{n=1}^{N} \beta_n q_n^{\,s}]^{1/s}. \tag{137}$$

Equations (136) are equivalent to the equations $q_n = [p_n/\lambda^*\beta_n]^{1/(s-1)}$ for $n = 1,\ldots,N$. Substitute these equations into equation (137) and obtain the following equations: $1 = \Sigma_{n=1}^{N} \beta_n q_n^{\,s} = \Sigma_{n=1}^{N} \beta_n[p_n/\lambda^*\beta_n]^{1/(s-1)}$. This equation can be solved for $\lambda^* = [S_{n=1}^{N} \beta_n^{\,1/(1-s)} p_n^{\,s/(s-1)}]^{(s-1)/s}$.[90] The optimal q_n^* are defined as $q_n^* = [p_n/\lambda^*\beta_n]^{1/(s-1)}$ for $n = 1,\ldots,N$. All of the equations in (136) and (137) will be satisfied by this λ^*, q^* solution.

Evaluate (136) and (137) at the optimal solution. Multiply both sides of equation n in (136) by q_n^* and sum the resulting N equations. This leads to the following equations:

$$\begin{aligned} c(p) &\equiv \Sigma_{n=1}^{N} p_n q_n^{\,*} \\ &= \lambda^*\Sigma_{n=1}^{N} \beta_n (q_n^{\,*})^s \\ &= \lambda^* \text{ using (137)} \\ &= [\Sigma_{n=1}^{N} \beta_n^{\,1/(1-s)} p_n^{\,s/(s-1)}]^{(s-1)/s}. \end{aligned} \tag{138}$$

It can be seen that the dual unit cost function $c(p)$ that corresponds to the CES utility function defined by (134) for $s \ne 0$ and $s \ne 1$ is proportional to a mean of order r in prices, where $r = s/(s-1)$. Thus, if $f(q)$ is the CES utility function defined by (134), then the corresponding elasticity of substitution is

$$\sigma = 1 - r = 1 - [s/(s-1)] = -1/(s-1) = 1/(1-s). \tag{139}$$

As s approaches 1 from below, σ approaches plus infinity. For $s = 0$, $\sigma = 1$ and we have Cobb–Douglas preferences. As s approaches minus infinity, s approaches 0 as a limiting case.[91]

In order to derive the system of inverse demand functions that correspond to the CES utility function $f(q)$ defined by (134), we make use of Wold's Identity, equations (17) which were $p^t/p^t\cdot q^t = \nabla f(q^t)/f(q^t)$. Upon defining the consumer's period t "income" as $e^t \equiv p^t\cdot q^t$, the CES system of *inverse demand functions* for period t is given by

$$p^t = e^t\nabla f(q^t)/f(q^t);\ t = 0,1,\ldots,T. \tag{140}$$

The system of inverse demand functions gives the period t price vector p^t as the prices that are consistent with q^t solving the consumer's period t utility maximization problem given that the consumer has "income" e^t to spend on the N commodities in the aggregate.

If consumers maximize the CES utility function defined by (134) when they face the positive period t price vector p^t and have $e^t > 0$ to spend on the N commodities, the utility-maximizing q^t will satisfy equations (140). If we evaluate equations (140) using the period t price and quantity data for periods $t = 0,1,\ldots,T$ and add error terms, we obtain the following system of equations:

$$p_n^{\,t} = e^t\beta_n(q_n^{\,t})^{s-1}/\Sigma_{i=1}^{N} \beta_i(q_i^{\,t})^s;\ t = 0,1,\ldots,T;\ n = 1,\ldots,N. \tag{141}$$

[89] Reservation prices will be discussed in Section 14 and in Chapter 8.

[90] Note that we require $s \ne 0$ and $s \ne 1$ in order for l^* to be well defined.

[91] The limiting case is the case of Leontief preferences.

Equations (141) is the consumer's *system of inverse demand functions*. Equations (141) are the counterparts to the consumer's system of (ordinary) demand functions defined earlier by equations (109). It can be seen that the expressions $\beta_n(q_n^t)^s/\Sigma_{i=1}^N \beta_i(q_i^t)^s$ are homogeneous of degree 0 in the parameters $\beta_1, \ldots, \beta_N$, so a normalization of these parameters is required for the identification of the β_n parameters. The normalization $\Sigma_{n=1}^N \beta_n = 1$ can be replaced by an equivalent normalization such as $\beta_N = 1$.[92]

Equations (141) are perfectly symmetric with equations (109), which gave us estimating equations for the system of ordinary consumer demand functions for a utility-maximizing consumer with CES preferences, except that the roles of prices and quantities have been interchanged. Equations (109) gave consumer demands q_n^t as functions of e^t and p^t, whereas equations (140) give us prices p_n^t that are consistent with utility maximization for CES preferences that are consistent with total expenditure equal to e^t and the utility-maximizing quantity vector q^t. If equations (109) fit the given price and quantity data perfectly, then equations (141) will also fit the given price and quantity data perfectly as well (and vice versa). However, with data that do not fit the CES model exactly, the two methods for fitting a CES utility function will usually not agree. We will discuss the problem of deciding which model is "best" later.

Equations (141) can be converted into a system of share equations where the period t expenditure shares s_n^t are functions of e^t and q^t: multiply both sides of equation n for period t by q_n^t/e^t to obtain the expenditure share s_n^t on the left-hand side of the resulting equation. The following system of share equations is obtained:

$$s_n^t = \beta_n(q_n^t)^s/\Sigma_{i=1}^N \beta_i(q_i^t)^s;\ t = 0,1, \ldots, T;\ n = 1, \ldots, N. \quad (142)$$

Equations (141) and (142) can be used as systems of estimating equations. Later, we will consider some alternative systems of estimating equations.

Take the logarithm of s_n^t defined by (142) and add the error term η_n^t to the right-hand side of equation n in period t. We obtain the following system of estimating equations:

$$lns_n^t = ln\beta_n + slnq_n^t - ln[\Sigma_{i=1}^N \beta_i(q_i^t)^s] + \eta_n^t;\ t = 0,1, \ldots, T;\ n = 1, \ldots, N \quad (143)$$

Equations (142) (which express the logarithm of shares as functions of quantities) are the counterparts to equations (120) (which expressed the logarithms of shares as functions of prices instead of quantities).

We can now repeat the differencing methods explained earlier when the task was to find estimates for the elasticity of substitution using the CES unit cost function as the starting point. Thus, the counterparts to the estimating equations defined earlier by (125) and (129) are now the following *double differenced systems of inverse demand estimating equations*:[93]

$$ds_n^t = sdq_n^t + e_n^t;\ t = 1, \ldots, T;\ n = 1, \ldots, N-1; \quad (144)$$

$$dp_n^t = (s-1)dq_n^t + \varepsilon_n^t;\ t = 1, \ldots, T;\ n = 1, \ldots, N-1 \quad (145)$$
$$= -\sigma^{-1}dq_n^t + e_n^t \text{ using (139).}$$

As was the case with the systems of estimating equations defined by (125) and (129), the systems of estimating equations defined by (144) and (145) will depend on the choice of a numeraire commodity. To avoid this problem, we can adapt the analysis surrounding equations (130)–(132) to the present situation. Thus, for each time period t, define the geometric average of s_n^t and q_n^t as $s_\bullet^t$ and $q_\bullet^t$, respectively for $t = 0,1, \ldots, T$. For each time period t, define the arithmetic average of η_n^t in equations (143) as $\eta_\bullet^t$ for $t = 0,1, \ldots, T$. Finally define the geometric average of β_n as $\beta_\bullet$. For each time period t, take the arithmetic average of both sides of equations (143) for all N observations in period t. The following equations are the result of these operations:

$$lns_\bullet^t = ln\beta_\bullet + slnq_\bullet^t - ln[\Sigma_{i=1}^N \beta_i(q_i^t)^s] + \eta_\bullet^t;\ t = 0,1, \ldots, T. \quad (146)$$

Take the Difference between lns_n^t defined by equations (143) and $lns_\bullet^t$ defined by (146). The following equations are obtained:

$$lns_n^t - lns_\bullet^t = ln\beta_n - ln\beta_\bullet + slnq_n^t - slnq_\bullet^t + \eta_n^t - \eta_\bullet^t;\ t = 0,1, \ldots, T;\ n = 1, \ldots, N. \quad (147)$$

Now taking the difference between the variables lns_n^t and $lns_\bullet^t$ with respect to time, we obtain the following estimating equations:[94]

$$lns_n^t - lns_n^{t-1} - lns_\bullet^t + lns_\bullet^{t-1} = s[lnq_n^t - lnq_n^{t-1} - lnq_\bullet^t + lnq_\bullet^{t-1}] + \varepsilon_n^t;\ t = 1, \ldots, T;\ n = 1, \ldots, N, \quad (148)$$

where $\varepsilon_n^t \equiv \eta_n^t - \eta_n^{t-1} - \eta_\bullet^t + \eta_\bullet^{t-1}$. Equations (148) are a system of estimating equations that is linear in the single parameter s.

Instead of starting with the share equations (142), one could start with the inverse demand functions defined by equations (141). Take logarithms of both sides of these equations and add error terms η_n^t. The following system of estimating equations is obtained:

$$lnp_n^t = ln\beta_n + (s-1)lnq_n^t - ln[\Sigma_{i=1}^N \beta_i(q_i^t)^s] + \eta_n^t;\ t = 0,1, \ldots, T;\ n = 1, \ldots, N \quad (149)$$

[92]The normalization on the β_n determines the units of measurement for utility. Since $S_{n=1}^N s_n^t = 1$ for $t = 0,1, \ldots, T$, the error terms will satisfy the constraints $\Sigma_{n=1}^N \eta_n^t = 0$ $t = 0,1, \ldots, T$, and thus the error terms pertaining to each time period cannot be distributed independently and so the estimating equations for one commodity n should be dropped from equations (141).

[93]We require that $s \neq 0$ and $s \neq 1$.

[94]Note that for each t, we have the following equalities: $0 = \Sigma_{n=1}^N [lns_n^t - lns_\bullet^t] = \Sigma_{n=1}^N [ln\beta_n - ln\beta_\bullet] = S_{n=1}^N [lnq_n^t - lnq_\bullet^t] = \Sigma_{n=1}^N [\eta_n^t - \eta_\bullet^t]$. Thus, for each t, the N equations for $lns_n^t - lns_\bullet^t$ for $n = 1, \ldots, N$ are linearly dependent, and hence any one of these N equations can be dropped. If the commodity N equations are dropped, then we use equations (148) as estimating equations only for $t = 1, \ldots, T$ and $n = 1, \ldots, N-1$. Under an appropriate stochastic specification, the estimate for s will not depend on which equation is dropped.

Define the geometric average of $p_n^{\,t}$ for period t as $p_\bullet^{\,t}$ for $t = 0,1, \ldots, T$. By applying the same definitions and techniques that led to equations (146)–(148), we obtain the following system of estimating equations:

$$\begin{aligned} lnp_n^{\,t} - lnp_n^{\,t-1} - lnp_\bullet^{\,t} + lnp_\bullet^{\,t-1} &= (s-1)[lnq_n^{\,t} \\ &\quad - lnq_n^{\,t-1} - lnq_\bullet^{\,t} + lnq_\bullet^{\,t-1}] + \varepsilon_n^{\,t};\ t = 1, \ldots, T; \\ &\quad n = 1, \ldots, N \qquad (150) \\ &= -\sigma^{-1}[lnq_n^{\,t} - lnq_n^{\,t-1} - lnq_\bullet^{\,t} + lnq_\bullet^{\,t-1}] + \varepsilon_n^{\,t} \text{ using (139),} \end{aligned}$$

where $\varepsilon_n^{\,t} \equiv \eta_n^{\,t} - \eta_n^{\,t-1} - \eta_\bullet^{\,t} + \eta_\bullet^{\,t-1}$. Equations (150) are a system of estimating equations that is linear in the single parameter σ^{-1}, which is the reciprocal of the elasticity of substitution between all pairs of commodities.[95]

From the previous materials, it can be seen that there is a bewildering array of alternative methods for estimating CES preferences. We have considered in some detail 12 methods. Equations (109) and (141) estimate the consumer's CES demand system and inverse demand system. In equations (109), quantities q^t are functions of total expenditure e^t and prices p^t for each period t; in equations (141), prices p^t are functions of e^t and q^t. The parameters of the CES unit cost function $c(p)$ defined by (108) are estimated using equations (109), while the parameters of the CES utility function $f(q)$ defined by (134) are estimated using equations (141). Equations (109) and (141) are our preferred specifications. The problem with the econometric specifications that involve shares as dependent variables is that shares by their very nature combine price and quantity information and so errors in either prices or quantities will show up in shares. Thus, a model involving shares as dependent variables could fit the data very well but the approximation errors or deviations from the CES exact model for either prices or quantities could offset each other in the shares. The model fits for (109) and (141) could be much worse than the model fits for any model involving shares. Thus, the share models will tend to give us an incomplete view of how well the CES model describes the actual data. Put another way, the use of shares does not make use of all available information on prices and quantities, whereas the models based on using (109) and (141) as estimating equations do use all available information, and thus these models are the best at showing us how well the CES model approximates reality. This observation means that the unit cost estimation models that use shares (110), (125), and (132) are less preferred than (109), and the utility function estimation models that use shares (142), (144), and (148) are less preferred than (141). Similarly, differencing the data throws out information on exactly how good the underlying CES model is at approximating the underlying price and quantity data. Thus, the unit cost function models using differences (125), (129), (132), and (133) are less preferred than (109), and the utility function models using differences (144), (145), (148), and (150) are less preferred than (141). If we reject share models and differenced models, we are left with the models defined by (109) and (141).

How can a choice be made between (109) and (141)? The answer to this question depends on the purpose for estimating CES preferences. If we want to predict q^t given e^t and p^t, then the model defined by equations (109) is the best choice. If we want to predict p^t given e^t and q^t, then (141) is the best choice. If the goal is to get a good estimate for the elasticity of substitution to use in the Lloyd–Moulton formula, then run both (109) and (141) and choose the model with the best fit. As was mentioned earlier, if (109) fits the data perfectly, then so will (141) (as well as the other 10 models under consideration). However, in reality, neither (109) nor (141) will fit the data perfectly. If the underlying preference function is approximately equal to a linear utility function (so that the products are highly substitutable), then the model defined by (141) will fit the data better than the model defined by (109). On the other hand, if preferences are close to being of the no substitution variable so that the unit cost function is almost linear, then the model defined by (109) will fit the data better than the model defined by (141). Choosing between (109) and (141) based on how well the two models fit the data seems to be a sensible strategy.[96]

11. The Allen Quantity Index

Make the same general assumptions on the utility function $f(q)$ that we made at the beginning of Section 2 so that $f(q)$ is a nonnegative, increasing, continuous, and concave function of q defined for $q \geq 0_N$.[97] Let $C(f(q),p)$ be the consumer's cost function that is dual to the aggregator function $f(q)$.[98] Let $p^t >> 0_N$ be the vector of observed prices that the consumer faces in period t for $t = 0,1$. Let $q^t >> 0_N$ be the vector of observed consumer choices for periods $t = 0,1$. We assume cost-minimizing behavior on the part of the consumer in periods 0 and 1 so that the following equations are satisfied:

$$\begin{aligned} C(f(q^t),p^t) &\equiv \min_q \{p^t \cdot q : f(q) \geq f(q^t); \\ &\quad q \geq 0_N\} = p^t \cdot q^t;\ t = 0,1. \qquad (151) \end{aligned}$$

The *Allen* (1949) *family of quantity indices*, $Q_A(q^0,q^1,p)$, is defined for an arbitrary positive reference price vector $p >> 0_N$ as follows:

$$Q_A(q^0,q^1,p) \equiv C(f(q^1),p)/C(f(q^0),p). \qquad (152)$$

The basic idea of the Allen quantity index dates back to Hicks (1941–1942), who observed that if the price vector p were held fixed and the quantity vector q is free to vary, then $C(f(q),p)$ is a perfectly valid cardinal measure of utility.[99]

As was the case with the true cost of living, the Allen definition simplifies considerably if the utility function happens

95 As usual, we need to drop the estimating equations for one of the N commodities since the error terms in (150) are not statistically independent because the data for each period satisfies the linear constraint $p^t \cdot q^t = e^t$ for $t = 0,1, \ldots, T$.

96 A possible disadvantage of using (109) or (141) to estimate σ is that nonlinear regression techniques have to be used in the estimation procedure. Thus, an attractive alternative would be to use either (133) or (150) to estimate s. These models are linear in the single unknown parameter.

97 In this section, we no longer assume that $f(q)$ is linearly homogeneous. The results in this section were established by Diewert (2009, 239–41).

98 Recall definition (1) in Section 2.

99 Samuelson (1974) called this a money metric measure of utility.

to be linearly homogeneous. In this case, for any $p >> 0_N$, (152) simplifies to

$$Q_A(q^0,q^1,p) = f(q^1)C(1,p)/f(q^0)C(1,p) = f(q^1)/f(q^0). \quad (153)$$

However, in the general case where the consumer has nonhomothetic preferences, we do not obtain the nice simplification given by (153).

As usual, it is useful to specialize the general definition of the Allen quantity index and let the reference price vector equal either the period 0 price vector p^0 or the period 1 price vector p^1:

$$Q_A(q^0,q^1,p^0) \equiv C(f(q^1),p^0)/C(f(q^0),p^0); \quad (154)$$
$$Q_A(q^0,q^1,p^1) \equiv C(f(q^1),p^1)/C(f(q^0),p^1). \quad (155)$$

Index number formulae that are exact for either of the theoretical indices defined by (154) and (155) do not seem to exist, at least for the case of nonhomothetic preferences that can be represented by a flexible functional form. However, we can find an index number formula that is exactly equal to the geometric mean of the Allen indices defined by (154) and (155), where the underlying preferences are represented by a flexible functional form. Before demonstrating this result, we need the following proposition:

Proposition 13: Let x and y be N- and M-dimensional vectors, respectively and let F^0 and F^1 be two general quadratic functions defined as follows:

$$F^0(x,y) \equiv a_0^{\,0} + a^0Tx + b^0Ty + (½)x^TA^0x + (½)y^TB^0y + x^TC^0y;\ A^0T = A^0;\ B^0T = B^0; \quad (156)$$
$$F^1(x,y) \equiv a_0^{\,1} + a^1Tx + b^1Ty + (½)x^TA^1x + (½)y^TB^1y + x^TC^1y;\ A^1T = A^1;\ B^1T = B^1, \quad (157)$$

where $a_0^{\,i}$ are the scalar parameters, a^i and b^i are the parameter vectors and A^i, B^i, and C^i are the parameter matrices for $i = 0,1$. Note that A^i and B^i are symmetric matrices. If $A^0 = A^1$, then the following equation holds for all x^0, x^1, y^0, and y^1:[100]

$$F^0(x^1,y^0) - F^0(x^0,y^0) + F^1(x^1,y^1) - F^1(x^0,y^1) = [\nabla_xF^0(x^0,y^0) + \nabla_xF^1(x^1,y^1)]\cdot[x^1 - x^0]. \quad (158)$$

Straightforward differentiation of the functions defined by (156) and (157) and substitution into (158) proves the proposition. The identity (158) is a *generalized quadratic identity*. This identity will prove to be useful, as will be seen later.

Suppose that the consumer's preferences can be represented by the general translog cost function, $C(u,p)$ defined by (77), with the restrictions (78)–(81). Shephard's Lemma implies that the period t expenditure shares, $s_n^{\,t}$, will satisfy the following equations:[101]

$$s_n^{\,t} = \partial lnC(u^t,p^t)/\partial lnp_n = a_n^{\,t} + b_n^{\,t}lnu^t + \Sigma_{i=1}^{N} a_{ni}^{\,t}lnp_i^{\,t};\ t = 0,1, \quad (159)$$

where $u^t \equiv f(q^t)$ for $t = 0,1$. Note that $\ln C(u,p)$ is quadratic in the variables $x_1 \equiv lnp_1, \ldots, x_N \equiv lnp_N$ and $y_1 \equiv \ln u$. Thus, we will be able to apply the generalized quadratic identity to $\ln C(u,p)$.

Recall that in Section 2, the Konüs–Laspeyres cost of living index was defined by $P_K(p^0,p^1,q^0) \equiv C[f(q^0),p^1]/C[f(q^0),p^0]$ and the Konüs–Paasche cost of living index was defined by $P_K(p^0,p^1,q^1) \equiv C[f(q^1),p^1]/C[f(q^1),p^0]$. Assuming that $C(u,p)$ is the translog cost function, we can obtain an exact formula for the geometric mean of $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$. The logarithm of this geometric mean is

$$\begin{aligned} &ln\{[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}\} \\ &= (½)lnP_K(p^0,p^1,q^0) + (½)lnP_K(p^0,p^1,q^1) \quad (160)\\ &= (½)ln[C(f(q^0),p^1)/C(f(q^0),p^0)] + (½)ln[C(f(q^1),p^1)/C(f(q^1),p^0)] \text{ using definitions (3) and (4)}\\ &= (½)ln[C(u^0,p^1)/C(u^0,p^0)] + (½)ln[C(u^1,p^1)/C(u^1,p^0)]\\ &= (½)\{lnC(u^0,p^1) - lnC(u^0,p^0) + lnC(u^1,p^1) - lnC(u^1,p^0)\}\\ &= (½)\Sigma_{n=1}^{N} \{[\partial lnC(u^0,p^0)/\partial lnp_n] + [\partial lnC(u^1,p^1)/\partial lnp_n]\}[lnp_n^{\,1} - lnp_n^{\,0}]\\ &\text{using (158) with } F^0(x,y) = F^1(x,y) \equiv lnC(y,x) \text{ with } y \equiv lnu \text{ and } x_n \equiv lnp_n \text{ for } n = 1,\ldots,N\\ &= (½)\Sigma_{n=1}^{N} [s_n^{\,0} + s_n^{\,1}][lnp_n^{\,1} - lnp_n^{\,0}] \text{ using (159)}\\ &= lnP_T(p^0,p^1,q^0,q^1), \end{aligned}$$

where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist Theil index number formula P_T defined in Chapter 4. The exact index number formula given by (160) is different from our earlier exact index number formula for P_T which was given by (88). The earlier result was $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$, where u^* was the geometric mean of u^0 and u^1. Our new result is

$$P_T(p^0,p^1,q^0,q^1) = [C(f(q^0),p^1)/C(f(q^0),p^0)]^{1/2} [C(f(q^1),p^1)/C(f(q^1),p^0)]^{1/2}. \quad (161)$$

Thus, P_T is also equal to the geometric mean of $C(f(q^0),p^1)/C(f(q^0),p^0)$ and $C(f(q^1),p^1)/C(f(q^1),p^0)$.

The implicit quantity index $Q_{T^*}(p^0,p^1,q^0,q^1)$ that corresponds to the Törnqvist–Theil price index P_T is defined as the value ratio, $p^1\cdot q^1/ p^0\cdot q^0$, divided by P_T. Thus, we have

$$\begin{aligned} Q_{T^*}(p^0,p^1,q^0,q^1) &\equiv [p^1\cdot q^1/p^0\cdot q^0]/P_T(p^0,p^1,q^0,q^1) \quad (162)\\ &= [C(f(q^1),p^1)/C(f(q^0),p^0)]/P_T(p^0,p^1,q^0,q^1) \text{ using (151)}\\ &= [C(f(q^1),p^1)/C(f(q^0),p^0)]/[\{C(f(q^0),p^1)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^1),p^0)\}]^{1/2} \text{ using (161)}\\ &= [\{C(f(q^1),p^0)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^0),p^1)\}]^{1/2}\\ &= [Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}, \end{aligned}$$

where the last equality follows from definitions (154) and (155). Thus, the observable implicit Törnqvist–Theil quantity index, $Q_{T^*}(p^0,p^1,q^0,q^1)$, is exactly equal to the geometric

[100] Balk (1998, 225–26) established this result using the Translog Lemma in Caves, Christensen, and Diewert (1982, 1412), which is simply a logarithmic version of (158).

[101] We need to assume that the points (u^0, p^0) and (u^1,p^1) are in the regularity region where the translog cost function C(u,p) is well-behaved.

mean of the two Allen quantity indices defined by (154) and (155).

Note that in general, the geometric mean of the two "natural" Allen quantity indices defined by (154) and (155) matches up with the geometric mean of the two "natural" Konüs price indices defined by (3) and (4); that is, using these definitions, we have

$$[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}[Q_A(q^0,q^1,p^0) \\ Q_A(q^0,q^1,p^1)]^{1/2} = C(f(q^1),p^1)/C(f(q^0),p^0) \\ = p^1{\cdot}q^1/p^0{\cdot}q^0. \quad (163)$$

Thus, in general, these two "natural" geometric mean price and quantity indices satisfy the product test. Under our translog assumptions, we have a special case of (163) where $Q_{T*}(p^0,p^1,q^0,q^1)$ is equal to $[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}$ and $P_T(p^0,p^1,q^0,q^1)$ is equal to $[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}$.[102] This result justifies the use of P_T and Q_{T*} even if the consumer does not have homothetic preferences. Hence, it indirectly justifies the use of the Fisher and Walsh price indices if consumers do not have homothetic preferences since these indices will approximate $P_T(p^0,p^1,q^0,q^1)$ to the second order around an equal price and quantity point.

12. Modeling Changes in Tastes

Suppose that the consumer's preference function changes going from period 0 to period 1. What is an appropriate theoretical concept for a price index under these conditions?

Suppose that the consumer's utility function is $f^0(q)$ in period 0 and $f^1(q)$ in period 1. Let $C^0(u,p)$ and $C^1(u,p)$ be the cost functions that correspond to these preferences for periods 0 and 1, respectively. A reasonable strategy under these circumstances is the following one:

- Calculate the Laspeyres–Konüs cost of living index using the preferences of period 0. This is the index $P_K(p^0,p^1,q^0) \equiv C^0(u^0,p^1)/C^0(u^0,p^0)$, where $u^0 = f(q^0)$ and q^0 satisfies $p^0{\cdot}q^0 = C^0(u^0,p^0)$.
- Calculate the Paasche–Konüs cost of living index using the preferences of period 1. This is the index $P_K(p^0,p^1,q^1) \equiv C^1(u^1,p^1)/C^1(u^1,p^0)$, where $u^1 = f(q^1)$ and q^1 satisfies $p^1{\cdot}q^1 = C^1(u^1,p^1)$.
- Take the geometric mean of $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$ as the final measure of price change over the two periods under consideration.

Make the additional assumption that the consumer's preferences can be modeled by translog cost functions in a region of regularity that includes $u^0 > 0$, $p^0 >> 0_N$ and $u^1 > 0$, $p^1 >> 0_N$. In this regularity region, the logarithms of the period t cost functions $C^t(u,p)$ are defined as follows:

$$lnC^0(u,p) \equiv F^0(x,y_1) \equiv a_0^{\,0} + \Sigma_{n=1}^{N} a_n^{\,0}x_n + b_1^{\,0}y_1 \\ + (½)x^TAx + (½)b_{11}^{\,0}(y_1)^2 + \Sigma_{n=1}^{N} c_n^{\,0}x_ny_1; \quad (164)$$

$$lnC^1(u,p) \equiv F^1(x,y_1) \equiv a_0^{\,1} + \Sigma_{n=1}^{N} a_n^{\,1}x_n \\ + b_1^{\,1}y_1 + (½)x^TAx + (½)b_{11}^{\,1}(y_1)^2 + \Sigma_{n=1}^{N} c_n^{\,1}x_ny_1, \quad (165)$$

where $A = A^T$, $x^T \equiv [x_1, \ldots, x_N] \equiv [lnp_1, \ldots, lnp_N]$ and $y_1 \equiv lnu$. Note that the parameters in (164) can be quite different from the parameters in (165) except that we assume that the $N(N + 1)/2$ a_{ik} parameters in the A matrix are the same in (164) and (165). It can be seen that the quadratic functions $F^0(x,y_1)$ and $F^1(x,y_1)$ are special cases of the functions $F^0(x,y)$ and $F^1(x,y)$ defined by (156) and (157) in the previous section. In order for $C^t(u,p)$ to be linearly homogeneous in p, we need to impose the restrictions $\Sigma_{n=1}^{N} a_n^{\,t} = 1$, $A1_N = 0_N$ and $\Sigma_{n=1}^{N} c_n^{\,t} = 0$ on the parameters for $t = 0,1$, where 1_N is a vector of ones of dimension N.

Shephard's Lemma implies that the period t expenditure shares, $s_n^{\,t}$, will satisfy the following equations:

$$s_n^{\,t} = \partial lnC(u^t,p^t)/\partial lnp_n = a_n^{\,t} + c_n^{\,t}lnu^t \\ + \Sigma_{k=1}^{N} a_{nk}lnp_k^{\,t};\ t = 0,1 \quad (166)$$

The logarithm of the geometric mean of $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$ is equal to the following expression:

$$ln\{[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}\} \\ = (½)lnP_K(p^0,p^1,q^0) + (½)lnP_K(p^0,p^1,q^1) \quad (167) \\ = (½)ln[C^0(u^0,p^1)/C^0(u^0,p^0)] + (½)ln[C^1(u^1,p^1)/C^1(u^1,p^0)] \\ = (½)\{lnC^0(u^0,p^1) - lnC^0(u^0,p^0) + lnC^1(u^1,p^1) - lnC^1(u^1,p^0)\} \\ = (½)\Sigma_{n=1}^{N} \{[\partial lnC^0(u^0,p^0)/\partial lnp_n] + [\partial lnC^1(u^1,p^1)/\partial lnp_n]\}[lnp_n^{\,1} - lnp_n^{\,0}] \text{ using (158)} \\ = (½)\Sigma_{n=1}^{N} [s_n^{\,0} + s_n^{\,1}][lnp_n^{\,1} - lnp_n^{\,0}] \text{ using (166)} \\ = lnP_T(p^0,p^1,q^0,q^1),$$

where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist–Theil index number formula P_T defined in Chapter 4 and $u^t \equiv f^t(q^t)$ for $t = 0,1$. Note that (167) implies the following equalities:

$$P_T(p^0,p^1,q^0,q^1) = [P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2} \\ = \{[C^0(u^0,p^1)/C^0(u^0,p^0)][C^1(u^1,p^1)/C^1(u^1,p^0)]\}^{1/2}, \quad (168)$$

where $u^t \equiv f^t(q^t)$ for $t = 0,1$. Thus, at least some forms of taste change can be accommodated by the use of the Törnqvist–Theil price index.

13. Conditional Cost of Living Indices

The models of consumer behavior considered in previous sections all assumed that the consumer maximized a utility function, $f(q)$, subject to a budget constraint of the form $p{\cdot}q = e$, where $e > 0$ is the total amount of "income" or expenditure that the consumer allocates to the purchase of the N goods and services under consideration. However, the utility of the consumer may be affected by other variables in addition to purchases of market goods and services that are represented by $q \equiv [q_1, \ldots, q_N]$. Thus, we now assume that utility is affected by an M dimensional vector of nonmarket *environmental*[103] *or*

[102] See Diewert (2009, 239–41).

[103] This is the terminology used by Pollak (1989, 181) in his model of the conditional cost of living concept.

demographic[104] *variables or public goods*, $z \equiv (z_1, z_2, \ldots, z_M)$. We suppose that the preferences of the household over different combinations of market commodities q and nonmarket variables z can be represented by the continuous utility function $f(q,z)$.[105] For periods $t = 0,1$, it is assumed that the observed household consumption vector $q^t \equiv (q_1^t, \ldots, q_N^t) > 0_N$ is a solution to the following household expenditure minimization problem:

$$\min_q \{p^t \cdot q : f(q,z^t) \geq u^t;\ q \geq 0_N\} \equiv C(p^t,u^t,z^t) = p^t \cdot q^t;\ t = 0,1; \quad (169)$$

where z^t is the environmental vector facing household h in period t, $u^t \equiv f(q^t,z^t)$ is the utility level achieved by household h during period t and C is the *conditional cost or expenditure function* that is dual to the utility function f.[106] Basically, these assumptions mean that the household has *stable preferences* over the same list of market commodities during the two periods under consideration and the household chooses its market consumption vector in the most cost-efficient way during each period, conditional on the environmental vector z^t that it faces during each period t.

With the previous assumptions in mind, the family of Pollak (1975, 142) *conditional cost of living index* between periods 0 and 1, conditional on the utility level u and the nonmarket vector z, is defined as follows:[107]

$$P_{Po}(p^0,p^1,u,z) \equiv C(p^1,u,z)/C(p^0,u,z). \quad (170)$$

In this definition, the household utility level u and the vector of nonmarket or environmental variables z are held constant in the numerator and denominator of the right-hand side of (170). Thus, only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index. Note that if z does not enter the consumer's utility function so that $f(q,z)$ is just $f(q)$, then $C(u,p,z)$ becomes $C(u,p)$ and the Pollak conditional cost of living indices collapses down to the Konüs family of true cost of living indices, $P_K(p^0,p^1,q)$, where $u = f(q)$.

The *Laspeyres–Pollak conditional cost of living index* is defined by (169) when $(u,z) = (u^0,z^0)$. Using (169) for $t = 0$, a feasibility argument establishes the following upper bound to $P_{Po}(p^0,p^1,u^0,z^0)$; that is, we have

$$P_{Po}(p^0,p^1,u^0,z^0) \leq p^1 \cdot q^0/p^0 \cdot q^0 = P_L(p^0,p^1,q^0,q^1), \quad (171)$$

where $P_L(p^0,p^1,q^0,q^1)$ is the ordinary Laspeyres price index for market commodities. The *Paasche–Pollak conditional cost of living index* is defined by (169) when $(u,z) = (u^1,z^1)$. Using (169) for $t = 1$, a feasibility argument establishes the following lower bound to $P_{Po}(p^0,p^1,u^1,z^1)$; that is, we have

$$P_{Po}(p^0,p^1,u^1,z^1) \geq p^1 \cdot q^1/p^0 \cdot q^1 = P_P(p^0,p^1,q^0,q^1), \quad (172)$$

where $P_P(p^0,p^1,q^0,q^1)$ is the ordinary Paasche price index for market commodities.[108]

It is possible to obtain two-sided bounds to a Pollak conditional cost of living index; that is, we have the following generalization of Proposition 1:

Proposition 14: There exists a number λ^* between 0 and 1 such that

$$P_L \leq P_{Po}[p^0,p^1,\lambda^*(q^0,z^0) + (1-\lambda^*)(q^1,z^1)] \leq P_P$$
$$or\ P_P \leq P_P[p^0,p^1,\lambda^*(q^0,z^0) + (1-\lambda^*)(q^1,z^1)] \leq P_L. \quad (173)$$

The proof of Proposition 14 is similar to the proof of Proposition 1; see Diewert (2001) for details.

There is one additional result on conditional cost of living indices that is very useful, and it involves the use of the generalized quadratic identity (158) and a generalized translog functional form for the conditional cost function $C(u,p,z)$. Suppose that the logarithm of the consumer's conditional cost function is defined as follows:

$$lnC(p,u,z) \equiv F(x,y) \equiv a_0 + a^T x + b^T y + (½)x^T A x + (½)y^T B y + x^T C y;\ A^T = A;\ B^T = B, \quad (174)$$

where $x^T \equiv [lnp_1, \ldots, lnp_N]$, $y^T \equiv [lnu, z_1, \ldots, z_M]$, a_0 is a scalar parameter, a and b are parameter vectors, A is an N by N symmetric matrix of parameters, B is an $M+1$ by $M+1$ symmetric matrix of parameters, and C is an N by $M+1$ matrix of parameters. In order to impose linear homogeneity in prices on $C(p,u,z)$, we require that the following restrictions on the parameters hold:

$$a^T 1_N = 1;\ A 1_N = 0_N \text{ and } C^T 1_N = 0_{M+1}. \quad (175)$$

Note that the demographic variables enter the right-hand side of (174) in a linear and quadratic fashion; this allows for the z_m variables to be discrete variables that can take on the value 0.[109] We assume that the period 0 and 1 price vectors, p^0 and p^1, are strictly positive and that $q^t >> 0_N$ solves the period t conditional cost minimization problem defined by (169) for $t = 0,1$. Thus, we have the following equations:

$$p^t \cdot q^t = C(p^t,u^t,z^t);\ t = 0,1. \quad (176)$$

Shephard's Lemma can be applied to these cost minimization problems, since the translog conditional cost function $C(p,u,z)$ defined by (174) is differentiable with respect to the components of p. Thus, we have the following equations:[110]

$$q_n^t = \partial C(p^t,u^t,z^t)/\partial p_n;\ n = 1, \ldots, N;\ t = 0,1 = [C(p^t,u^t,z^t)/p_n^t]\partial lnC(p^t,u^t,z^t)/\partial lnp_n. \quad (177)$$

[104] Caves, Christensen, and Diewert (1982, 1409) used the terms *demographic variables* or *public goods* to describe the vector of conditioning variables z in their generalized model of the Konüs price index or cost of living index. Weather variables could also be included in the z vector.

[105] We initially assume that $f(q,z)$ is jointly continuous in (q,z), increasing in the components of q and concave in the components of q.

[106] Conditional cost functions were first defined by Pollak (1975, 142).

[107] See also Caves, Christensen, and Diewert (1982, 1409).

[108] The bounds (171) and (172) can be found in Caves, Christensen, and Diewert (1982, 1409–10).

[109] Thus, the number of children in a household is a discrete variable that can take on the value 0. If we entered the corresponding variable as z_1 on the right-hand side of (174) as the logarithm of the number of children, the definition of $C(p,u,z)$ would break down.

[110] See equations (83) in Section 7. We require that (p^t,u^t,z^t) be in the regularity set where $C(p,u,z)$ is positive and increasing in the components of p and u and concave in p holding u and z fixed.

Using definition (174), equations (177) can be rearranged to read as follows:

$$s_n^{\,t} = \partial lnC(p^t,u^t,z^t)/\partial lnp_n = a_n + \Sigma_{k=1}^{N} a_{nk} lnp_k^{\,t} + \Sigma_{m=1}^{M} c_{nm} z_m^{\,t};\ n = 1, \ldots, N;\ t = 0,1. \quad (178)$$

Now take the logarithm of the geometric mean of the two conditional indices $P_{Po}(p^0,p^1,u^0,z^0)$ and $P_{Po}(p^0,p^1,u^1,z^1)$. We find that

$$\begin{aligned} & ln\{[P_{Po}(p^0,p^1,u^0,z^0)P_{Po}(p^0,p^1,u^1,z^1)]^{1/2}\} \quad (179) \\ &= (½)[lnC(p^1,u^0,z^0) - lnC(p^0,u^0,z^0) + lnC(p^1,u^1,z^1) - lnC(p^0,u^1,z^1)] \\ &= (½)\Sigma_{n=1}^{N} [(\partial lnC(p^0,u^0,z^0)/\partial lnp_n) + (\partial lnC(p^1,u^1,z^1)/\partial lnp_n)][lnp_n^{\,1} - lnp_n^{\,0}] \text{ using definition (174) and the generalized quadratic identity (158)} \\ &= (½)\Sigma_{n=1}^{N} [s_n^{\,1} + s_n^{\,0}][lnp_n^{\,1} - lnp_n^{\,0}] \text{ using (178)} \\ &= lnP_T(p^0,p^1,q^0,q^1), \end{aligned}$$

where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist–Theil index number formula P_T defined in Chapter 4. Note that (179) implies the following equalities:[111]

$$\begin{aligned} P_T(p^0,p^1,q^0,q^1) &= [P_{Po}(p^0,p^1,u^0,z^0) P_{Po}(p^0,p^1,u^1,z^1)]^{1/2} \quad (180) \\ &= \{[C(p^1,u^0,z^0)/C(p^0,u^0,z^0)][C(p^1,u^1,z^1)/C(p^0,u^1,z^1)]\}^{1/2}. \end{aligned}$$

Thus, the Törnqvist–Theil price index has many useful interpretations.

14. Reservation Prices and New and Disappearing Products

New products appear and old products disappear at substantial annual rates in most economies in the world today. This creates substantial problems for national statistical offices that are responsible for producing CPIs, since traditional index number theory is based on matching prices for identical products over time. Thus up to now, our treatment of the different approaches to index number theory has assumed that the number of consumer goods and services available to the public has remained constant over the two periods being compared. This implicit assumption is not an accurate reflection of reality: In practice, perhaps 1–2 percent of all consumer products appear or disappear each month. The economic approach to index number theory can be helpful in providing a framework for treating this lack of matching problem.

The basic idea for the treatment of new products in a cost of living type price index is as follows. Assume that the consumer has the same preferences over continuing and new and disappearing products over periods 0 and 1. For a product that is not available in one of the two periods under consideration, the quantity consumed is obviously equal to zero units. The corresponding prices for these products that are present in only one of the two periods are missing. It turns out that if we can somehow estimate *reservation prices* for these missing products in the two periods under consideration, then normal index number theory using the economic approach to index number theory can be applied. The reservation price for a missing product is the price that is just high enough to induce purchasers of the product to demand zero units of it. This reservation price approach for the treatment of new goods was developed by Hicks (1940, 114). Hofsten (1952, 95–97) extended the approach of Hicks to cover the case of disappearing goods as well.

In Chapter 8, we will consider several alternative methods that have been suggested in the literature to estimate reservation prices.[112] In the present section, we will use maximum overlap price indices to form approximations to reservation prices, and we will derive some theoretical bias estimates for these approximate reservation prices. A *maximum overlap index*[113] is one that constructs a price index using just the products that are present in the two periods under consideration. Typically, the maximum overlap price index will be biased compared to the "true" cost of living index, which uses reservation prices. This bias in the deflator translates into a corresponding bias in the real output aggregate. We will evaluate this bias in the context of a statistical agency that uses a maximum overlap Törnqvist–Theil price index.[114]

Consider two periods 0 and 1. There are three classes of commodities. Class 1 products are present in both periods with positive prices and quantities for all N products in this group. Denote the period t price and quantity vectors for this group of products as $p_1^{\,t} \equiv [p_{11}^{\,t}, \ldots, p_{1N}^{\,t}] >> 0_N$ and $q_1^{\,t} \equiv [q_{11}^{\,t}, \ldots, q_{1N}^{\,t}] >> 0_N$ for $t = 0,1$.

Class 2 products are the *new* goods and services that are not available in period 0 but are available in period 1. Denote the period 0 price and quantity vectors for this group of K products as $p_2^{\,0*} \equiv [p_{21}^{\,0*}, \ldots, p_{2K}^{\,0*}] >> 0_N$ and $q_2^{\,0} \equiv [q_{11}^{\,0}, \ldots, q_{1K}^{\,0}] = 0_N$. The prices in the vector $p_2^{\,0*}$ are the positive reservation prices that make the demand for these products in period 0 equal to 0. These reservation prices have to be estimated somehow. The period 1 price and quantity vectors for these K products are $p_2^{\,1} \equiv [p_{21}^{\,1}, \ldots, p_{2K}^{\,1}] >> 0_N$ and $q_2^{\,1} \equiv [q_{21}^{\,1}, \ldots, q_{2K}^{\,1}] >> 0_N$, and these vectors are observable.

Class 3 products are the *disappearing* goods and services that were available in period 0 but are not available in period 1. Denote the period 0 price and quantity vectors for this group of M products as $p_3^{\,0} \equiv [p_{31}^{\,0}, \ldots, p_{3M}^{\,0}] >> 0_N$ and $q_3^{\,0}$

[111] This result is a special case of a more general result established by Caves, Christensen, and Diewert (1982, 1410). Their result also allows for taste change between the periods.

[112] These methods include Feenstra's (1994) CES methodology, the Diewert and Feenstra (2019) (2022) methodology that involves the estimation of the preference function that is exact for the Fisher ideal index and methodologies based on experimental economics. See Brynjolfsson et al. (2019) (2021) and Diewert, Fox, and Schreyer (2022) on the experimental approach.

[113] This type of index dates back to Marshall (1887). Keynes (1930, 94) called it the highest common factor method, while Triplett (2004, 18) called it the overlapping link method. See Diewert (1993c, 52–56) for additional material on the early history of the new goods problem.

[114] The material in this section is mostly due to de Haan and Krsinich (2012) (2014). Diewert, Fox, and Schreyer (2017c) extended the de Haan and Krisinich analysis to bias estimates if the Laspeyres, Paasche, or Fisher maximum overlap indices are used in place of the Törnqvist–Theil price index.

$\equiv [q_{31}^0, \ldots, q_{3M}^0] >> 0_N$. The period 1 price and quantity vectors for these M products are $p_3^{1*} \equiv [p_{31}^{1*}, \ldots, p_{3M}^{1*}] >> 0_N$ and $q_3^1 \equiv [q_{31}^1, \ldots, q_{3M}^1] = 0_N$. The prices in the vector p_3^{1*} are the positive reservation prices that make the demand for these products in period 1 equal to 0. Again, these reservation prices have to be estimated somehow.

Define the *true expenditure shares* for product n in Group 1 for periods 0 and 1, s_{1n}^0 and s_{1n}^1, as the following fractions of total expenditure in period 0 or 1:

$$s_{1n}^0 \equiv p_{1n}^0 q_{1n}^0/[p_1^0 \cdot q_1^0 + p_2^{0*} \cdot q_2^0 + p_3^0 \cdot q_3^0]; \quad n = 1, \ldots, N; \tag{181}$$
$$= p_{1n}^0 q_{1n}^0/[p_1^0 \cdot q_1^0 + p_3^0 \cdot q_3^0] \text{ since } q_2^0 = 0_N;$$
$$s_{1n}^1 \equiv p_{1n}^1 q_{1n}^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1 + p_3^{1*} \cdot q_3^1]; \quad n = 1, \ldots, N; \tag{182}$$
$$= p_{1n}^1 q_{1n}^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1] \text{ since } q_3^1 = 0_N.$$

Note that these shares can be calculated using observable data; that is, these shares do not depend on the imputed prices p_2^{0*} and p_3^{1*}.

Define the *true expenditure shares* for product k in Group 2 for periods 0 and 1, s_{2k}^0 and s_{2k}^1, as follows:

$$s_{2k}^0 \equiv p_{2k}^0 q_{2k}^0/[p_1^0 \cdot q_1^0 + p_2^{0*} \cdot q_2^0 + p_3^0 \cdot q_3^0]; \quad k = 1, \ldots, K; \tag{183}$$
$$= p_{2k}^0 q_{2k}^0/[p_1^0 \cdot q_1^0 + p_3^0 \cdot q_3^0] \text{ since } q_2^0 = 0_N;$$
$$= 0; \text{ since } q_{2k}^0 = 0;$$
$$s_{2k}^1 \equiv p_{2k}^1 q_{2k}^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1 + p_3^{1*} \cdot q_3^1]; \quad k = 1, \ldots, K; \tag{184}$$
$$= p_{2k}^1 q_{2k}^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1] \text{ since } q_3^1 = 0_N.$$

Note that these shares can also be calculated using observable data.

Define the *true expenditure shares* for product m in Group 3 for periods 0 and 1, s_{3m}^0 and s_{3m}^1, as follows:

$$s_{3m}^0 \equiv p_{3m}^0 q_{3m}^0/[p_1^0 \cdot q_1^0 + p_2^{0*} \cdot q_2^0 + p_3^0 \cdot q_3^0]; \quad m = 1, \ldots, M; \tag{185}$$
$$= p_{3m}^0 q_{3m}^0/[p_1^0 \cdot q_1^0 + p_3^0 \cdot q_3^0] \text{ since } q_2^0 = 0_N;$$
$$s_{3m}^1 \equiv p_{3m}^1 q_{3m}^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1 + p_3^{1*} \cdot q_3^1]; \quad m = 1, \ldots, M; \tag{186}$$
$$= p_{3m}^1 q_{3m}^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1] \text{ since } q_3^1 = 0_N;$$
$$= 0 \text{ since } q_{3m}^1 = 0.$$

Note that these shares can also be calculated using observable data.

Now define the expenditure shares for product Group 1 using just the products that are in Group 1. These are the shares that are relevant for the maximum overlap indices which will be defined shortly. The *maximum overlap share* for product n in period t, s_{1nO}^t, is defined as follows:

$$s_{1nO}^t \equiv p_{1n}^t q_{1n}^t/p_1^t \cdot q_1^t; \quad t = 0,1; \; n = 1, \ldots, N. \tag{187}$$

These maximum overlap shares are also observable. It can be seen that the following relationships hold between the true Group 1 shares and the maximum overlap Group 1 shares:[115]

$$s_{1n}^0 = s_{1nO}^0 p_1^0 \cdot q_1^0/[p_1^0 \cdot q_1^0 + p_3^0 \cdot q_3^0]; \quad n = 1, \ldots, N; \tag{188}$$
$$= s_{1nO}^0[1 - \Sigma_{m=1}^M s_{3m}^0];$$
$$s_{1n}^1 = s_{1nO}^1 p_1^1 \cdot q_1^1/[p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1]; \quad n = 1, \ldots, N; \tag{189}$$
$$= s_{1nO}^1[1 - \Sigma_{k=1}^K s_{2k}^1].$$

Let P_{TO} denote the *Törnqvist maximum overlap index*. The logarithm of this index is defined as follows:

$$lnP_{TO} \equiv \Sigma_{n=1}^N (1/2)(s_{1nO}^0 + s_{1nO}^1) ln(p_{1n}^1/p_{1n}^0). \tag{190}$$

The logarithm of the *true Törnqvist index*, P_T, is defined as follows:

$$lnP_T \equiv \Sigma_{n=1}^N ½(s_{1n}^0 + s_{1n}^1) ln(p_{1n}^1/p_{1n}^0) + \Sigma_{k=1}^K ½(s_{2k}^0 + s_{2k}^1) ln(p_{2k}^1/p_{2k}^{0*}) + \Sigma_{m=1}^M ½(s_{3m}^0 + s_{3m}^1) ln(p_{3m}^{1*}/p_{3m}^0) \tag{191}$$
$$= \Sigma_{n=1}^N ½(s_{1n}^0 + s_{1n}^1) ln(p_{1n}^1/p_{1n}^0) + \Sigma_{k=1}^K ½(0 + s_2k^1) ln(p_2k^1/p_2k^{0*})$$
$$+ \Sigma_{m=1}^M ½(s_{3m}^0 + 0) ln(p_{3m}^{1*}/p_{3m}^0) \text{ using (183) and (186)}$$
$$= \Sigma_{n=1}^N ½\{s_{1nO}^0[1 - \Sigma_{m=1}^M s_{3m}^0] + s_{1nO}^1[1 - \Sigma_{k=1}^K s_{2k}^1]\} ln(p_{1n}^1/p_{1n}^0)$$
$$+ \Sigma_{k=1}^K ½(s_{2k}^1) ln(p_{2k}^1/p_{2k}^{0*}) + \Sigma_{m=1}^M ½(s_{3m}^0) ln(p_{3m}^{1*}/p_{3m}^0) \text{ using (188) and (189)}$$
$$= lnP_{TO} + ½\Sigma_{k=1}^K s_{2k}^1[ln(p_{2k}^1/p_{2k}^{0*}) - \Sigma_{n=1}^N s_{1nO}^1 \, ln(p_{1n}^1/p_{1n}^0)]$$
$$+ ½\Sigma_{m=1}^M s_{3m}^0[ln(p_{3m}^{1*}/p_{3m}^0) - \Sigma_{n=1}^N s_{1nO}^0 \, ln(p_{1n}^1/p_{1n}^0)] \text{ using (190)}$$
$$= lnP_{TO} + ln\kappa + ln\mu,$$

where the logarithms of the terms κ and μ are defined as

$$ln\kappa \equiv (1/2)\Sigma_{k=1}^K s_{2k}^1[ln(p_{2k}^1/p_{2k}^{0*}) - \Sigma_{n=1}^N s_{1nO}^1 \, ln(p_{1n}^1/p_{1n}^0)] \tag{192}$$
$$= (1/2)\Sigma_{k=1}^K s_{2k}^1[ln(p_{2k}^1/p_{2k}^{0*}) - lnP_{JO}^1];$$
$$ln\mu \equiv (1/2)\Sigma_{m=1}^M s_{3m}^0[ln(p_{3m}^{1*}/p_{3m}^0) - \Sigma_{n=1}^N s_{1nO}^0 \, ln(p_{1n}^1/p_{1n}^0)] \tag{193}$$
$$= (1/2)\Sigma_{m=1}^M s_{3m}^0[ln(p_{3m}^{1*}/p_{3m}^0) - lnP_{JO}^0],$$

where the (weighted) *Jevons index* using the maximum overlap share weights of period 1 is P_{JO}^1 and the (weighted)

[115] These relationships were developed by de Haan and Krsinich (2012, 31–32).

Jevons index using the maximum overlap share weights of period 0 is P_{JO}^{0}; that is, the logarithm of these two indices is defined as follows:[116]

$$lnP_{JO}^{1} \equiv \Sigma_{n=1}^{N} s_{1nO}^{1}\, ln(p_{1n}^{1}/p_{1n}^{0});\; lnP_{JO}^{0} \equiv \Sigma_{n=1}^{N} s_{1nO}^{0}\, ln(p_{1n}^{1}/p_{1n}^{0}). \quad (194)$$

Exponentiating both sides of (191) leads to the following relationship between the "true" cost of living index P_T and the price index P_{TO} that is defined over products that are available in both periods:[117]

$$P_T = P_{TO} \times \kappa \times \mu. \quad (195)$$

The term κ defined by (192) can be regarded as a measure of the *reduction* in the true cost of living due to the introduction of new products. The period 0 imputed price for new product k, p_{2k}^{0*}, is likely to be higher than the actual price for new product k in period 1 adjusted for general inflation, p_{2k}^{1}/P_{JO}^{1}, and thus κ is likely to be less than 1. The bigger the share of new products in period 1, $S_{k=1}^{K}\, s_{2k}^{1}$, the more κ will be less than 1. Note that the logarithmic contribution of each new product to the reduction in the true cost of living can be measured using the additive decomposition that definition (192) provides.

The inflation adjustment term μ defined by (193) can be regarded as a measure of the *increase* in the true cost of living due to the disappearance of existing products. The period 1 imputed price for disappearing product m, p_{3m}^{1*}, is likely to be higher than the actual price for product m in period 0 adjusted for general inflation, $p_{3m}^{0}P_{JO}^{0}$, and thus μ is likely to be greater than 1. The bigger the share of disappearing products in period 0, $\Sigma_{m=1}^{M}\, s_{3m}^{0}$, the more μ will be greater than 1.

The decomposition defined by (191) is also useful in the context of defining *imputed carry-backward or carry-forward prices* for products that may be new or unavailable. Recall that the imputed reservation prices in period 0 are the prices p_{2k}^{0*} and the imputed reservation prices in period 1 are the prices p_{3m}^{1*}. Rough estimates or more precise econometric estimates have to be made for these reservation prices. However, it is possible to use available information on prices and quantities for periods 0 and 1 in order to define the following *carry-backward prices* p_{2kb}^{0} for the missing products in period 0 and the following *carry-forward prices* p_{3mf}^{1} for the missing products in period 1:

$$p_{2kb}^{0} \equiv p_{2k}^{1}/P_{JO}^{1} \textit{ for } k = 1, \ldots, K \text{ and } p_{3mf}^{1} \equiv p_{3m}^{0}P_{JO}^{0} \textit{ for } m = 1, \ldots, M. \quad (196)$$

Thus, the inflation-adjusted carry-forward price defined by (196) for the missing product m in period 1 takes the observed price for product m in period 0, p_{3m}^{0}, and adjusts it for general inflation for the group of products that are present in both periods 0 and 1 using the weighted maximum overlap Jevons index P_{JO}^{0}. Similarly, the inflation-adjusted carry-backward price defined by (195) for the missing product k in period 0 takes the observed price for product k in period 1, p_{2k}^{1}, and deflates it by the weighted Jevons maximum overlap price index, P_{JO}^{1}. These inflation-adjusted imputed prices are more reasonable than the *constant carry-forward prices*, p_{3m}^{0}, or *constant carry-backward prices*, p_{2k}^{1}, which are frequently used to fill in the missing prices. From (190), (191), and (189), it can be seen that if the reservation prices are equal to their inflation-adjusted carry-forward prices (so that $p_{3m}^{1*} = p_{3mf}^{1}$ for $m = 1, \ldots, M$) and inflation-adjusted carry-backward prices (so that $p_{2k}^{0*} = p_{2kb}^{0}$ for $k = 1, \ldots, K$), then the true Törnqvist index P_T will equal its maximum overlap counterpart, P_{TO}.

However, in general, economic theory suggests that the reservation prices will be greater than their inflation-adjusted carry-forward or carry-backward prices. Thus, we define the following *margin terms*, κ_k and μ_m,, which express how much higher each reservation price is from its inflation-adjusted carry-forward or carry-backward price counterpart:

$$1 + \kappa_k \equiv p_{2k}^{0*}/p_{2kb}^{0};\; k = 1, \ldots, K; \quad (197)$$

$$1 + \mu_m \equiv p_{3m}^{1*}/p_{3mf}^{1};\; m = 1, \ldots, M. \quad (198)$$

Now substitute definitions (195)–(198) into (191), and we obtain the following *exact relationship* between the true Törnqvist index P_T and its maximum overlap counterpart P_{TO}:

$$ln(P_T/P_{TO}) = -\Sigma_{k=1}^{K}\,(1/2)s_{2k}^{1}\, ln(1 + \kappa_k) + \Sigma_{m=1}^{M}\,(1/2)s_{3m}^{0}\, ln(1 + \mu_m). \quad (199)$$

Exponentiate both sides of (199) and subtract 1 from both sides of the resulting expression. Define the right-hand side of the resulting expression as the function $g(\kappa_1, \ldots, \kappa_K, m_1, \ldots, m_M)$ and approximate g by taking the first-order Taylor series approximation to g evaluated at $0 = \kappa_1 = \ldots = \kappa_K = \mu_1 = \ldots = \mu_M$. The resulting approximation to $(P_T/P_{TO}) - 1$ is the following one:[118]

$$(P_T/P_{TO}) - 1 \approx \Sigma_{m=1}^{M}\,(1/2)s_{3m}^{0}\mu_m - \Sigma_{k=1}^{K}\,(1/2)s_{2k}^{1}\kappa_k. \quad (200)$$

The period 0 and 1 value aggregates for the goods and services in the group of $N + K + M$ commodities under consideration, V^0 and V^1, are defined as follows:

$$V^0 \equiv p_1^{0}\cdot q_1^{0} + p_3^{0}\cdot q_3^{0};\; V^1 \equiv p_1^{1}\cdot q_1^{1} + p_2^{1}\cdot q_2^{1}. \quad (201)$$

The "true" *implicit Törnqvist quantity index* Q_T is defined as the value ratio, V^1/V^0, deflated by the "true" Törnqvist price index, P_T; that is, we have

$$Q_T \equiv [V^1/V^0]/P_T. \quad (202)$$

[116] These indices could also be described as Cobb–Douglas indices. The indices defined by (194) have also been described as geometric Paasche and geometric Laspeyres indices, respectively.

[117] This formula was first derived by de Haan and Krsinich (2012, 31–32) (2014, 344). They obtained imputed prices for the missing products by using hedonic regressions, which will be studied in some detail in Chapter 8.

[118] This formula is similar in spirit to the highly simplified approximate new goods bias formulae obtained by Diewert (1987, 779) (1998, 51–54).

Statistical agencies can use maximum overlap Törnqvist–Theil price indices to deflate final demand aggregates in order to construct aggregate quantity or volume indices.[119] Thus, in our context, the *maximum overlap Törnqvist–Theil quantity index*, Q_{TO}, is defined as follows:

$$Q_{TO} \equiv [V^1/V^0]/P_{TO}. \tag{203}$$

The *reciprocal* of the bias in Q_{TO} relative to its true counterpart Q_T can be measured by the ratio Q_T/Q_{TO}:

$$Q_T/Q_{TO} = P_{TO}/P_T, \tag{204}$$

where we have used definitions (202) and (203) to derive (204). An exact expression for the logarithm of P_{TO}/P_T can be obtained from (199):

$$ln(P_{TO}/P_T) = \Sigma_{k=1}^{K} (1/2)s_{2k}^{1}\, ln(1 + \kappa_k) - \Sigma_{m=1}^{M} (1/2)s_{3m}^{0}\, ln(1 + \mu_m). \tag{205}$$

Exponentiate both sides of (205), and subtract 1 from both sides of the resulting expression. Define the right-hand side of the resulting expression as the function $h(\kappa_1, \ldots, \kappa_K, m_1, \ldots, m_M)$, and approximate h by taking the first-order Taylor series approximation to h evaluated at $0 = \kappa_1 = \ldots = \kappa_K = \mu_1 = \ldots = \mu_M$. The resulting approximation to $(Q_T/Q_{TO}) - 1$ is the following one:

$$(Q_T/Q_{TO}) - 1 \approx \Sigma_{k=1}^{K} (1/2)s_{2k}^{1}\kappa_k - \Sigma_{m=1}^{M} (1/2)s_{3m}^{0}\mu_m. \tag{206}$$

Thus, if there are no disappearing goods, the right-hand side of (206) becomes $\Sigma_{k=1}^{K} (1/2)s_{2k}^{1}\kappa_k$, and this number is a measure of the downward bias in the maximum overlap Törnqvist quantity index for the value aggregate in percentage points. That is, (206) gives the downward bias in welfare from ignoring new goods and services.

For analogous bias formulae for price and quantity aggregates that are constructed using maximum overlap Laspeyres, Paasche, or Fisher indices, see Diewert, Fox, and Schreyer (2017b).

15. Becker's Theory of the Allocation of Time

Peter Hill (1999), in discussing the classic study by Nordhaus (1997) on the price of light, raised the issue as to how should a cost of living index treat *household production* where consumers combine purchased market goods or "inputs" to produce finally demanded "commodities" that yield utility:[120]

> There is another area in which the definition of a COL requires further clarification and precision. From what is utility derived? Households do not consume many of the goods and services they purchase directly but use them to produce other goods or services from which they derive utility. In a recent stimulating and important paper, Nordhaus has used light as a case study. Households purchase items such as lamps, electric fixtures and fittings, light bulbs and electricity to produce light, which is the product they consume directly. . . . The light example is striking because Nordhaus provides a plausible case for arguing that the price of light, measured in lumens, has fallen absolutely (at least in US dollars) and dramatically over the last two centuries as a result of major inventions, discoveries and "tectonic" improvements in the technology of producing light.
>
> The question that arises is whether goods and services that are essentially *inputs* into the production of other goods and services should be treated in a COL as if they provided utility directly. In principle, a COL should include the shadow, or imputed, prices, of the outputs from these processes of production and not the prices of the inputs. . . . There is a need to clarify exactly how this issue is to be dealt with in a COL index.
>
> Peter Hill (1999, 5)

In this section, we address the issues raised by Hill by using the model of household production of final demand commodities that was postulated by Becker (1965) many years ago. Becker's model illustrates not only how household production of the type mentioned by Hill can be integrated into a cost of living framework but also indicates the important role that the *allocation of household time* plays in a more realistic model of household behavior. In order to measure *welfare change* more accurately, it is necessary to model how a household manages its allocation of time during the two periods under consideration.

In Becker's model of consumer behavior, a household (consisting of a single individual for simplicity) purchases q_n units of *market commodity n* and combines it with a household input of time, t_n, to produce $Q_n = f^n(q_n,t_n)$ units of a *finally demanded commodity* for $n = 1,2, \ldots,N$, where f^n is *the household production function* for the nth finally demanded commodity.[121] Thus, using Becker's theory, the purchase of market goods and services alone does not provide utility for the household; these market purchases must be combined with household time in order to provide utility. Some examples of Becker's finally demanded commodities (or *basic commodities* to use his terminology) are as follows:

- Making a meal: The inputs are the ingredients used, the use of utensils and possibly a stove and time required to make the meal and the output is the prepared meal.
- Eating a meal: The inputs are the prepared meal and time spent eating and the output is a consumed meal.

[119] The US Bureau of Labor Statistics uses the Törnqvist price index as its target index for its chained CPI. Typically, there are no adjustments for new and disappearing products, so these Törnqvist price indices are essentially maximum overlap price indices.

[120] See also Hill (2009).

[121] More complicated household production functions could be introduced, but the present assumptions will suffice to show how household production can be modeled in a COLI framework using exact index number formulae. For additional work on Becker's theory of the allocation of time and household production, see Pollak and Wachter (1975) (1977), Diewert (2001), Abraham and Mackie (2005), Hill (2009), Landefeld, Fraumeni, and Vojtech (2009), Schreyer and Diewert (2014), and Diewert, Fox, and Schreyer (2018).

- Cleaning a house: The inputs are cleaning utensils, soapy water, polish and time and the output is a clean house.
- Gardening services: The inputs are tools used in the yard, fuel (if power tools are used) and time and the output is a beautiful yard.
- Reading a book: The inputs are computer services or a physical book and time and the output is a book which has been read.

Activities 1, 3, and 4 listed here are examples of basic commodities, which could be *purchased* by the household; that is, a cook could be hired to prepare a meal, a house cleaning service could be hired to clean the house, and a gardening service could be hired to maintain the yard in good condition. These activities could be called examples of household *work activities*. Activities 2 and 5 are examples of *leisure activities*, where the utility generated by the activity cannot be outsourced. We will see subsequently why this distinction between the two types of household production can be important.

We follow Becker's example and assume that the household production functions, $f^n(q_n,t_n)$, are linearly homogeneous.[122] If $p_n > 0$ is the price for a unit of q_n and $w > 0$ is the price of household time, then the *unit cost functions* $c^n(p_n,w)$ that correspond to the $f^n(q_n,t_n)$ can be defined as follows:

$$c^n(p_n,w) \equiv \min_{q_n,t_n}\{p_nq_n + wt_n : f^n(q_n,t_n) \geq 1; \quad q_n \geq 0;\ t_n \geq 0\};\ n = 1,\ldots,N. \qquad (207)$$

Assume that the household faces the prices $p^\tau \equiv [p_1{}^\tau,\ldots,p_N{}^\tau] >> 0_N$ and $w^\tau > 0$ for periods $\tau = 0,1$. Further assume that the period τ observed purchases of commodity n, $q_n{}^\tau$, and time allocated to its consumption in period τ, $t_n{}^\tau$, solve the cost minimization problems, $\min_{q_n,t_n}\{p_n{}^\tau q_n + w^\tau t_n : f^n(q_n,t_n) \geq f^n(q_n{}^\tau,t_n{}^\tau);\ q_n \geq 0;\ t_n \geq 0\}$ for $n = 1,\ldots,N$ and $\tau = 0,1$. In view of the linear homogeneity of the household production functions, f^n, we obtain the following equalities:

$$p_n{}^\tau q_n{}^\tau + w^\tau t_n{}^\tau = c^n(p_n{}^\tau,w^t)f^n(q_n{}^\tau,t_n{}^\tau) = P_n{}^\tau Q_n{}^\tau; \quad t = 0,1;\ n = 1,\ldots,N, \qquad (208)$$

where the period τ *basic prices and quantities* for the nth household activity are defined as follows:[123]

$$P_n{}^\tau \equiv c^n(p_n{}^\tau,w^\tau);\ Q_n{}^\tau \equiv f^n(q_n{}^\tau,t_n{}^\tau); \quad \tau = 0,1;\ n = 1,\ldots,N. \qquad (209)$$

At this point, the theory of exact index numbers can be used in order to obtain empirical estimates for the unobserved $P_n{}^\tau$ and $Q_n{}^\tau$. Pick an index number formula that is exact for a certain functional form for either $c^n(p_n,w)$ or $f^n(q_n,t_n)$. For example, pick the Fisher price index, $P_F(p_n{}^0,w^0;p_n{}^1,w^1;q_n{}^0,t_n{}^0;q_n{}^1,t_n{}^1)$, which is exact for certain flexible functional forms[124] for either the nth unit cost function $c^n(p_n,w)$ or the nth household production function $f^n(q_n,t_n)$ for $n = 1,\ldots,N$. The basic prices and quantities for period 0 are defined as follows:[125]

$$P_n{}^0 \equiv 1;\ Q_n{}^0 \equiv p_n{}^0q_n{}^0 + w^0t_n{}^0;\ n = 1,\ldots,N. \qquad (210)$$

The basic prices and quantities for period 1 are defined as follows:

$$P_n{}^1 \equiv P_F(p_n{}^0,w^0;p_n{}^1,w^1;q_n{}^0,t_n{}^0;q_n{}^1,t_n{}^1); \quad Q_n{}^1 \equiv [p_n{}^1q_n{}^1 + w^1t_n{}^1]/P_n{}^1;\ n = 1,\ldots,N. \qquad (211)$$

The $P_n{}^\tau$ and $Q_n{}^\tau$ defined by (210) and (211) will be consistent with equations (208) provided that the c^n or f^n have the functional forms that are exact for the Fisher index. For future reference, note that the following equations will hold:

$$P^\tau\cdot Q^\tau \equiv \Sigma_{n=1}{}^N P_n{}^\tau Q_n{}^\tau = \Sigma_{n=1}{}^N [p_n{}^\tau q_n{}^\tau + w^\tau t_n{}^\tau] = p^\tau\cdot q^\tau + w^\tau[\Sigma_{n=1}{}^N t_n{}^\tau];\ \tau = 0,1 \qquad (212)$$

where $P^\tau \equiv [P_1{}^\tau,\ldots,P_N{}^\tau]$, $Q^\tau \equiv [Q_1{}^\tau,\ldots,Q_N{}^\tau]$, $p^\tau \equiv [p_1{}^\tau,\ldots,p_N{}^\tau]$ and $q^\tau \equiv [q_1{}^\tau,\ldots,q_N{}^\tau]$ for $\tau = 0,1$.

We return to Becker's model of the allocation of time. In addition to spending time on the N household production activities, Becker assumed that the household provides $t_L > 0$ hours of labor market services at the after tax wage rate of $w_L > 0$. Becker also assumed that the household spends the amount of Y of nonlabor income on the purchase of market goods and services.[126] Finally, Becker assumed that the consumer-worker has preferences over different combinations of the finally demanded commodities, $Q_1,\ldots,Q_N$, that are summarized by the (macro) *utility function*, $U(Q_1,\ldots,Q_N) \equiv U[f^1(q_1,t_1),\ldots,f^N(q_N,t_N)]$.[127] In addition to the household budget constraint, $\Sigma_{n=1}{}^N p_nq_n \leq Y + w_Lt_L$, the household has to satisfy the *time constraint*, $\Sigma_{n=1}{}^N t_n + t_L = H$, where H is the number of hours available in the period under consideration.

Let $p^\tau \equiv [p_1{}^\tau,\ldots,p_N{}^\tau] >> 0_N$ and $w_L{}^\tau > 0$ be the observed prices for purchases of market goods and services for period t, let $t^\tau \equiv [t_1{}^\tau,\ldots,t_N{}^\tau] >> 0_N$ be the household's period τ vector of time inputs into the household production functions and let $t_L{}^\tau > 0$ be the observed household labor supply for periods $\tau = 0,1$. We assume that q^τ, t^τ, and $t_L{}^\tau$ solve the following period τ household-constrained utility maximization problem for $\tau = 0,1$:[128]

$$\max_{q_1,\ldots,q_N,\ldots,t_N,t_L}\{U[f^1(q_1,t_1),\ldots,f^N(q_N,t_N)] : Y^\tau + w_L{}^\tau t_L - \Sigma_{n=1}{}^N p_n{}^\tau q_n \geq 0;\ H - \Sigma_{n=1}{}^N t_n - t_L \geq 0\}. \qquad (213)$$

[122] In addition, following Schreyer and Diewert (2014), we assume that the household production functions are nonnegative, once differentiable, concave, and increasing in q_n and t_n. Becker (1965, 496) assumed that the household production functions f^n were of the Leontief, no substitution type.

[123] Becker (1965, 497) called P_n the *full price* for consuming a unit of the nth final commodity; that is, it is the sum of the prices of the goods and time used to produce a unit of the finally demanded commodity Q_n.

[124] See Section 5.

[125] Definitions (210) and (211) make specific cardinalizations for measuring the unobserved outputs of the N household production functions.

[126] If w_Lt_L (equal to after-tax labour earnings) is large enough, it could be the case that Y is negative; that is, some of the household labour earnings are saved. This does not affect Becker's theory.

[127] The utility function U is assumed to be once differentiable, linearly homogeneous, concave, and increasing in the $Q_1,\ldots,Q_N$.

[128] We have omitted the nonnegativity constraints $t_n \geq 0$, $t_L \geq 0$ and $q_n \geq 0$ for $n = 1,\ldots,N$ from (212) to save space. Since we have assumed a strictly positive solution to (212) for each time period τ, these nonnegativity constraints will not be binding and hence can be ignored in what follows.

We assume that the inequality constraints in (213) are satisfied as equalities when evaluated at the q^τ, t^τ, and t_L^τ solutions to (213). This means that the following equations hold:

$$\Sigma_{n=1}^{N} p_n^\tau q_n^\tau = Y^\tau + w_L^\tau t_L^\tau;\ \tau = 0,1; \qquad (214)$$
$$w_L^\tau[\Sigma_{n=1}^{N} t_n^\tau] = w_L^\tau[H - t_L^\tau];\ \tau = 0,1. \qquad (215)$$

Equations (212) will also hold with $w^\tau = w_L^\tau$ for $\tau = 0,1$, as will be seen later. These equations along with (214) and (215) imply that the following equations will hold:

$$P^\tau\cdot Q^\tau = \Sigma_{n=1}^{N}[p_n^\tau q_n^\tau + w_L^\tau t_n^\tau] = Y^\tau + w_L^\tau t_L^\tau + w_L^\tau[H - t_L^\tau] = Y^t + w_L^\tau H \equiv F^\tau;\ \tau = 0,1, \qquad (216)$$

where F^τ is Becker's *full income*.[129] To see why the consumer's regular budget constraint and time constraint can be combined into a single constraint, form the Lagrangian $L^\tau(q,t,t_L,\lambda,\omega)$ for the constrained maximization problem defined by (213) for $\tau = 0$ or 1:

$$L^\tau(q,t,t_L,\lambda,\omega) \equiv U[f^1(q_1,t_1), \ldots, f^N(q_N,t_N)] + \lambda[Y^\tau + w_L^\tau t_L - \Sigma_{n=1}^{N} p_n^\tau q_n] + \omega[H - \Sigma_{n=1}^{N} t_n - t_L];\ \tau = 0,1. \qquad (217)$$

Under our regularity conditions on the functions U and $f^1, \ldots, f^N$, there will exist positive Lagrange multipliers, $\lambda^\tau > 0$ and $\omega^\tau > 0$ such that the observed period τ solution to the period τ constrained maximization problem defined by (213), q^τ, t^τ, and t_L^τ, will satisfy the following first-order conditions:

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n] = \lambda^\tau p_n^\tau;\quad n = 1, \ldots, N;\ \tau = 0,1; \qquad (218)$$
$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = \omega^\tau;\quad n = 1, \ldots, N;\ \tau = 0,1; \qquad (219)$$
$$0 = \lambda^\tau w_L^\tau - \omega^\tau;\ \tau = 0,1. \qquad (220)$$

Equations (220) show that $w^t = \lambda^\tau w_L^\tau$ for $\tau = 0,1$. These equations justify Becker's statement that the household budget constraint and the corresponding time constraint can be combined into a single constraint. Using (220), equations (219) become the following equations:

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = \lambda^\tau w_L^\tau;\quad n = 1, \ldots, N;\ \tau = 0,1. \qquad (221)$$

For each τ, take equation n in (218) and multiply both sides by q_n^τ. Take equation n in (221) and multiply both sides by t_n^τ. For each τ and n, add these equations. Using the linear homogeneity of $\partial f^n(q_n,t_n)/\partial t_n$ and using definitions (209) with $w^\tau \equiv w_L^\tau$, which imply that $Q_n^\tau \equiv f^n(q_n^\tau,t_n^\tau)$ for each n, we obtain the following equations:

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_n]Q_n^\tau = \lambda^\tau[p_n^\tau q_n^\tau + w_L^\tau t_n^\tau];\quad n = 1, \ldots, N;\ \tau = 0,1; \qquad (222)$$
$$= \lambda^\tau[P_n^\tau Q_n^\tau] \text{ using (208) with } w^\tau \equiv w_L^\tau.$$

For each τ, sum the N equations in (222). Using the linear homogeneity of $U(Q_1, \ldots, Q_N)$ and equations (216), we obtain the following equations:

$$U(Q_1^\tau, \ldots, Q_N^\tau) = \lambda^\tau P^\tau\cdot Q^\tau;\ t = 0,1 \qquad (223)$$
$$= \lambda^\tau F^\tau \text{ using definitions (216).}$$

Equations (223) can be solved for the Lagrange multipliers λ^τ. The solutions are $\lambda^\tau = U(Q_1^\tau, \ldots, Q_N^\tau)/P^\tau\cdot Q^\tau$ for $\tau = 0,1$. Substitute these values for λ^τ back into equations (222). After some rearrangement, we obtain the following equations, which are *Wold's Identity equations* applied to the macro utility function $U(Q_1, \ldots, Q_N)$:

$$P^\tau/P^\tau\cdot Q^\tau = \nabla_Q U(Q^\tau)/U(Q^\tau);\ t = 0,1. \qquad (224)$$

Recall that the P^τ and Q^τ are well defined by equations (210) and (211) with $w^0 \equiv w_L^0$ and $w^1 \equiv w_L^1$. At this stage, we can assume a functional form for the macro utility function $U(Q_1, \ldots, Q_N) = U(Q)$, which has an exact index number formula associated with it. Thus, assume that $U(Q)$ can be approximated by the homogeneous quadratic utility function, $U(Q) \equiv [Q^TAQ]^{1/2}$, where the symmetric matrix A has one positive eigenvalue with a strictly positive eigenvector and the other eigenvalues of A are either equal to 0 or negative. Then the Fisher index is exact for this functional form. The nominal growth of full consumption going from period 0 to 1 is equal to the nominal growth of full income, $F^1/F^0 = P^1\cdot Q^1/P^0\cdot Q^0$, and the real growth of household full consumption is equal to the Fisher ideal quantity index, $Q_F(P^0,P^1,Q^0,Q^1)$.[130] The appropriate consumer price index under these conditions is the Fisher ideal price index, $P_F(P^0, P^1,Q^0,Q^1)$.

In the aforementioned model of consumer behavior, the household price of time for period τ turns out to be the *after tax wage rate*, w_L^τ. But there are many households that do not offer market labor services; that is, individuals who are retired or who are simply not in the labor force. How can we value household time in this situation? It is possible to modify Becker's model of the consumer-worker household to deal with non-worker households. Make the same assumptions as in the model explained previously with one exception: we assume that the Nth household production activity is one where the household time input, t_N, can be replaced by hiring market services, s_N, at the price $w_N > 0$. Thus, if the Nth activity is yard maintenance, time spent maintaining the yard can be replaced by hiring a service that will undertake the necessary work. Thus, the production function for the Nth activity is $Q_N = f^N(q_N,t_N + s_N)$.[131]

[129] "This suggests dropping the approach based on explicitly considering separate goods and time constraints and substituting one in which the total resource constraint necessarily equaled the maximum money income achievable, which will be simply called 'full income'" (Gary Becker [1965, 497]).

[130] The period 0 and 1 levels of household real consumption are set equal to $U^0 \equiv P^0\cdot Q^0 = p^0\cdot q^0 + w_L^0[\Sigma_{n=1}^{N} t_n^0]$ and $U^1 \equiv U^0\times Q_F(P^0,P^1,Q^0,Q^1) = U^0\times[P^0\cdot Q^1 P^1\cdot Q^1/P^0\cdot Q^0 P^1\cdot Q^0]^{1/2}$, respectively.

[131] Thus, we are assuming that personal yard work and hired yard work are perfect substitutes.

Let $p^\tau \equiv [p_1^\tau, \ldots, p_N^\tau] >> 0_N$ and $w_S^\tau > 0$ be the observed prices for purchases of market goods and services for period t and let $t^\tau \equiv [t_1^\tau, \ldots, t_N^\tau] >> 0_N$ be the household's period τ vector of time inputs into the household production functions and for periods τ = 0,1. Let $q_S^\tau > 0$ be the household's purchases of market labor services for activity N for $t = 0,1$. We assume that q^t, t^τ, and q_S^τ solve the following period τ household-constrained utility maximization problem:[132]

$$\max_{q_1,\ldots,q_N,t_1,\ldots,t_N,q_S} \{U[f^1(q_1,t_1), \ldots, f^{N-1}(q_{N-1},t_{N-1}), f^N(q_N,t_N + q_S)] : \quad (225)$$
$$Y^\tau - w_S^\tau q_S - \Sigma_{n=1}^N p_n^\tau q_n \geq 0;\ H - \Sigma_{n=1}^N t_n \geq 0\};\ \tau = 0,1.$$

We assume that the functions $U, f^1, \ldots, f^N$ satisfy the same regularity conditions as in the Becker model here. Thus, the two constraints in (225) will hold as equalities. Hence, we will have $Y^\tau = \Sigma_{n=1}^N p_n^\tau q_n^\tau + w_S^\tau q_S^\tau$, $w_S^\tau H = w_S^\tau S_{n=1}^N t_n^\tau$ for τ = 0,1 as well as the following equations:

$$\Sigma_{n=1}^N p_n^\tau q_n^\tau + w_S^\tau q_S^\tau + S_{n=1}^N w_S^\tau t_n^\tau = Y^\tau + w_S^\tau H \equiv F^\tau;\ \tau = 0,1, \quad (226)$$

where the new *period τ full income* F^τ is equal to period τ nonlabor income Y^τ plus the value of period τ household time H valued at the period τ market service wage for the Nth activity, w_S^τ.

Form the Lagrangians $L^t(q,q_S,t,\lambda,\omega)$ for the constrained maximization problems defined by (225) for τ = 0,1:

$$L^\tau(q,q_S,t,\lambda,\omega) \equiv U[f^1(q_1,t_1), \ldots, f^N(q_N,t_N + q_S)] + \lambda[Y^\tau - \Sigma_{n=1}^N p_n^\tau q_n - w_S^\tau q_S] + \omega[H - \Sigma_{n=1}^N t_n - t_L];\ \tau = 0,1. \quad (227)$$

Under our regularity conditions on the functions U and $f^1, \ldots, f^N$, there will exist positive Lagrange multipliers, $\lambda^\tau > 0$ and $\omega^\tau > 0$, such that the observed period τ solution, q^τ, q_S^τ and t^τ, to the period τ constrained maximization problem defined by (225) will satisfy the following first-order conditions:

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n] = \lambda^\tau p_n^\tau;\ n = 1, \ldots, N-1;\ \tau = 0,1; \quad (228)$$

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau + q_S^\tau)/\partial q_N] = \lambda^\tau p_N^\tau;\ \tau = 0,1; \quad (229)$$

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = \omega^\tau;\ n = 1, \ldots, N-1;\ \tau = 0,1; \quad (230)$$

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau + q_S^\tau)/\partial t_N] = \omega^\tau;\ \tau = 0,1; \quad (231)$$

$$[\partial U(Q_1^\tau, \ldots, Q_N^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau + q_S^\tau)/\partial q_S] = \lambda^\tau w_S^\tau;\ \tau = 0,1. \quad (232)$$

For τ = 0 or 1, it can be seen that the derivatives on the left-hand sides of (231) and (232) are identical. Hence, the right-hand sides are equal and we obtain the equations $\omega^\tau = \lambda^\tau w_S^\tau$ for τ = 0,1. Substitute these solutions for the ω^τ into equations (230) and (231) and we obtain the following equations:

$$[\partial U(Q^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = \lambda^\tau w_S^\tau;\ n = 1, \ldots, N-1;\ \tau = 0,1; \quad (233)$$

$$[\partial U(Q^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau + q_S^\tau)/\partial t_N] = \lambda^\tau w_S^\tau;\ t = 0,1. \quad (234)$$

For τ = 0,1 and $n = 1, \ldots, N-1$, multiply both sides of equation n in (228) by q_n^τ and both sides of equation n in (233) by t_n^τ, and add the resulting two equations. Using the linear homogeneity of $f^n(q_n,t_n)$, we have $q_n^\tau[\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n] + t_n^\tau[\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = f^n(q_n^\tau,t_n^\tau)$. Thus, we obtain the following equations:

$$[\partial U(Q^\tau)/\partial Q_n]f^n(q_n^\tau,t_n^\tau) = \lambda^\tau[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau];\ n = 1, \ldots, N-1;\ \tau = 0,1. \quad (235)$$

For each τ = 0,1 and $n = 1, \ldots, N-1$, equation n in equations (235) can be solved for λ^τ, and this value for λ^τ can be substituted back into equations n in (228) and (223). After suitable rearrangement, the following equations are obtained:

$$[\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n]/f^n(q_n^\tau,t_n^\tau) = p_n^\tau/[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau];\ n = 1, \ldots, N-1;\ \tau = 0,1; \quad (236)$$

$$[\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n]/f^n(q_n^\tau,t_n^\tau) = w_S^\tau/[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau];\ n = 1, \ldots, N-1;\ \tau = 0,1. \quad (237)$$

For each $n = 1, \ldots, N-1$ and for τ =0,1, equations (236) and (237) are the Wold's Identity equations (14) for the household production function $f^n(q_n,t_n)$. Thus, we can approximate f^n by a homogeneous quadratic utility function and use the Fisher price and quantity indices to estimate $Q_n^0 \equiv f^n(q_n^0,t_n^0)$ and $Q_n^1 \equiv f^n(q_n^1,t_n^1)$ for $n = 1, \ldots, N-1$; that is, define Q_n^τ and the companion prices $P_n^t \equiv c^h(p_n^\tau,w_S^\tau)$ as follows:

$$P_n^0 \equiv 1 \equiv c^n(p_n^0,w_S^0);\ Q_n^0 \equiv p_n^0 q_n^0 + w_S^0 t_n^0 \equiv f^n(q_n^0,t_n^0);\ n = 1, \ldots, N-1; \quad (238)$$

$$P_n^1 \equiv P_F(p_n^0,w^0;p_n^1,w^1;q_n^0,t_n^0;q_n^1,t_n^1) \equiv c^n(p_n^1,w_S^1);\ Q_n^1 \equiv [p_n^1 q_n^1 + w^1 t_n^1]/P_n^1 \equiv f^n(q_n^1,t_n^1);\ n = 1, \ldots, N-1. \quad (239)$$

Now use equations (229), (231), and (232) and repeat these operations for $f^N(q_N,t_N + q_S)$ and obtain the following counterparts to (236)–(239):

$$[\partial f^N(q_N^\tau,t_N^\tau + q_S^\tau)/\partial q_N]/f^N(q_N^\tau,t_N^\tau + q_S^\tau) = p_N^\tau/[p_N^\tau q_N^\tau + w_S^\tau(t_N^\tau + q_S^\tau)];\ t = 0,1; \quad (240)$$

$$[\partial f^N(q_N^\tau,t_N^\tau)/\partial t_N]/f^N(q_N^\tau,t_N^\tau) = w_S^\tau(t_N^\tau + q_S^\tau)/[p_N^\tau q_N^\tau + w_S^\tau(t_N^\tau + q_S^\tau)];\ t = 0,1. \quad (241)$$

$$P_N^0 \equiv 1 \equiv c^N(p_N^0,w_S^0);\ Q_N^0 \equiv p_N^0 q_N^0 + w_S^0(t_N^0 + q_S^0) \equiv f^N(q_N^0,t_N^0 + q_S^0); \quad (242)$$

[132] We have omitted the nonnegativity constraints $q_S \geq 0$, $t_n \geq 0$, and $q_n \geq 0$ for $n = 1, \ldots, N$ from (225) to save space. Since we have assumed a strictly positive solution to (225) for each time period τ, these nonnegativity constraints will not be binding and hence can be ignored in what follows.

$$P_N^{\,1} \equiv P_F(p_N^{\,0},w_S^{\,0};p_N^{\,1},w_S^{\,1};q_N^{\,0},t_N^{\,0} + q_S^{\,0};q_N^{\,1},t_N^{\,1} + q_S^{\,1}) \equiv c^N(p_N^{\,1},w_S^{\,1}); \tag{243}$$
$$Q_N^{\,1} \equiv p_N^{\,1}q_N^{\,1} + w_S^{\,1}(t_N^{\,1} + q_S^{\,1})]/P_N^{\,1} \equiv f^N(q_N^{\,1},t_N^{\,1} + q_S^{\,1}).$$

Definitions (240)–(243) can be substituted back into equations (228)–(235) in order to derive the following equations:

$$[\partial U(Q_1^{\,\tau}, \ldots, Q_N^{\,\tau})/\partial Q_n]Q_n^{\,\tau} = \lambda^\tau[p_n^{\,\tau}q_n^{\,\tau} + w_S^{\,\tau}t_n^{\,\tau}] = \lambda^\tau[P_n^{\,\tau}Q_n^{\,\tau}];\ n = 1, \ldots, N-1;\ \tau = 0,1; \tag{244}$$
$$[\partial U(Q_1^{\,\tau}, \ldots, Q_N^{\,\tau})/\partial Q_N]Q_N^{\,\tau} = \lambda^\tau[p_N^{\,\tau}q_N^{\,\tau} + w_S^{\,\tau}(t_n^{\,\tau} + q_S^{\,\tau})] = \lambda^\tau[P_N^{\,\tau}Q_N^{\,\tau}]\ \tau = 0,1. \tag{245}$$

For each τ, sum the N equations in (244) and (245). Using the linear homogeneity of $U(Q_1, \ldots, Q_N)$ and the definition (226) for period τ full income F^τ, we obtain the following equations:

$$U(Q_1^{\,\tau}, \ldots, Q_N^{\,\tau}) = \lambda^\tau P^\tau \cdot Q^\tau = \lambda^\tau F^\tau;\ t = 0,1. \tag{246}$$

Equations (246) can be solved for the Lagrange multipliers, λ^τ for $\tau = 0,1$. We obtain $l^t = U(Q_1^{\,\tau}, \ldots, Q_N^{\,\tau})/P^\tau \cdot Q^\tau$ for $\tau = 0,1$. Substitute these values for λ^τ back into equations (244) and (245). After some rearrangement, we obtain the following equations, which are *Wold's Identity equations* applied to the macro utility function $U(Q_1, \ldots, Q_N)$:

$$P^\tau/P^\tau \cdot Q^\tau = \nabla_Q U(Q^\tau)/U(Q^\tau);\ t = 0,1. \tag{247}$$

Recall that P^τ and Q^τ are well defined by equations (238), (239), (242), and (243). Now assume a functional form for the macro utility function $U(Q_1, \ldots, Q_N) = U(Q)$, which has an exact index number formula associated with it. Thus, assume that $U(Q)$ can be approximated by the homogeneous quadratic utility function, $U(Q) \equiv [Q^TAQ]^{1/2}$, where the symmetric matrix A has one positive eigenvalue with a strictly positive eigenvector and the other eigenvalues of A are either equal to 0 or negative. Then the Fisher price and quantity indices are exact for this functional form. The nominal growth of full consumption going from period 0 to 1 is equal to the nominal growth of full income, $F^1/F^0 = P^1 \cdot Q^1/P^0 \cdot Q^0$, where F^τ are defined by (226) and the real growth of household full consumption is equal to the Fisher ideal quantity index, $Q_F(P^0,P^1,Q^0,Q^1)$.[133] The appropriate consumer price index under these conditions is the Fisher ideal price index, $P_F(P^0,P^1,Q^0,Q^1)$.

Here are the important points that emerge from our analysis of the aforementioned two models for the household's allocation of time:[134]

- Depending on the type of household, the valuation of household time is either the *after tax wage rate* for the household or the *price of market services* that can substitute for household work.
- It is possible to use "normal" index number theory to provide price and volume indices for utility-maximizing households that face both a budget constraint and a time constraint.

However, there are many problems with the two models of household behavior that were considered earlier:

- The first model did not take into account the possible *disutility of providing market labor supply*, while neither model did not take into account the possible *disutility of providing household work*.[135] Taking possible disutility into account greatly complicates the analysis. In particular, the scaling of the utility functions, F and $f^1, \ldots, f^N$ is no longer straightforward.
- In more realistic models of household behavior, *corner solutions* to the household utility maximization problems emerge as realistic possibilities.[136]
- In more realistic models of household behavior, it is possible to identify the "correct" prices of time to value household labor supply, household time in leisure activities, and household time in work activities, but econometric estimation is required.[137] This means that it will be difficult for statistical agencies to deal with these difficulties in practical settings.
- There are also problems in forming household utility functions when there are multiple persons in the household.[138]
- Finally, the household production functions for work- and leisure-type activities could be subject to *technological change*. In this case, it will be necessary to measure the constant quality outputs produced by the household production functions directly instead of using the indirect methods that rely on inputs that were used in the previous models.

In spite of these difficulties, there is no doubt that the allocation of time plays an important role in determining household welfare. Hopefully, future research will address some of the previous problems.

16. Aggregate Cost of Living Indices

In previous sections, we have considered the theory of the cost of living index for only a single consumer or household. In this section, we consider some of the problems involved

[133] The period 0 and 1 levels of household real full consumption are set equal to $U^0 \equiv F^0 = P^0 \cdot Q^0 = p^0 \cdot q^0 + w_S^{\,0}q_S^{\,0} + w_S^{\,0}[\Sigma_{n=1}^{\;N}\ t_n^{\,0}]$ and $U^1 \equiv U^0 \times Q_F(P^0,P^1,Q^0,Q^1) = U^0 \times [P^0 \cdot Q^1 P^1 \cdot Q^1/P^0 \cdot Q^0 P^1 \cdot Q^0]^{1/2}$, respectively.

[134] These two models are considered in more detail by Schreyer and Diewert (2014). Schreyer (2022) considers models along the lines considered here.

[135] The utility function $U[f^1(q_1,t_1), \ldots, f^N(q_N,t_N)]$ should be replaced by $U[f^1(q_1,t_1), \ldots, f^N(q_N,t_N),t_L]$ for the Becker model, where $U[f^1(q_1,t_1), \ldots, f^N(q_N,t_N),t_L]$ is decreasing as labour supply t_L increases. For the second model, the utility function $U[f^1(q_1,t_1), \ldots, f^{N-1}(q_{N-1},t_{N-1}),f^N(q_N,t_N + q_S)]$ should be replaced by $U[f^1(q_1,t_1), \ldots, f^{N-1}(q_{N-1},t_{N-1}),f^N(q_N,t_N + q_S),t_N]$ where this function could be decreasing in the household's supply of time spent t_N on final demand activity N.

[136] A corner solution to a household utility maximization problem is one where the nonnegativity constraints in the consumer's constrained utility maximization problem is binding (that is, some q_n or t_n are equal to 0) and hence cannot be ignored. See Diewert, Fox, and Schreyer (2018) for the analysis of corner solutions.

[137] See Diewert, Fox, and Schreyer (2018) for approaches to the econometric estimation problems. The econometrics of consumer demand models where there are two constraints instead of a single budget constraint is not a well-developed area.

[138] There are also complications due to changes in the composition of households over time resulting from demographic changes.

in the construction of a cost of living index when there are many households or regions in the economy and the goal is the production of a national index. Later, we allow for an arbitrary number of households, H, so in principle, each household in the economy under consideration could have its own consumer price index. However, in practice, it will be necessary to group households into various classes and within each class, it will be necessary to assume that the group of households in the class behaves as if it were a single household in order to apply the economic approach to index number theory.[139] Our partition of the economy into H household classes can also be given a regional interpretation: Each household class can be interpreted as a group of households within a region of the country under consideration.

In this section, we will consider an economic approach to the CPI that was initiated by Pollak (1980) (1981), who called his index concept a *social cost of living index*. It is a straightforward extension of the *Konüs Cost of Living Index* (COLI) for an individual household to a group of households.

Suppose that there are H households (or regions) in the economy and suppose further that there are N commodities in the economy in periods 0 and 1 that households consume in the two periods. Denote an N-dimensional vector of commodity consumption in a given period by $q \equiv (q_1,q_2, \ldots, q_N)$ as usual. Denote the vector of period t market prices faced by household h by $p_h^{\,t} \equiv (p_{h1}^{\,t},p_{h2}^{\,t}, \ldots,p_{hN}^{\,t})$ for $t = 0,1$. Denote the corresponding observed consumption vector for household h in period t by $q_h^{\,t} \equiv (q_{h1}^{\,t},q_{h2}^{\,t}, \ldots,q_{hN}^{\,t})$ for $t = 0,1$. Note that we are *not* assuming that each household faces the same vector of commodity prices. The preferences of household h over different combinations of market commodities q is represented by the continuous utility function $f^h(q)$ for $h = 1,2, \ldots,H$.[140] For periods $t = 0,1$ and for households $h = 1,2, \ldots,H$, it is assumed that the observed household h consumption vector $q_h^{\,t} \equiv (q_{h1}^{\,t}, \ldots,q_{hN}^{\,t})$ is a solution to the following household h expenditure minimization problem:

$$\min_q \{p_h^{\,t}\cdot q : f^h(q) \geq u_h^{\,t}\} \equiv C^h(u_h^{\,t},p_h^{\,t}) = p_h^{\,t}\cdot q_h^{\,t}; \quad t = 0,1;\ h = 1,2, \ldots H, \tag{248}$$

where $u_h^{\,t} \equiv f^h(q_h^{\,t})$ is the utility level achieved by household h during period t and C^h is the cost or expenditure function that is dual to the utility function f^h. Basically, these assumptions mean that each household has *stable preferences* over the same list of commodities during the two periods under consideration, the same households appear in each period, and each household chooses its consumption bundle in the most cost-efficient way during each period. Let p^t be defined as the period t price vector of dimension HN that combines all of the household-specific period t observed price vectors $p_1^{\,t}, \ldots,p_H^{\,t}$ into one big price vector, and let q^t be the companion economy-wide quantity vector that combines all of the observed period t quantity vectors $q_1^{\,t}, \ldots,q_H^{\,t}$ into a single vector of dimension HN. Let q be a reference quantity vector of dimension HN; that is, $q \equiv [q_{11}, \ldots,q_1N;q_{21}, \ldots,q_2N; \ldots;q_{H1}, \ldots,q_{HN}]$.

With the previous definitions in mind, the family of *social cost of living indices* or *aggregate Konüs cost of living indices* for the group of households under consideration is defined as follows:[141]

$$P_K(p^0,p^1,q) \equiv \Sigma_{h=1}^{H}\, C^h(f^h(q_h),p_h^{\,1})/\Sigma_{h=1}^{H}\, C^h(f^h(q_h),p_h^{\,0}). \tag{249}$$

The numerator on the right-hand side of (249) is the sum over households of the minimum cost, $C^h(u_h,p_h^{\,1})$, for household h to achieve the reference utility level $u_h \equiv f^h(q_h)$ given that the household h faces the period 1 vector of prices $p_h^{\,1}$. The denominator on the right-hand side of (249) is the sum over households of the minimum cost, $C^h(u_h,p_h^{\,0})$, for household h to achieve the *same* reference utility level u_h, given that the household faces the period 0 vector of prices $p_h^{\,0}$. Thus, in the numerator and denominator of (249), only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index.

We now specialize the general definition (249) by replacing the general utility vector u by either the period 0 vector of household utilities $u^0 \equiv (u_1^{\,0},u_2^{\,0}, \ldots u_H^{\,0})$ or the period 1 vector of household utilities $u^1 \equiv (u_1^{\,1},u_2^{\,1}, \ldots u_H^{\,1})$. The choice of the base period vector of utility levels leads to the *Laspeyres–Konüs cost of living index*, $P_K(p^0,p^1,q^0)$, while the choice of the period 1 vector of utility levels leads to the *Paasche–Konüs cost of living index*, $P_K(p^0,p^1,q^1)$. It turns out that these two indices satisfy some inequalities, which are counterparts to the inequalities (3) and (4) in Section 2.

$$\begin{aligned} P_K(p^0,p^1,q^0) &\equiv \Sigma_{h=1}^{H}\, C^h(u_h^{\,0},p_h^{\,1})/\Sigma_{h=1}^{H}\, C^h(u_h^{\,0},p_h^{\,0}), \\ &\quad \text{where } u_h^{\,0} \equiv f^h(q_h^{\,0}) \text{ for } h = 1, \ldots,H \qquad (250) \\ &= \Sigma_{h=1}^{H}\, C^h(u_h^{\,0},p_h^{\,1})/\Sigma_{h=1}^{H}\, p_h^{\,0}\cdot q_h^{\,0} \text{ using (248) for } t = 0^{142} \\ &\leq \Sigma_{h=1}^{H}\, p_h^{\,1}\cdot q_h^{\,0}/\Sigma_{h=1}^{H}\, p_h^{\,0}\cdot q_h^{\,0} \\ &\quad \text{since } q_h^{\,0} \text{ is feasible for the cost minimization problem } C^h(u_h^{\,0},p_h^{\,1}) \text{ for } h = 1,2, \ldots,H \\ &\equiv P_L(p^0,p^1,q^0,q^1), \end{aligned}$$

where $P_L(p^0,p^1,q^0,q^1)$ is defined to be the economy-wide observable (in principle) *Laspeyres price index*, $\Sigma_{h=1}^{H}\, p_h^{\,1}\cdot q_h^{\,0}/\Sigma_{h=1}^{H}\, p_h^{\,0}\cdot q_h^{\,0} = p^1\cdot q^0/\, p^0\cdot q^0$, which treats each household consumption vector as a separate commodity so that p^0, p^1, and q^0 are HN-dimensional vectors.

The inequality (250) says that the theoretical Laspeyres–Konüs cost of living index, $P_K(p^0,p^1,q^0)$, is bounded from above by the observable Laspeyres price index P_L. In a similar manner, specializing definition (249), *the Paasche–Konüs*

[139] The problems associated with grouping households will be discussed in Section 18.

[140] As usual, we assume that each $f^h(q)$ is continuous, concave, and increasing in the components of q.

[141] See Pollak (1980, 276) (1981, 328) (1989, 182) and Diewert (1983, 190–92) (2001, 170) for additional materials on social cost of living indices.

[142] It can be seen that $P_K(p^0,p^1,q^0)$ is also equal to a weighted average of the individual Laspeyres Konüs cost of living indices; that is, $P_K(p^0,p^1,q^0) = \Sigma_{h=1}^{H}\, S_h^{\,0} C^h(u_h^{\,0},p_h^{\,1})/C^h(u_h^{\,0},p_h^{\,0})$, where $S_h^{\,0} \equiv p_h^{\,0}\cdot q_h^{\,0}/\Sigma_{i=1}^{H}\, p_i^{\,0}\cdot q_i^{\,0}$ for $h = 1, \ldots,H$. Since the weights for the individual household cost of living indices are equal to the household's share of total nominal consumption in period 0, $P_K(p^0,p^1,q^0)$ is a *plutocratic aggregate cost of living index* to use the terminology of Prais (1959). Prais (1959) defined a *democratic COLI* as $\Sigma_{h=1}^{H}\, (1/H)C^h(u_h^{\,0},p_h^{\,1})/C^h(u_h^{\,0},p_h^{\,0})$.

cost of living index, $P_K(p^0,p^1,q^1)$, satisfies the following inequality:

$$P_K(p^0,p^1,q^1) \equiv \Sigma_{h=1}^{H} C^h(u_h^1,p_h^1)/\Sigma_{h=1}^{H} C^h(u_h^1,p_h^0), \text{ where } u_h^1 \equiv f^h(q_h^1,e_h^1) \text{ for } h = 1,\ldots,H \quad (251)$$
$$= \Sigma_{h=1}^{H} p_h^1\cdot q_h^1 /\Sigma_{h=1}^{H} C^h(u_h^1,p_h^0) \text{ using (248) for } t = 1^{143}$$
$$\geq \Sigma_{h=1}^{H} p_h^1\cdot q_h^1/\Sigma_{h=1}^{H} p_h^0\cdot q_h^1 \text{ using feasibility arguments}$$
$$\equiv P_P(p^0,p^1,q^0,q^1),$$

where $P_P(p^0,p^1,q^0,q^1)$ is defined as the observable (in principle) *Paasche price index*, $\Sigma_{h=1}^{H} p_h^1\cdot q_h^1/\Sigma_{h=1}^{H} p_h^0\cdot q_h^1 = p^1\cdot q^1/p^0\cdot q^1$. The inequality (251) says that the theoretical Paasche–*Konüs* cost of living index, $P_K(p^0,p^1,q^1)$, is bounded from below by the observable Paasche price index P_P.

It is possible to find two-sided bounds for a Konüs cost of living index; that is, we have the following proposition:

Proposition 15: Under suitable continuity assumptions on preferences, there exists a number λ^* between 0 and 1 such that

$$P_L \leq P_K(p^0,p^1,\lambda^* q^0 + (1-\lambda^*)q^1) \leq P_P \text{ or } P_P \leq P_K(p^0,p^1,\lambda^* q^0 + (1-\lambda^*)q^1) \leq P_L, \quad (252)$$

where $P_L \equiv p^1\cdot q^0/ p^0\cdot q^0$ and $P_P \equiv p^1\cdot q^1/ p^0\cdot q^1$. The proof of Proposition 15 is similar to the proof of Proposition 1; see Diewert (2001, 173) for details.

This result tells us that *the theoretical aggregate Konüs cost of living index CPI* $P_K(p^0,p^1,q^*)$ lies between the observable Laspeyres index P_L and the Paasche index P_P, where $q^* \equiv \lambda^* q^0 + (1-\lambda^*)q^1$ is an intermediate quantity vector that lies between q^0 and q^1. Hence, if P_L and P_P are not too different, a good approximation to a theoretical aggregate cost of living index will be the *Fisher index* $P_F(p^0,p^1,q^0,q^1)$ defined as $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)P_P(p^0,p^1,q^0,q^1)]^{1/2}$. This Fisher price index is computed just like the usual Fisher price index, except that each commodity in each region (or for each household) is regarded as a separate commodity.

It is possible to obtain an alternative estimator for an aggregate cost of living index if stronger assumptions on household preferences are made. Thus, assume that the preferences of household h are represented by the linearly homogeneous utility function $f^h(q_h) \equiv [q_h^T A^h q_h]^{1/2}$, where A^h is a symmetric matrix which satisfies the regularity conditions discussed in Section 5 for $h = 1,\ldots,H$. Under these assumptions, the Fisher price and quantity indices will be exact for these preferences; see Section 5. Let $c^h(p_h) = c^h(p_{h1},\ldots,p_{hN})$ be the unit cost function that corresponds to $f^h(q_h)$ for $h = 1,\ldots,H$. Assuming utility-maximizing behavior on the part of each household, the following equations will be satisfied:

$$p_h^t\cdot q_h^t = f^h(q_h^t)c^h(p_h^t);\ h = 1,\ldots,H;\ t = 0,1. \quad (253)$$

Now use Fisher price and quantity indices to estimate household quantity and price levels, $Q_h^t \equiv f^h(q_h^t)$ and $P_h^t \equiv c^h(p_h^t)$, for $t = 0,1$ and $h = 1,\ldots,H$ as follows:

$$P_h^0 \equiv 1 \equiv c^h(p_h^0);\ Q_h^0 \equiv p_h^0\cdot q_h^0 \equiv f^h(q_h^0);\ h = 1,\ldots,H; \quad (254)$$
$$P_h^1 \equiv P_F(p_h^0,p_h^1,q_h^0,q_h^1) \equiv c^h(p_h^1);\ Q_h^1 \equiv [p_h^1\cdot q_h^1]/P_h^1;\ h = 1,\ldots,H. \quad (255)$$

Under our new assumption of homothetic preferences for each household, definition (250) for the *Laspeyres–Konüs cost of living index* $P_K(p^0,p^1,q^0)$ simplifies into the following equations:

$$P_K(p^0,p^1,q^0) \equiv \Sigma_{h=1}^{H} C^h(u_h^0,p_h^1)/\Sigma_{h=1}^{H} C^h(u_h^0,p_h^0), \text{ where } u_h^0 \equiv f^h(q_h^0) \text{ for } h = 1,\ldots,H \quad (256)$$
$$= \Sigma_{h=1}^{H} u_h^0 c^h(p_h^1)/\Sigma_{h=1}^{H} u_h^0 c^h(p_h^0) \text{ since } C^h(u_h,p_h) = u_h c(p_h) \text{ for each } h$$
$$= \Sigma_{h=1}^{H} P_h^1 Q_h^0/\Sigma_{h=1}^{H} P_h^0 Q_h^0 \text{ using (254) and (255)}$$
$$= P_L(P^0,P^1,Q^0,Q^1),$$

where $P^t \equiv [P_1^t,\ldots,P_H^t]$ and $Q^t \equiv [Q_1^t,\ldots,Q_H^t]$ for $t = 0,1$ and $P_L(P^0,P^1,Q^0,Q^1)$ is the ordinary Laspeyres price index using the aggregate household prices and quantities for the two periods under consideration as the price and quantity variables.

Similarly, definition (251) for the *Paasche–Konüs cost of living index* $P_K(p^0,p^1,q^1)$ simplifies into the following equations:

$$P_K(p^0,p^1,q^1) \equiv \Sigma_{h=1}^{H} C^h(u_h^1,p_h^1)/\Sigma_{h=1}^{H} C^h(u_h^1,p_h^0), \text{ where } u_h^1 \equiv f^h(q_h^1) \text{ for } h = 1,\ldots,H \quad (257)$$
$$= \Sigma_{h=1}^{H} u_h^1 c^h(p_h^1)/\Sigma_{h=1}^{H} u_h^1 c^h(p_h^0) \text{ since } C^h(u_h,p_h) = u_h c(p_h) \text{ for each } h$$
$$= \Sigma_{h=1}^{H} P_h^1 Q_h^1/\Sigma_{h=1}^{H} P_h^0 Q_h^1 \text{ using (254) and (255)}$$
$$\equiv P_P(P^0,P^1,Q^0,Q^1),$$

where $P_P(P^0,P^1,Q^0,Q^1)$ is the ordinary Paasche price index using the aggregate household prices and quantities for the two periods under consideration as the price and quantity variables.

The aggregate price indices $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$ defined by (256) and (257) are equally plausible measures of overall consumer price inflation between periods 0 and 1 and so it is reasonable to take an average of these two indices to obtain a "final" estimate of inflation between the two periods. As usual, the geometric average leads to an index that will satisfy a time reversal test. Thus, we have

$$[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2} = [P_L(P^0,P^1,Q^0,Q^1) P_P(P^0,P^1,Q^0,Q^1)]^{1/2} \equiv P_F(P^0,P^1,Q^0,Q^1), \quad (258)$$

where $P_F(P^0,P^1,Q^0,Q^1)$ is the *Fisher index* defined over the aggregate household prices and quantities for the two periods under consideration. It is actually a two-stage Fisher

[143] It can be verified that $P_K(p^0,p^1,q^1)$ is equal to the following weighted harmonic average of the individual Paasche Konüs cost of living indices: $P_K(p^0,p^1,q^1) = \{\Sigma_{h=1}^{H} S_h^1[C^h(u_h^1,p_h^1)/C^h(u_h^1,p_h^0)]^{-1}\}^{-1}$, where $S_h^1 \equiv p_h^1\cdot q_h^1/\Sigma_{i=1}^{H} p_i^1\cdot q_i^1$ for $h = 1,\ldots,H$.

index where the first stage of aggregation uses the price and quantity data for each household to construct household-specific Fisher price and quantity levels for each household. The two-stage Fisher price index $P_F(P^0,P^1,Q^0,Q^1)$ defined by (258) can be compared to the single-stage Fisher price index $P_F(p^0,p^1,q^0,q^1)$ defined earlier as the geometric mean of $P_L(p^0,p^1,q^0,q^1)$ and $P_P(p^0,p^1,q^0,q^1)$ defined by (250) and (251). Using the results listed in Section 8, we know that the single-stage Fisher index will approximate its two-stage counterpart to the second order around an equal price and quantity point. Thus, normally, we would not expect much difference between these alternative measures of overall consumer price inflation.

In the following section, we turn our attention to the definition of aggregate quantity indices.

17. Aggregate Allen Quantity Indices

Recall the definition of the Allen quantity index for a single household defined in Section 11. In this section, we will generalize this index concept to cover the case of many households.

Make the same assumptions on households and their preference functions that were made at the beginning of the previous section. Again assume that the observed household h consumption vector $q_h^t \equiv (q_{h1}^t, \ldots, q_{hN}^t)$ is a solution to the following household h expenditure minimization problem defined by (248) for $t = 0,1$ and $h = 1, \ldots, H$. Using the same notation that was used at the beginning of the previous section, the family of *aggregate Allen quantity indices* for the group of households under consideration is defined as follows:

$$Q_A(q^0,q^1,p) \equiv \Sigma_{h=1}^H C^h(f^h(q_h^1),p_h)/\Sigma_{h=1}^H C^h(f^h(q_h^0),p_h) = \Sigma_{h=1}^H C^h(u_h^1,p_h)/\Sigma_{h=1}^H C^h(u_h^0,p_h), \quad (259)$$

where $u_h^t \equiv f^h(q_h^t)$ for $t = 0,1$ and $h = 1, \ldots, H$ and $p \equiv [p_1, \ldots p_H]$ is an NH-dimensional vector of reference prices.

Note that in the numerator and denominator of the last equation in (259), only the household utility variables are different, which is appropriate for an overall measure of household utility which in turn is an overall quantity or volume measure. Note also that if $H = 1$, definition (259) reduces to the definition of an Allen (1949) quantity index.

We now specialize the general definition (259) by replacing the reference price vector p by either the period 0 economy-wide price vector p^0 or the period 1 economy-wide price vector p^1. Thus, define the *Laspeyres aggregate Allen quantity index* by $Q_A(q^0, q^1,p^0)$ and the *Paasche aggregate Allen quantity index* by $Q_A(q^0, q^1,p^1)$. It turns out that these two indices satisfy some inequalities, which are counterparts to the inequalities (3) and (4) discussed in Section 2. Thus, choosing $p = p^0$ leads to the following index:

$$Q_A(q^0,q^1,p^0) \equiv \Sigma_{h=1}^H C^h(f^h(q_h^1),p_h^0)/\Sigma_{h=1}^H C^h(f^h(q_h^0),p_h^0) \quad (260)$$

$$= \Sigma_{h=1}^H C^h(f^h(q_h^1),p_h^0)/\Sigma_{h=1}^H p_h^0 \cdot q_h^0 \text{ using (248) for } t = 0^{144}$$

$$\leq \Sigma_{h=1}^H p_h^0 \cdot q_h^1/\Sigma_{h=1}^H p_h^0 \cdot q_h^0$$

since q_h^1 is feasible for the cost minimization problem $C^h(f^h(q_h^1),p_h^0)$ for $h = 1,2, \ldots, H$

$$\equiv Q_L(p^0,p^1,q^0,q^1),$$

where $Q_L(p^0,p^1,q^0,q^1)$ is defined as the observable (in principle) *Laspeyres quantity index*, $\Sigma_{h=1}^H p_h^0 \cdot q_h^1/\Sigma_{h=1}^H p_h^0 \cdot q_h^0 = p^0 \cdot q^1/ p^0 \cdot q^0$, which treats each household consumption vector as a separate commodity so that p^0, q^0, and q^1 are HN-dimensional vectors.

The inequality (260) says that the theoretical Laspeyres Allen aggregate quantity index, $Q_A(q^0, q^1,p^0)$, is bounded from above by the observable Laspeyres quantity index Q_L. In a similar manner, specializing definition (259) by setting $p = p^1$, *the Paasche–Allen aggregate quantity index*, $Q_A(q^0, q^1, p^1)$, satisfies the following inequality:

$$Q_A(q^0,q^1,p^1) \equiv \Sigma_{h=1}^H C^h(f^h(q_h^1),p_h^1)/\Sigma_{h=1}^H C^h(f^h(q_h^0),p_h^1) \quad (261)$$

$$= \Sigma_{h=1}^H p_h^1 \cdot q_h^1/\Sigma_{h=1}^H C^h(f^h(q_h^0),p_h^1) \text{ using (248) for } t = 1^{145}$$

$$\geq \Sigma_{h=1}^H p_h^1 \cdot q_h^1/\Sigma_{h=1}^H p_h^1 \cdot q_h^0$$

since q_h^0 is feasible for the cost minimization problem $C^h(f^h(q_h^0),p_h^1)$ for $h = 1,2, \ldots, H$

$$\equiv Q_P(p^0,p^1,q^0,q^1),$$

where $Q_P(p^0,p^1,q^0,q^1)$ is defined as the observable (in principle) *Paasche quantity index*, $\Sigma_{h=1}^H \mathrm{p}_h^1 \cdot \mathrm{q}_h^1/\Sigma_{h=1}^H p_h^1 \cdot q_h^0 = p^1 \cdot q^1/ p^1 \cdot q^0$. The inequality (261) says that the theoretical Paasche–Allen aggregate quantity index, $Q_A(q^0, q^1, p^1)$, is bounded from below by the observable Paasche quantity index $Q_P \equiv p^1 \cdot q^1/ p^1 \cdot q^0$.

As usual, it is possible to find two-sided bounds for a relevant Allen aggregate quantity index; that is, we have the following proposition:

Proposition 16: Under our regularity conditions, there exists a number λ^* between 0 and 1 such that

$$Q_L \leq Q_A(q^0,q^1,\lambda^* p^0 + (1 - \lambda^*)p^1) \leq Q_P \text{ or}$$
$$Q_P \leq Q_A(q^0,q^1,\lambda^* p^0 + (1 - \lambda^*)p^1) \leq Q_L, \quad (262)$$

where $Q_L \equiv p^0 \cdot q^1/ p^0 \cdot q^0$ and $Q_P \equiv p^1 \cdot q^1/ p^1 \cdot q^0$. The proof of Proposition 16 is similar to the proof of Proposition 1.

This result tells us that *the theoretical aggregate Allen quantity index*, $Q_A(q^0, q^1,\lambda^* p^0 + (1 - \lambda^*) p^1)$, lies between the observable Laspeyres and Paasche quantity indices, Q_L and Q_P, where the reference price vector is the intermediate price vector, $\lambda^* p^0 + (1 - \lambda^*) p^1$. Hence, if Q_L and Q_P are not too different, a good approximation to a theoretical aggregate quantity index will be the *single-stage Fisher quantity index* $Q_F(p^0,p^1,q^0,q^1)$ defined as $[p^0 \cdot q^1 p^1 \cdot q^1/p^0 \cdot q^0 p^1 \cdot q^0]^{1/2}$. This

[144] It can be seen that $Q_A(q^0,q^1, p^0)$ is equal to a weighted average of the individual household Laspeyres Allen quantity indices; that is, $Q_A(q^0,q^1, p^0) = \Sigma_{h=1}^H S_h^0 C^h(u_h^1,p_h^0)/C^h(u_h^0,p_h^0)$, where $S_h^0 \equiv p_h^0 \cdot q_h^0/\Sigma_{i=1}^H p_i^0 \cdot q_i^0$ for $h = 1, \ldots, H$. Since the weights for the individual household quantity indices are equal to the household's share of total nominal consumption in period 0, $Q_A(q^0,q^1,p^0)$ can be interpreted as a *plutocratic aggregate quantity index*. A *democratic aggregate quantity index* can be defined as $\Sigma_{h=1}^H (1/H)[C^h(u_h^1,p_h^0)/C^h(u_h^0,p_h^0)]$.

[145] It can be seen that $Q_A(q^0,q^1,p^1)$ is equal to a weighted harmonic average of the individual household Paasche Allen quantity indices; that is, $Q_A(q^0,q^1,p^1) = \{\Sigma_{h=1}^H S_h^1 [C^h(u_h^1,p_h^0)/C^h(u_h^0,p_h^0)]^{-1}\}^{-1}$.

single-stage Fisher quantity index is computed just like the usual Fisher quantity index, except that each commodity in each region (or for each household) is regarded as a separate commodity.

The two special cases of the family of aggregate Allen quantity indices defined by (260) and (261) are connected to the two special cases of family of Konüs cost of living indices defined by (250) and (251) in the previous section. Using these definitions, it is straightforward to show that the following two relationships hold:

$$P_K(p^0,p^1,q^0)Q_A(q^0,q^1,p^1) = \Sigma_{h=1}^{H} C^h(f^h(q_h^1),p_h^1)/\Sigma_{h=1}^{H} C^h(f^h(q_h^0),p_h^0) = \Sigma_{h=1}^{H} p_h^1 \cdot q_h^1/\Sigma_{h=1}^{H} p_h^0 \cdot q_h^0; \quad (263)$$

$$P_K(p^0,p^1,q^1)Q_A(q^0,q^1,p^0) = \Sigma_{h=1}^{H} C^h(f^h(q_h^1),p_h^1)/\Sigma_{h=1}^{H} C^h(f^h(q_h^0),p_h^0) = \Sigma_{h=1}^{H} p_h^1 \cdot q_h^1/\Sigma_{h=1}^{H} p_h^0 \cdot q_h^0. \quad (264)$$

Thus, the aggregate Laspeyres–Konüs price index $P_K(p^0,p^1,q^0)$ times the aggregate Paasche–Allen quantity index $Q_A(q^0, q^1, p^1)$ equals the aggregate value ratio for the group of households, $p^1 \cdot q^1/ p^0 \cdot q^0$, and the aggregate Paasche–Konüs price index $P_K(p^0,p^1,q^1)$ times the aggregate Laspeyres Allen quantity index $Q_A(q^0, q^1,p^0)$ also equals the aggregate value ratio, $p^1 \cdot q^1/ p^0 \cdot q^0$.

As was the case in the previous section, it is possible to obtain an alternative estimator for an aggregate quantity index if stronger assumptions on household preferences are made. Thus, as in the previous section, assume that the preferences of household h are represented by the linearly homogeneous utility function $f^h(q_h) \equiv [q_h^T A^h q_h]^{1/2}$, where A^h is a symmetric matrix, which satisfies the regularity conditions discussed in Section 5 for $h = 1, \ldots, H$. Under these assumptions, the individual household Fisher price and quantity indices, $P_F(p_h^0,p_h^1,q_h^0,q_h^1)$ and $Q_F(p_h^0,p_h^1,q_h^0,q_h^1)$, will be exact for these preferences. As in the previous section, let $c^h(p_h) = c^h(p_{h1}, \ldots, p_{hN})$ be the unit cost function that corresponds to $f^h(q_h)$ for $h = 1, \ldots, H$. Assuming utility-maximizing behavior on the part of each household, equations (253)–(255) will be satisfied.

Under the aforementioned homothetic utility function assumptions on household preferences, definition (260) for the Laspeyres Allen aggregate quantity index, $Q_A(q^0, q^1,p^0)$, simplifies into the following expression:

$$Q_A(q^0,q^1,p^0) \equiv \Sigma_{h=1}^{H} C^h(u_h^1,p_h^0)/\Sigma_{h=1}^{H} C^h(u_h^0,p_h^0), \text{ where } u_h^t \equiv f^h(q_h^t) \text{ for } h = 1, \ldots, H \text{ and } t = 0,1 \quad (265)$$

$$= \Sigma_{h=1}^{H} u_h^1 c^h(p_h^0)/\Sigma_{h=1}^{H} u_h^0 c^h(p_h^0) \text{ since } C^h(u_h,p_h) = u_h c(p_h) \text{ for each } h$$

$$= \Sigma_{h=1}^{H} P_h^0 Q_h^1/\Sigma_{h=1}^{H} P_h^0 Q_h^0 \text{ using (254) and (255)}$$

$$= Q_L(P^0,P^1,Q^0,Q^1),$$

where $P^t \equiv [P_1^t, \ldots, P_H^t]$ and $Q^t \equiv [Q_1^t, \ldots, Q_H^t]$ for $t = 0,1$ and $Q_L(P^0,P^1,Q^0,Q^1)$ is the ordinary Laspeyres quantity index using the aggregate household prices and quantities, P^t and Q^t, for the two periods under consideration as the household aggregate price and quantity variables.

Similarly, definition (261) for the Paasche–Allen aggregate quantity index $Q_A(q^0, q^1, p^1)$ simplifies into the following expression:

$$Q_A(q^0,q^1,p^1) \equiv \Sigma_{h=1}^{H} C^h(u_h^1,p_h^1)/\Sigma_{h=1}^{H} C^h(u_h^0,p_h^1) \quad (266)$$

$$= \Sigma_{h=1}^{H} u_h^1 c^h(p_h^1)/\Sigma_{h=1}^{H} u_h^0 c^h(p_h^1) \text{ since } C^h(u_h,p_h) = u_h c(p_h) \text{ for each } h$$

$$= \Sigma_{h=1}^{H} P_h^1 Q_h^1/\Sigma_{h=1}^{H} P_h^1 Q_h^0 \text{ using (254) and (255)}$$

$$\equiv Q_P(P^0,P^1,Q^0,Q^1),$$

where $Q_P(P^0,P^1,Q^0,Q^1)$ is the ordinary Paasche quantity index using the aggregate household prices and quantities for the two periods under consideration as the price and quantity variables.

The aggregate quantity indices $Q_A(q^0, q^1,p^0)$ and $Q_A(q^0, q^1, p^1)$ defined by (265) and (266) are equally plausible measures of overall consumer quantity or volume growth between periods 0 and 1 and so it is reasonable to take an average of these two indices to obtain a "final" estimate of aggregate quantity growth between the two periods. As usual, the geometric average leads to an index that will satisfy a time reversal test. Thus, we have

$$[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2} = [Q_L(P^0,P^1,Q^0,Q^1)Q_P(P^0,P^1,Q^0,Q^1)]^{1/2} \equiv Q_F(P^0,P^1,Q^0,Q^1), \quad (267)$$

where $Q_F(P^0,P^1,Q^0,Q^1)$ is the *Fisher quantity index* defined over the aggregate household prices and quantities for the two periods under consideration. It is a two-stage Fisher index where the first stage of aggregation uses the price and quantity data for each household to construct household-specific Fisher price and quantity levels for each household. The two-stage Fisher quantity index $Q_F(P^0,P^1,Q^0,Q^1)$ defined by (267) can be compared to the single-stage Fisher quantity index $Q_F(p^0,p^1,q^0,q^1)$ defined as the geometric mean of $Q_L(p^0,p^1,q^0,q^1) \equiv p^0 \cdot q^1/ p^0 \cdot q^0$ and $Q_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/p^1 \cdot q^0$. Using the results listed in Section 8, we know that the single-stage Fisher quantity index will approximate its two-stage counterpart to the second order around an equal price and quantity point. Thus, normally, we would not expect much difference between these alternative measures of overall real aggregate consumption growth.

18. Social Welfare Functions and Inequality Indices

Equations (265) and (266) have some interesting implications. These equations give the following decompositions for an aggregate quantity index: $Q_A(q^0,q^1,p^0) = \Sigma_{h=1}^{H} u_h^1 c^h(p_h^0)/\Sigma_{h=1}^{H} u_h^0 c^h(p_h^0)$ and $Q_A(q^0, q^1, p^1) = \Sigma_{h=1}^{H} u_h^1 c^h(p_h^1)/\Sigma_{h=1}^{H} u_h^0 c^h(p_h^1)$. The numerators in these equations can be interpreted as aggregate period 1 quantity levels and the denominators as aggregate period 0 quantity levels. These quantity levels have the same general form; that is, the period t aggregate quantity level Q^t is equal to a weighted sum of the period t household utility levels so that $Q^t \equiv \Sigma_{h=1}^{H} \omega_h u_h^t$ for $t = 0,1$, where the weights ω_h are fixed nonnegative numbers. Functions like $\Sigma_{h=1}^{H} \omega_h u_h^t$ are called *social welfare functions* in the economics literature. Thus, the two aggregate Allen indices can be regarded as specific examples where the indices are equal to ratios of social welfare functions.

Choosing the appropriate weights for a social welfare function is a nontrivial problem, which has not been completely resolved in the economics literature but there is a demand for statistical agencies to produce measures of social welfare that take into account possible inequality in the distribution of income between households.[146] We will not go into great detail on the complex issues surrounding the measurement of social welfare but we will indicate some of the problems that are associated with the construction of indices of social welfare.

The first problem that needs to be addressed is that the individual household utility measures have to be made cardinally comparable in some way. Recall the assumptions made on household preferences made before equations (253). In order to construct meaningful measures for the levels of social welfare, it is necessary to make stronger assumptions; that is, we now assume that the preferences of household h are represented by the linearly homogeneous utility function $f^h(q_h) \equiv [q_h^T A q_h]^{1/2}$ for each h where A is a symmetric matrix, which satisfies the regularity conditions discussed in Section 5. Thus, under these stronger assumptions, we are now assuming that the household preference functions are *identical* across households for $h = 1, \ldots, H$. Under these assumptions, the Fisher price and quantity indices will be exact across households within a time period as well as across time periods. Let $c(p_h) = c(p_{h1}, \ldots, p_{hN})$ be the unit cost function that corresponds to $f(q_h)$ for $h = 1, \ldots, H$. Assuming utility-maximizing behavior on the part of each household, the following equations should be satisfied:

$$p_h^t \cdot q_h^t = f(q_h^t)c(p_h^t);\ h = 1, \ldots, H;\ t = 0,1. \qquad (268)$$

Now use Fisher price and quantity indices to estimate household quantity and price levels, $Q_h^t \equiv f(q_h^t)$ and $P_h^t \equiv c(p_h^t)$, for $t = 0,1$ and $h = 1, \ldots, H$ as follows:

$$P_1^0 \equiv 1 \equiv c(p_1^0);\ Q_1^0 \equiv p_1^0 \cdot q_1^0 \equiv f(q_1^0) \equiv u_1^0; \qquad (269)$$

$$P_h^0 \equiv P_F(p_1^0, p_h^0, q_1^0, q_h^0) \equiv c(p_h^0)\ ;\ Q_h^0 \equiv p_h^0 \cdot q_h^0 / P_h^0 \equiv f(q_h^0) \equiv u_h^0\ ;\ h = 2, \ldots, H; \qquad (270)$$

$$P_h^1 \equiv P_F(p_1^0, p_h^1, q_1^0, q_h^1) \equiv c(p_h^1)\ ;\ Q_h^1 \equiv [p_h^1 \cdot q_h^1]/P_h^1 \equiv f(q_h^1) \equiv u_h^1;\ h = 1, \ldots, H. \qquad (271)$$

Thus, household 1 in period 0 acts as a *numeraire household*; the Fisher price and quantity indices for the other households in periods 0 and 1 are computed relative to household 1 in period 0.[147] Once the cardinally comparable utility levels u_h^t have been computed using definitions (269)–(271), they can be used to determine the level of *social welfare* in each period t. For example, the period t level of social welfare could be defined as $Q^t \equiv \Sigma_{h=1}^H \omega_h u_h^t$ for $t = 0,1$, where the weights ω_h are somehow chosen by the statistical office.

However, it has proven to be difficult to come up with consensus social welfare weights for ω_h. A simple solution is to set $\omega_h = 1$ for $h = 1, \ldots, H$. The resulting function is the *utilitarian social welfare function*. However, this function shows no concern of the distribution of utility across all households. An allocation of the economy's real expenditures on consumer goods and services that gave most of the total group expenditure to one household would generate the same level of social welfare using the utilitarian function as the distribution that divided the total real expenditures equally across households. In order to address distributional issues, it is necessary to introduce nonlinear social welfare functions.

Atkinson (1970, 257) introduced the following *mean of order r social welfare function*:[148]

$$W^r(u_1, \ldots, u_H) \equiv [\Sigma_{h=1}^H (1/H)(u_h)^r]^{1/r}, \qquad (272)$$

where $r \le 1$ and $r \ne 0$.[149] Note that $W^r(u_1, \ldots, u_H)$ is a measure of per capita utility rather than a measure of total utility for the period under consideration. Using the earlier materials on CES utility functions, we know that $W^r(u_1, \ldots, u_H) \equiv W^r(u)$ is a linearly homogeneous, concave increasing function of the household utility levels, $u \equiv [u_1, \ldots, u_H]$. When $r = 1$, $W^1(u) = S_{h=1}^H (1/H)u_h$ which is *per capita utility*. As r approaches minus infinity, $W^r(u_1, \ldots, u_H)$ approaches $\min_h \{u_h : h = 1, \ldots, H\}$, which is the social welfare function advocated by Rawls (1971).[150]

It proves to be useful to compare an Atkinson measure of social welfare $W^r(u_1, \ldots, u_H)$ with per capita utility for each period. *Period t per capita utility* is defined as follows:

$$u_A^t \equiv \Sigma_{h=1}^H (1/H)u_h^t \equiv W^1(u_1^t, \ldots, u_H^t);\ t = 0,1. \qquad (273)$$

Thus, per capita utility is a special case of the Atkinson family of social welfare measures with $r = 1$. For a general $r < 1$, Atkinson's (1970, 250) period *t equally distributed equivalent real income per head*, u_E^t, is defined (implicitly) by the following equation:

$$W^r(u_1^t, \ldots, u_H^t) = W^r(u_E^t 1_H);\ t = 0,1 \qquad (274)$$

$$= u_E^t W^r(1_H) \text{ using the linear homogeneity property of } W^r(u_1, \ldots, u_H)$$

$$= u_E^t \text{ using definition (272) which implies } W^r(1_H) = 1.$$

Thus, actual social welfare in period t, $W^r(u_1^t, \ldots, u_H^t)$, is set equal to a level of social welfare where each household gets the same level of utility, u_E^t. Hardy, Littlewood, and Polyá (1934, 26) show that the mean of order r function, $W^r(u_1, \ldots, u_H)$, is increasing in r provided that not all u_h are the same and nondecreasing in r in general. Since $r < 1$, $W^r(u_1^t, \ldots, u_H^t) \le W^1(u_1^t, \ldots, u_H^t)$ for $t = 0,1$. Using these

[146] See Hays, Martin, and Mkandawire (2019).

[147] This is known as a "star" approach to the construction of multilateral indices, and the resulting indices will depend on the choice of the numeraire household. We will introduce more symmetric methods for making multilateral comparisons in Chapter 7.

[148] Atkinson worked with continuous distributions of nominal incomes, whereas we work with discrete distributions of real incomes. Fleurbaey (2009, 1032) has a discrete version of Atkinson's approach, which is similar to the approach presented here except that nominal incomes are used in place of our real incomes. Finally, Jorgenson and Schreyer (2017, S466) use a version of the approach presented here except they assume all households face the same prices.

[149] As usual, if $r = 0$, define the logarithm of $W^0(u_1, \ldots, u_H)$ as $\Sigma_{h=1}^H (1/H) \ln u_h$.

[150] See also Blackorby and Donaldson (1978).

inequalities and definitions (273) and (274), we have the following inequalities:

$$u_E{}^t/u_A{}^t \le 1;\ t = 0,1. \qquad (275)$$

Kolm's (1969, 186) period *t index of relative injustice* or Atkinson's (1970, 257) and Sen's (1973, 42) period *t relative inequality index*, I^t, is defined as follows:

$$I^t \equiv 1 - (u_E{}^t/u_A{}^t) \ge 0;\ t = 0,1. \qquad (276)$$

Thus, if household utility levels in period t are identical, $u_E{}^t$ will equal $u_A{}^t$ and period t inequality I^t will equal 0. If r is a very large negative number and one or more households in period t has a very low utility level, then $u_E{}^t$ will be close to 0 and I^t will be close to 1, the maximum amount of inequality that can occur.

Define the *period t equality index* as

$$E^t \equiv u_E{}^t/u_A{}^t;\ t = 0,1. \qquad (277)$$

Thus, the closer E^t is to its maximum value 1, the more equal is the distribution of real consumption in the group of households under consideration. Since period t Atkinson welfare is equal to $W^r(u_1{}^t, \ldots, u_H{}^t) = u_E{}^t$, we can write period t welfare as the product of per capita real consumption, $u_A{}^t$, times E^t:[151]

$$W^r(u_1{}^t, \ldots, u_H{}^t) = u_A{}^t E^t\ t = 0,1. \qquad (278)$$

A practical problem with this approach for measuring social welfare is that it is necessary to pick a specific value for r in order to implement it.[152] Since the results will depend on which r is chosen and since there is no general consensus on which r to choose, statistical agencies have largely not produced practical measures of social welfare. Thus, we will conclude this section by considering one more approach to the production of social welfare indices: an approach that, at first glance, does not require choosing parameters for the social welfare function.

Our final approach to the measurement of social welfare relies on a discrete version of the Gini (1921) coefficient. We first convert the household utility levels $u_h{}^t$ defined by (269)–(271) into *household shares of total utility* $\sigma_h{}^t$ for each time period:

$$\sigma_h{}^t \equiv u_h{}^t/\Sigma_{i=1}{}^H u_i{}^t;\ h = 1, \ldots, H;\ t = 0,1. \qquad (279)$$

Now order the households so that household 1 has the lowest utility in period t, household 2 has the next lowest utility, and so on. Thus, for each period t, the shares $s_h{}^t$ will satisfy the following inequalities:

$$\sigma_1{}^t \le \sigma_2{}^t \le \ldots \le \sigma_H{}^t;\ t = 0,1. \qquad (280)$$

The *area under the cumulative distribution function of the share variables* $\sigma_h{}^t$ is proportional to A^t defined as follows:

$$\begin{aligned} A^t &\equiv \sigma_1{}^t + (\sigma_1{}^t + \sigma_2{}^t) + (\sigma_1{}^t + \sigma_2{}^t + \sigma_3{}^t) + \ldots \\ &\quad + (\Sigma_{h=1}{}^{H-1} \sigma_h{}^t) + (\Sigma_{h=1}{}^H \sigma_h{}^t);\ t = 0,1 \qquad (281) \\ &= H\sigma_1{}^t + (H-1)s_2{}^t + (H-2)\sigma_3{}^t + \ldots + 2\sigma_{H-1}{}^t + \sigma_H{}^t. \end{aligned}$$

Consider the following linear programming problem:

$$\max_{\sigma_1,\cdots,\sigma_H} \{H\sigma_1 + (H-1)\sigma_2 + (H-2)\sigma_3 + \ldots + 2\sigma_{H-1} + \sigma_H : 0 \le \sigma_1{}^t \le \sigma_2{}^t \le \ldots \le \sigma_H{}^t;\ \Sigma_{h=1}{}^H \sigma_h = 1\}. \qquad (282)$$

The solution to this problem is $\sigma_h = 1/H$ for $h = 1, \ldots, H$. Substitute this solution into the objective function in (282) and this will determine the maximum value A^* for the objective function in (282):

$$A^* \equiv [H + (H-1) + (H-2) + \ldots + 2 + 1][1/H] = [H(H+1)/2][1/H] = (H+1)/2. \qquad (283)$$

Define the period *t Gini index of equality* for the distribution of household utilities, E^{t*}, as

$$E^{t*} \equiv A^t/A^* \le 1;\ t = 0,1, \qquad (284)$$

where A^t is defined by (281) and A^* is defined by (283). The inequalities $A^t/A^* \le 1$ follow since A^t is necessarily less than the maximum possible value for A^t, which is A^*. The period *t Gini coefficient* or *Gini index of inequality* for the discrete income distribution, G^t, is defined as

$$G^t \equiv 1 - E^{t*};\ t = 0,1. \qquad (285)$$

The Gini coefficient as a measure of inequality in *nominal* income distributions is well understood and well accepted in economic measurement circles. The algebra stated here simply adapts it as a measure of inequality for *real* income distributions. There are no additional parameters that have to be determined by the official statistician.[153]

The final step is to use E^{t*} to adjust per capita real consumption $u_A{}^t$ defined by definitions (273) for inequality in the real income distribution; that is, define period t welfare, W^t, as

$$W^t \equiv u_A{}^t E^{t*} = u_A{}^t(1 - G^t);\ t = 0,1. \qquad (286)$$

Thus, for each period t, per capita real consumption for the group under consideration, $u_A{}^t$, is multiplied by the Gini equality index E^{t*} to give an estimate of social welfare for the group that takes into account the distribution of real incomes within the group. Since the Gini coefficient is a

[151] See Atkinson (1970, 250) and Fleurbaey (2009, 1032) for this type of decomposition applied to nominal incomes, and see Jorgenson and Schreyer (2017, S470) for this type of decomposition applied to real incomes.

[152] For alternative social welfare functions that require exogenous parameterization, see Diewert (1985, 77–82), Fleurbaey (2009, 1032–36), and Jorgenson and Schreyer (2017).

[153] However, the fact that the economic statistician using the Gini equality index to adjust per capita real income for inequality does not have to pick a particular value of r as is the case if an Atkinson social welfare function is used to measure inequality does not imply that the use of the Gini coefficient methodology is free of value judgments. The social welfare function defined by (286) does imply specific judgments about the relative welfare of the individuals in the welfare comparison; see Atkinson (1970, 257).

generally accepted measure of inequality, the social welfare estimates defined by (286) are likely to be acceptable to the public.[154]

However, there are a number of practical measurement problems that are not addressed in the material here:

- Real income distributions (or more accurately, distributions of real consumption over households in a country) do not exist. Thus, the real "income" distribution described here may have to be approximated by a corresponding nominal distribution of household consumption expenditures for a period. This approximation may be satisfactory if all households in the group under consideration face approximately the same prices.
- Some households have more members than other households, but the theory outlined here implicitly assumed that all households had the same size. This problem can be addressed by the use of *household equivalence scales* but some measurement error will be introduced by their use.[155] For references to the literature on alternative household equivalence scales, see Fleurbaey (2009, 1051–52), Jorgenson and Slesnick (1987), and Jorgenson and Schreyer (2017, S462–S65).
- The services of consumer durables should be included in household consumption.[156] Most nominal income (or consumption) distributions for countries ignore the services provided by household durable goods. In particular, the services provided by OOH are typically missing in published income distributions.[157] This is a serious omission.
- Finally, adjustments to household nominal expenditures should be made for households that receive goods and services provided by governments and charitable organizations at no cost or at highly subsidized prices. These subsidized goods and services should be valued at comparable market prices.[158]

19. The Matching of Prices Problem

The economic approach to index number theory starts out by developing a theory of individual household behavior. With the exception of the material in Section 14, our analysis of the economic approach has assumed that prices faced by households were all positive in the two periods being compared and the quantities purchased by each household during the two periods were also positive. However, individual households rarely purchase positive amounts of the same commodities in two consecutive periods. The shorter is the time period, the greater will be this *lack of matching problem*. Part of the problem is due to the existence of seasonal commodities and part is due to the fact that consumers can store goods purchased in one period and consume them over multiple periods and the economic approach to index number theory does not take the storage problem into consideration. In recent years, an increasing number of firms have used *dynamic pricing*; that is, they vary the prices of their products by introducing deeply discounted prices at random intervals. Thus, individuals can purchase these discounted products in one period and gradually consume them over multiple periods.

There are a number of ways to address this lack of matching problem:

- Make the reference time period longer; that is, move from a weekly index to a monthly index or move from a monthly index to a quarterly index.
- Instead of defining products narrowly (that is, by a product code and by a particular point of purchase), group similar products together and use *broadly defined unit value prices* instead of narrowly defined unit value prices. This reduces the number of products in scope for the index from N to a number considerably less than N and this will increase the number of "matched" products.
- Aggregate households that are "similar" into a group of households and apply the economic approach to the group.
- Acknowledge that the economic approach is difficult to implement at the level of individual households and apply the fixed basket approach to index number theory that was developed in Chapter 2 to groups of households.

We will address each of the aforementioned points in turn.

There are a few countries that construct quarterly CPIs but most countries find that a monthly CPI seems to satisfy most user needs. Thus, moving from a monthly CPI to a quarterly CPI is not feasible for most countries. Moving to weekly or daily CPIs is likely to encounter severe lack of matching problems if they are constructed at the individual level.

The problem with moving from narrowly defined products to more broadly defined products is that *unit value bias* or *quality adjustment bias* is likely to result. It is difficult to quantify the tradeoff between obtaining more product matches versus increased unit value bias.

The economic approach to index number theory can be applied to a group of households under some restrictive assumptions. Suppose we have a group of similar households which have the same homothetic preferences. In particular, suppose we have H households and N commodities

[154] For related work on the use of the Gini coefficient in measures of welfare, see Sen (1976, 30–31) and Fleurbaey (2009, 1034–35).

[155] The simplest way to deal with households that differ in the number of members is to divide their utility, say u_h^t for household h in period t, by n_h^t, which is the number of household members. Then, when constructing the distribution of utilities for period t, replace u_h^t by n_h^t copies of per person utility, $u_h^t/n_h^t \equiv u_h^{t*}$. This crude adjustment of utility for household composition neglects the fact that multiple person households can share the services of the durable goods owned by the household. A *household equivalence scale* for household h in period t is a *household efficiency factor* a_h^t, which is equal to 1 if $n_h^t = 1$ and if $n_h^t > 1$, $a_h^t > 1$. The new adjusted per person utility for the household u_h^{t*} is set equal to unadjusted per person utility, u_h^t/n_h^t, times the household efficiency factor a_h^t. Thus, the new adjusted for composition per person household utility is $u_h^{t*} \equiv u_h^t a_h^t/n_h^t \geq u_h^t/n_h^t$. Thus, when constructing the distribution of utilities for period t, replace u_h^t by n_h^t copies of the *composition adjusted per person utility*, $u_h^{t*} = u_h^t a_h^t/n_h^t$. Our suggested approach to adjusting social welfare measures for household composition is more or less the same as the procedure suggested by Jorgenson and Schreyer (2017, S466).

[156] Christensen and Jorgenson (1969) advocated this inclusion many years ago and provided estimates for the United States.

[157] various approaches to the measurement of the services provided by consumer durables will be considered in Chapter 10.

[158] Thus, there is a difference between *household expenditures* (final consumption expenditures in the System of National Accounts) and *actual individual consumption*, which includes social transfers in kind such as free or subsidized services such as health, education, and housing services provided by governments at free or below market prices by government agencies. The latter concept is the correct concept to use in welfare measures.

and the unit cost function for each household is $c(p) \equiv (p^T Bp)^{1/2}$, where B is an N by N symmetric matrix with one positive eigenvalue with a strictly positive eigenvector and the remaining eigenvalues are nonpositive. We know that the Fisher price and quantity indices for each household are exact for this functional form. Let the utility function that corresponds to this unit cost function be $f(q)$. Let $p_h^t >> 0_N$ and $q_h^t > 0_N$ be the "observed" price and quantity vectors for household h in period t for $h = 1,\ldots,H$ and $t = 0,1$.[159] Assuming cost-minimizing behavior for each household in each period and using Shephard's Lemma, the following equations will hold, where $u_h^t \equiv f(q_h^t)$ for $h = 1,\ldots,H$ and $t = 0,1$:

$$q_h^t \equiv \nabla_p c(p_h^t)u_h^t = Bp_h^t u_h^t / c(p_h^t); \quad h = 1,\ldots,H \text{ and } t = 0,1. \qquad (287)$$

Define the period t *aggregate quantity vector* q^t and *aggregate utility level* u^t as follows:

$$q^t \equiv \Sigma_{h=1}^H q_h^t;\ u^t \equiv \Sigma_{h=1}^H u_h^t;\ t = 0,1. \qquad (288)$$

Our final assumption is that all households in each period t face the same vector of prices p^t:

$$p_h^t = p^t;\ h = 1,\ldots,H \text{ and } t = 0,1. \qquad (289)$$

Using (287)–(289), we have the following equations:

$$q^t \equiv \Sigma_{h=1}^H q_h^t = \Sigma_{h=1}^H Bp^t u_h^t / c(p^t) = Bp^t[\Sigma_{h=1}^H u_h^t]/c(p^t) = Bp^t u^t / c(p^t);\ t = 0,1. \qquad (290)$$

Thus, q^t, p^t, and u^t satisfy the Shephard's Lemma equations (287), where q^t, p^t, and u^t have replaced q_h^t, p_h^t, and u_h^t. Thus, the period t aggregate price and quantity vectors, p^t and q^t, along with the aggregate utility level u^t for $t = 0$ and 1 will be exact for the following Fisher aggregate quantity index:

$$u^1/u^0 = [p^0 \cdot q^1 p^1 \cdot q^1 / p^0 \cdot q^0 p^1 \cdot q^0]^{1/2}. \qquad (291)$$

Thus, according to this hypotheses, the aggregate data will satisfy the same equations as the micro data. These assumptions justify treating the data for the group as if it were generated by a single utility-maximizing household. This result is better than having no result at all but it does rest on two restrictive assumptions: (i) identical homothetic preferences and (ii) all members of the group face the same vector of prices in each period. Thus, if we apply this theory, we should try to group households so that they are demographically similar (so that their preferences can be better represented by the same preference function) and so that they face similar prices (so grouping households by location is also a useful thing to do).[160] Jorgenson and Schreyer summarized the need to group households in the following quotation:

> Another, related measurement issue is the level of detail at which distributional measures are put in place. Ideally, the equivalence scales are directly applied to household-level information. In practice, another simplifying assumption is often used in empirical measurements. Rather than applying equivalence scales (and, as will be discussed below, price indices) at the level of individual households, groups of households are the object of measurement in the simplified case. Each group is treated like a single, homogenous household.
>
> Dale Jorgenson and Paul Schreyer (2017, S464)

Finally, it is possible to fall back on our very first approach to index number theory that was explained in Chapter 2. This theory works as follows: a group of households collectively purchase the vector of goods and services q^t in periods $t = 0,1$. The corresponding unit value price vector for period t is p^t for $t = 0,1$. Two equally reasonable measures of price inflation for this group of purchasers are the Laspeyres and Paasche price indices, $P_L \equiv p^1 \cdot q^0 / p^0 \cdot q^0$ and $P_P \equiv p^1 \cdot q^1 / p^0 \cdot q^1$. Since both indices are equally plausible, it makes sense to take an average of the two to obtain a point estimate of the price inflation facing this group of purchasers. The Fisher index is perhaps the "best" average because it ends up satisfying the time reversal test. A similar theory works well for measuring the growth of consumption at constant prices. If we use the base period prices as weights, the Laspeyres quantity index, $Q_L \equiv p^0 \cdot q^1 / p^0 \cdot q^0$ is a reasonable measure and if we use the current period prices as weights, the Paasche quantity index, $Q_P \equiv p^1 \cdot q^1 / p^1 \cdot q^0$ is another reasonable measure for the growth of consumption at constant prices. Again, it is reasonable to take a symmetric average of these two measures to end up with a point estimate for real consumption growth. The Fisher quantity index is again "best" because it satisfies the time reversal test.

[159] We have assumed that all prices are positive, but some quantities are allowed to equal 0. We assume that positive reservation prices are used for products that are not consumed by a household in some period.

[160] This last point helps to justify applying the above methodology to the customers of a particular retail outlet.

Annex: Proofs of Propositions

Proof of Proposition 1: Define $g(\lambda)$ for $0 \le \lambda \le 1$ by $g(\lambda) \equiv P_K(p^0, p^1, (1-\lambda)\, q^0 + \lambda\, q^1)$. Note that $g(0) = P_K(p^0,p^1,q^0)$ and $g(1) = P_K(p^0,p^1,q^1)$. There are 24 = (4)(3)(2)(1) possible a priori inequality relations that are possible between the four numbers $g(0)$, $g(1)$, P_L, and P_P. However, the inequalities (3) and (4) imply that $g(0) \le P_L$ and $P_P \le g(1)$. This means that there are only six possible inequalities between the four numbers:

$$g(0) \le P_L \le P_P \le g(1); \qquad \text{(A1)}$$

$$g(0) \le P_P \le P_L \le g(1); \qquad \text{(A2)}$$

$$g(0) \le P_P \le g(1) \le P_L; \qquad \text{(A3)}$$

$$P_P \le g(0) \le P_L \le g(1); \qquad \text{(A4)}$$

$$P_P \le g(1) \le g(0) \le P_L; \qquad \text{(A5)}$$

$$P_P \le g(0) \le g(1) \le P_L. \qquad \text{(A6)}$$

Using the assumptions that (a) the consumer's utility function f is continuous over its domain of definition; (b) the utility function is increasing in the components of q and hence is subject to local nonsatiation and (c) the price vectors p^t have strictly positive components, it is possible to use Debreu's (1959, 19) Maximum Theorem (see also Diewert (1993a, 112–13) for a statement of the Theorem) to show that the consumer's cost function $C(f(q),p^t)$ will be continuous in the components of q. Thus, using definition (2), it can be seen that $P_K(p^0,p^1,q)$ will also be continuous in the components of the vector q. Hence, $g(\lambda)$ is a continuous function of λ and assumes all intermediate values between $g(0)$ and $g(1)$. By inspecting the inequalities (A1)–(A6), it can be seen that we can choose λ between 0 and 1, λ^* say, such that $P_L \le g(l^*) \le P_P$ for case (A1) or such that $P_P \le g(\lambda^*) \le P_L$ for cases (A2) to (A6). Thus, at least one of the two inequalities in (5) holds.

Proof of Proposition 2: Using assumptions (ii) and (iv), $q^t >> 0_N$ solves the concave programming problem $\max_q \{f(q) : p^t{\cdot}q \le e^t;\ q \ge 0_N\}$ for $t = 0,1$. Since q^t is strictly positive, the nonnegativity constraints $q \ge 0_N$ are not binding and hence, using the differentiability assumptions (iii), the following Lagrangian conditions are necessary and sufficient for q^t to solve the period t constrained maximization problem in (13):

$$\nabla f(q^t) = 1_t p^t;\ t = 0,1; \qquad \text{(A7)}$$

$$p^t{\cdot}q^t = e^t. \qquad \text{(A8)}$$

Take the inner product of both sides of (A7) with q^t and solve the resulting equation for λ_t. The solution for $t = 0,1$ is $\lambda_t = q^t{\cdot}\nabla f(q^t)/p^t{\cdot}q^t > 0$.[161] Substitute this solution for λ_t into equation t in (A7). After suitable rearrangement, we obtain the equations $p^t/p^t{\cdot}q^t = \nabla f(q^t)/q^t{\cdot}\nabla f(q^t)$ for $t = 0,1$.

Proof of Proposition 3: Let $u^t = f(q^t)$ for $t = 0,1$. By assumption (iii), q^t solves the cost minimization problem defined by $C(u^t,p^t)$ for $t = 0,1$. Thus, q^t is a feasible solution for the following cost minimization problem where the general price vector $p >> 0_N$ has replaced the specific period t price vector p^t:

$$C(u^t,p) \equiv \min_q \{p{\cdot}q : f(q) \ge u^t;\ q \ge 0_N\};$$
$$t = 0,1 \le p{\cdot}q^t, \qquad \text{(A9)}$$

where the inequality follows, since q^t is a feasible (but not necessarily an optimal) solution for the cost minimization problem defined by $C(u^t,p)$. Since by assumption (iii), q^t is a solution to the cost minimization problem defined by $C(u^t,p^t)$, we must have the following equalities:

$$C(u^t,p^t) = p^t{\cdot}q^t;\ t = 0,1. \qquad \text{(A10)}$$

Define the function $g^t(p) \equiv C(u^t,p) - p{\cdot}q^t$ for $t = 0,1$. Since $C(u^t,p)$ is a concave function in p and since the linear function $-p{\cdot}q^t$ is also concave in p, it can be seen that $g^t(p)$ is also a concave function of p for $t = 0,1$. The inequalities (A9) and equalities (A10) show that $g^t(p)$ achieves a global maximum at $p = p^t$ for $t = 0,1$. Since $C(u^t,p)$ is differentiable with respect to the components of p at $p = p^t$, the following first-order necessary conditions for maximizing $C(u^t,p)$ with respect to the components of p must hold:

$$\nabla_p g(p^t) = \nabla_p C(u^t,p^t) - q^t = 0_N;\ t = 0,1. \qquad \text{(A11)}$$

Equations (A11) can be rearranged to give the following equations:

$$q^t = \nabla_p C(u^t,p^t);\ t = 0,1. \qquad \text{(A12)}$$

To establish the uniqueness of q^t, let q^{t*} be any other solution to the cost minimization problem defined by $C(u^t,p^t)$ for $t = 0,1$. Repeat the aforementioned proof to show that $q^{t*} = \nabla_p C(u^t,p^t)$ for $t = 0,1$. Thus, $q^t = q^{t*}$ for $t = 0,1$ and the solution to the cost minimization problem defined by $C(u^t,p^t)$ is unique for $t = 0,1$.[162]

Proof of Proposition 4: Let $f^*(q)$ be a given increasing linearly homogeneous function which is twice continuously differentiable along the ray λq^*, where $\lambda > 0$ and $q^* >> 0_N$. We assume that $f^*(q^*) > 0$. Since $f^*(q)$ is linearly homogeneous, we have

$$f^*(\lambda q^*) = \lambda f^*(q^*) \text{ for all } \lambda > 0. \qquad \text{(A13)}$$

Differentiate both sides of (A13) with respect to λ and evaluate the resulting derivatives at $\lambda = 1$. We obtain the following equation:

$$f^*(q^*) = \nabla f^*(q^*)^T q^* = \Sigma_{n=1}^N q_n^* \partial f^*(q^*)/\partial q_n. \qquad \text{(A14)}$$

[161] We assume that at least one component of $\nabla f(q^t)$ is positive for $t = 0,1$.

[162] This method of proof was developed by McKenzie (1956). Shephard (1953) (1970) was the first to derive this result starting with a differentiable cost function. However, Hotelling (1932, 594) stated a version of the result in the context of profit functions and Hicks (1946, 331) and Samuelson (1953, 15–16) established the result starting with a differentiable utility or production function. For a more complete exposition of the technical details and references to the literature, see Diewert (1993a, 107–17).

Thus, if the first-order partial derivatives of $f^*(q^*)$ are known numbers, then the number $f^*(q^*)$ is also known and is equal to $q^*T\nabla f^*(q^*) = \Sigma_{n=1}^{N} q_n^* \partial f^*(q^*)/\partial q_n$.

Now partially differentiate both sides of (A13) with respect to q_n for $n = 1, \ldots, N$. The following equations are obtained for all $\lambda > 0$:

$$[\partial f^*(\lambda q^*)/\partial(\lambda q_n)][\partial(\lambda q_n)/\partial\lambda] = \lambda \partial f^*(\lambda q^*)/\partial(\lambda q_n) = \lambda \partial f^*(q^*)/\partial q_n \; n = 1, \ldots, N. \quad \text{(A15)}$$

Let $f_n^*(q) \equiv \partial f^*(q)/\partial q_n$ denote the function that is the partial derivative of $f^*(q)$ with respect to q_n for $n = 1, \ldots, N$. Using this notation, equations (A15) simplify to the following equations:

$$f_n^*(\lambda q^*) = f_n^*(q^*) \text{ for all } \lambda > 0;\; n = 1, \ldots, N. \quad \text{(A16)}$$

Thus, the first-order partial derivative functions $f_n^*(q)$ of a linearly homogeneous function $f^*(q)$ are homogeneous of degree 0. Now by differentiating both sides of equations (A16) with respect to λ and evaluating the resulting second-order partial derivatives $f_{nk}^*(\lambda q^*)$ at $\lambda = 1$, we obtain the following system of equations:

$$\Sigma_{k=1}^{N} f_{nk}^*(q^*) q_n^* = 0;\; n = 1, \ldots, N, \quad \text{(A17)}$$

where $f_{nk}^*(q^*) \equiv \partial^2 f^*(q)/\partial q_n \partial q_k$ for $n,k = 1, \ldots, N$. The N equations (A17) can be rewritten more succinctly using matrix notation as the following matrix equation:

$$\nabla^2 f^*(q^*) q^* = 0_N. \quad \text{(A18)}$$

Since $f^*(q)$ is assumed to be twice continuously differentiable at $q = q^*$, Young's Theorem in advanced calculus implies that the matrix of second-order derivatives, $\nabla^2 f^*(q^*)$, is a symmetric matrix so that $\partial^2 f^*(q)/\partial q_n \partial q_k = \partial^2 f^*(q)/\partial q_k \partial q_n$ for all $n, k = 1, \ldots, N$. Using matrix notation once again, this means that

$$[\nabla^2 f^*(q^*)]^T = \nabla^2 f^*(q^*). \quad \text{(A19)}$$

The $1 + N + N^2$ numbers $f^*(q^*)$, $\nabla f^*(q^*)$, and $\nabla^2 f^*(q^*)$ are regarded as given numbers or parameters in what follows. From the previous derivations, we see that these numbers are not independent: equation (A14), $f^*(q^*) = \nabla f^*(q^*)^T q^*$, implies that if the N components in the vector of first-order partial derivatives $\nabla f^*(q^*)$ are given numbers, then the level of the function $f^*(q)$ evaluated at the point q^* is determined by these numbers. Similarly, the symmetry conditions (A19) imply that if the N^2 second-order partial derivatives of $f^*(q^*)$ are calculated, then these numbers are not independent of each other either. If the $N(N-1)/2$ components of $\nabla^2 f^*(q^*)$ in the upper triangle of this matrix are given (so that $\partial^2 f^*(q)/\partial q_n \partial q_k$ for $1 \le n < k \le N$ are given numbers), then the $N(N-1)/2$ numbers in the lower triangle of this matrix are also determined. Furthermore, the N restrictions given by equations (A18) mean that if the upper triangle second-order partial derivatives are given (which means that the lower triangle second-order partial derivatives are also given), then the main diagonal second-order partial derivatives (the N derivatives $\partial^2 f^*(q)/\partial q_n \partial q_n$ for $n = 1, \ldots, N$) are also determined (provided that the components of the q^* vector are all positive). Thus, the assumption of linear homogeneity of $f^*(q)$ (along with the assumption that second-order partial derivatives of $f^*(q)$ exist and are continuous at $q = q^*$) implies that there are only $N(N-1)/2$ independent parameters instead of N^2 parameters in the matrix $\nabla^2 f^*(q^*)$.

Define the utility function $f(q)$ over the set $S \equiv (q : q \ge 0_N;\; Aq \ge 0_N;\; q^T A q > 0)$ as

$$f(q) \equiv (q^T A q)^{1/2}, \text{ where } A = A^T. \quad \text{(A20)}$$

To show that $f(q)$ is a flexible functional form at $q = q^* >> 0_N$, we need to solve the following equations for the components of the N by N matrix $A \equiv [a_{nk}]$, where $a_{nk} = a_{kn}$ for $1 \le n < k \le N$:

$$f(q^*) = f^*(q^*); \quad \text{(A21)}$$

$$\nabla f(q^*) = \nabla f^*(q^*); \quad \text{(A22)}$$

$$\nabla^2 f(q^*) = \nabla^2 f^*(q^*). \quad \text{(A23)}$$

Define matrix A as follows:

$$A \equiv f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T. \quad \text{(A24)}$$

Note that this matrix A is symmetric; that is, $A = A^T$. Use matrix A to define $f(q) \equiv (q^T A q)^{1/2}$ and compute $f(q^*)^2$:

$$\begin{aligned} f(q^*)^2 &= q^*TAq^* \quad \text{(A25)} \\ &= q^*T[f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T]q^* \\ &\quad \text{using definition } (A24) \\ &= q^*T\nabla f^*(q^*)\nabla f^*(q^*)^T q^* \text{ using } (A18) \\ &= f^*(q^*)^2 \text{ using } (A14). \end{aligned}$$

Take positive square roots of both sides of (A25) and the resulting equation is (A21). Now calculate the vector of first-order partial derivatives of $f(q)$ defined by (A20) and (A24) and evaluate these derivatives at $q = q^*$:

$$\begin{aligned} \nabla f(q^*) &= Aq^*/(q^*TAq^*)^{1/2} \quad \text{(A26)} \\ &= [f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T]q^*/f^*(q^*) \\ &\quad \text{using } (A24) \text{ and } (A25) \\ &= 0_N + \nabla f^*(q^*)[\nabla f^*(q^*)^T q^*]/f^*(q^*) \text{ using } (A18) \\ &= \nabla f^*(q^*) \text{ using } (A14). \end{aligned}$$

Thus, equations (A22) are satisfied. Finally, calculate the matrix of second-order partial derivatives of $f(q)$ defined by (A20) and (A24) and evaluate these derivatives at $q = q^*$. Differentiating the first line in (A26) leads to the following matrix equation:

$$\begin{aligned} \nabla^2 f(q^*) &= \{A/(q^*TAq^*)^{1/2}\} - \{Aq^*q^*TA/(q^*TAq^*)^{3/2}\} \quad \text{(A27)} \\ &= [f^*(q^*)]^{-1}\{f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T\} - \{Aq^*q^*TA/(q^*TAq^*)^{3/2}\} \text{ using } (A24) \text{ and } (A25) \end{aligned}$$

$= \nabla^2 f^*(q^*) + [f^*(q^*)]^{-1}[\nabla f^*(q^*)\nabla f^*(q^*)^T] - [f^*(q^*)]^{-1}[\nabla f^*(q^*)\nabla f^*(q^*)^T]$ using (A25) and (A26)

$= \nabla^2 f^*(q^*)$.

Thus, equations (A23) are satisfied and $f(q) \equiv (q^TAq)^{1/2}$ is a flexible functional form.[163] Note that this functional form has the minimum number of free parameters (which is $N(N + 1)/2$) that is required to satisfy the $1 + N + N^2$ equations (A21)–(A23). In the literature on flexible functional forms, such a function is called a *parsimonious flexible functional form*.

Proof of Proposition 5: Let $c(p) = (p^TBp)^{1/2}$, where $B = B^T$ and B has one positive eigenvalue with a strictly positive eigenvector and the remaining $N - 1$ eigenvalues of B are negative. The function $c(p)$ is well defined over the set $S^* \equiv \{p: p\ ^3\ 0_N;\ Bp \geq 0_N;\ p^TBp > 0\}$. Under our eigenvalue assumptions, a result obtained by Diewert and Hill (2010) will imply that $c(p)$ is a concave function over the set S^*. It will also be increasing, linearly homogeneous, and positive over S^*. Let $q^* >> 0_N$ and suppose also that $B^{-1}q^* >> 0_N$. Let $f(q)$ be the utility function that is dual to $c(p)$. Then, $f(q^*)$ can be defined by the following modification of definition (50) in the main text:[164]

$$f(q^*) = 1/\max_p \{c(p) : p{\cdot}q^* = 1;\ p\in S^*\}. \qquad \text{(A28)}$$

Consider the maximization problem on the right-hand side of (A28). If we temporarily drop the constraints $p\in S^*$, then the resulting problem is

$$\max_p \{(p^TBp)^{1/2} : p{\cdot}q^* = 1\}. \qquad \text{(A29)}$$

The first-order necessary conditions for an interior maximum for the constrained maximization problem (A29) are equivalent to the following conditions:

$$Bp^* = \lambda^* q^*; \qquad \text{(A30)}$$
$$p^*{\cdot}q^* = 1. \qquad \text{(A31)}$$

Since B^{-1} exists under our assumptions, $p^* = 1^*B^{-1}q^*$. Substitute this equation into (A31) and solve the resulting equation, $\lambda^*q^*TB^{-1}q^* = 0$ for $\lambda^* = 1/q^*TB^{-1}q^*$, which is positive since q^* and $B^{-1}q^*$ are strictly positive vectors by our assumptions. Thus, $p^* = \lambda^*B^{-1}q^* = B^{-1}q^*/q^*TB^{-1}q^*$. It can be seen that this p^* is the global maximizer for the problem defined by (A29) under our regularity conditions on B. Thus, we have

$$\max_p \{(p^TBp)^{1/2} : p{\cdot}q^* = 1\} = (p^*TBp^*)^{1/2} = (q^*TB^{-1}q^*)^{-1/2}. \qquad \text{(A32)}$$

Since $B^{-1}q^* >> 0_N$ and $\lambda^* > 0$, $p^* = \lambda^*B^{-1}q^* >> 0_N$. From (A30), $Bp^* = \lambda^*q^* >> 0_N$. Thus, p^* also solves the maximization problem on the right-hand side of (A28) since p^* belongs to S^*. Thus, we have[165]

$$f(q^*) = 1/\max_p \{c(p) : p{\cdot}q^* = 1;\ p\in S^*\} = 1/(q^*TB^{-1}q^*)^{-1/2} = (q^*TB^{-1}q^*)^{1/2}. \qquad \text{(A33)}$$

Proof of Proposition 6: Let $A \equiv [a_{ik}]$ be an N by N symmetric matrix with element a_{ik} in row i and column k so that $A = A^T$. Suppose $r \neq 0$, $q >> 0_N$ and define $f(q)$ as follows:[166]

$$f(q) = f(q_1, \ldots, q_N) \equiv [\Sigma_{i=1}^N \Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{1/r}. \qquad \text{(A34)}$$

Denote the nth first-order partial derivative of $f(q)$ as $f_n(q) \equiv \partial f(q)/\partial q_n$ for $n = 1, \ldots, N$. Assuming that $\Sigma_{i=1}^N\Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}$ is positive, $f_n(q)$ is equal to the following expression:

$$f_n(q) = (1/r)[\Sigma_{i=1}^N \Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{(1/r)-1} r[\Sigma_{k=1}^N a_{nk}q_n^{(r/2)-1}q_k^{r/2}]\ n = 1, \ldots, N \qquad \text{(A35)}$$
$$= [f(q)]^{1-r}[\mathring{a}_{k=1}^N a_{nk}q_n^{(r/2)-1}q_k^{r/2}].$$

Denote the second-order partial derivative of $f(q)$ with respect to q_n and q_m as $f_{nm}(q) \equiv \partial^2 f(q)/\partial q_n \partial q_m$ for $n = 1, \ldots, N$ and $m = 1, \ldots, N$. For $n < m$, $f_{nm}(q)$ is equal to the following expression:

$$f_{nm}(q) = [(1/r) - 1][\Sigma_{i=1}^N \Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{(1/r)-2r} [\Sigma_{k=1}^N a_{nk}q_n^{(r/2)-1}q_k^{r/2}][\Sigma_{k=1}^N a_{mk}q_m^{(r/2)-1}q_k^{r/2}] + [f(q)]^{1-r}[^r/2][a_{nm}q_n^{(r/2)-1}q_m^{(r/2)-1}]\ 1 \leq n < m \leq N \qquad \text{(A36)}$$
$$= (1 - r)[f(q)]^{-1}f_n(q)f_m(q) + (r/2)a_{nm}q_n^{(r/2)-1}q_m^{(r/2)-1}.$$

As was seen in the proof of Proposition 4, because the $f(q)$ defined by (A34) is linearly homogeneous, we need only to choose a_{nm} to satisfy equations (A22) and the upper triangle of equations (A23) in order to prove that $f(q)$ is a flexible functional form; that is, for $q^* >> 0_N$. we need a_{nm} to satisfy the following equations:[167]

$$f_n(q^*) = f^*_n(q^*);\ n = 1, \ldots, N; \qquad \text{(A37)}$$
$$f_{nm}(q^*) = f^*_{nm}(q^*);\ 1 \leq n < m \leq N. \qquad \text{(A38)}$$

Temporarily assume that we have found a set of a_{nm} so that equations (A37) and the following equation are satisfied:

$$f(q^*) = f^*(q^*). \qquad \text{(A39)}$$

[163] The previous proof of flexibility is an adaptation of the proof of flexibility for this functional form in Diewert (1974b, 125). See also Diewert (1976, 140–42) for an alternative proof.

[164] See Blackorby and Diewert (1979) for additional material on local duality theorems.

[165] This seems to be the model considered by Konüs and Byushgens (1926, 171).

[166] In order to ensure that $f(q)$ is well defined for any $r \neq 0$, we require that $\Sigma_{i=1}^N\Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2} > 0$. If each $a_{ik} \geq 0$ and at least one $a_{ik} > 0$, then for $q >> 0_N$, $\Sigma_{i=1}^N\Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}$ will be greater than 0. However, as will be seen later in the proof, $\Sigma_{i=1}^N\Sigma_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}$ can be positive without assuming that each $a_{ik} \geq 0$.

[167] We assume that the exogenous $f^*(q^*)$ and $\nabla f^*(q^*)$ satisfy the positivity restrictions $\nabla f^*(q^*) >> 0_N$, and hence $f^*(q^*) = q^*T\nabla f^*(q^*) > 0$.

Evaluate the second-order partial derivatives of $f(q)$ at q^* using equations (A36) and set the nm^{th} partial derivative of $f(q)$ equal to the corresponding nm^{th} partial derivative of $f^*(q^*)$. Using equations (A37) and (A39), these equations become the following equations:

$$f^*_{nm}(q^*) = (1-r)[f^*(q^*)]^{-1}f^*_n(q^*)f^*_m(q^*) + (r/2)a_{nm}(q_n^*)^{(r/2)-1}(q_m^*)^{(r/2)-1};\ 1 \le n < m \le N. \quad \text{(A40)}$$

The $N(N-1)/2$ equations (A40) determine a_{nm} for $1 \le n < m \le N$. Define $a_{mn} = a_{nm}$ for $1 \le n < m \le N$. Thus, all of the a_{nm} are determined except for the a_{nn} for $n = 1, \ldots, N$. Again, assume that $f(q^*) = f^*(q^*)$, evaluate equations (A35) at $q = q^*$, and set the resulting first-order partial derivatives of $f(q^*)$ equal to the corresponding given first-order partial derivatives of $f^*(q^*)$. We obtain the following N equations:

$$f^*_n(q^*) = [f^*(q^*)]^{1-r}[\Sigma_{k=1}^N a_{nk}(q_n^*)^{(r/2)-1}(q_k^*)^{r/2}];\ n = 1, \ldots, N. \quad \text{(A41)}$$

The N equations (A41) determine the a_{nn} for $n = 1, \ldots, N$. It turns out that this solution for a_{nm} enables $f(q)$ defined by (A34) to satisfy all of the equations (A21)–(A23). Thus, $f(q)$ is a flexible functional form.[168] Note that the resulting $f(q)$ will be positive and the first-order derivatives of $f(q)$ will be positive in a neighborhood around q^* due to the continuity of the function $f(q)$ defined by (A34). Finally, note that if $r = 2$, then $f(q) = (q^TAq)^{1/2}$, and so the proof of Proposition 6 provides an alternative proof for Proposition 4.

Proof of Proposition 7: Let $r \neq 0$ and define $f^r(q)$ by (53). The assumption that $q^t >> 0_N$ solves the constrained utility maximization problem $\max_q \{f^r(q) : p^t \cdot q \le e^t;\ q \in S\}$, where S is an open convex set means that q^t is not on the boundary of S, and hence q^t will satisfy the first-order conditions for the problem $\max_q \{f^r(q) : p^t \cdot q \le e^t\}$ for $t = 0,1$. The first-order necessary conditions for these problems (which are equivalent to the Wold's Identity conditions (16)) are the following conditions:

$$p_n^t/e^t = p_n^t/p^t \cdot q^t = f^r_n(q^t)/f^r(q^t) = [f^r(q^t)]^{-r}[\Sigma_{k=1}^N a_{nk}(q_n^t)^{(r/2)-1}(q_k^t)^{r/2}];\ n = 1, \ldots, N;\ t = 0,1, \quad \text{(A42)}$$

where we have used equations (A35) to establish the last equation in (A42). Using equations (A42), we obtain the following expressions for the shares s_n^t:

$$s_n^t = p_n^t q_n^t/e^t = [f^r(q^t)]^{-r}[\Sigma_{k=1}^N a_{nk}(q_n^t)^{(r/2)}(q_k^t)^{r/2}];\ n = 1, \ldots, N;\ t = 0,1. \quad \text{(A43)}$$

Now substitute s_n^t defined by (A43) into (54), the definition of $Q^r(p^0,p^1,q^0,q^1)$:

$$\begin{aligned} Q^r(p^0,p^1,q^0,q^1) &\equiv \{\Sigma_{n=1}^N s_n^0(q_n^1/q_n^0)^{r/2}\}^{1/r} \{\Sigma_{n=1}^N s_n^1(q_i^1/q_n^0)^{-r/2}\}^{-1/r} \quad \text{(A44)}\\ &= [f^r(q^0)]^{-1}\{\Sigma_{n=1}^N \Sigma_{k=1}^N a_{nk}(q_n^0)^{(r/2)}(q_k^1)^{r/2}\}^{1/r}[f^r(q^1)] \{\Sigma_{n=1}^N \Sigma_{k=1}^N a_{nk}(q_n^0)^{(r/2)}(q_k^1)^{r/2}\}^{-1/r}\\ &= f^r(q^1)/f^r(q^0). \end{aligned}$$

[168] This method of proof was developed by Diewert (1976, 140–41).

Proof of Proposition 9: Consider the following constrained maximization problem:

$$\max_p \{c^r(p);\ e^t = p \cdot q^t;\ p \in S^*\}. \quad \text{(A45)}$$

Since S^* is an open set, the first-order necessary conditions for $p^* \in S^*$ to solve (A45) is that there exist λ^* such that the following equations are satisfied:

$$\nabla c^r(p^*) = \lambda^* q^t; \quad \text{(A46)}$$
$$p^* \cdot q^t = e^t. \quad \text{(A47)}$$

By premultiplying both sides of (A46) by p^{*T} we obtain the equation $\lambda^* p^{*T} q^t = p^{*T}\nabla c^r(p^*) = c^r(p^*)$, where the last equality follows from the linear homogeneity of $c^r(p)$. Thus, $\lambda^* = c^r(p^*)/p^* \cdot q^t = c^r(p^*)/e^t$, where the last equation follows from (A47). Substituting $\lambda^* = c^r(p^*)/e^t$ into (A47) gives the equation $\nabla c^r(p^*) = [c^r(p^*)/e^t]q^t$, which in turn can be written as follows:

$$q^t \equiv e^t \nabla c^r(p^*)/c^r(p^*). \quad \text{(A48)}$$

But from (64), we have $q^t \equiv e^t \nabla c^r(p^t)/c^r(p^t)$. Thus, if we set p^* to p^t, equation (A48) will be satisfied. We also have $p^t \cdot q^t = e^t p^t \cdot \nabla c^r(p^t)/c^r(p^t) = e^t c^r(p^t)/c^r(p^t) = e^t$, so equation (A47) is satisfied if $p^* = p^t$. If we define $\lambda^* = c^r(p^t)/e^t$, then (A46) with $p^* = p^t$ becomes $\nabla c^r(p^*) = [c^r(p^t)/e^t]q^t$ which is (A48) and so $p^* \equiv p^t$ and $\lambda^* = c^r(p^t)/e^t$ satisfy equations (A46) and (A47). Thus, p^t is a candidate to solve (A45) since it satisfies the first-order necessary conditions for an interior solution for (A45).

Next, we show that p^t actually solves the constrained maximization problem defined by (A45). Define $\lambda^* \equiv c^r(p^t)/e^t$ and define the function $g(p)$ as follows:

$$g(p) \equiv c^r(p) + \lambda^*[e^t - p \cdot q^t]. \quad \text{(A49)}$$

Since $c^r(p)$ is concave over S^* by assumption and the function $\lambda^*[e^t - p \cdot q^t]$ is linear in p (and hence concave everywhere), $g(p)$ is a differentiable concave function over S^*. Hence, the first-order Taylor series approximation to $g(p)$ around the point p^t will be coincident with or lie above the function; that is, we have the following inequality:

$$g(p) \le g(p^t) + \nabla g(p^t)(p - p^t) \text{ for all } p \in S^*. \quad \text{(A50)}$$

Substituting definition (A49) into (A50) and noting that $\nabla g(p^t) = \nabla c^r(p^t) - \lambda^* q^t = 0_N$ (using (A46) with $p^* = p^t$ and $\lambda^* = c^r(p^t)/e^t$), we find that (A50) becomes

$$c^r(p) + \lambda^*[e^t - p \cdot q^t] \le c^r(p^t) + \lambda^*[e^t - p^t \cdot q^t];\ p \in S^*. \quad \text{(A51)}$$

But this inequality does not take into account the constraint $e^t = p \cdot q^t$. If we impose this additional constraint on p, the inequality (A51) becomes

$$c^r(p) \le c^r(p^t);\ p \in S^* \text{ and } p \cdot q^t = e^t. \quad \text{(A52)}$$

Thus, p^t solves the constrained maximization problem (A45), and we have

$$c^r(p^t) = \max_p \{c^r(p);\ e^t = p \cdot q^t;\ p \in S^*\}. \qquad (A53)$$

Now use definition (63) with $e = e^t$ to define $f^{r*}(q^t)$, and we obtain the following result using (A53):

$$f^{r*}(q^t) = e^t/\max_p \{c^r(p);\ e^t = p \cdot q^t;\ p \in S^*\} = e^t/c^r(p^t). \qquad (A54)$$

(A54) establishes (65). Now consider the following local utility maximization problem

$$\max_q \{f^{r*}(q) : p^t \cdot q = e^t;\ q \in S\}, \qquad (A55)$$

where $f^{r*}(q)$ is defined as

$$f^{r*}(q) = e^t/\max_p \{c^r(p);\ e^t = p \cdot q;\ p \in S^*\}. \qquad (A56)$$

Let $q \in S$, and we suppose that q also satisfies the consumer's period t budget constraint, $p^t \cdot q = e^t$. Let p^* be a solution to $\max_p \{c^r(p);\ e^t = p \cdot q;\ p \in S^*\}$. Thus, we have

$$c^r(p^*) = \max_p \{c^r(p);\ e^t = p \cdot q;\ p \in S^*\} \geq c^r(p^t), \qquad (A57)$$

since $p^t \cdot q = e^t$, and hence p^t is a feasible solution for the constrained maximization problem. Using (A54), (A56), and (A57), we have $f^{r*}(q^t) \geq f^{r*}(q)$ for all q belonging to S such that $p^t \cdot q = e^t$. Thus, q^t solves the local utility maximization problem (A55).

Proof of Proposition 10: The proof of the previous proposition showed that q^t solves the local utility maximization problem, $\max_q \{f^{r*}(q);\ p^t \cdot q = e^t;\ q \in S\}$, for $t = 0,1$.

Conditions (68) (Shephard's Lemma) and definition (59) imply that the following equations will hold:

$$q_n^t/p^t \cdot q^t = c^r_n(p^t)/c^r(p^t) = [c^r(p^t)]^{-r} [\Sigma_{k=1}^N b_{nk} (p_n^t)^{(r/2)-1}(p_k^t)^{r/2}];\ n = 1, \ldots, N;\ t = 0,1. \qquad (A58)$$

Using equations (A58), we obtain the following expressions for the shares s_n^t:

$$s_n^t = p_n^t q_n^t/p^t \cdot q^t = [c^r(p^t)]^{-r} [\Sigma_{k=1}^N b_{nk} (p_n^t)^{(r/2)}(p_k^t)^{r/2}];\ n = 1, \ldots, N;\ t = 0,1. \qquad (A59)$$

Now substitute s_n^t defined by (A59) into (69), the definition of $P^r(p^0,p^1,q^0,q^1)$:

$$P^r(p^0,p^1,q^0,q^1) \equiv \{\Sigma_{n=1}^N s_n^0 (p_n^1/p_n^0)^{r/2}\}^{1/r} \{\Sigma_{n=1}^N s_n^1 (p_n^1/p_n^0)^{-r/2}\}^{-1/r} \qquad (A60)$$

$$= [c^r(p^0)]^{-1} \{\Sigma_{n=1}^N \Sigma_{k=1}^N b_{nk} (p_n^0)^{(r/2)}(p_k^1)^{r/2}\}^{1/r} [c^r(p^1)] \{\Sigma_{n=1}^N \Sigma_{k=1}^N b_{nk} (p_n^0)^{(r/2)}(p_k^1)^{r/2}\}^{-1/r}$$

$$= c^r(p^1)/c^r(p^0).$$

Proof of Proposition 11: Let $p \equiv [p_1, \ldots, p_N] >> 0_N$. Ignoring the constraints $q \geq 0_N$, the first-order necessary (and sufficient) conditions for $q^* >> 0_N$ and $\lambda^* > 0$ to solve the unit cost minimization problem defined by (96) are

$$p_n = \lambda^* \partial f(q^*)/\partial q_n = \lambda^* \alpha_n f(q^*)/q_n^*;\quad n = 1, \ldots, N; \qquad (A61)$$

$$1 = f(q^*). \qquad (A62)$$

Substituting (A62) into (A61), we get the N equations $p_n = \lambda^* \alpha_n/q_n^*$ for $n = 1, \ldots, N$ which can be rearranged to give us the following equations:

$$q_n^* = \lambda^* \alpha_n/p_n;\ n = 1, \ldots, N. \qquad (A63)$$

Now substitute equations (A63) into equation (A62) and using definition (94) for f, we get the following single equation involving λ^*:

$$1 = \alpha_0 \Pi_{n=1}^N [\lambda^* a_n/p_n]^{\alpha_n} = \lambda^* \alpha_0 \Pi_{n=1}^N [\alpha_n]^{\alpha_n} \Pi_{n=1}^N [1/p_n]^{\alpha_n}. \qquad (A64)$$

Therefore, we have the following expression for λ^*:

$$\lambda^* = [\alpha_0 \Pi_{n=1}^N [\alpha_n]^{\alpha_n}]^{-1} \Pi_{n=1}^N [p_n]^{\alpha_n} = \mathrm{k}\, \Pi_{n=1}^N p_n^{\alpha_n} > 0, \qquad (A65)$$

where the constant κ is defined as $\kappa \equiv [\alpha_0 \Pi_{n=1}^N [\alpha_n]^{\alpha_n}]^{-1}$. Substitute λ^* defined by (A65) back into equations (A63) and we obtain the q^* solution to the cost minimization problem defined by (96):

$$q_n^* = \kappa\, [\Pi_{n=1}^N p_n^{\alpha_n}] \alpha_n/p_n;\ n = 1, \ldots, N. \qquad (A66)$$

Thus, the optimized objective function for (96) is equal to the following expression:

$$c(p) = \Sigma_{n=1}^N p_n q_n^* \qquad (A67)$$

$$= \Sigma_{n=1}^N p_n \kappa [\Pi_{n=1}^N p_n^{\alpha_n}] \alpha_n/p_n \text{ using } (A66)$$

$$= \kappa [\Pi_{n=1}^N p_n^{\alpha_n}][\Sigma_{n=1}^N \alpha_n]$$

$$= \kappa\, \Pi_{n=1}^N p_n^{\alpha_n} \text{ using } (95).$$

Thus, $c(p)$ is defined by (97).

Proof of Proposition 12: If $r = 0$, then the CES preferences collapse to Cobb–Douglas preferences, which will imply that $s^0 = s^1$, and thus the Sato vartia index collapses to the Konüs Byushgens index which was studied in Section 9. Hence, we assume $r \neq 0$ and define the consumer's unit cost function by (108). Let $p^0 >> 0_N$, $p^1 >> 0_N$ and define q^0 and q^1 using Shephard's Lemma, equations (109). We assume that $q^0 >> 0_N$ and $q^1 >> 0_N$ and hence the share vectors s^0 and s^1 defined by equations (110) also satisfy $s^0 >> 0_N$ and $s^1 >> 0_N$. Given these positivity conditions, equations (110) can be rewritten as follows:

$$\Sigma_{n=1}^N \alpha_n (p_n^t)^r = \alpha_i (p_i^t)^r/s_i^t;\ t = 0,1;\ i = 1, \ldots, N. \qquad (A68)$$

By taking the logarithm of both sides of (A68) we obtain the following equations:

$$ln[\Sigma_{n=1}^N \alpha_n (p_n^t)^r] = ln\alpha_i + r\,lnp_i^t - lns_i^t;\ t = 0,1;\ i = 1, \ldots, N. \qquad (A69)$$

The consumer's true cost of living index is $c(p^1)/c(p^0) = \alpha_0 [\Sigma_{n=1}^N \alpha_n (p_n^1)^r]^{1/r}/\alpha_0 [\Sigma_{n=1}^N \alpha_n (p_n^0)^r]^{1/r}$, which equals $[\Sigma_{n=1}^N \alpha_n (p_n^1)^r]^{1/r}/[\Sigma_{n=1}^N \alpha_n (p_n^0)^r]^{1/r}$. Raising both sides of this equation to the power r and taking the logarithm of the resulting equation leads to the following equation:

$$ln\{[c(p^1)/c(p^0)]^r\} = ln[\Sigma_{n=1}^N \alpha_n (p_n^1)^r] - ln[\Sigma_{n=1}^N \alpha_n (p_n^0)^r]. \quad \text{(A70)}$$

From (118), the logarithm of $P_{SV}(p^0,p^1,q^0,q^1)^r$ is defined as follows:

$$ln\{P_{SV}(p^0,p^1,q^0,q^1)^r\} = r\Sigma_{n=1}^N w_i^*[lnp_i^1 - lnp_i^0]/\Sigma_{n=1}^N w_i^*, \quad \text{(A71)}$$

where $w_i^* \equiv [s_i^1 - s_i^0]/[lns_i^1 - lns_i^0]$ if $s_i^1 \neq s_i^0$ and $w_i^* \equiv s_i^0$ if $s_i^1 = s_i^0$. Now equate (A71) to (A70) and after suitable rearrangement, we obtain the following equation:

$$r\Sigma_{n=1}^N w_i^*[lnp_i^1 - lnp_i^0] = \Sigma_{n=1}^N w_i^* ln[\Sigma_{n=1}^N \alpha_n (p_n^1)^r] - \Sigma_{n=1}^N w_i^* ln[\Sigma_{n=1}^N \alpha_n (p_n^0)^r] \quad \text{(A72)}$$
$$= \Sigma_{n=1}^N w_i^*[ln\alpha_i + rlnp_i^1 - lns_i^1] - \Sigma_{n=1}^N w_i^*[ln\alpha_i + rlnp_i^0 - lns_i^0] \text{ using } (A69)$$
$$= r\Sigma_{n=1}^N w_i^*[lnp_i^1 - lnp_i^0] - \Sigma_{n=1}^N w_i^*[lns_i^1 - lns_i^0]$$
$$= r\Sigma_{n=1}^N w_i^*[lnp_i^1 - lnp_i^0] - \Sigma_{n=1}^N [s_i^1 - s_i^0]$$
$$= r\Sigma_{n=1}^N w_i^*[lnp_i^1 - lnp_i^0] \text{ since } \Sigma_{n=1}^N s_i^1 = \Sigma_{n=1}^N s_i^0 = 1.$$

The last equality follows because if $s_i^0 \neq s_i^1$, then $w_i^*[lns_i^1 - lns_i^0] = \{[s_i^1 - s_i^0]/[lns_i^1 - lns_i^0]\}[lns_i^1 - lns_i^0] = s_i^1 - s_i^0$. If $s_i^1 = s_i^0$, then $w_i^* = s_i^0$ but $lns_i^1 - lns_i^0 = 0$ so $w_i^*[lns_i^1 - lns_i^0] = 0 = s_i^1 - s_i^0$. Thus, we have shown that $\ln\{[c(p^1)/c(p^0)]^r\} = \ln\{P_{SV}(p^0,p^1,q^0,q^1)^r\}$ and thus that $c(p^1)/c(p^0) = P_{SV}(p^0,p^1,q^0,q^1)$.

We note that the Sato vartia quantity index $Q_{SV}(p^0,p^1,q^0,q^1)$ can be defined by interchanging prices and quantities in the definition of the Sato vartia price index; that is, define $Q_{SV}(p^0,p^1,q^0,q^1) \equiv P_{SV}(q^0,q^1,p^0,p^1)$. The aforementioned proof can be adapted to show that $f(q^1)/f(q^0) = Q_{SV}(p^0,p^1,q^0,q^1)$, where $f(q)$ is defined by (134). In order to prove this result, we require that $s < 1$ and $s \neq 0$.

References

Abraham, Katharine G., and Christopher Mackie, eds. 2005. *Beyond the Market; Designing Nonmarket Accounts for the United States*. Washington, DC: The National Academies Press.

Afriat, Sidney N. 1967. "The Construction of Utility Functions from Finite Expenditure Data." *International Economic Review* 8: 67–77.

Afriat, Sidney N. 1972. "The Theory of International Comparisons of Real Income and Prices." In *International Comparisons of Prices and Outputs*, edited by D. John Daly, 13–69. New York: National Bureau of Economic Research.

Allen, Roy G. D. 1938. *Mathematical Analysis for Economists*. London: Macmillan.

Allen, Roy G. D. 1949. "The Economic Theory of Index Numbers." *Economica* 16: 197–203.

Alterman, William F., W. Erwin Diewert, and Robert C. Feenstra. 1999. *International Trade Price Indexes and Seasonal Commodities*. Washington, DC: Bureau of Labor Statistics.

Arrow, Kenneth J., Hollis B. Chenery, Bagicha S. Minhas, and Robert M. Solow. 1961. "Capital-Labor Substitution and Economic Efficiency." *Review of Economics and Statistics* 63: 225–50.

Atkinson, Anthony B. 1970. "On the Measurement of Inequality." *Journal of Economic Theory* 2: 244–63.

Balk, Bert M. 1998. *Industrial Price, Quantity and Productivity Indexes*. Boston: Kluwer Academic Publishers.

Becker, Gary S. 1965. "A Theory of the Allocation of Time." *Economic Journal* 75: 493–517.

Blackorby, Charles, and W. Erwin Diewert. 1979. "Expenditure Functions, Local Duality, and Second Order Approximations." *Econometrica* 47: 579–601.

Blackorby, Charles, and David Donaldson. 1978. "Measures of Relative Equality and Their Meaning in Terms of Social Welfare." *Journal of Economic Theory* 18: 59–80.

Brynjolfsson, Erik, Avinash Collis, W. Erwin Diewert, Felix Eggers, and Kevin J. Fox. 2018. "The Digital Economy, GDP and Household Welfare: Theory and Evidence." Paper presented at the Sixth IMF Statistical Forum, Measuring Economic Welfare in the Digital Age: What and How? IMF, Washington, DC, November 19–20.

Brynjolfsson, Erik, Avinash Collis, W. Erwin Diewert, Felix Eggers, and Kevin J. Fox. 2019. "The Digital Economy, GDP and Household Welfare: Theory and Evidence." NBER Working Paper 25695, National Bureau of Economic Research, Cambridge, MA.

Brynjolfsson, Erik, Avinash Collis, W. Erwin Diewert, Felix Eggers, and Kevin J. Fox. 2020a. "Measuring the Impact of Free Goods on Real Household Consumption." *American Economic Association Papers and Proceedings* 110: 25–30.

Brynjolfsson, Erik, Avinash Collis, W. Erwin Diewert, Felix Eggers, and Kevin J. Fox. 2020b. "Measuring the Impact of Free New Goods on Real Household Consumption." Discussion Paper 20–01, Vancouver School of Economics, University of British Columbia, Vancouver, BC, Canada, V6T 1L4.

Caves, Douglas W., Laurits R. Christensen, and W. Erwin Diewert. 1982. "The Economic Theory of Index Numbers and the Measurement of Input, Output and Productivity." *Econometrica* 50: 1393–414.

Christensen, Laurits R., and Dale W. Jorgenson. 1969. "The Measurement of US Real Capital Input, 1929–1967." *Review of Income and Wealth* 15 (4): 293–320.

Christensen, Laurits R., Dale W. Jorgenson, and Lawrence J. Lau. 1971. "Conjugate Duality and the Transcendental Logarithmic Production Function." *Econometrica* 39: 255–56.

Christensen, Laurits R., Dale W. Jorgenson, and Lawrence J. Lau. 1975. "Transcendental Logarithmic Utility Functions." *American Economic Review* 65: 367–83.

Cobb, Charles W., and Paul H. Douglas. 1928. "A Theory of Production." *American Economic Review* 18: 139–65.

de Haan, Jan, and Frances Krsinich. 2012. "The Treatment of Unmatched Items in Rolling Year GEKS Price Indexes: Evidence from New Zealand Scanner Data." Paper presented at the Economic Measurement Workshop 2012, University of New South Wales, November 23.

de Haan, Jan, and Frances Krsinich. 2014. "Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes." *Journal of Business and Economic Statistics* 32 (3): 341–58.

Debreu, Gerard. 1959. *Theory of Value*. New York: John Wiley and Sons.

Denny, Mark. 1974. "The Relationship Between Functional Forms for the Production System." *Canadian Journal of Economics* 7: 21–31.

Diewert, W. Erwin. 1971. "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function." *Journal of Political Economy* 79: 481–507.

Diewert, W. Erwin. 1973. “Afriat and Revealed Preference Theory.” *Review of Economic Studies* 40: 419–26.
Diewert, W. Erwin. 1974a. “Applications of Duality Theory.” In *Frontiers of Quantitative Economics*, edited by Michael D. Intriligator and David A. Kendrick, vol. II, 106–71. Amsterdam: North-Holland.
Diewert, W. Erwin. 1974b. “Functional Forms for Revenue and Factor Requirements Functions.” *International Economic Review* 15: 119–30.
Diewert, W. Erwin. 1976. “Exact and Superlative Index Numbers.” *Journal of Econometrics* 4: 114–45.
Diewert, W. Erwin. 1978. “Superlative Index Numbers and Consistency in Aggregation.” *Econometrica* 46: 883–900.
Diewert, W. Erwin. 1983. “The Theory of the Cost of Living Index and the Measurement of Welfare Change.” In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette, 163–233. Ottawa: Statistics Canada, reprinted as in *Price Level Measurement*, edited by W. Erwin Diewert, 79–147. Amsterdam: North-Holland, 1990.
Diewert, W. Erwin. 1985. “The Measurement of Waste and Welfare in Applied General Equilibrium Models.” In *New Developments in Applied General Equilibrium Analysis*, edited by John Piggott and John Whalley, 42–103. Cambridge: Cambridge University Press.
Diewert, W. Erwin. 1987. “Index Numbers.” In *The New Pa1grave: A Dictionary of Economics*, edited by John Eatwell, Murray Milgate, and Peter Newman, 767–80. London: The Macmillan Press.
Diewert, W. Erwin. 1993a. “Duality Approaches to Microeconomic Theory.” In *Essays in Index Number Theory in Volume I, Contributions to Economic Analysis 217*, edited by W. Erwin Diewert and Alice O. Nakamura, 105–75. Amsterdam: North Holland.
Diewert, W. Erwin. 1993b. “Symmetric Means and Choice Under Uncertainty.” In *Essays in Index Number Theory*, edited by W. Erwin Diewert and Alice O. Nakamura, vol. 1, 355–433. Amsterdam: North-Holland.
Diewert, W. Erwin. 1993c. “The Early History of Price Index Research.” In *Essays in Index Number Theory*, edited by W. Erwin Diewert and Alice O. Nakamura, vol. 1, 33–65. Amsterdam: North-Holland.
Diewert, W. Erwin. 1996. “Price and Volume Measures in the System of National Accounts.” In *The New System of National Economic Accounts*, edited by Johnson Kendrick, 237–85. Norwell, MA: Kluwer Academic Publishers.
Diewert, W. Erwin. 1998. “Index Number Issues in the Consumer Price Index.” *Journal of Economic Perspectives* 12 (1): 47–58.
Diewert, W. Erwin. 2001. “The Consumer Price Index and Index Number Purpose.” *Journal of Economic and Social Measurement* 27: 167–248.
Diewert, W. Erwin. 2002. “The Quadratic Approximation Lemma and Decompositions of Superlative Indexes.” *Journal of Economic and Social Measurement* 28: 63–88.
Diewert, W. Erwin. 2009. “Cost of Living Indexes and Exact Index Numbers.” In *Quantifying Consumer Preferences*, edited by Daniel Slottje, 207–46. London: Emerald Group Publishing.
Diewert, W. Erwin, and Robert C. Feenstra. 2019. “Estimating the Benefits of New Products: Some Approximations.” Discussion Paper 19–02, Vancouver School of Economics, University of British Columbia, Vancouver, BC, Canada, V6T 1L4.
Diewert, W. Erwin, and Robert C. Feenstra. 2022. “Estimating the Benefits of New Products.” In *Big Data for Twenty-First-Century Economic Statistics*, edited by Katharine G. Abraham, Ron S. Jarmin, Brian C. Moyer, and Matthew D. Shapiro, 437–73. Chicago: University of Chicago Press.
Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2017a. “The Allocation and Valuation of Time.” Discussion Paper 17–04, School of Economics, University of British Columbia, Vancouver, BC, Canada, V6N 1Z1.
Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2017b. “The Digital Economy, New Products and Consumer Welfare.” Discussion Paper 17–09, Vancouver School of Economics, University of British Columbia, Vancouver, BC, Canada, V6T 1L4.
Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2017c. “The Digital Economy, New Products and Consumer Welfare.” Vancouver School of Economics Discussion Paper 17–09, University of British Columbia.
Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2018. “The Allocation and Valuation of Time.” Vancouver School of Economics Discussion Paper 18–10, University of British Columbia, Vancouver, Canada.
Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2019. “Experimental Economics and the New Commodities Problem.” Discussion Paper 19–03, Vancouver School of Economics, University of British Columbia, Vancouver, BC, Canada, V6T 1L4.
Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2022. “Experimental Economics and the New Commodities Problem.” *Review of Income and Wealth*, Vol. 68, December 2022: 895-905.
Diewert, W. Erwin, and Robert J. Hill. 2010. “Alternative Approaches to Index Number Theory.” In *Price and Productivity Measurement*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 263–78. Victoria, Canada: Trafford Press.
Feenstra, Robert C. 1994. “New Product varieties and the Measurement of International Prices.” *American Economic Review* 84 (1): 157–77.
Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton-Mifflin.
Fleurbaey, Marc. 2009. “The Quest for a Measure of Social Welfare.” *Journal of Economic Literature* 47: 1029–75.
Gini, Corrado. 1921. “Measurement of Inequality of Incomes.” *Economic Journal* 31: 124–26.
Haan, Jan de. 2017. “Quality Change, Hedonic Regression and Price Index Construction.” Paper presented at Ottawa Group Meeting, 2017.
Haan, Jan de, and Frances Krsinich. 2012. “The Treatment of Unmatched Items in Rolling Year GEKS Price Indexes: Evidence from New Zealand Scanner Data.” Paper presented at the Economic Measurement Group Workshop 2012, Australian School of Business, University of New South Wales, November 23.
Haan, Jan de, and Frances Krsinich. 2014. “Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes.” *Journal of Business and Economic Statistics* 32 (3): 341–58.
Hardy, Godfrey Harold, John Edensor Littlewood, and George Polyá. 1934. *Inequalities*. Cambridge: Cambridge University Press.
Hays, Richard, Josh Martin, and Walter Mkandawire. 2019. “GDP and Welfare: A Spectrum of Opportunity.” ESCoE Discussion Paper No. 2019–16, Economic Statistics Centre of Excellence National Institute of Economic and Social Research, 2 Dean Trench St, London SW1P 3HE.
Hicks, John Richard. 1940. “The Valuation of the Social Income.” *Economica* 7: 105–24.
Hicks, John Richard. 1941–42. “Consumers’ Surplus and Index Numbers.” *The Review of Economic Studies* 9: 126–37.
Hicks, John Richard. 1946. *Value and Capital*. 2nd ed. Oxford: Clarendon Press.
Hill, Robert J. 2006. “Superlative Indexes: Not All of Them Are Super.” *Journal of Econometrics* 130: 25–43.
Hill, T. Peter. 1999. “COL Indexes and Inflation Indexes.” Paper tabled at the 5th Meeting of the Ottawa Group on Price Indexes, Reykjavik, Iceland, August 25–27.
Hill, T. Peter. 2009. “Consumption of Own Production and Cost of Living Indexes.” In *Price and Productivity Measurement; Studies in Income and Wealth, CRIW/NBER*, edited

by W. Erwin Diewert, John Greenlees, and Charles Hulten, 429–44. Chicago: University of Chicago Press.

Hofsten, E. von. 1952. *Price Indexes and Quality Change*. London: George Allen and Unwin.

Hotelling, Harold. 1932. "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions." *Journal of Political Economy* 40: 577–616.

Jorgenson, Dale W., and Paul Schreyer. 2017. "Measuring Individual Economic Well-Being and Social Welfare Within the Framework of the System of National Accounts." *The Review of Income and Wealth* 63 (S2): S460–S77.

Jorgenson, Dale W., and Daniel T. Slesnick. 1987. "Aggregate Consumer Behavior and Household Equivalence Scales." *Journal of Business and Economic Statistics* 5 (32): 1987.

Keynes, John M. 1930. *Treatise on Money*. Vol. 1. London: Macmillan.

Kolm, Serge-Christophe. 1969. "The Optimal Production of Social Justice." In *Public Economics*, edited by Julius Margolis and H. Guitton, 173–200. London: MacMillan.

Konüs, Alexander Alexandrovich. 1924. "The Problem of the True Index of the Cost of Living." Translated in *Econometrica* 7 (1939): 10–29.

Konüs, Alexander Alexandrovich, and Sergei Sergeyevich Byushgens. 1926. "K probleme pokupatelnoi cili deneg." *Voprosi Konyunkturi* 2: 151–72.

Landefeld, Steven, Barbara Fraumeni, and Cindy Vojtech. 2009. "Accounting for Nonmarket Production: A Prototype Satellite Account Using the American Time Use Survey." *Review of Income and Wealth* 55 (2): 205–25.

Lau, Lawrence J. 1979. "On Exact Index Numbers." *Review of Economics and Statistics* 61: 73–82.

Lloyd, Peter J. 1975. "Substitution Effects and Biases in Nontrue Price Indexes." *American Economic Review* 65: 301–13.

Marshall, Alfred. 1887. "Remedies for Fluctuations of General Prices." *Contemporary Review* 51: 355–75.

McFadden, Daniel. 1966. "Cost, Revenue and Profit Functions: A Cursory Review." IBER Working Paper No. 86, University of California, Berkeley.

McFadden, Daniel. 1978. "Cost, Revenue and Profit Functions." In *Production Economics: A Dual Approach*, edited by Melvyn Fuss and Daniel McFadden, vol. 1, 3–109. Amsterdam: North-Holland.

McKenzie, Lionel W. 1956. "Demand Theory without a Utility Index." *Review of Economic Studies* 24: 184–89.

Moulton, R. Brent. 1996. "Constant Elasticity Cost-of-Living Index in Share Relative Form." Bureau of Labor Statistics, Washington, DC, December.

Nordhaus, D. William. 1997. "Do Real Output and Real Wage Measures Capture Reality? The History of Lighting Suggests Not." In *The Economics of New Goods*, edited by Timothy F. Bresnahan and Robert J. Gordon, 29–66. Chicago: University of Chicago Press.

Olsson, Carl-Axel. 1971. "The Cobb-Douglas or the Wicksell Function?" *Economy and History* 14 (1): 64–69.

Pollak, A. Robert. 1971. "The Theory of the Cost of Living Index." Bureau of Labor Statistics Working Paper 11. Washington, DC, June.

Pollak, A. Robert. 1975. "Subindexes in the Cost of Living Index." *International Economic Review* 16: 135–50.

Pollak, A. Robert. 1980. "Group Cost-of-Living Indexes." *American Economic Review* 70: 273–78.

Pollak, A. Robert. 1981. "The Social Cost-of-Living Index." *Journal of Public Economics* 15: 311–36.

Pollak, A. Robert. 1983. "The Theory of the Cost-of-Living Index." In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette, 87–161. Ottawa: Statistics Canada; also reprinted as in Robert. A. Pollak. 1989. *The Theory of the Cost-of-Living Index*, 3–52. Oxford: Oxford University Press; also reprinted as in *Price Level Measurement*, edited by W. Erwin Diewert, 5–77. Amsterdam: North-Holland, 1990.

Pollak, A. Robert. 1989. "The Treatment of the Environment in the Cost of Living Index." In *The Theory of the Cost-of-Living Index*, edited by Robert A. Pollak, 181–85. Oxford: Oxford University Press.

Pollak, A. Robert, and Michael L. Wachter. 1975. "The Relevance of the Household Production Function and Its Implications for the Allocation of Time." *Journal of Political Economy* 83: 255–77.

Pollak, A. Robert, and Michael L. Wachter. 1977. "Reply: Pollak and Wachter on the Household Production Approach." *Journal of Political Economy* 85: 1083–86.

Prais, Sigbert John. 1959. "Whose Cost of Living?" *The Review of Economic Studies* 26: 126–34.

Rawls, John. 1971. *A Theory of Justice*. Cambridge MA: Harvard University Press.

Samuelson, A. Paul. 1953. "Prices of Factors and Goods in General Equilibrium." *Review of Economic Studies* 21: 1–20.

Samuelson, A. Paul. 1974. "Complementarity—An Essay on the 40th Anniversary of the Hicks-Allen Revolution in Demand Theory." *Journal of Economic Literature* 12: 1255–89.

Samuelson, A. Paul, and Subramanian Swamy. 1974. "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis." *American Economic Review* 64: 566–93.

Sato, Kazuo. 1976. "The Ideal Log-Change Index Number." *Review of Economics and Statistics* 58: 223–28.

Schreyer, Paul. 2022. "Accounting for Free Digital Services and Household Production – An Application to Facebook (Meta)." Eurostat Review on National Accounts and Macroeconomic Indicators, 2022 Edition: 7-26.

Schreyer, Paul, and W. Erwin Diewert. 2014. "Household Production, Leisure and Living Standards." In *Measuring Economic Sustainability and Progress*, edited by Dale W. Jorgenson, J. Steven Landefeld, and Paul Schreyer, 89–114. Chicago IL: University of Chicago Press.

Sen, Amartya. 1973. *On Economic Inequality*. Oxford: Clarendon Press.

Sen, Amartya. 1976. "Real National Income." *The Review of Economic Studies* 43: 19–39.

Shapiro, D. Matthew, and David W. Wilcox. 1997. "Alternative Strategies for Aggregating Prices in the CPI." *Federal Reserve Bank of St. Louis Review* 79 (3): 113–25.

Shephard, Ronald William. 1953. *Cost and Production Functions*. Princeton: Princeton University Press.

Shephard, Ronald William. 1970. *Theory of cost and Production Functions*. Princeton: Princeton University Press.

Theil, Henri. 1967. *Economics and Information Theory*. Amsterdam: North-Holland Publishing.

Törnqvist, Leo., and Erik Törnqvist. 1937. "Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?" *Ekonomiska Samfundets Tidskrift* 39: 1–39 reprinted as in *Collected Scientific Papers of Leo Törnqvist*, 121–60. Helsinki: The Research Institute of the Finnish Economy, 1981.

Triplett, Jack. 2004. "Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes." Directorate for Science, Technology and Industry, DSTI/DOC(2004)9, OECD, Paris.

Uzawa, Hirofumi. 1962. "Production Functions with Constant Elasticities of Substitution." *Review of Economic Studies* 29: 291–99.

vartia, O. Yrjö. 1976. "Ideal Log-Change Index Numbers." *Scandinavian Journal of Statistics* 3: 121–26.

Walsh, Correa Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Walsh, Correa Moylan. 1921. *The Problem of Estimation*. London: P. S. King & Son.

Wold, Herman. 1944. "A Synthesis of Pure Demand Analysis, Part 3." *Skandinavisk Aktuarietidskrift* 27: 69–120.

Wold, Herman. 1953. *Demand Analysis*. New York: John Wiley.

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ELEMENTARY INDICES* 6

1. Introduction

In all countries, the calculation of a CPI proceeds in two (or more) stages. In the first stage of calculation, *elementary price indices* are calculated for the *elementary expenditure aggregates* of a CPI. In the second and higher stages of aggregation, these elementary price indices are combined to obtain higher level indices using information on the expenditures on each elementary aggregate as weights. An *elementary aggregate* consists of the expenditures by a specified group of consumers on a relatively homogeneous set of products defined within the consumption classification used in the CPI.

At the first stage of aggregation, one of two possible situations can occur:

- Detailed price and quantity (or price and value) information on all transacted products in the elementary aggregate is available for the time period under consideration.[1]
- Price information is available only for the products in the aggregate under consideration. Moreover, the price information may be collected only for a *sample* of the entire set of product prices that are in scope.

At higher levels of aggregation, typically price and quantity (or value) information is available. Thus, for higher levels of aggregation and for situations where detailed price and quantity information is available at the first stage of aggregation, the materials in previous chapters can be applied; that is, Lowe, Laspeyres, Paasche, and Fisher indices can be used at higher levels of aggregation and at the elementary level if detailed price and quantity information is available. However, for situations where quantity or value information is not available, most of the index number theory outlined in previous chapters is not directly applicable. In this case, an elementary price index relies only on price data. The situation where only price information is available will be the focus of this chapter. However, some elementary indices can be constructed using price and quantity (or expenditure) data, and so some attention will be paid to this situation as well.[2]

The question of what is the most appropriate formula to use to construct an elementary price index is considered in this chapter.[3] The quality of a CPI depends heavily on the quality of the first stage of aggregation elementary indices, which are the basic building blocks from which CPIs are constructed.

CPI compilers have to select *representative products* within an elementary aggregate and then collect a sample of prices for each of the representative products, usually from a sample of different outlets. The individual products whose prices are actually collected are described as the *sampled products*. Their prices are collected over successive time periods. An elementary price index is therefore typically calculated from two sets of *matched price observations*. In this chapter, we will assume that there are no missing observations and no changes in the quality of the products sampled so that the two sets of prices are perfectly matched. In the following chapter, we will consider alternative strategies when there are multiple time periods and missing observations; that is, in Chapter 7, we will discuss *multilateral index number theory*. In Chapter 8, the treatment of new and disappearing goods and services and the related problems associated with measuring *quality change* will be discussed.

Before we define the elementary indices used in practice, we will first consider in Section 2 what a suitable definition for an *ideal elementary index* is. An *ideal index* will make use of expenditure data (as well as price data) even though it cannot always be implemented in practice due to lack of expenditure and quantity data. The problems involved in aggregating transaction prices for the same product over *time* are also discussed in this section. In general, the discussion in Section 2 provides a theoretical target index that uses both price and quantity information. "Practical" elementary price indices that are constructed using only information on prices will be discussed in subsequent sections.

Section 3 provides some additional discussion about the problems involved in picking a suitable level of disaggregation for the elementary aggregates. Should the elementary aggregates have a regional dimension in addition to a product dimension? Should prices be collected from retail outlets or from households? These are the types of questions discussed in this section.

Section 4 introduces the main elementary index formulae that are used in practice, and Section 5 develops some

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*This chapter draws heavily on Chapter 20 of the *Consumer Price Index Manual*; see ILO, IMF, OECD, UNECE, Eurostat, World Bank (2004; 355–371). The author thanks Elizabeth Abraham, Carsten Boldsen, Yuri Dikhanov, Kevin Fox, Heidi Ertl, Robert Hill, Ronald Johnson, Claude Lamboray, Cindy MacDonald, Marshall Reinsdorf, Mark Ruddock, Chihiro Shimizu, Mick Silver, Jasmin Whelan, and Grant Yake for their helpful comments.

[1] With the increased availability of scanner data both for retail outlets and for individual consumers, the first situation is increasingly likely. Also it may be the case that the statistical office will have access to price and quantity data on deliveries to households from regulated electricity and telecom firms. In the annex to this chapter, we will use such a data set for the UK fixed-line telecom sector in order to show how the various elementary indices to be considered below perform in practice.

[2] Thus, scanner data are increasingly applied at the elementary level by national statistical agencies. The use of scanner data can lead to chain drift problems, which will be addressed in the following chapter.

[3] The material in this chapter draws heavily on the contributions of Dalén (1992), Balk (1994) (2002) (2008), and Diewert (1995) (2002).

numerical relationships between the various "practical" indices. These relationships will be illustrated for a particular data set in Annex A to this chapter.

Section 6 develops the axiomatic or *test approach* to bilateral elementary indices when only price information is available.

Section 7 contains some material on the importance of the time reversal test.

Section 8 concludes with an overview of the various results.

Annex A looks at the problems that arise when households have to pay a fixed fee to gain access to various products or services that a firm sells. For the most part, these access fees are not very large, so their treatment in a CPI does not make a material difference. However, in the case of telecommunication services, alternative treatments of access fees lead to very different price (and quantity) indices, as will be seen in the annex. Also, as mentioned earlier, the numerical relationships between the various elementary indices that are developed in Section 5 will be illustrated in Annex A with actual telecom data from the United Kingdom.

Annex B lists the objections to the use of the Carli index made by Robert Hill in his testimony to the UK House of Lords on the use of the Carli index in the UK's Retail Price Index.

2. Ideal Elementary Indices

The aggregates covered by a CPI are usually arranged in the form of a tree-like hierarchy, such as the Classification of Individual Consumption by Purpose (COICOP). An aggregate is a set of economic transactions pertaining to a set of products and a set of economic agents over a specified time period. Every economic transaction relates to the change of ownership of a specific, well-defined product (good or service) at a particular place and date and comes with a quantity and a price. A price index for an aggregate is typically calculated as a weighted average of the price indices for the subaggregates, the (expenditure or sales) weights, and type of average being determined by the index formula. One can descend in such a hierarchy as far as available information allows the weights to be decomposed. The lowest level aggregates are called *elementary* aggregates. They are basically of two types:

- Those for which all detailed price and quantity information is available
- Those for which the statistician, considering the operational cost and the response burden of getting detailed price and quantity information about all the transactions, decides to make use of a representative sample of products and respondents

As indicated earlier, the practical relevance of studying this topic is large. Since the elementary aggregates form the building blocks of a CPI, the choice of an inappropriate formula at this level can have a tremendous impact on the overall index.

In this section, it will be assumed that detailed price and quantity information for all transactions pertaining to the elementary aggregate for the two time periods under consideration is available. This assumption allows us to define an *ideal elementary aggregate*. Subsequent sections will relax this strong assumption about the availability of detailed price and quantity data on transactions, but in any case, it is useful to have a theoretically ideal target for the "practical" elementary index.

The detailed price and quantity data, although perhaps not available to the statistician, are, in principle, available in the outside world. It is frequently the case that at the respondent level (that is, at the outlet or firm level), some aggregation of the individual transaction information has been executed, usually in a form that suits the respondent's financial or management information system. This respondent-determined level of information could be called the *basic information level*. This is, however, not necessarily the finest level of information that could be made available to the price statistician. One could always ask the respondent to provide more disaggregated information. For instance, instead of monthly data one could ask for weekly data; or, whenever appropriate, one could ask for regional instead of global data; or, one could ask for data according to a finer product classification. The only natural barrier to further disaggregation is the individual transaction level.[4]

It is now necessary to discuss a problem[5] that arises when detailed data on *individual transactions* are available, either at the level of the individual household or at the level of an individual outlet. Recall that in previous chapters, the price and quantity indices, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, were introduced. These (bilateral) price and quantity indices decomposed the value ratio V^1/V^0 into a price change part $P(p^0,p^1,q^0,q^1)$ and a quantity change part $Q(p^0,p^1,q^0,q^1)$. In this framework, it was taken for granted that the period t price and quantity for product i, p_i^t, and q_i^t were well defined.[6] However, these definitions are not straightforward since individual consumers may purchase the *same* item during period *t at different prices*. Similarly, if we look at the sales of a particular shop or outlet that sells to consumers*, the same item may sell at very different prices during the course of the period*. Hence, before a traditional bilateral price index of the form $P(p^0,p^1,q^0,q^1)$ considered in previous chapters can be applied, there is a nontrivial *time aggregation problem* that must be solved in order to obtain the basic prices p_i^t and q_i^t which are the components of the price vectors p^0 and p^1 and the quantity vectors q^0 and q^1.

Walsh[7] and Davies (1924) (1932) suggested a solution to this time aggregation problem: the appropriate quantity at

[4] The material in this section is based on Balk (1994).

[5] This time aggregation problem was discussed briefly in Chapter 2.

[6] Note that the period of time t could represent any period of time: a quarter, a month, a week, a day, or an hour.

[7] Walsh explained his reasoning as follows: "Of all the prices reported of the same kind of article, the average to be drawn is the arithmetic; and the prices should be weighted according to the relative mass quantities that were sold at them" (Correa Moylan Walsh (1901; 96)). "Some nice questions arise as to whether only what is consumed in the country, or only what is produced in it, or both together are to be counted; and also there are difficulties as to the single price quotation that is to be given at each period to each commodity, since this, too, must be an average. Throughout the country during the period a commodity is not sold at one price, nor even at one wholesale price in its principal market. various quantities

this very first stage of aggregation is the *total quantity purchased* of the narrowly defined item and the corresponding price is the value of purchases of this item divided by the total amount purchased, which is a *narrowly defined unit value*. In more recent times, most researchers have adopted the Walsh and Davies solution to the time aggregation problem.[8] Note that this solution to the time aggregation problem has the following advantages:

- The quantity aggregate is intuitively plausible, being the total quantity of the narrowly defined item purchased by the household (or sold by the outlet) during the time period under consideration.
- The product of the price times quantity equals the total value purchased by the household (or sold by the outlet) during the time period under consideration.

We will adopt this solution to the time aggregation problem as our concept for the price and quantity at this preliminary stage of aggregation.

Having decided on an appropriate theoretical definition of price and quantity for an item at the very lowest level of aggregation (that is, a narrowly defined unit value and the total quantity sold of that item at the individual outlet), we now consider how to aggregate these narrowly defined elementary prices and quantities into an overall elementary aggregate. Suppose that there are N lowest level items or specific products in this chosen elementary category. Denote the period t quantity of item n by $q_n^{\,t}$ and the corresponding time-aggregated unit value price by $p_n^{\,t}$ for $t = 0,1$ and for items $n = 1,2,\ldots,N$. Define the period t quantity and price vectors as $q^t \equiv [q_1^{\,t},q_2^{\,t},\ldots,q_N^{\,t}]$ and $p^t \equiv [p_1^{\,t},p_2^{\,t},\ldots,p_N^{\,t}]$ for $t = 0,1$. It is now necessary to choose a theoretically ideal index number formula $P(p^0,p^1,q^0,q^1)$ that will aggregate the individual item prices into an overall aggregate price relative to the N items in the chosen elementary aggregate. However, this problem of choosing a functional form for $P(p^0,p^1,q^0,q^1)$ is *identical to the overall index number problem* that was addressed in previous chapters. In the previous chapters, four different approaches to index number theory were studied that led to specific index number formulae as being "best" from each perspective. From the viewpoint of *fixed basket approaches*, it was found that the Fisher (1922) and Walsh (1901) price indices, P_F and P_W, appeared to be the "best." From the viewpoint of the *test approach*, the Fisher index appeared to be the "best." From the viewpoint of the *stochastic approach* to index number theory, the Törnqvist–Theil index number formula P_T was considered as the "best." Finally, from the viewpoint of the *economic approach* to index number theory, the Walsh price index P_W, the Fisher ideal index P_F, and the Törnqvist–Theil index number formula P_T were all regarded as equally desirable. It was also shown that the same three index number formulae numerically approximate each other very closely under certain conditions, and so it will not matter very much which of these alternative indices is chosen.[9] Hence, the *theoretically ideal elementary index number formula* is taken to be one of the three formulae $P_F(p^0,p^1,q^0,q^1)$, $P_W(p^0,p^1,q^0,q^1)$, or $P_T(p^0,p^1,q^0,q^1)$, where the period t quantity of item n, $q_n^{\,t}$, is the total quantity of that narrowly defined item purchased by the household during period t (or sold by the outlet during period t), and the corresponding price for item n is $p_n^{\,t}$, the time aggregated unit value, for $t = 0,1$ and for items $n = 1,2,\ldots,N$.

In the following sections, various "practical" elementary price indices will be defined. These indices do not have quantity weights and thus are functions only of the price vectors p^0 and p^1. Thus, when a practical elementary index number formula, say $P_E(p^0,p^1)$, is compared to an ideal elementary price index, say the Fisher price index $P_F(p^0,p^1,q^0,q^1)$, then obviously P_E will differ from P_F because the *prices are not weighted according to their economic importance* in the practical elementary formula. It is useful to list the following possible sources of difference between a practical elementary price index $P_E(p^0,p^1)$ and an ideal target index:

- *Weighting bias* or, more generally, *formula bias*—that is, a price index of the form $P_E(p^0,p^1)$—is not able to weight prices according to the economic importance of the product in the consumer's total expenditures on the group of products under consideration.[10]
- *Sampling bias*—that is, the statistical agency may not be able to collect information on all N products in the elementary aggregate—that is, only a sample of the N prices may be collected.[11]
- *Time aggregation bias*—that is, even if a price for a narrowly defined item is collected by the statistical agency, it may not be equal to the theoretically appropriate time-aggregated unit value price.[12]
- *Item aggregation bias* or *unit value bias*. The statistical agency may classify certain distinct products as being essentially equivalent, and thus the unit value aggregate for this group of aggregated products may not take into account possible significant quality differences in the group of aggregated products. For example, products that are thought to be very similar and are sold in the same units of measurement could be treated as a single product.[13]

of it are sold at different prices, and the full value is obtained by adding all the sums spent (at the same stage in its advance toward the consumer), and the average price is found by dividing the total sum (or the full value) by the total quantities" (Correa Moylan Walsh (1921a; 88)).

[8] See, for example, Szulc (1987; 13), Dalén (1992; 135), Reinsdorf (1994), Diewert (1995; 20–21), Reinsdorf and Moulton (1997), and Balk (2002).

[9] Theorem 5 in Diewert (1978; 888) showed that P_F, P_T, and P_W will approximate each other to the second order around an equal price and quantity point. However, if there are violent fluctuations in prices and quantities, a second-order approximation to any one of these formulae may not be very accurate.

[10] For materials on how to measure formula bias, see Diewert (1998), White (1999) (2000), and Chapter 7.

[11] This is a specialized topic with a long history. It will not be covered in this volume.

[12] Many statistical agencies send price collectors to various outlets on certain days of the month to collect list prices of individual items. Usually, price collectors do not work on weekends when many sales take place, and thus the collected prices may not be fully representative of all transactions that occur. Thus, these collected prices can be regarded only as approximations to the time aggregated unit values for those items.

[13] For materials on unit value bias, see Diewert and von der Lippe (2010) and Silver (2010) (2011) and the additional references in these papers.

- *Aggregation over agents* or *aggregation over entities bias* or *aggregation over outlets bias*. The unit value for a particular item may be constructed by aggregating overall households in a region or a certain demographic class or by aggregating overall outlets or shops that sell the item in a particular region.[14]
- *New and disappearing products bias*—that is, $P_E(p^0,p^1)$ measures price change only over matched products for the two periods being compared; new products and disappearing products are ignored in standard elementary indices that depend only on prices.[15]

Approximations to the numerical differences between various elementary indices of the form $P_E(p^0,p^1)$ and various superlative indices will be developed in Chapter 7.

In the following section, the problems of aggregation and classification will be discussed in more detail.

3. Aggregation and Classification Problems for Elementary Aggregates

Hawkes and Piotrowski (2003) noted that the definition of an elementary aggregate involves aggregation over *four* possible dimensions:[16]

- A *time* dimension—that is, the item unit value could be calculated for all item transactions for a year, a month, a week, or a day.
- A *spatial* dimension—that is, the item unit value could be calculated for all item transactions in the country, province, state, city, neighborhood, or individual location.
- A *product* dimension—that is, the item unit value could be calculated for all item transactions in a broad general category (for example, food), in a more specific category (for example, margarine), for a particular brand (ignoring package size), or for a particular narrowly defined item (for example, a particular AC Nielsen universal product code).
- A *sectoral* (or *entity* or *economic agent*) dimension—that is, the item unit value could be calculated for a particular class of households or a particular class of outlets.

Each of the aforementioned dimensions for choosing the domain of definition for an elementary aggregate will be discussed in turn.

As the time period is compressed, several problems emerge:

- Purchases (by households) and sales (by outlets) become erratic and sporadic. Thus, the frequency of unmatched purchases or sales from one period to the next increases and in the limit (choose the time period to be one minute), nothing will be matched, and bilateral index number theory fails at the individual consumer level.[17]
- As the time period becomes shorter, chained indices exhibit more "drift"; that is, if the data at the end of a chain of periods reverts to the data in the initial period, the chained index does not revert back to unity. As was discussed in Section 8 of Chapter 2, it is only appropriate to use chained indices when the underlying price and quantity data exhibit relatively smooth trends. When the time period is short, seasonal fluctuations[18] and periodic sales and advertising campaigns[19] can cause prices and quantities to oscillate (or "bounce" to use Szulc's (1983; 548) term), and hence it is not appropriate to use chained indices under these circumstances. If Fixed-Base indices are used in this short time period situation, then the results will usually depend very strongly on the choice of the base period. In the seasonal context, not all products may even be in the marketplace during the chosen base period.[20] All of these problems can be mitigated by choosing a longer time period so that trends in the data will tend to dominate the short-term fluctuations.
- As the time period contracts, virtually all goods become *durable*, in the sense that they yield services not only for the period of purchase but for subsequent periods. Thus, the period of purchase or acquisition becomes different from the periods of use, leading to many complications.[21]
- As the time period contracts, users will usually not be particularly interested in the short-term fluctuations of the resulting index, and there will be demands for smoothing the necessarily erratic results. Put another way, users will desire a way of summarizing the weekly or daily movements in the index into monthly or quarterly movements in prices. Hence, from the viewpoint of meeting the needs

[14] For materials on possible methods to measure outlet substitution bias, see Diewert (1998). The problems associated with measuring aggregation over consumers' bias were noted in the final sections of Chapter 5.

[15] This problem was addressed in Section 14 of Chapter 5. It will be addressed in more detail in Chapters 7 and 8.

[16] Hawkes and Piotrowski (2003; 31) combined the spatial and sectoral dimensions into the spatial dimension. They also acknowledged the pioneering work of Theil (1954), who identified three dimensions of aggregation: aggregation over individuals, aggregation over products, and aggregation over time. It should be noted that William Hawkes was a pioneer in realizing the importance of scanner data for the construction of CPIs; see Hawkes (1997). Other important contributors include Reinsdorf (1996), Silver (1995), Silver and Heravi (2001) (2003) (2005), de Haan and van der Grient (2011), Ivancic, Diewert, and Fox (2011), and de Haan and Krsinich (2014).

[17] This problem was noted in Section 19 of Chapter 5. David Richardson (2003; 51) also made this point: "Defining items with a finer granularity, as is the case if quotes in different weeks are treated as separate items, results in more missing data and more imputations." However, high-frequency CPIs could be successfully constructed if aggregation over households or outlets is permitted.

[18] See Chapter 9 for a monthly seasonal example where chained month-to-month indices exhibit significant drift.

[19] See Feenstra and Shapiro (2003) for an example of a weekly superlative index that exhibits massive chain drift. Substantial chain drift can also occur using monthly indices; see Szulc (1983) (1987). See Richardson (2003; 50–51) and Ivancic, Diewert, and Fox (2011) for additional discussions of the issues involved in choosing weekly unit values versus monthly unit values.

[20] See Chapter 9 for suggested solutions to these seasonality problems.

[21] See Chapter 10 for more material on the possible CPI treatment of durable goods.

of users, there may be relatively little demand for high-frequency indices.

In view of these considerations, it is recommended that the index number time period be at least four consecutive weeks or a month.[22]

It is also necessary to choose the spatial dimension of the elementary aggregate. Should item prices in each city or region be considered as separate aggregates, or should a national item aggregate be constructed? Obviously, if it is desired to have regional CPIs that aggregate up to a national CPI, then it will be necessary to collect item prices by region. However, it is not clear how fine the "regions" should be. It could be as fine as a grouping of households in a postal code or to individual outlets across the country.[23] There does not seem to be a clear consensus on what the optimal degree of spatial disaggregation should be.[24] Each statistical agency will have to make its own judgments on this matter, taking into account the costs of data collection and the demands of users for a spatial dimension for the CPI.

How detailed should the product dimension be? The possibilities range from regarding all products in a general category as being equivalent to the other extreme, where only a product in a particular package size made by a particular manufacturer or service provider is regarded as being equivalent. All things being equal, Triplett (2004) stressed the advantages of matching products at the most detailed level possible, since this will prevent quality differences from clouding the period-to-period price comparisons. This is sensible advice, but then what are the drawbacks to working with the finest possible product classification? The major drawback is that the finer the classification is, the more difficult it will be to *match* the item purchased or sold in the base period to the same item in the current period. Hence, the finer the product classification, the smaller will be the number of matched price comparisons that are possible.[25] This would not be a problem if the unmatched prices followed the same trend as the matched ones in a particular elementary aggregate, but in at least some circumstances, this will not be the case.[26] Thus, the finer the classification system is, the more work (in principle) there will be for the statistical agency to quality adjust or impute the prices that do not match. Choosing a relatively coarse classification system can lead to a very cost-efficient system of quality adjustment (that is, essentially no explicit quality adjustment or imputation is done for the prices that do not exactly match), but it may not be very accurate. The statistical agency will have to balance the theoretical purity of a very fine classification system with the possible loss of product matches.

The final issue in choosing a classification scheme is the issue of choosing a sectoral dimension; that is, should the unit value for a particular item be calculated for a particular outlet or a particular household or for a class of outlets or households? Before this question can be answered, it is necessary to ask whether the individual outlet or the individual household is the appropriate finest level of entity classification. If the economic approach to the CPI is taken, then the individual household is the appropriate finest level of entity classification.[27] However, if the time period is short, a single household will not work very well as the basic unit of entity observation due to the sporadic nature of many purchases by an individual household; that is, there will be tremendous difficulties in matching prices across periods for individual households. However, for a grouping of "similar" households that is sufficiently large, it does become feasible in theory to use the grouped household as the entity classification rather than the outlet as is usually done. This is not usually done because of the costs and difficulties involved in collecting individual household data on prices and expenditures.[28] Thus, price information is usually collected from retail establishments or outlets that sell mainly to households. Matching problems are mitigated

[22] If there is very high inflation in the economy (or even hyperinflation), then it may be necessary to move to weekly or even daily indices. Also, it should be noted that some index number theorists feel that new theories of consumer behavior should be developed that could utilize weekly or daily data: "Some studies have endorsed unit values to reduce high frequency price variation, but this implicitly assumes that the high frequency variation represents simply noise in the data and is not meaningful in the context of a COLI. That is debatable. We need to develop a theory that confronts the data, not truncate the data to fit the theory" (Jack E. Triplett (2003; 153)). However, until such new theories are adequately developed, it seems pragmatic to define the item unit values over months or quarters rather than days or weeks.

[23] Iceland no longer uses regional weights but uses individual outlets as the primary geographical unit; see Gudnason (2003; 18).

[24] Hawkes and Piotrowski note that it is quite acceptable to use national elementary aggregates when making international comparisons between countries: "When we try to compare egg prices across geography, however, we find that lacing across outlets won'*t* work, because the eyelets on one side of the shoe (or outlets on one side of the river) don'*t* match up with those on the other side. Thus, in making interspatial comparisons, we have no choice but to aggregate outlets all the way up to the regional (or, in the case of purchasing power parities, national) level. We have no hesitation about doing this for interspatial comparisons, but we are reluctant to do so for intertemporal ones. Why is this?" William J. Hawkes and Frank W. Piotrowski (2003; 31–32). An answer to their question is that it is preferable to match like with like as closely as possible, which leads statisticians to prefer the finest possible level of aggregation, which, in the case of intertemporal comparisons, would be the individual household or the individual outlet. However, in making cross-region comparisons, matching is not possible unless regional item aggregates are formed, as Hawkes and Piotrowski pointed out earlier.

[25] This is part of the *matching problem* discussed at the end of Chapter 5.

[26] Silver and Heravi (2001) (2003; 286) (2005) and Koskimäki and vartia (2001) stressed this point and presented empirical evidence to back up their point. Feenstra (1994) and Balk (2000) used the assumption of CES preferences to deal with the new products problem. Their approaches will be discussed in Chapter 8.

[27] This point has been made emphatically by two authors in a book on scanner data and price indices: "In any case, unit values across stores are not the prices actually faced by households and do not represent the per period price in the COLI, even if the unit values are grouped by type of retail outlet" (Jack E. Triplett (2003; 153–154)). "Furthermore, note that the relationship being estimated is not a proper consumer demand function but rather an 'establishment sales function.' Only after making further assumptions – for example, fixing the distribution of consumers across establishments – is it permissible to jump to demand functions" (Eduardo Ley (2003; 380)).

[28] However, it is possible to collect accurate household data in certain circumstances; see Gudnason (2003), who pioneered a receipts methodology for collecting household price and expenditure data in Iceland. Also, in the future, as monetary transactions are replaced by debit and credit card transactions, it will become possible to construct individual household estimates of real consumption, provided that product codes are included in the transaction records.

Table 6.1 Proportion of Transactions in 2000 That Could Be Matched to 1998

	COICOP	COICOP	AC Nielsen	AC Nielsen
	5 digit	**7 digit**	**Brand**	**UPC**
Country	1.000	1.000	0.982	0.801
Province	1.000	1.000	0.975	0.774
AC Nielsen Region	1.000	1.000	0.969	0.755
Individual Outlet	0.904	0.904	0.846	0.617

using this strategy (but not eliminated) because the retail outlet generally sells the same items on a continuing basis.

If expenditures by all households in a region are aggregated together, will they equal sales by the retail outlets in the region? Under certain conditions, the answer to this question is yes. The conditions are that the outlets do not sell any items to purchasers who are not local households (no regional exports or sales to local businesses or governments), and that the regional households do not make any purchases of consumption items other than from the local outlets (no household imports or transfers of products to local households by governments). Obviously, these restrictive conditions will not be met in practice, but they may hold as a first approximation.

The effects of *regional aggregation* and *product aggregation* can be examined owing to a study by Koskimäki and Ylä-Jarkko (2003). This study utilized scanner data for the last week in September 1998 and September 2000 on butter, margarine, and other vegetable fats, vegetable oils, soft drinks, fruit juices, and detergents. This information was provided by the AC Nielsen company for Finland. At the finest level of item classification (the AC Nielsen Universal Product Code), the number of individual items in the sample was 1,028. The total number of outlets in the sample was 338. Koskimäki and Ylä-Jarkko considered four levels of spatial disaggregation:

- The entire country (1 level)
- Provinces (4 levels)
- AC Nielsen regions (15 levels)
- Individual outlets (338 levels)

They also considered four levels of product disaggregation:

- The COICOP 5-digit classification (6 levels)
- The COICOP 7-digit classification (26 levels)
- The AC Nielsen brand classification (266 levels)
- The AC Nielsen individual Universal Product Code (1,028 distinct products)

In order to illustrate the ability to match products over the two-year period as a function of the degree of fineness of the classification, Koskimäki and Ylä-Jarkko (2003; 10) presented a table that shows that the proportion of transactions that could be matched across the two years fell steadily as the fineness of the classification scheme increased. At the highest level of aggregation (the national and COICOP 5 digit), all transactions could be matched over the two-year period, but at the finest level of aggregation (338 outlets times 1,028 individual products or 347,464 classification cells in all), only 61.7 percent of the value of transactions in 2000 could be matched back to their 1998 counterparts. Their Table 7 is reproduced as Table 6.1 here.

For each of these 16 levels of product and regional disaggregation, for the products that were available in September of 1998 and 2000, Koskimäki and Ylä-Jarkko (2003; 9) calculated Laspeyres and Fisher price indices. They found substantial differences in these indices as the degree of disaggregation increased.

Another study on the effects of alternative methods of unit value aggregation over outlets (that is, treat each unit value for each product in each store as a unique product versus aggregating products over stores and chains) was conducted by Ivancic and Fox (2013). They used 65 weeks of scanner data on the sales of different types of instant coffee sold by four supermarket chains in Australia in 110 stores. The data were collected between February 1997 and April 1998. They contain information on 110 stores that belong to four supermarket chains located in the metropolitan area of one of the major capital cities in Australia. These stores accounted for over 80 percent of grocery sales in the various capital cities of Australia during this period. After data exclusions, 436,103 weekly observations on 157 coffee items were used in their study.[29] Their results on alternative methods of aggregation can be summarized as follows:

> The results show that when non-superlative index numbers are used to calculate price change, aggregation choices can have a huge impact. However, the issue of aggregation seems to become relatively trivial when the standard Fisher and Törnqvist superlative indices are used, with an extremely close range of estimates of price change found across different aggregation methods. This result seems to provide further support for the use of these superlative indices over the use of non-superlative indices to estimate price change.
> Lorraine Ivancic and Kevin J. Fox (2013; 643)

The non-superlative index numbers[30] were chained Laspeyres and Paasche indices, and the superlative indices were chained Fisher and Törnqvist–Theil indices.

Thus, the problem of determining the "best" unit value to insert into an index number formula is far from settled. We will look at this problem again in Chapter 11.

Another issue that arises in the context of defining exactly what prices and quantities should be entered into an index number formula is the following one: Should statistical

[29] Their paper also lists some related studies.

[30] The weekly unit values by product were aggregated into monthly unit values.

agencies exclude sale prices? In general, this is not a recommended practice since very large amounts of a product can be sold at a sale price. Fox and Syed (2016; 404) found that the exclusion of sale prices can introduce a substantial bias. They also found that even when sale prices are included, they are systematically underweighted, but the underweighting remains fairly stable over time so that inflation measurement is not significantly affected. They also found evidence that the typical practice of using data from an incomplete period in constructing unit values can lead to an upward bias in the resulting price index.[31]

4. Some Elementary Indices That Have Been Suggested Over the Years

Suppose that there are N products in a chosen elementary category. Denote the period t price of item n by p_n^t for t = 0,1 and for items $n = 1,2,...,N$. As usual, define the period t price vector as $p^t \equiv [p_1^t,p_2^t,...,p_N^t]$ for $t = 0,1$.

The first simple elementary index number formula was derived by the French economist Dutot (1738):

$$P_D(p^0,p^1) \equiv [\Sigma_{n=1}^N (1/N)\, p_n^1]/[\Sigma_{n=1}^N (1/N)\, p_n^0] = [\Sigma_{n=1}^N p_n^1]/[\Sigma_{n=1}^N p_n^0] = p^1 \cdot 1_N / p^0 \cdot 1_N. \quad (1)$$

Thus, the Dutot elementary price index is equal to the arithmetic average of the N period 1 prices divided by the arithmetic average of the N period 0 prices.

The second simple elementary index number formula was developed by the Italian economist Carli (1764):

$$P_C(p^0,p^1) \equiv \Sigma_{n=1}^N (1/N)(p_n^1/p_n^0). \quad (2)$$

Thus, the Carli elementary price index is equal to the *arithmetic* average of the N item price ratios or price relatives, p_n^1/p_n^0. This formula was already encountered in our study of the unweighted stochastic approach to index numbers; recall definition (2) in Chapter 4.

The third simple elementary index number formula was introduced by the English economist Jevons (1865):

$$P_J(p^0,p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}. \quad (3)$$

Thus, the Jevons elementary price index is equal to the *geometric* average of the N item price ratios or price relatives, p_n^1/p_n^0. Again, this formula was introduced as formula (4) in our discussion of the unweighted stochastic approach to index number theory in Chapter 4.

The fourth elementary index number formula P_H is the *harmonic* average of the N item price relatives, and it was first suggested in passing as an index number formula by Jevons (1865; 121) and Coggeshall (1887):

$$P_H(p^0,p^1) \equiv [\Sigma_{n=1}^N (1/N)(p_n^1/p_n^0)^{-1}]^{-1}. \quad (4)$$

[31] Diewert, Fox, and de Haan (2016) also found this effect. The direction of this bias may be due to an increasing frequency of end-of-month or quarter sales.

Finally, the fifth elementary index number formula is the geometric average of the Carli and harmonic formulae; that is, it is *the geometric mean of the arithmetic and harmonic means* of the N price relatives:

$$P_{CSWD}(p^0,p^1) \equiv [P_C(p^0,p^1)\, P_H(p^0,p^1)]^{1/2}. \quad (5)$$

This index number formula was first suggested by Fisher (1922; 472) as his formula 101. Fisher also observed that, empirically for his data set, P_{CSWD} was very close to the Jevons index, P_J, and these two indices were his "best' unweighted index number formulae. In more recent times, Carruthers, Sellwood, and Ward (1980; 25) and Dalén (1992; 140) also proposed P_{CSWD} as an elementary index number formula.

It should be noted that the Jevons index is now the most commonly used elementary index (when only price information is available). The Dutot and Carli formulae are used by a few statistical agencies.

Having defined the most commonly used elementary formulae, the question now arises: Which formula is the "best"? Obviously, this question cannot be answered until desirable properties for elementary indices are developed. This will be done in a systematic manner in Section 6 (using the test approach), but in the present section, one desirable property for an elementary index will be noted. This is the *time reversal test*, which was noted earlier in Chapters 2 and 3. In the present context, this test for the elementary index $P(p^0,p^1)$ becomes

$$P(p^0,p^1)P(p^1,p^0) = 1. \quad (6)$$

This test says that if the prices in period 2 revert to the initial prices of period 0, then the product of the price change going from period 0 to 1, $P(p^0,p^1)$, times the price change going from period 1 to 2, $P(p^1,p^0)$, should equal unity; that is, under the stated conditions, we should end up where we started.[32] It can be verified that the Dutot, Jevons, and Carruthers, Sellwood, Ward, and Dalén indices, P_D, P_J, and P_{CSWD}, all satisfy the time reversal test but that the Carli and Harmonic indices, P_C and P_H, fail this test. In fact, these last two indices fail the test in the following *biased* manner:

$$P_C(p^0,p^1)\, P_C(p^1,p^0) \geq 1, \quad (7)$$
$$P_H(p^0,p^1)\, P_H(p^1,p^0) \leq 1, \quad (8)$$

with strict inequalities holding in (7) and (8) provided that the period 1 price vector p^1 is not proportional to the period 0 price vector p^0.[33] Thus, the Carli index will generally have an *upward bias,* while the harmonic index will generally have a *downward bias*. Fisher (1922; 66 and 383) was quite definite in his condemnation of the Carli index due to its upward bias.[34] Because it fails the time reversal test, the

[32] This test can also be viewed as a special case of Walsh's (1901) multiperiod identity test (63) in Chapter 2.

[33] These inequalities follow from the fact that a harmonic mean of N positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901;517) or Fisher (1922; 383–384). This inequality is a special case of Schlömilch's Inequality; see Hardy, Littlewood, and Pólya (1934; 26).

[34] See also Szulc (1987; 12) and Dalén (1992; 139). Dalén (1994; 150–151) provided some nice intuitive explanations for the upward bias of the Carli index.

Carli index is not used in compiling elementary price indices for the HICP, which is the official Eurostat index used to compare consumer prices across the European Union countries.

In the following section, some numerical relationships between the five elementary indices defined in this section will be established. Then in the subsequent section, a more comprehensive list of desirable properties for elementary indices will be developed and the five elementary formulae will be evaluated in the light of these properties or tests.

5. Numerical Relationships between Some Elementary Indices

It can be shown[35] that the Carli, Jevons, and Harmonic elementary price indices satisfy the following inequalities:

$$P_H(p^0,p^1) \le P_J(p^0,p^1) \le P_C(p^0,p^1); \tag{9}$$

that is, the Harmonic index is always equal to or less than the Jevons index, which in turn is always equal to or less than the Carli index. In fact, the strict inequalities in (9) will hold provided that the period 0 vector of prices, p^0, is not proportional to the period 1 vector of prices, p^1.

The inequalities (9) do not tell us by how much the Carli index will exceed the Jevons index and by how much the Jevons index will exceed the Harmonic index. Hence, in the remainder of this section, some approximate relationships between the five indices defined in the previous section will be developed that will provide some practical guidance on the relative magnitudes of each of the indices.

The first approximate relationship that will be derived is between the Jevons index P_J and the Dutot index P_D. For each period t, define the *arithmetic mean* of the N prices pertaining to that period as follows:

$$p^{t*} \equiv \Sigma_{n=1}^{N} (1/N)\, p_n^{\,t};\ t = 0,1. \tag{10}$$

Now define (implicitly) the *multiplicative deviation* of the nth price in period t relative to the mean price in that period, $e_n^{\,t}$, as follows:

$$p_n^{\,t} = p^{t*}(1 + e_n^{\,t});\ n = 1,\ldots,N;\ t = 0,1. \tag{11}$$

Note that (10) and (11) imply that the deviations $e_n^{\,t}$ sum to 0 in each period; that is, we have

$$\Sigma_{n=1}^{N} e_n^{\,t} = 0;\ t = 0,1. \tag{12}$$

Note that the Dutot index can be written as the ratio of the mean prices, p^{1*}/p^{0*}; that is, we have

$$P_D(p^0,p^1) = p^{1*}/p^{0*}. \tag{13}$$

Now substitute equations (11) into the definition of the Jevons index (3):

$$\begin{aligned} P_J(p^0,p^1) &= \Pi_{n=1}^{N} [p^{1*}(1 + e_n^{\,1})/p^{0*}(1 + e_n^{\,0})]^{1/N} \\ &= [p^{1*}/p^{0*}]\, \Pi_{n=1}^{N} [(1 + e_n^{\,1})/(1 + e_n^{\,0})]^{1/N} \\ &= P_D(p^0,p^1)\, f(e^0,e^1) \text{ using definition (1),} \end{aligned} \tag{14}$$

where $e^t \equiv [e_1^{\,t},\ldots,e_N^{\,t}]$ for $t = 0$ and 1, and the function f is defined as follows:

$$f(e^0,e^1) \equiv \Pi_{N=1}^{N} [(1 + e_n^{\,1})/(1 + e_n^{\,0})]^{1/N}. \tag{15}$$

Expand f(e^0,e^1) by a second-order Taylor series approximation around $e^0 = 0_N$ and $e^1 = 0_N$. Using (12), it can be verified[36] that we obtain the following second-order approximate relationship between P_J and P_D:

$$\begin{aligned} P_J(p^0,p^1) &\approx P_D(p^0,p^1)[1 + (1/2N)e^0\cdot e^0 - (1/2N)e^1\cdot e^1] \\ &= P_D(p^0,p^1)[1 + (1/2)var(e^0) - (1/2)var(e^1)], \end{aligned} \tag{16}$$

where var(e^t) is the variance of the period t multiplicative deviations; that is, for $t = 0,1$:

$$\begin{aligned} var(e^t) &\equiv (1/N)\Sigma_{n=1}^{N} (e_n^{\,t} - e^{t*})^2 \\ &= (1/N)\Sigma_{n=1}^{N} (e_n^{\,t})^2 \text{ since } e^{t*} = 0 \text{ using (12)} \\ &= (1/N)e^t\cdot e^t. \end{aligned} \tag{17}$$

Under normal conditions,[37] the variance of the deviations of the prices from their means in each period is likely to be approximately constant, and so under these conditions, the Jevons price index will approximate the Dutot price index to the second order.

Note that with the exception of the Dutot formula, the remaining four elementary indices defined in Section 4 are functions of the relative prices of the N items being aggregated.[38] This fact is used in order to derive some approximate

[35] Each of the three indices P_H, P_J, and P_C is a mean of order r, where r equals –1, 0, and 1, respectively, and so the inequalities follow from Schlömilch's inequality.

[36] This approximate relationship was first obtained by Carruthers, Sellwood, and Ward (1980; 25).

[37] If there are significant changes in the overall inflation rate, some studies indicate that the variance of deviations of prices from their means can also change. Also if N is small, then there will be sampling fluctuations in the variances of the prices from period to period, leading to random differences between the Dutot and Jevons indices. If prices are normalized to equal 1 in period 0, this amounts to choosing particular units of measurement for the N products. In this case, var(e^0) = 0, and the approximation (16) becomes the inequality $P_J(p^0,p^1) < P_D(p^0,p^1)$ if var(e^1) > 0. In this case where normalized prices are used, the Dutot index becomes a Carli index which has an upward bias relative to the Jevons index. Annex A shows that this bias can be substantial.

[38] The Dutot index can be rewritten as a function of relative prices and shares that depend only on period 0 prices as follows: $P_D(p^0,p^1) = \Sigma_{n=1}^{N} \Sigma_n^{\,0}(p_n^{\,1}/p_n^{\,0})$, where $\Sigma_n^{\,0} \equiv p_n^{\,0}/\Sigma_{i=1}^{N} p_i^{\,0}$ for $n = 1,\ldots,N$; see IMF, ILO, Eurostat, UNECE, OECD, and World Bank (2020; 180). This publication also notes the following problem with the use of the Dutot formula: "Even when the varieties are fairly homogeneous and measured in the same units, the Dutot's implicit weights may still not be satisfactory. More weight is given to the price changes for the more expensive varieties, but in practice, they may well account for only small shares of the total expenditure within the aggregate. Consumers are unlikely to buy

relationships between these four elementary indices. Thus, define the *nth price relative* as

$$r_n \equiv p_n^{\,1}/p_n^{\,0};\; n = 1,\ldots,N. \tag{18}$$

Define the *arithmetic mean of the n price relatives* as

$$r^* \equiv (1/N)\Sigma_{n=1}^{N}\, r_n = P_C(p^0,p^1), \tag{19}$$

where the last equality follows from definition (2) of the Carli index. Finally, define (implicitly) the *deviation* e_n of the nth price relative r_n from the arithmetic average of the N price relatives r^* as follows:

$$r_n = r^*(1 + e_n);\; n = 1,\ldots,N. \tag{20}$$

Note that (19) and (20) imply that the deviations e_n sum to 0; that is, we have

$$\Sigma_{n=1}^{N}\, e_n = 0. \tag{21}$$

Now substitute equations (20) into the definitions of P_C, P_J, P_H, and P_{CSWD}, (2)–(5), in order to obtain the following representations for these indices in terms of the vector of deviations, $e \equiv [e_1,\ldots,e_N]$:[39]

$$P_C(p^0,p^1) = \Sigma_{n=1}^{N}\,(1/N)r_n = r^* \equiv r^* f_C(e); \tag{22}$$

$$P_J(p^0,p^1) = \prod_{n=1}^{N} r_n^{\,1/N} = r^*\prod_{n=1}^{N}(1 + e_n)^{1/N} \equiv r^* f_J(e); \tag{23}$$

$$P_H(p^0,p^1) = [\Sigma_{n=1}^{N}\,(1/N)(r_n)^{-1}]^{-1} = r^*[\Sigma_{n=1}^{N}\,(1/N)(1 + e_n)^{-1}]^{-1} \equiv r^* f_H(e); \tag{24}$$

$$P_{CSWD}(p^0,p^1) = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = r^*[f_C(e)f_H(e)]^{1/2} \equiv r^* f_{CSWD}(e), \tag{25}$$

where the last equation serves to define the deviation functions $f_C(e)$, $f_J(e)$, $f_H(e)$, and $f_{CSWD}(e)$. The second-order Taylor series approximations to each of these functions[40] around the point $e = 0_N$ are

$$f_C(e) \approx 1; \tag{26}$$

$$f_J(e) \approx 1 - (1/2N)e{\cdot}e = 1 - (1/2)var(e); \tag{27}$$

$$f_H(e) \approx 1 - (1/N)e{\cdot}e = 1 - var(e); \tag{28}$$

$$f_{CSWD}(e) \approx 1 - (1/2N)e{\cdot}e = 1 - (1/2)var(e), \tag{29}$$

where we have made repeated use of (21) in deriving these approximations.[41] Thus, to the second order, the Carli index P_C will *exceed* the Jevons and Carruthers, Sellwood, Ward, and Dalén indices, P_J and P_{CSWD}, by $(1/2)r^*$var(e), which is r^* times one half the variance of the N price relatives $p_n^{\,1}/p_n^{\,0}$. Similarly, to the second order, the Harmonic index P_H will *lie below* the Jevons and Carruthers, Sellwood, Ward, and Dalén indices, P_J and P_{CSWD}, by r^* times one half the variance of the N price relatives $p_n^{\,1}/p_n^{\,0}$.

Thus, empirically, it is expected that the Jevons and Carruthers, Sellwood, Ward, and Dalén indices will be very close to each other.[42] Using the previous approximation result (16), it is expected that the Dutot index P_D will also be fairly close to P_J and P_{CSWD}, with some fluctuations over time due to changing variances of the period 0 and 1 deviation vectors, e^0 and e^1. Thus, it is expected that these three elementary indices will give much the same numerical answers in empirical applications. On the other hand, the Carli index can be expected to be substantially *above* these three indices, with the degree of divergence growing as the variance of the N price relatives grows. Similarly, the Harmonic index can be expected to be substantially *below* the three middle indices, with the degree of divergence growing as the variance of the N price relatives grows.

6. The Test Approach to Elementary Indices

Recall that in Chapter 3, the axiomatic approach to bilateral price indices $P(p^0,p^1,q^0,q^1)$ was developed. In the present section, the elementary price index $P(p^0,p^1)$ depends only on the period 0 and 1 price vectors, p^0 and p^1, respectively, so that the elementary price index does not depend on the period 0 and 1 quantity vectors, q^0 and q^1. One approach to obtaining new tests or axioms for an elementary index is to look at the 20 or so axioms that were listed in Chapter 3 for bilateral price indices $P(p^0,p^1,q^0,q^1)$ and adapt those axioms to the present context; that is, use the old bilateral tests for $P(p^0,p^1,q^0,q^1)$ that do not depend on the quantity vectors q^0 and q^1 as tests for an elementary index $P(p^0,p^1)$.[43] This approach will be utilized in the present section.

The first eight tests or axioms are reasonably straightforward and uncontroversial:

T1: *Continuity*: $P(p^0,p^1)$ is a continuous function of the N positive period 0 prices $p^0 \equiv [p_1^{\,0},\ldots,p_N^{\,0}]$ and the N positive period 1 prices $p^1 \equiv [p_1^{\,1},\ldots,p_N^{\,1}]$.

T2: *Identity*: $P(p,p) = 1$; that is, if the period 0 price vector equals the period 1 price vector, then the index is equal to unity.

T3: *Monotonicity in Current Period Prices*: $P(p^0,p^1) < P(p^0,p)$ if $p^1 < p$; that is, if any period 1 price increases, then the price index increases.

T4: *Monotonicity in Base Period Prices*: $P(p^0,p^1) > P(p,p^1)$ if $p^0 < p$; that is, if any period 0 price increases, then the price index decreases.

varieties at high prices if the same varieties are available at lower prices" (IMF, ILO, Eurostat, UNECE, OECD, and World Bank (2020; 180–181)).

[39] Note that the vector of deviations e defined by equations (20) is different from the deviation vectors e^0 and e^1 defined by equations (11).

[40] From (22), it can be seen that f_C(e) is identically equal to 1 so that (26) will be an exact equality rather than an approximation.

[41] These second-order approximations were developed by Dalén (1992; 143) for the case $r^* = 1$ and by Diewert (1995; 29) for the case of a general r^*.

[42] Reinsdorf and Triplett (2009; 63) noted that for the case $N = 2$, $P_{CSWD}(p^0,p^1) = P_J(p^0,p^1)$. This paper and Diewert (1993) provide a review of early approaches to index number theory and the construction of a CPI.

[43] This was the approach used by Diewert (1995; 5–17), who drew on the earlier work of Eichhorn (1978; 152–160) and Dalén (1992).

T5: *Proportionality in Current Period Prices*: $P(p^0,\lambda p^1) = \lambda P(p^0,p^1)$ if $\lambda > 0$; that is, if all period 1 prices are multiplied by the positive number λ, then the initial price index is also multiplied by λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1) = \lambda^{-1} P(p^0,p^1)$ if $\lambda > 0$; that is, if all period 0 prices are multiplied by the positive number λ, then the initial price index is multiplied by $1/\lambda$.

T7: *Mean Value Test*: $\min_n\{p_n^1/p_n^0 : n = 1,\ldots,N\} \leq P(p^0,p^1) \leq \max_n\{p_n^1/p_n^0 : n = 1,\ldots,N\}$; that is, the price index lies between the smallest and largest price relatives.

T8: *Symmetric Treatment of Outlets*: $P(p^0,p^1) = P(p^{0*},p^{1*})$, where p^{0*} and p^{1*} denote the *same* permutation of the components of p^0 and p^1; that is, if we change the ordering of the outlets (or households) from which we obtain the price quotations for the two periods, then the elementary index remains unchanged.

Eichhorn (1978; 155) showed that Tests 1, 2, 3, and 5 imply Test 7, so that not all of these tests are logically independent.

The following tests are more controversial and not necessarily accepted by all price statisticians.

T9: *The Price Permutation Test*: $P(p^0,p^1) = P(p^{0*},p^{1**})$, where p^{0*} and p^{1**} denote possibly *different* permutations of the components of p^0 and p^1; that is, if the ordering of the price quotes for both periods is changed in possibly different ways, then the elementary index remains unchanged.

Obviously, T8 is a special case of T9, where the two permutations of the initial ordering of the prices are restricted to be the same. Thus, T9 implies T8. Test T9 was developed by Dalén (1992; 138). He justified this test by suggesting that the price index should remain unchanged if outlet prices "bounce" in such a manner that the outlets are just exchanging prices with each other over the two periods.[44]

The following test was also proposed by Dalén (1992) in the elementary index context:

T10: *Time Reversal*: $P(p^1,p^0) = 1/P(p^0,p^1)$; that is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index.

It is difficult to accept an index that gives a different answer if the ordering of time is reversed.

T11: *Circularity*: $P(p^0,p^1)P(p^1,p^2) = P(p^0,p^2)$; that is, the price index going from period 0 to 1 times the price index going from period 1 to 2 equals the price index going from period 0 to 2 directly.

[44]Since a typical official CPI consists of approximately 600 to 1000 separate strata where an elementary index needs to be constructed for each stratum, it can be seen that many strata will consist of quite heterogeneous items. Thus, for a fruit category, some of the N items whose prices are used in the elementary index will correspond to quite different types of fruit with quite different prices. Randomly permuting these prices in periods 0 and 1 will lead to very odd price relatives in many cases, which may cause the overall index to behave badly unless the Jevons or Dutot formula is used.

The circularity and identity tests imply the time reversal test; (just set $p^2 = p^0$). The circularity property would seem to be a very desirable property: It is a generalization of a property that holds for a single price relative.

Elementary price indices may be calculated as direct price indices by comparing the prices of the current period with those of a fixed price reference period or as chained short-term indices obtained by multiplying the monthly (or quarterly) price indices into a long-term price index. Many statistical offices chose to calculate the elementary price indices by chaining the short-term (monthly or quarterly) indices because this has some practical advantages when dealing with replacements in the sample. For elementary indices calculated as chained short-term price indices, it is crucial that the index meets the circularity test.

T12: *Commensurability*: $P(\lambda_1 p_1^0,\ldots,\lambda_N p_N^0; \lambda_1 p_1^1,\ldots,\lambda_N p_N^1) = P(p_1^0,\ldots,p_N^0; p_1^1,\ldots,p_N^1) = P(p^0,p^1)$ for all $\lambda_1 > 0,\ldots,\lambda_N > 0$; that is, if we change the units of measurement for each product in each outlet, then the elementary index remains unchanged.

In the bilateral index context, virtually every price statistician accepts the validity of this test. However, in the elementary context, this test is more controversial. If the N items in the elementary aggregate are all very homogeneous, then it makes sense to measure all of the items in the same units. Hence, if we change the unit of measurement in this homogeneous case, then test T12 should restrict each of the λ_n to be the same number (say λ), and test T12 becomes the following test:

$$P(\lambda p^0,\lambda p^1) = P(p^0,p^1) \textit{ for all } p^0 >> 0_N, p^1 >> 0_N \textit{ and } \lambda > 0. \tag{30}$$

Note that (30) will be satisfied if tests T5 and T6 are satisfied.

However, in actual practice, elementary strata may not be very homogeneous: There may be thousands of individual items in each elementary aggregate, and the hypothesis of item homogeneity may not be warranted. Under these circumstances, it is important that the elementary index satisfies the commensurability test, since the units of measurement of the heterogeneous items in the elementary aggregate are arbitrary and hence *the price statistician can change the index simply by changing the units of measurement for some of the items*.[45]

This completes the listing of the tests for an elementary index. There remains the task of evaluating how many tests are passed by each of the five elementary indices defined in Section 2.

The following results hold:

- The Jevons elementary index P_J satisfies *all* of the above tests.
- The Dutot index P_D satisfies all of the tests with the exception of the important Commensurability Test T12, which it fails.

[45]The empirical example in Annex A shows that changing the units of measurement for the Dutot index makes a huge difference.

- The Carli and Harmonic elementary indices, P_C and P_H, fail the price permutation test T9, the time reversal test T10, and the circularity test T11 but pass the other tests.
- The geometric mean of the Carli and Harmonic elementary indices, P_{CSWD}, fails only the price permutation test T9 and the circularity test T11.[46]

Since the Jevons elementary index P_J satisfies *all* of the tests, it emerges as being "best" from the viewpoint of the axiomatic approach to elementary indices.

The Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails. If there are heterogeneous items in the elementary aggregate, this is a rather serious failure, and hence price statisticians should be careful in using this index under these conditions. If the N items under consideration are all measured in the same units and the products are close substitutes so that the product prices vary in a proportional manner over time, then the Dutot index could be used.[47] But if prices vary almost proportionally over time, then almost any reasonable index number formula will pick up the common factor of proportionality. The empirical example in Annex A shows that if there are systematic divergent trends in prices, then the Dutot index can change dramatically as the units of measurement are changed.

The use of the Dutot, Carli, and Harmonic indices should be avoided.

The geometric mean of the Carli and Harmonic elementary indices fail only the price permutation test T9 and the circularity test T11. The failure of test T9 is probably not a fatal failure, and P_{CSWD} will usually be numerically close to P_J and so it will be close to satisfying the circularity test.[48]

The Carli and Harmonic elementary indices, P_C and P_H, fail the price permutation test T9, the time reversal test T10, and the circularity test T11 and pass the other tests. The failure of the time reversal test T10 (with an upward bias for the Carli and a downward bias for the Harmonic) is a rather serious failure, and so price statisticians should not use these indices.

In the following section, we present an argument due originally to Irving Fisher on why it is desirable for an index number formula to satisfy the time reversal test.

7. Fisher's Rectification Procedure and the Time Reversal Test

There is a problem with the Carli and Harmonic indices that was first pointed out by Irving Fisher:[49] The rate of price change measured by the index number formula between two periods is dependent on the period that is regarded as the base period. Thus, the Carli index, $P_C(p^0,p^1)$, as defined by (2), takes period 0 as the base period and calculates (one plus) the rate of price change between periods 0 and 1.[50] Instead of choosing period 0 to be the base period, we could equally choose period 1 to be the base period and measure a reciprocal inflation rate going *backward* from period 1 to period 0, and this *backward measured inflation rate* would be $\Sigma_{n=1}^{N} (1/N)(p_n^{\,0}/p_n^{\,1})$. In order to make this backward inflation rate comparable to the forward inflation rate, we then take the reciprocal of $\Sigma_{n=1}^{N} (1/N)(p_n^{\,0}/p_n^{\,1})$, and thus the overall inflation rate going from period 0 to 1 using period 1 as the base period is the following *Backward Carli index* P_{BC}:[51]

$$P_{BC}(p^0,p^1) \equiv [\Sigma_{n=1}^{N} (1/N)(p_n^{\,1}/p_n^{\,0})^{-1}]^{-1} = P_H(p^0,p^1); \qquad (31)$$

that is, the Backward Carli index equals the Harmonic index $P_H(p^0,p^1)$ defined earlier by (4).

If the forward and backward methods of computing price change between periods 0 and 1 using the Carli formula were equal, then we would have the following equality:[52]

$$P_C(p^0,p^1) = P_H(p^0,p^1). \qquad (32)$$

Fisher argued that a good index number formula should satisfy (32) since the end result of using the formula should not depend on which period was chosen as the base period.[53] This seems to be a persuasive argument: If for whatever reason, a particular formula is favored, where the base period 0 is chosen to be the period that appears before the comparison period 1, then the same arguments that justify the forward-looking version of the index number formula can be used to justify the backward-looking version. If the forward and backward versions of the index agree with one another, then it does not matter which version is used, and this equality provides a powerful argument in favor of using the formula. If the two versions do not agree, then rather than picking the forward version over the backward version, a more symmetric procedure would be to take an average of the forward- and backward-looking versions of the index formula.

Fisher provided an alternative way for justifying the equality of the two indices in equation (32). He argued that the forward-looking inflation rate using the Carli formula

[46] But using the approximations given by (27) and (29), P_{CSWD} will satisfy circularity approximately.

[47] Evans (2012; 4) compared the Slovenian CPI with its corresponding HICP and found very little difference over the period 1998–2011. The Slovenian national CPI used Dutot indices at the elementary level and the Slovenian HICP used Jevons indices at the elementary level.

[48] This is the case for the numerical example in the annex.

[49] "Just as the very idea of an index number implies a set of commodities, so it implies two (and only two) times (or places). Either one of the two times may be taken as the 'base'. Will it make a difference which is chosen? Certainly it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other point, *no matter which of the two is taken as the base*" (Irving Fisher (1922; 64)).

[50] Instead of calculating price inflation between periods 0 and 1, period 1 can be replaced by any period t that follows period 1; that is, p^1 in the Carli formula $P_C(p^0,p^1)$ can be replaced by p^t and then the index $P_C(p^0,p^t)$ measures price change between periods 0 and t. The arguments concerning $P_C(p^0,p^1)$ that follow apply equally well to $P_C(p^0,p^t)$.

[51] Fisher (1922; 118) termed the backward-looking counterpart to the usual forward-looking index the *time antithesis* of the original index number formula. Thus, P_H is the time antithesis to P_C. The Harmonic index defined by (4) is also known as the Coggeshall (1887) index.

[52] Of course, equation (32) is *not* satisfied.

[53] "The justification for making this rule is twofold: (1) no reason can be assigned for choosing to reckon in one direction which does not also apply to the opposite, and (2) such reversibility does apply to any *individual* commodity. If sugar costs twice as much in 1918 as in 1913, then necessarily it costs half as much in 1913 as in 1918" (Irving Fisher (1922; 64)).

is $P_C(p^0,p^1) = \Sigma_{n=1}^N (1/N)(p_n^1/p_n^0)$. As noted earlier, the backward-looking inflation rate using the Carli formula is $\Sigma_{n=1}^N (1/N)(p_n^0/p_n^1) = P_C(p^1,p^0)$. Fisher[54] argued that the product of the forward-looking and backward-looking indices should equal unity; that is, a good formula *should* satisfy the following equality (which is equivalent to (32)):

$$P_C(p^0,p^1)P_C(p^1,p^0) = 1. \tag{33}$$

But (33) is the usual *time reversal test* that was listed in the previous section. Thus, Fisher provided a reasonably compelling case for the satisfaction of this test.

As we have seen in Section 4,[55] the problem with the Carli formula is that it not only fails satisfy the equalities (32) or (33) but also *fails* (33) with the following definite inequality:

$$P_C(p^0,p^1)P_C(p^1,p^0) > 1, \tag{34}$$

unless the price vector p^1 is proportional to p^0 (so that $p^1 = \lambda p^0$ for some scalar $\lambda > 0$), in which case (33) will hold. The main implication of the inequality (34) is that the use of *the Carli index will tend to give higher measured rates of inflation* than a formula that satisfies the time reversal test (using the same data set and the same weighting).

Fisher showed how the downward bias in the backward-looking Carli index P_H and the upward bias in the forward-looking Carli index P_C could be cured. The Fisher *time rectification procedure*[56] as a general procedure for obtaining a bilateral index number formula that satisfies the time reversal test works as follows. Given a bilateral price index P, Fisher (1922; 119) defined the *time antithesis* $P°$ for P as follows:

$$P^{\circ}(p^0,p^1,q^0,q^1) \equiv 1/P(p^1,p^0,q^1,q^0). \tag{35}$$

Thus, $P°$ is equal to the reciprocal of the price index that has reversed the role of time, $P(p^1,p^0,q^1,q^0)$. Fisher (1922; 140) then showed that the geometric mean of P and $P°$, say $P^* \equiv [P \times P°]^{1/2}$, satisfies the time reversal test, $P^*(p^0,p^1,q^0,q^1)P^*(p^1,p^0,q^1,q^0) = 1$.

In the present context, P_C is only a function of p^0 and p^1, but the same rectification procedure works, and the time antithesis of P_C is the harmonic index P_H. Applying the Fisher rectification procedure to the Carli index, the resulting rectified Carli formula, P_{RC}, turns out to equal the Carruthers, Sellwood, and Ward (1980) and the Dalén elementary index P_{CSWD} defined earlier by (5):

$$P_{RC}(p^0,p^1) \equiv [P_C(p^0,p^1)P_{BC}(p^0,p^1)]^{1/2} = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = P_{CSWD}(p^0,p^1). \tag{36}$$

Thus, P_{CSWD} is the geometric mean of the forward-looking Carli index P_C and its backward-looking counterpart $P_{BC} = P_H$, and, of course, P_{CSWD} will satisfy the time reversal test.[57]

8. Conclusion

The main results in this chapter can be summarized as follows:

- In order to define a "best" elementary index number formula, it is necessary to have a target index number concept. In Section 2, it is suggested that normal bilateral index number theory applies at the elementary level as well as at higher levels, and hence the target concept should be one of the Fisher, Törnqvist, or Walsh formulae.
- When aggregating the prices of the same narrowly defined item within a period, the narrowly defined unit value is a reasonable target price concept.
- The axiomatic approach to traditional elementary indices (that is, no quantity or value weights are available) supports the use of the Jevons formula under most circumstances.[58] The Carruthers, Sellwood, and Ward formula can be used as an alternative to the Jevons formula, but both will give much the same numerical answers.
- The Carli index has an upward bias (with respect to satisfying the time reversal test), and the Harmonic index has a downward bias.
- All five unweighted elementary indices are not really satisfactory. A much more satisfactory approach would be to collect quantity or value information along with price information and form sample superlative indices as the preferred elementary indices. However, if a chained superlative index is calculated, it should be examined for chain drift; that is, a chained index should only be used if the data are relatively smooth and subject to long-term trends rather than short-term fluctuations.[59]

[54] "Putting it in still another way, more useful for practical purposes, the forward and backward index number multiplied together should give unity" (Irving Fisher (1922; 64)).

[55] Recall the inequalities (7) and (8) above.

[56] Actually, Walsh (1921b; 542) showed Fisher (1921) how to rectify a formula so it would satisfy the factor reversal test and Fisher (1922) simply adapted the methodology of Walsh to the problem of rectifying a formula so that it would satisfy the time reversal test.

[57] See Figure A6.2 in the annex where it will be seen that for our empirical example, the Jevons index cannot be distinguished from the Carruthers, Sellwood, Ward and Dalen index.

[58] One exception to this advice is when a price can be 0 in one period and positive in another comparison period. In this situation, the Jevons index will fail and the corresponding item will have to be ignored in the elementary index. The problems raised by missing prices will be considered at greater length in the subsequent chapters on multilateral methods and strongly seasonal products.

[59] If the price and quantity data are subject to large fluctuations, then multilateral methods should be used instead of a bilateral index number formula. Multilateral methods will be discussed in Chapter 7.

Annex A

Alternative Approaches to the Treatment of Access Charges

An interesting CPI problem arises when there is a fixed access charge for the right of consumers to purchase products or services from a supplier. Examples of such charges are annual club memberships, annual fees for the use of a credit card, and fixed charges for access to telecommunication services. In this annex, we will outline three different approaches that could be used by consumer price statisticians to deal with these charges, which are independent of the actual consumption of the goods and services that the payment of a fixed charge allows consumers to purchase.

The notation that is used in this annex is similar to that used in Section 2 with some new notation for the fixed charge. Thus, let $p^t \equiv [p_1^t,. . .,p_N^t]$ and $q^t \equiv [q_1^t,. . .,q_N^t]$ be the period t price and quantity vectors for the purchases of the goods or services that the payment of the access charge $P^t > 0$ allows the consumer or group of consumers to purchase for $t = 0,1$.

Define e^t as the *period t expenditure* on the actual goods and services purchased and v^t as the *value of period t total expenditures* on the group of products, which is equal to e^t plus the period t access fixed charge P^t. It is also useful to define the period t fixed cost *margin* m^t as the ratio of P^t to e^t. Thus, we have the following definitions:

$$e^t \equiv p^t{\cdot}q^t \equiv \Sigma_{n=1}^N p_n^t q_n^t;\ v^t \equiv p^t{\cdot}q^t + P^t = e^t + P^t;\ m^t \equiv P^t/e^t;\ t = 0,1. \qquad \text{(A.1)}$$

In the analysis that follows, we will look at some "practical" price indices and compare their magnitudes. However, before we define these indices, it is useful to look at three alternative utility maximization models that help motivate the alternative practical indices.

Suppose the "consumer" has the utility function $f(q)$. The first utility maximization model that we will consider is a "traditional" model that treats the period t fixed charge as a charge on the "income" that the consumer allocates to the N products in the group of products under consideration. Thus, the *Model 1* period t utility maximization problem for the subgroup of products under consideration is the following one:

$$\max_q \{f(q): p^t{\cdot}q \le v^t - P^t = e^t;\ q \ge 0_N\}. \qquad \text{(A.2)}$$

If the CPI were constructed in only a single stage, then Model 1 is a "practical" model that price statisticians could use to guide the construction of the national CPI. However, a typical CPI is constructed by aggregating over both product groupings and outlets or households. In order to implement the Model 1 approach, price statisticians would have to keep track of the various fixed charges that occur for various outlets and product groups as well as collecting the basic price and quantity information. The CPI subindices that would be computed using this approach would also have to include (separately) information on the fixed charges by product group. The national accounts division of the national statistical agency would not be able to take a CPI subindex and use it for deflation purposes if that subgroup of products included substantial fixed charges; that is, the period t CPI subindex would be appropriate for deflating the actual product expenditures e^t, but *the subindex would not be appropriate for deflating actual group expenditures* (including the fixed charges) P^t.

The second utility maximization problem treats the access charge as a separate product that gives utility to consumers even if they do not consume any products or services that the access charge enables. The new utility function is $f^*(q,Q)$, where $Q = 1$ represents the contribution of access to overall utility for the subgroup of product under consideration. Thus, the *Model 2* period t utility maximization problem for the subgroup of products under consideration is the following one:

$$\max_q \{f^*(q,1): p^t{\cdot}q + P^t \le v^t;\ q \ge 0_N\}. \qquad \text{(A.3)}$$

This way of thinking about fixed charges in the telecommunications context is used by national regulators. The approach taken to the treatment of access charges is of some importance in measuring the productivity of telecommunications firms, as will be seen in the example that follows. The advantage of this approach is that the CPI index that is constructed using this framework will be suitable for national accounts deflation purposes; that is, the period t subindex that is a result of using this approach can be used to deflate total period t expenditures v^t on the product class.

The third utility maximization problem allocates the period t fixed charge P^t in proportional to expenditure manner across the "usage" prices p^t. Recall that (A.1) defined the period t margin m^t as P^t/e^t. The margin is treated in much the same way as a general sales tax is treated; that is, it is added on to the period t usage prices p^t. Thus, the *Model 3* period t utility maximization problem for the subgroup of products under consideration is the following one:[60]

$$\max_q \{f(q): (1 + m^t)p^t{\cdot}q \le v^t;\ q \ge 0_N\}. \qquad \text{(A.4)}$$

When price statisticians apply the economic approach to index number theory, it is assumed that the observed period t quantity vector q^t solves the corresponding period t utility maximization problem. It is also assumed that the first inequality constraint in problems (A.2)–(A.4) holds with equality. Thus, if q^t solves problem (A.2) for period t, then $p^t{\cdot}q^t = v^t - P^t = e^t$ for $t = 0,1$; if q^t solves problem (A.3) for period t, then $p^t{\cdot}q^t = v^t - P^t = e^t$ for $t = 0,1$; and if q^t solves problem (A.4) for period t, then $(1 + m^t)p^t{\cdot}q^t = v^t$ for $t = 0,1$. Using the definitions for m^t, e^t, and v^t in (A.1), it can be seen that $(1 + m^t)p^t{\cdot}q^t = [1 + (P^t/e^t)]p^t{\cdot}q^t = [1 + (P^t/p^t{\cdot}q^t)]p^t{\cdot}q^t = p^t{\cdot}q^t + P^t = v^t$ for $t = 0,1$. Thus, for all three utility maximization problems, it is assumed that the various equalities in definitions (A.1) are satisfied.

We use these alternative models of economic behavior to motivate the definitions of the alternative Laspeyres and Paasche indices. Next, we will define the Laspeyres and Paasche indices that correspond to the three models and compare their magnitudes.

[60] Models 1 and 3 will not work if $q^t = 0_N$ for some period t. If this case occurs empirically, then Model 2 or some other model will have to be used.

The Laspeyres and Paasche indices comparing the prices of period 1 to the corresponding prices of period 0 *using the Model 1 framework*, P_{L1} and P_{P1}, respectively, are defined as follows:

$$P_{L1} \equiv p^1 \times q^0 / p^0 \cdot q^0, \quad \text{(A.5)}$$
$$P_{P1} \equiv p^1 \times q^1 / p^0 \cdot q^1. \quad \text{(A.6)}$$

The Laspeyres and Paasche indices comparing the prices of period 1 to the corresponding prices of period 0 *using the Model 2 framework*, P_{L2} and P_{P2}, respectively, are defined as follows:

$$P_{L2} \equiv [p^1 \cdot q^0 + P^1]/[p^0 \cdot q^0 + P^0] \quad \text{(A.7)}$$
$$= [P_{L1} + (P^1/e^0)]/[1 + (P^0/e^0)] \text{ dividing the numerator and denominator by } e^0,$$

$$P_{P2} \equiv [p^1 \cdot q^1 + P^1]/[p^0 \cdot q^1 + P^0] \quad \text{(A.8)}$$
$$= [1 + (P^1/e^1)]/[P_{P1}^{-1} + (P^0/e^1)] \text{ dividing the numerator and denominator by } e^1.$$

We also used definitions (A.5) and (A.6) in deriving the second lines of (A.7) and (A.8).

Using definitions (A.5) and (A.7), it is possible to compare P_{L1} to P_{L2}:

$$P_{L1} - P_{L2} = P_{L1} - \{[P_{L1} + (P^1/e^0)]/[1 + (P^0/e^0)]\} \quad \text{(A.9)}$$
$$= [1 + (P^0/e^0)]^{-1}[P_{L1}\{1 + (P^0/e^0)\} - P_{L1} - (P^1/e^0)]$$
$$= [1 + m^0]^{-1}[P_{L1}(P^0/e^0) - (P^1/e^0)]$$
$$= [1 + m^0]^{-1}[P_{L1}(P^0/e^0) - (P^1/P^0)(P^0/e^0)]$$
$$= [m^0/(1 + m^0)][P_{L1} - (P^1/P^0)].$$

Thus, if the Laspeyres price index P_{L1} for the N products that are made available by paying the access charge in each period is equal to one plus the growth rate in the access charges, P^1/P^0, then P_{L1} will be equal to P_{L2} (which is the Laspeyres price index that treats the access charge as a normal product). If P_{L1} is greater than P^1/P^0, then P_{L1} will be greater than P_{L2}; if P_{L1} is less than P^1/P^0, then P_{L1} will be less than P_{L2}. If m^0 is large and the difference between P_{L1} and P^1/P^0 is also large, then the difference between P_{L1} and P_{L2} can be substantial. This case can occur in the case of a telecommunications subindex.[61]

Using definitions (A.6) and (A.8), it is possible to compare P_{P1} to P_{P2}, but the resulting formula is a bit more complicated:

$$P_{P1}^{-1} - P_{P2}^{-1} = P_{P1}^{-1} - \{[P_{P1}^{-1} + (P^0/e^1)]/[1 + (P^1/e^1)]\} \quad \text{(A.10)}$$
$$= [1 + (P^1/e^1)]^{-1}[P_{P1}^{-1}\{1 + (P^1/e^1)\} - P_{P1}^{-1} - (P^0/e^1)]$$
$$= [1 + m^1]^{-1}[P_{P1}^{-1}(P^1/e^1) - (P^0/e^1)]$$
$$= [1 + m^1]^{-1}[P_{P1}^{-1}(P^1/e^1) - (P^0/P^1)(P^1/e^1)]$$
$$= [m^1/(1 + m^1)][P_{P1}^{-1} - (P^1/P^0)^{-1}].$$

By multiplying both sides of (A.10) by $P_{P1}P_{P2}$, the following expression is obtained:

$$P_{P2} - P_{P1} = [m^1/(1 + m^1)] P_{P2}[1 - P_{P1}(P^1/P^0)^{-1}] \quad \text{(A.11)}$$
$$= [m^1/(1 + m^1)] P_{P2}[P^1/P^0]^{-1} [(P^1/P^0) - P_{P1}].$$

Finally, By multiplying both sides of (A.11) by –1, we obtain the following counterpart to (A.9):

$$P_{P1} - P_{P2} = [m^1/(1 + m^1)] P_{P2}[P^1/P^0]^{-1} [P_{P1} - (P^1/P^0)]. \quad \text{(A.12)}$$

Thus, if the Paasche price index P_{P1} for the N products that are made available by paying the access charge in each period is equal to one plus the growth rate in the access charges, P^1/P^0, then P_{P1} will be equal to P_{P2} (which is the Paasche price index that treats the access charge as a normal product). If P_{P1} is greater than P^1/P^0, then P_{P1} will be greater than P_{P2}; if P_{P1} is less than P^1/P^0, then P_{P1} will be less than P_{P2}. If m^1 is large and the difference between P_{P1} and P^1/P^0 is also large, then the difference between P_{P1} and P_{P2} can be substantial.[62]

We turn now to the Model 3 framework. The Laspeyres and Paasche indices comparing the prices of period 1 to the corresponding prices of period 0 *using the Model 3 framework*, P_{L3} and P_{P3}, respectively, are defined as follows:

$$P_{L3} \equiv (1 + m^1)p^1 \cdot q^0/(1 + m^0)p^0 \cdot q^0 \quad \text{(A.13)}$$
$$= [(1 + m^1)/(1 + m^0)] P_{L1} \text{ dividing the numerator and denominator by } e^0;$$

$$\text{(A.14) } P_{P3} \equiv (1 + m^1)p^1 \cdot q^1/(1 + m^0)p^0 \cdot q^1$$
$$= (1 + m^1)/[(1 + m^0)P_{P1}^{-1}] \text{ dividing the numerator and denominator by } e^1;$$
$$= [(1 + m^1)/(1 + m^0)]P_{P1}.$$

It is very easy to compare P_{L3} to P_{L1} and compare P_{P3} to P_{P1}. Using definitions (A.13) and (A.14), we have

$$P_{L3}/P_{L1} = P_{P3}/P_{P1} = (1 + m^1)/(1 + m^0). \quad \text{(A.15)}$$

Thus, P_{L3} will equal P_{L1} and P_{P3} will equal P_{P1} if $m^1 \equiv P^1/e^1$ is equal to $m^0 \equiv P^0/e^0$ or if $P^1/P^0 = e^1/e^0$. P_{L3} will be greater than P_{L1} and P_{P3} will be greater than P_{P1} if $m^1 > m^0$ or if $P^1/P^0 > e^1/e^0$. These results are very straightforward and easy to understand.

[61] Our analysis for the case of Laspeyres price indices also applies to other fixed basket indices; that is, simply replace the base period quantity vector q^0 by the fixed basket quantity vector q^* and apply our analysis pertaining to the differences between the various Laspeyres indices. The definitions for e^0, v^0, and m^0 become $e^0 \equiv p^0 \cdot q^*$, $v^0 \equiv e^0 + P^0$, and $m^0 \equiv P^0/e^0$. P_{L1} becomes $p^1 \cdot q^*/p^0 \cdot q^*$, P_{L2} becomes $[p^1 \times q^* + P^1]/[p^0 \cdot q^* + P^0]$, and P_{L3} (which will be defined below) becomes $(1 + m^1)p^1 \cdot q^*/(1 + m^0)p^0 \cdot q^*$, where $e^1 \equiv p^1 \times q^*$, $v^1 \equiv e^1 + P^1$, and $m^1 \equiv P^1/e^1$.

[62] Note that the conditions for "bias" between P_{L1} and P_{L2} and for "bias" between P_{P1} and P_{P2} are very similar in structure.

Table A.1 Fixed-Line UK Retail Telecommunications Revenues and Quantities

Year t	v_1^t	v_2^t	v_3^t	v_4^t	v_5^t	q_1^t	q_2^t	q_3^t	q_4^t	q_5^t	e^t	v^t
2010	935	293	849	824	3259	65134	4850	5642	14736	23752	2901	6160
2011	787	237	675	742	3375	56083	4570	4471	13066	23872	2441	5816
2012	723	198	566	659	3706	51985	4111	3902	11506	24462	2146	5852
2013	673	155	488	620	3964	46191	3455	3351	10681	24970	1936	5900
2014	577	132	430	620	4148	40766	3015	2940	9028	25549	1759	5907
2015	498	123	369	604	4462	35586	2749	2735	8855	26075	1594	6056
2016	428	111	270	596	4776	30471	2169	2811	7826	26482	1405	6181
2017	362	89	228	543	4969	24705	1550	2587	6126	26661	1222	6191

The more interesting comparisons are between P_{L3} and P_{L2} and between P_{P3} and P_{P2}. For the Laspeyres comparisons, using (A.7) and (A.13), we have

$$\begin{aligned} P_{L2} - P_{L3} &= \{[P_{L1} + (P^1/e^0)]/[1 + (P^0/e^0)]\} \\ &\quad - \{(1 + m^1)P_{L1}/(1 + m^0)\} \qquad (A.16) \\ &= [1 + m^0]^{-1}[P_{L1} + (P^1/e^0) - (1 + \{P^1/e^1\})P_{L1}] \\ &= [1 + m^0]^{-1}[(P^1/e^0) - (P^1/e^1)P_{L1}] \\ &= m^1[1 + m^0]^{-1}[(e^1/e^0) - P_{L1}]. \end{aligned}$$

Thus, if the usage expenditure ratio, e^1/e^0, is equal to the Laspeyres price index for the available products or services, P_{L1}, then P_{L2} will equal P_{L3}. In the telecommunications context, typically usage expenditures will grow more rapidly than the usage Laspeyres price index so that e^1/e^0 will be much greater than P_{L1}, which will imply that P_{L2} will be greater than P_{L3} using (A.16). If m^1 is also large, then P_{L2} will be substantially greater than P_{L3}.[63] In the telecommunications context, the choice of index number method will matter, as will be shown in the empirical example.

Using definitions (A.8) and (A.14), we have the following equality:

$$\begin{aligned} P_{P2}^{-1} - P_{P3}^{-1} &= \{[P_{P1}^{-1} + (P^0/e^1)]/[1 + (P^1/e^1)]\} \\ &\quad - \{(1 + m^0)P_{P1}^{-1}/(1 + m^1)\} \qquad (A.17) \\ &= [1 + m^1]^{-1}\{P_{P1}^{-1} + (P^0/e^1) - P_{P1}^{-1} - P_{P1}^{-1}(P^0/e^0)\} \\ &= [1 + m^1]^{-1}[(P^0/e^1) - P_{P1}^{-1}(P^0/e^0)] \\ &= m^0[1 + m^1]^{-1}[(e^1/e^0)^{-1} - P_{P1}^{-1}]. \end{aligned}$$

Divide both sides of (A.17) by P_{P3}^{-1} in order to obtain the following equalities:

$$\begin{aligned} P_{P3}/P_{P2} - 1 &= m^0[1 + m^1]^{-1}[(e^1/e^0)^{-1} - P_{P1}^{-1}]P_{P3} \qquad (A.18) \\ &= m^0[1 + m^1]^{-1}[(e^1/e^0)^{-1} - P_{P1}^{-1}](1 + m^1)(1 + m^0)^{-1}P_{P1} \\ &\qquad \text{using (A.14)} \\ &= m^0(1 + m^0)^{-1}[P_{P1}(e^1/e^0)^{-1} - 1] \\ &= m^0(1 + m^0)^{-1}(e^1/e^0)^{-1}[P_{P1} - (e^1/e^0)]. \end{aligned}$$

By multiplying both sides of (A.18) by $-P_{P2}$, we obtain the following equality:

$$P_{P2} - P_{P3} = m^0(1 + m^0)^{-1}(e^1/e^0)^{-1}P_{P2}[(e^1/e^0) - P_{P1}]. \qquad (A.19)$$

Thus, if the usage expenditure ratio, e^1/e^0, is equal to the Paasche price index for the available products or services, P_{P1}, then P_{P2} will equal P_{P3}. As noted earlier, in the telecommunications context, typically usage expenditures will grow more rapidly than the usage Paasche price index so that e^1/e^0 will be much greater than P_{P1}, which will imply that P_{P2} will be greater than P_{P3} using (A.19). If m^0 is also large, then P_{P2} will be substantially greater than P_{P3}.[64] Thus, again, in the telecommunications context, the choice of index number method will matter.

For empirical evidence on the huge differences in actual national indices that the alternative treatment of access charges can make in the telecommunications context, we draw on the UK data that are provided in the recent study by Abdirahman, Coyle, Heys, and Stewart (2020).[65] The UK retail telecom revenues for fixed lines $v_n^t \equiv p_n^t q_n^t$ and the corresponding quantities q_n^t for the years 2010–2017 are listed in Table A.1. These data are not "pure" CPI data in that they do not refer to the purchases by households but instead refer to all retail purchases. However, these data will serve as an example that will show that the above three alternative treatment of access charges can lead to significantly different price (and quantity) indices.

The revenues in Table A1 are expressed in millions of UK Pounds. The five "products" and their units of measurement for the corresponding quantities are as follows:

- 1 = UK geographic calls in millions of minutes
- 2 = International calls in millions of minutes
- 3 = Calls to mobile phones in millions of minutes
- 4 = Other calls in millions of minutes

[63] Note that $e^1/e^0 = P_{L1}Q_{P1}$, where $Q_{P1} \equiv p^1 \cdot q^1/p^1 \cdot q^0$ is the Paasche quantity index for usage expenditures. Thus, (A.16) can be rewritten as $P_{L2} - P_{L3} = m^1[1 + m^0]^{-1}P_{L1}[Q_{P1} - 1]$. Thus, if $Q_{P1} > 1$, then $P_{L2} > P_{L3}$.

[64] Note that $e^1/e^0 = P_{P1}Q_{L1}$, where $Q_{L1} \equiv p^0 \cdot q^1/p^0 \cdot q^0$ is the Laspeyres quantity index for usage expenditures. Thus, (A.19) can be rewritten as $P_{P2} - P_{P3} = m^0(1 + m^0)^{-1}(e^1/e^0)^{-1}P_{P2}P_{P1}[Q_{L1} - 1]$. Thus, if $Q_{L1} > 1$, then $P_{P2} > P_{P3}$.

[65] Their recent study extends their earlier important study; see Abdirahman, Coyle, Heys, and Stewart (2017). These papers make clear that the alternative treatment of access charges makes a big difference not only to price indices but also to the measurement of national output, consumption, and productivity.

Table A.2 Normalized Prices and Quantities for the UK Fixed-Line Retail Sector

Year t	p_1^{t*}	p_2^{t*}	p_3^{t*}	p_4^{t*}	p_5^{t*}	q_1^{t*}	q_2^{t*}	q_3^{t*}	q_4^{t*}	q_5^{t*}	m^t
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**935.00**	**293.00**	**849.00**	**824.00**	**3259.00**	**1.1234**
2011	**0.9776**	**0.8584**	**1.0033**	**1.0156**	**1.0304**	**805.07**	**276.08**	**672.79**	**730.62**	**3275.47**	**1.3826**
2012	**0.9689**	**0.7973**	**0.9640**	**1.0243**	**1.1042**	**746.25**	**248.36**	**587.17**	**643.39**	**3356.42**	**1.7269**
2013	**1.0150**	**0.7426**	**0.9678**	**1.0381**	**1.1570**	**663.07**	**208.72**	**504.25**	**597.25**	**3426.12**	**2.0475**
2014	**0.9860**	**0.7247**	**0.9720**	**1.2282**	**1.1833**	**585.20**	**182.14**	**442.41**	**504.82**	**3505.57**	**2.3582**
2015	**0.9749**	**0.7406**	**0.8966**	**1.2198**	**1.2472**	**510.84**	**166.07**	**411.56**	**495.15**	**3577.74**	**2.7993**
2016	**0.9785**	**0.8471**	**0.6383**	**1.3619**	**1.3144**	**437.41**	**131.03**	**423.00**	**437.61**	**3633.58**	**3.3993**
2017	**1.0208**	**0.9505**	**0.5857**	**1.5852**	**1.3583**	**354.64**	**93.64**	**389.29**	**342.55**	**3658.14**	**4.0663**

- 5 = Fixed-line access charges; units are the number of lines in thousands

Note that $e^t \equiv v_1^t + v_2^t + v_3^t + v_4^t$ is the total revenue or expenditure for year t on the various types of calls made from fixed lines in the United Kingdom and $v^t \equiv e^t + v_5^t$ is the total expenditure including access charges v_5^t. The ratio of access charges in year t to the corresponding total call revenues is the *margin* $m^t \equiv v_5^t/e^t$, which is listed in Table A2. From Table A2, it can be seen that m^t increases steadily from 1.12 in 2010 to 4.07 in 2017. Thus, the treatment of access charges is likely to make a substantial difference to any telecom price index based on these data.

The *unit value prices* for each product can be constructed using the information in Table A1; that is, we have $p_n^t \equiv v_n^t/q_n^{t*}$ for $n = 1,. . .,5$ and $t = 2010,. . .,2017$. In order to see more clearly how the prices of the various telecom products have changed over the sample period, normalize the unit value prices to 1 in the base year 2010; that is, define the *normalized prices and quantities*, p_n^{t*} and q_n^{t*}, as follows:[66]

$$p_n^{t*} \equiv p_n^t/p_n^{2010};\ q_n^{t*} \equiv q_n^t p_n^{2010};\ n = 1,. . .,5;\quad t = 2010,. . .,2017. \tag{A.20}$$

Table A.2 lists the normalized prices and quantities for the five products along with the margin series, $m^t \equiv v_5^t/e^t$.

It can be seen that relative prices and relative quantities vary considerably over the sample period. This will lead to dispersion among alternative index number formulae. We utilize the data in the previous tables to compute alternative indices for each of the three approaches outlined earlier for the treatment of access charges.[67]

For the Approach 1 indices, we ignore the access charges and simply compute the alternative indices using only the prices and quantities for the first four products. In Table A.3, the "unweighted" price indices[68] that were defined in Section 4 are listed. The Fixed-Base Harmonic, Caruthers, Sellwood, Ward, Dalén, and Carli indices, P_H^t, P_{CWSD}^t, and P_C^t and their chained counterparts, P_{HCH}^t, P_{CWSDCH}^t, and P_{CCH}^t, are listed in this table. The Fixed-Base and chained Dutot and Jevons indices coincide, and so these indices are simply listed as P_D^t and P_J^t in Table A.3. These indices were calculated using $p_n^t = v_n^t/q_n^t$, where v_n^t and q_n^t are listed in Table A.1. All of these indices with the exception of the Dutot index are independent of the units of measurement. Instead of using the original units of measurement to calculate the Dutot index, we could "standardize" the unit value prices using the normalized prices $p_n^{t*} \equiv p_n^t/p_n^{2010}$ listed in Table A.2 and calculate a new Dutot index using the normalized prices.[69] It turns out that this new Dutot index P_{DN}^t using normalized prices in place of the original prices is equal to the Fixed-Base Carli index P_C^t, so we did not list P_{DN}^t in Table A.3. For all of the index number formulae that appear in Table A.3 with the exception of the Dutot index P_D^t, it does not matter whether we use the prices and quantities listed in Table A.1 or their normalized counterparts listed in Table A.2.

It can be seen that the Dutot index using normalized prices, $P_{DN}^t = P_C^t$, ends up well above its Jevons index counterpart, P_J^t, when $t = 2017$. In Section 5, we indicated that under certain circumstances, the Jevons and Dutot indices should be approximately equal; see the approximate equality (16). Using our current notation, the approximate equality (16) becomes the following one:

$$P_J^t \approx P_{DN}^t[1 + (1/2)\mathrm{var}(\varepsilon^{2010}) - (1/2)\mathrm{var}(\varepsilon^t)];\quad t = 2011, 2012, . . ., 2017, \tag{A.21}$$

where $p_A^t \equiv (1/4)(p_1^{t*} + p_2^{t*} + p_3^{t*} + p_4^{t*})$, $\varepsilon_n^t \equiv (p_n^{t*}/p_A^t) - 1$ for $n = 1,2,3,4$, $\varepsilon^t \equiv [\varepsilon_1^t, \varepsilon_2^t, \varepsilon_3^t, \varepsilon_4^t]$ and $\mathrm{var}(\varepsilon^t) \equiv (1/4)\Sigma_{n=1}^4 (\varepsilon_n^t)^2$ for $t = 2010,. . .,2017$. Since the normalized prices p_n^{t*} all equal 1 when $t = 2010$, we see that $\mathrm{var}(e^{2010}) = 0$. Moreover, because p_3^{t*} trends down and p_4^{t*} trends up as t increases, $\mathrm{var}(\varepsilon^t)$ is increasing over time and hence, using the above approximate equality, it can be seen that P_J^t will tend to be less than P_{DN}^t and the gap will grow over time as $\mathrm{var}(\varepsilon^t)$ increases. Thus, we have an explanation for why the gap between P_J^t and $P_{DN}^t = P_C^t$ increases over time.[70]

[66] If we change the units of measurement of prices, then we have to change the corresponding units of measurement for quantities in the opposite direction in order to preserve values.

[67] We will also consider a fourth approach which is relevant for producer price indices.

[68] The term "unweighted" really means "equally weighted." These indices do not make any use of quantity or value information. Thus, they do not take into account the economic importance of each product. This is not a problem if expenditure shares are roughly equal but typically this is not the case.

[69] Thus, define $P_{DN}^t \equiv [p_1^{t*} + p_2^{t*} + p_3^{t*} + p_4^{t*}]/[p_1^{2010*} + p_2^{2010*} + p_3^{2010*} + p_4^{2010*}] = [p_1^{t*} + p_2^{t*} + p_3^{t*} + p_4^{t*}]/[4] = (1/4)\Sigma_{n=1}^4 (p_n^t/p_n^{2010}) \equiv P_C^t$, where the second equality follows from $p_n^{2010*} = 1$ for $n = 1,2,3,4$. Thus, the Dutot index using normalized prices in place of the initial prices is equal to the Fixed-Base Carli index, P_C^t, for $t = 2010,. . .,2017$.

[70] As we have seen above, using normalized prices in the Dutot formula converts the Fixed-Base Dutot index into a Fixed-Base Carli index.

Table A.3 Approach 1 Unweighted Price Indices

Year t	P_D^t	P_J^t	P_H^t	P_{CSWD}^t	P_C^t	P_{HCH}^t	P_{CSWDCH}^t	P_{CCH}^t
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**0.9733**	**0.9616**	**0.9594**	**0.9616**	**0.9637**	**0.9594**	**0.9616**	**0.9637**
2012	**0.9404**	**0.9345**	**0.9302**	**0.9344**	**0.9386**	**0.9319**	**0.9345**	**0.9370**
2013	**0.9358**	**0.9328**	**0.9241**	**0.9324**	**0.9409**	**0.9294**	**0.9328**	**0.9362**
2014	**0.9705**	**0.9610**	**0.9440**	**0.9607**	**0.9777**	**0.9544**	**0.9611**	**0.9677**
2015	**0.9314**	**0.9427**	**0.9278**	**0.9428**	**0.9580**	**0.9355**	**0.9427**	**0.9499**
2016	**0.8445**	**0.9213**	**0.8882**	**0.9217**	**0.9565**	**0.8972**	**0.9204**	**0.9442**
2017	**0.8851**	**0.9742**	**0.9153**	**0.9736**	**1.0355**	**0.9447**	**0.9731**	**1.0024**

The large differences between the Dutot index using the original units of measurement, $P_D^{t,}$, and the version of the Dutot index that uses normalized prices, P_{DN}^t (which turns out to be equal to the Fixed-Base Carli index P_C^t), indicates that *the Dutot formula should be used with extreme caution* even if there are common units of measurement for the individual products in scope for the index.

From Table A.3, it can be seen that the Jevons index is approximately equal to both the Fixed-Base and chained Carruthers, Ward, Sellwood, and Dalén indices; that is, we have the following approximate equalities that are consistent with the analysis in Section 5:

$$P_J^t \approx P_{CSWD}^t \approx P_{CSWDCH}^t; \quad t = 2011, 2012, \ldots, 2017. \tag{A.22}$$

From Table A.3, it can be seen that the following inequalities hold:

$$P_H^t < P_J^t < P_C^t; P_{HCH}^t < P_J^t < P_{CCH}^t; \quad t = 2011, 2012, \ldots, 2017. \tag{A.23}$$

These inequalities are consistent with the inequalities (9) discussed in Section 5.

Note that in 2017, the Dutot index P_D^{2017} was equal to 0.8851, while the Fixed-Base Carli index P_C^{2017} was equal to 1.0355. Thus, $P_C^{2017}/P_D^{2017} = 1.0355/0.8851 = 1.170$. Thus, there is a 17.0 percent spread between these indices listed in Table A.3, which is substantial. The choice of an unweighted index number formula matters.

The fact that the Jevons indices P_J^t approximate the Carruthers, Sellwood, Ward, and Dalén indices P_{CSWD}^t can be demonstrated in another way. From Diewert (1978; 893), it is known that the Fisher index number formula, $P_F(p^1,p^t,q^1,q^t)$, approximates the Törnqvist–Theil index, $P_T(p^1,p^t,q^1,q^t)$, to the second order around a point where $p^1 = p^t$ and $q^1 = q^t$. It is obvious that the Törnqvist–Theil index collapses down to the Jevons index $P_J^t = P_J(p^1,p^t) \equiv \prod_{n=1}^N (p_{tn}/p_{1n})^{1/N}$ if each expenditure share in periods 1 and t is equal to $1/N$. Reinsdorf and Triplett (2009; 63) and Diewert (2013; 6) showed that if all expenditure shares in periods 1 and t are equal to $1/N$, then the Fisher index collapses down to the Carruthers, Sellwood, Ward, and Dalén index $P_{CSWD}(p^1,p^t) = P_{CSWD}^t$. Thus, using the Diewert (1978; 893) second-order approximation result, it can be seen that $P_J(p^1,p^t)$ will approximate $P_{CSWD}(p^1,p^t)$ to the second order around any point, where $p^1 = p^t$.[71] In Chapter 5, the Walsh (1901; 398) (1921a; 97) index was defined as follows:[72]

$$\begin{aligned} P_W(p^1,p^t,q^1,q^t) &\equiv \Sigma_{n=1}^N p_{tn}(q_{tn}q_{1n})^{1/2}/\Sigma_{n=1}^N p_{1n}(q_{tn}q_{1n})^{1/2} \\ &= \Sigma_{n=1}^N (p_{tn}/p_{1n})^{1/2} (p_{tn}q_{tn}p_{1n}q_{1n})^{1/2}/\Sigma_{n=1}^N (p_{1n}/p_{tn})^{1/2} (p_{tn}q_{tn}p_{1n}q_{1n})^{1/2} \\ &= \Sigma_{n=1}^N (p_{tn}/p_{1n})^{1/2} (s_{tn}s_{1n})^{1/2}/\Sigma_{n=1}^N (p_{1n}/p_{tn})^{1/2} (s_{tn}s_{1n})^{1/2}, \end{aligned} \tag{A.24}$$

where $s_{1n} \equiv p_{1n}q_{1n}/p^1\cdot q^1$ and $s_{tn} \equiv p_{tn}q_{tn}/p^t\cdot q^t$ for $n = 1,\ldots,N$ are the period 1 and t expenditure shares. If we again assume that all expenditure shares in periods 1 and t are equal to $1/N$, then the Walsh index collapses down to the following *Dikhanov elementary index* $P_{DI}(p^1,p^t)$:[73]

$$P_{DI}(p^1,p^t) \equiv \Sigma_{n=1}^N (p_{tn}/p_{1n})^{1/2} /\Sigma_{n=1}^N (p_{1n}/p_{tn})^{1/2}. \tag{A.25}$$

Diewert's 1978 second-order approximation result also applies to Walsh and Fisher indices, so it carries over in the present special case where expenditure shares are assumed to be equal and constant across periods. Thus, $P_{DI}(p^1,p^t)$ will approximate $P_J(p^1,p^t)$ and $P_{CSWD}(p^1,p^t)$ to the second order around any point where $p^t = \lambda p^1$.[74]

We turn to the *Approach 1 weighted indices* for our UK telecom data set. Denote the year t Fixed-Base Laspeyres, Paasche, and Fisher indices by P_{L1}^t, P_{P1}^t, and P_{F1}^t and their chained counterparts by P_{LCH1}^t, P_{PCH1}^t, and P_{FCH1}^t. These indices are listed in Table A.4.

Note that the weighted indices listed in Table A.4 are generally higher than their unweighted counterparts listed in Table A.3. The chained Laspeyres indices are always above their chained Paasche counterparts, but this is not always the case for the Fixed-Base Laspeyres and Paasche indices. Note also

Hence, the divergence is explained by the fact that a geometric mean of numbers that are not all equal (the Jevons index) will always be less than the corresponding arithmetic mean (the Dutot index using normalized prices which is the Fixed-Base Carli index). Recall that the indices other than the Dutot index are invariant to the units of measurement.

[71] This second-order approximation result also holds if $p^t = \lambda p^1$ for any scalar $\lambda > 0$.

[72] All prices and quantities are assumed to be positive.

[73] Yuri Dikhanov in a private communication suggested this approximation to the Walsh index.

[74] Using the data for our telecom example, the Dikhanov indices P_{DI}^t were as follows; 1.0000, 0.9616, 0.9345, 0.9327, 0.9609, 0.9427, 0.9214, and 0.9740. These numbers are very close to their P_J^t and P_{CWSD}^t counterparts.

Table A.4 Approach 1 Laspeyres, Paasche, and Fisher Indices

Year t	P_{L1}^{t}	P_{P1}^{t}	P_{LCH1}^{t}	P_{PCH1}^{t}	P_{F1}^{t}	P_{FCH1}^{t}
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**0.9839**	**0.9825**	**0.9839**	**0.9825**	**0.9832**	**0.9832**
2012	**0.9658**	**0.9644**	**0.9661**	**0.9648**	**0.9651**	**0.9655**
2013	**0.9802**	**0.9811**	**0.9805**	**0.9797**	**0.9807**	**0.9801**
2014	**1.0243**	**1.0259**	**1.0274**	**1.0249**	**1.0251**	**1.0262**
2015	**0.9979**	**1.0066**	**1.0034**	**1.0009**	**1.0022**	**1.0022**
2016	**0.9746**	**0.9832**	**0.9931**	**0.9790**	**0.9789**	**0.9860**
2017	**1.0466**	**1.0355**	**1.0690**	**1.0481**	**1.0411**	**1.0585**

Table A.5 Approach 2 Laspeyres, Paasche, and Fisher Indices, P^t/P^{2010} and e^t/e^{2010}

Year t	P_{L2}^{t}	P_{P2}^{t}	P_{LCH2}^{t}	P_{PCH2}^{t}	P_{F2}^{t}	P_{FCH2}^{t}	P^t/P^{2010}	e^t/e^{2010}
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**1.0112**	**1.0126**	**1.0112**	**1.0126**	**1.0119**	**1.0119**	**1.0356**	**0.8414**
2012	**1.0565**	**1.0671**	**1.0611**	**1.0658**	**1.0618**	**1.0634**	**1.1372**	**0.7397**
2013	**1.1051**	**1.1276**	**1.1137**	**1.1203**	**1.1163**	**1.1170**	**1.2163**	**0.6674**
2014	**1.1558**	**1.1877**	**1.1659**	**1.1722**	**1.1716**	**1.1691**	**1.2728**	**0.6063**
2015	**1.1943**	**1.2506**	**1.2198**	**1.2282**	**1.2221**	**1.2240**	**1.3691**	**0.5495**
2016	**1.2343**	**1.3185**	**1.2797**	**1.2870**	**1.2757**	**1.2834**	**1.4655**	**0.4843**
2017	**1.2996**	**1.3946**	**1.3419**	**1.3465**	**1.3463**	**1.3442**	**1.5247**	**0.4212**

that the spread between the six weighted indices listed in Table A.4 for 2017 is much smaller than the corresponding spread between the unweighted indices in Table A.3: The highest index value was 1.0690 for the chained Laspeyres index and the lowest index value was 1.0355 for the Fixed-Base Paasche index. Thus, the index spread in 2017 was 1.0690/1.0355 = 1.032 or a 3.2 percent spread, which is far smaller than the unweighted index spread in 2017, which was 17.0 percent.

Since the Paasche and Laspeyres indices have equal justifications, we prefer the Fisher index, which is an average of these two indices which satisfies the time reversal test. To choose between the Fixed-Base Fisher and its chained counterpart, we look at the spread between the Laspeyres and Paasche indices in 2017. For the Fixed-Base versions of these indices, the spread is equal to $P_{LFB}^{2017}/P_{PFB}^{2017} = 1.0466/1.0355 =$ 1.011 or 1.1 percent. For the chained versions of these indices, the spread is equal to $P_{LCH}^{2017}/P_{PCH}^{2017} = 1.0690/1.0481 = 1.020$ or 2.0 percent. Since the spread is smaller for the Fixed-Base indices, we prefer the Fixed-Base indices over the chained indices, and hence our preferred index for the present data set is the Fisher Fixed-Base index, P_{FFB}^{t}.

For the *Approach 2 weighted indices*, we treat the total access charges $v_5^t \equiv P^t$ as the aggregate price of access in year t,[75] and we set the corresponding year t quantity, Q^t, equal to 1. The prices and quantities for products 1–4 are the p_n^t and q_n^t that are listed in Table A.1. The price of access, $P^t = v_5^t$, is listed in Table A.1. Denote the resulting year t Fixed-Base Laspeyres, Paasche, and Fisher indices by P_{L2}^{t}, P_{P2}^{t}, and P_{F2}^{t} and their chained counterparts by P_{LCH2}^{t}, P_{PCH2}^{t}, and P_{FCH2}^{t}. These indices are listed in Table A.5. We also list (one plus) the rate of growth in the access charges, P^t/P^{2010}, and (one plus) the rate of growth in expenditures on products 1–4, e^t/e^{2010}. Note that P^t/P^{2010} increases rapidly over time, while e^t/e^{2010} decreases rapidly.

Bringing access charges into the scope of the index has led to a general increase in the weighted index numbers. The Fixed-Base Fisher index for Approach 1 ended up at 1.0481 in 2017, whereas the Fixed-Base Fisher index for Approach 2 ended up at 1.3442. This is a very large difference. The Fixed-Base Laspeyres index ended up at 1.2996, while the counterpart Fixed-Base Paasche index ended up at 1.3946. The corresponding chained indices ended up at 1.3419 and 1.3465. Thus, for Approach 2, we prefer the chained Fisher index over its Fixed-Base counterpart since the spread between the Laspeyres and Paasche indices is much smaller for the chained indices. However, the two Fisher indices were very close to each other, and they ended up at 1.3463 and 1.3442, so in this case, it does not matter which Fisher index is chosen.

Recall equation (A.9), which established the following relationship between the year t Approach 1 Laspeyres index, P_{L1}^{t}, and the Approach 2 Laspeyres index, P_{L2}^{t}: $P_{L1}^{t} - P_{L2}^{t} = [m^{2010}/(1 + m^{2010})][P_{L1}^{t} - (P^t/P^{2010})]$. From Tables A.4 and A.5, it can be seen that $P_{L1}^{t} < P^t/P^{2010}$ for all $t > 2010$, and thus $P_{L1}^{t} < P_{L2}^{t}$ for $t = 2011,\ldots,2017$. Similarly, (A.12) established the following relationship between the year t Approach 1 Paasche index, P_{P1}^{t}, and the Approach 2 Paasche index, P_{P2}^{t}: $P_{P1}^{t} - P_{P2}^{t} = [m^t/(1 + m^t)]P_{P2}^{t}\,[P^t/P^{2010}]^{-1}\,[P_{P1}^{t} - (P^t/P^{2010})]$. From Tables A.4 and A.5, it can be seen that $P_{P1}^{t} < P^t/P^{2010}$ for all

[75] This is only an approximation to Model 2 defined by (A.3) since the UK data are aggregate retail sales data rather than individual household consumption data. Also Model 2 defined by (A.3) is a model that applies to a single household; we have neglected the complications that arise when aggregating over households.

Table A.6 Approach 3 Laspeyres, Paasche, and Fisher Indices

Year t	$P_{L3}{}^t$	$P_{P3}{}^t$	$P_{LCH3}{}^t$	$P_{PCH3}{}^t$	$P_{F3}{}^t$	$P_{FCH3}{}^t$
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**1.1040**	**1.1024**	**1.1040**	**1.1024**	**1.1032**	**1.1032**
2012	**1.2403**	**1.2385**	**1.2407**	**1.2391**	**1.2394**	**1.2399**
2013	**1.4068**	**1.4081**	**1.4072**	**1.4061**	**1.4074**	**1.4066**
2014	**1.6199**	**1.6225**	**1.6249**	**1.6209**	**1.6212**	**1.6229**
2015	**1.7854**	**1.8010**	**1.7953**	**1.7909**	**1.7932**	**1.7931**
2016	**2.0191**	**2.0369**	**2.0575**	**2.0282**	**2.0280**	**2.0428**
2017	**2.4972**	**2.4706**	**2.5506**	**2.5008**	**2.4839**	**2.5256**

Table A.7 Approach 4 Laspeyres, Paasche, and Fisher Indices

Year t	$P_{L4}{}^t$	$P_{P4}{}^t$	$P_{LCH4}{}^t$	$P_{PCH4}{}^t$	$P_{F4}{}^t$	$P_{FCH4}{}^t$
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**1.0085**	**1.0097**	**1.0085**	**1.0097**	**1.0091**	**1.0091**
2012	**1.0390**	**1.0484**	**1.0427**	**1.0470**	**1.0437**	**1.0449**
2013	**1.0737**	**1.0927**	**1.0800**	**1.0857**	**1.0832**	**1.0829**
2014	**1.1084**	**1.1316**	**1.1135**	**1.1178**	**1.1199**	**1.1156**
2015	**1.1298**	**1.1733**	**1.1479**	**1.1541**	**1.1513**	**1.1510**
2016	**1.1544**	**1.2209**	**1.1904**	**1.1953**	**1.1872**	**1.1929**
2017	**1.2115**	**1.2796**	**1.2419**	**1.2438**	**1.2451**	**1.2428**

$t > 2010$, and thus $P_{P1}{}^t < P_{P2}{}^t$ for $t = 2011,...,2017$. These inequalities also imply that $P_{F1}{}^t < P_{F2}{}^t$ for $t = 2011,...,2017$. Thus, due to the very rapid growth in access charges over the sample period, the Approach 2 Laspeyres, Paasche, and Fisher indices will be much larger than their Approach 1 counterparts.

For the *Approach 3 weighted indices*, the access charges are spread across products 1–4 in a proportional manner. Thus, define $1 + m^t \equiv v^t/e^t$ and $p_n{}^{t**} \equiv (1 + m^t)p_n{}^{t*}$ for $n = 1,2,3,4$ and $t = 2010,...,2017$. The corresponding quantities are the $q_n{}^{t*}$ listed in Table A.2.[76] Denote the Approach 3 year t Fixed-Base Laspeyres, Paasche, and Fisher indices by $P_{L3}{}^t$, $P_{P3}{}^t$, and $P_{F3}{}^t$ and their chained counterparts by $P_{LCH3}{}^t$, $P_{PCH3}{}^t$, and $P_{FCH3}{}^t$. These indices are listed in Table A.6.

Allocating the access charges across the first four type of call products leads to a very large increase in the weighted index numbers. The Fixed-Base Fisher indices for Approach 1 and 2 end up at 1.0481 and 1.3442, respectively, in 2017, whereas the Fixed-Base Fisher index for Approach 3 ends up at 2.4839. These differences are very large. The Approach 3 Fixed-Base Laspeyres and Paasche spread in 2017 was smaller than the corresponding spread in their chained counterparts, so the Fixed-Base Fisher index $P_{F3}{}^t$ is our preferred weighted index for this approach.

Using our current notation, the equalities in (A.15) translate into the following equalities:

$$P_{L3}{}^t/P_{L1}{}^t = P_{P3}{}^t/P_{P1}{}^t = (1 + m^t)/(1 + m^{2010});\quad t = 2010,...,2017. \tag{A.26}$$

From Table A.2, we see that m^t is monotonically increasing. Thus, using (A.26), it can be seen that the inequalities $P_{L3}{}^t > P_{L1}{}^t$ and $P_{P3}{}^t > P_{P1}{}^t$ for $t > 2010$ must hold.

Using our current notation, (A.16) can be rewritten as follows:

$$P_{L3}{}^t - P_{L2}{}^t = m^t[1 + m^{2010}]^{-1}[P_{L1}{}^t - (e^t/e^{2010})];\quad t = 2010,...,2017. \tag{A.27}$$

Tables A.4 and A.5 list the usage expenditure ratios (e^t/e^{2010}) and the Approach 1 Laspeyres indices $P_{L1}{}^t$. Using these series, it can be seen that $P_{L1}{}^t > e^t/e^{2010}$ for $t > 2010$. Thus, using (A.27), we must have $P_{L3}{}^t > P_{L2}{}^t$ for $t > 2010$.

Using our current notation, (A.19) can be rewritten as follows:

$$P_{P3}{}^t - P_{P2}{}^t = m^{2010}[1 + m^{2010}]^{-1} P_{P2}{}^t[P_{P1}{}^t - (e^t/e^{2010})];\quad t = 2010,...,2017. \tag{A.28}$$

Tables A.4 and A.5 list the usage expenditure ratios (e^t/e^{2010}) and the Approach 1 Paasche indices $P_{P1}{}^t$, and it can be seen that $P_{P1}{}^t > e^t/e^{2010}$ for $t > 2010$. Thus, using (A.28), we must have $P_{P3}{}^t > P_{P2}{}^t$ for $t > 2010$. It follows that it is also the case that $P_{F3}{}^t > P_{F2}{}^t$ for $t > 2010$.

Finally, we consider *Approach 4*. This approach is used when constructing producer price indices for the telecom sector in the regulation literature that attempts to measure the Total Factor Productivity of the sector. In this approach, the number of line connections is used as the output measure for access charges.[77] Thus, this approach simply uses $v_n{}^t$ and $q_n{}^t$ that are listed in Table A.1 (and the implied prices $p_n{}^t \equiv v_n{}^t/q_n{}^t$ for $n = 1,...,5$) in the usual index number formulae that are considered in this annex. Denote the Approach 4 year t Fixed-Base Laspeyres, Paasche, and Fisher indices by $P_{L4}{}^t$, $P_{P4}{}^t$, and $P_{F4}{}^t$ and their chained counterparts by $P_{LCH4}{}^t$, $P_{PCH4}{}^t$, and $P_{FCH4}{}^t$. These indices are listed in Table A.7.

Using the Approach 4 methodology, it can be seen that the Fixed-Base Paasche index grows more rapidly than the corresponding Fixed-Base Laspeyres index. The addition of product 5 to the first four products has caused this somewhat unusual phenomenon. The price of product 5 increases 1.36 fold over the sample period, which is much higher than a weighted average of the prices of the first four products; that is, $P_{L1}{}^t$ and $P_{P1}{}^t$ increased 1.047 fold and 1.036 fold, respectively, over the sample period. At the same time, $q_5{}^t$ increased, while $q_1{}^t$–$q_4{}^t$ decreased substantially over the sample period. Under these conditions, $P_{P4}{}^t$ will increase more rapidly than $P_{L4}{}^t$. Table A.7 also indicates that the spread between $P_{L4}{}^{2017}$ and $P_{P4}{}^{2017}$ is larger than the spread between the chained indices $P_{L4CH}{}^{2017}$ and $P_{P4CH}{}^{2017}$. Under

[76] Instead of using $p_n{}^{t**} \equiv (1 + m^t)p_n{}^{t*}$ and $q_n{}^{t*}$ for $n = 1,..,4$ from Table A.2 as the primary data that are used in the various Laspeyres, Paasche and Fisher indices, we could use $(1 + m^t)p_n{}^t$ and $q_n{}^t$ for $n = 1,..,4$ from Table A.1 as the primary data. The indices remain the same since the Laspeyres, Paasche, and Fisher indices are invariant to changes in the units of measurement.

[77] See, for example, Lawrence and Diewert (2006; 218) where the distributor's number of line connections is regarded as an output of the firm. Their paper is concerned with electricity distribution but the same methodology is used for telecommunication firms.

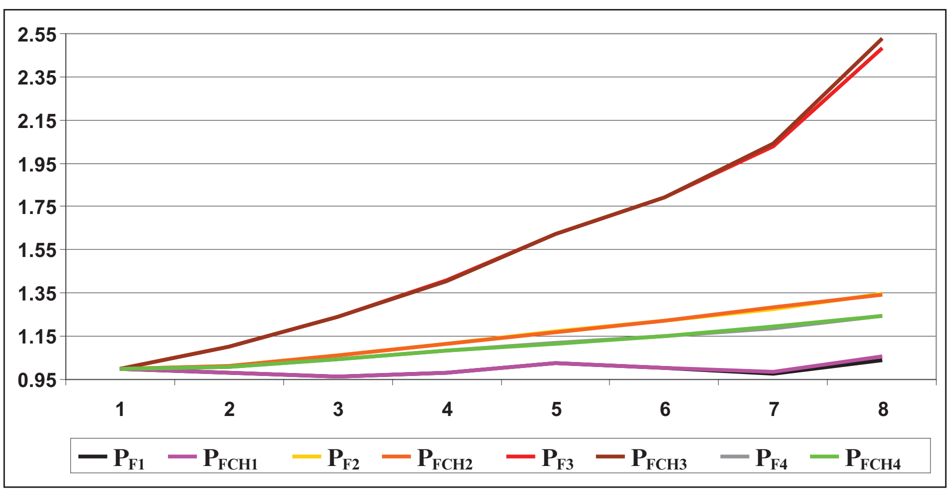
Table A.8 Fixed-Base and Chained Fisher Indices for All Four Approaches

Year t	P_{F1}^{t}	P_{FCH1}^{t}	P_{F2}^{t}	P_{FCH2}^{t}	P_{F3}^{t}	P_{FCH3}^{t}	P_{F4}^{t}	P_{FCH4}^{t}
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**0.9832**	**0.9832**	**1.0119**	**1.0119**	**1.1032**	**1.1032**	**1.0091**	**1.0091**
2012	**0.9651**	**0.9655**	**1.0618**	**1.0634**	**1.2394**	**1.2399**	**1.0437**	**1.0449**
2013	**0.9807**	**0.9801**	**1.1163**	**1.1170**	**1.4074**	**1.4066**	**1.0832**	**1.0829**
2014	**1.0251**	**1.0262**	**1.1716**	**1.1691**	**1.6212**	**1.6229**	**1.1199**	**1.1156**
2015	**1.0022**	**1.0022**	**1.2221**	**1.2240**	**1.7932**	**1.7931**	**1.1513**	**1.1510**
2016	**0.9789**	**0.9860**	**1.2757**	**1.2834**	**2.0280**	**2.0428**	**1.1872**	**1.1929**
2017	**1.0411**	**1.0585**	**1.3463**	**1.3442**	**2.4839**	**2.5256**	**1.2451**	**1.2428**

Table A.9 Approach 4 Unweighted Price Indices

Year t	P_{D}^{t}	P_{J}^{t}	P_{H}^{t}	P_{CSWD}^{t}	P_{C}^{t}	P_{HCH}^{t}	P_{CSWDCH}^{t}	P_{CCH}^{t}
2010	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**	**1.0000**
2011	**0.9920**	**0.9750**	**0.9728**	**0.9749**	**0.9770**	**0.9728**	**0.9749**	**0.9770**
2012	**0.9941**	**0.9662**	**0.9605**	**0.9661**	**0.9717**	**0.9629**	**0.9662**	**0.9694**
2013	**1.0083**	**0.9739**	**0.9629**	**0.9734**	**0.9841**	**0.9697**	**0.9738**	**0.9780**
2014	**1.0403**	**1.0019**	**0.9838**	**1.0012**	**1.0188**	**0.9950**	**1.0019**	**1.0088**
2015	**1.0349**	**0.9970**	**0.9779**	**0.9967**	**1.0158**	**0.9891**	**0.9970**	**1.0048**
2016	**0.9986**	**0.9892**	**0.9498**	**0.9882**	**1.0280**	**0.9660**	**0.9882**	**1.0108**
2017	**1.0403**	**1.0412**	**0.9792**	**1.0379**	**1.1001**	**1.0133**	**1.0400**	**1.0674**

Figure A6.1 Alternative Approach Fisher Indices



these conditions, we prefer the chained Fisher index P_{FCH4}^{t} over its Fixed-Base counterpart P_{F4}^{t}. However, Table A.7 indicates that the difference between the Fixed-Base and chained Fisher indices is negligible using Approach 4.

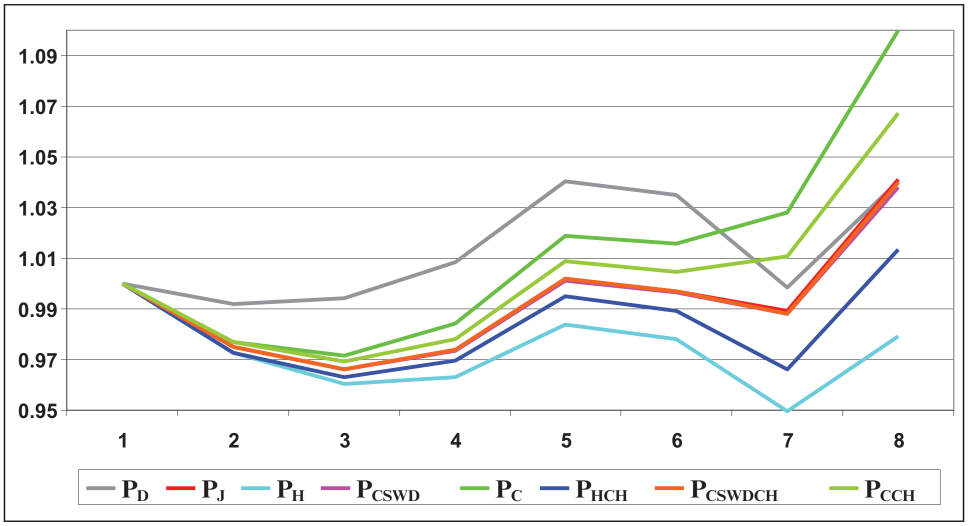
Table A.8 lists the Fixed-Base and chained Fisher indices for all four approaches.

From Table A.8, it can be seen that the Approach 1 Fisher indices (which ignored the access charges) generate the lowest increase in prices, followed by the Approach 4 indices (which include access charges as a regular product with the quantity set equal to the number of lines), followed by the Approach 2 indices (which include access charges but hold the corresponding quantity fixed at unity), and finally followed by the Approach 3 Fisher indices (which spread the access charges across the other products). These alternative approach Fisher indices are plotted in Figure A6.1.

It can be seen that the differences between Fixed-Base and chained Fisher indices for each approach are small, but the differences between the four approaches is very large indeed. Thus, in the case of fixed-line telecommunications services, the choice of an approach to the treatment of access charges is important.

In the case where quantity or expenditure weights are not available, the choice of an elementary index number formula is also important for the telecommunications sector; recall Table A.3, which listed the unweighted indices using

Figure A6.2 Approach 4 Unweighted Indices



the prices of products 1–4. To conclude this annex, we list the same unweighted indices as were listed in Table A.3 but using the prices of products 1–5 in Table A.9.

The 2017 spread in these unweighted indices is 1.1001/0.9792 = 1.123 or 12.3 percent. Recall that the corresponding index spread for the Approach 1 unweighted price indices was 17.0 percent, so the addition of product 5 has lowered the spread significantly. These indices used the prices that correspond to the values and quantities listed in Table A.1. Recall that the Dutot index using normalized prices, $P_{DN}{}^t$, was equal to the chained Carli index, $P_{CCH}{}^t$, listed in Table A.3. A similar result holds here: $P_{DN}{}^t$ is equal to $P_{CCH}{}^t$ listed in Table A.8. The indices listed in Table A.8 are plotted in Figure A6.2. It can be seen that $P_J{}^t$, $P_{CSWD}{}^t$, and $P_{CSWDCH}{}^t$ cannot be distinguished in Figure A6.2.[78] These series are in the middle of the listed indices, with the chained Carli and Carli indices, $P_{CCH}{}^t$ and $P_C{}^t$, well above the middle series and the chained Harmonic and Harmonic indices, $P_{HCH}{}^t$ and $P_H{}^t$, well below the middle series. The Dutot series $P_D{}^t$ is initially well above the other series, but it joins up with the middle series at the end of the sample period. The Dutot index $P_{DN}{}^t$ using the normalized prices listed in Table A.2 coincides with the Fixed-Base Carli index $P_C{}^t$. Thus, there is a substantial difference in the Dutot indices as the units of measurement change. The remaining indices are invariant to changes in the units of measurement.

Some of the conclusions that can be drawn from this annex are as follows:

- Unweighted elementary indices can differ substantially depending on which formula is used.
- The Carli Fixed-Base and chained indices are not recommended due to their failure of the time reversal test with a built-in upward bias.
- The Dutot index is also not recommended due to its lack of invariance to changes in the units of measurement. Even if the units of measurement are the same, the empirical example shows that changing the units of measurement can make a huge difference.
- The approach used to allocate access charges can make a substantial difference to the CPI in the case of regulated network industries where access charges can be substantial.[79]
- In the case of regulated industries, price, and quantity data will often be available to the price statistician.[80] In this case, weighted indices are preferred over unweighted indices because they take into account the economic importance of the various outputs of the regulated industry. The example in this annex shows that there can be significant differences between weighted and unweighted indices.

[78] Recall the approximate equalities (27) and (29) in Section 5.

[79] This point was noted by Abdirahman, Coyle, Heys, and Stewart (2017) (2020).

[80] Unfortunately, data submitted to regulators is usually quarterly data which presents challenges in the context of producing a monthly CPI. However, national income accountants have to produce quarterly CPIs and perhaps more importantly, national accounts price indices can be revised. Hence, as better information becomes available to the price statistician, better (revised) indices can be produced.

Annex B

Additional Problems Associated with the Use of the Carli Index

Robert Hill (2018) submitted some testimony to the United Kingdom's House of Lords Economic Affairs Committee on the use of the Carli Index in the UK's Retail Price Index. His points 3–5 listed here deal with problems associated with the use of the Carli index. Since his testimony is not easily accessible and some of his points were not made in this chapter, the first five points in his testimony are quoted here.

1. I am responding to the latest call for evidence from the House of Lords Economic Affairs Committee in my capacity as a researcher in the field of price indices. I am a British citizen based at University of Graz in Austria, where I am Professor of Macroeconomics. I served on the Expert Advisory Group for Paul Johnson's report on UK Consumer Prices Statistics. I have also served as an advisor to Eurostat on the treatment of OOH in the HICP.
2. In this statement I will focus on what I think are the two most serious problems with the RPI. These are its use of the Carli formula at the elementary level and its treatment of OOH.
3. Irving Fisher warned against using the Carli formula in his 1922 book on index numbers. Carli fails the time reversal test and suffers from a systematic upward bias. For example, if prices change from periods 1 to 2, but then in period 3 return to their original period 1 levels, a chained Carli index will always find that the price level is higher in period 3 than in period 1 (except in the special case where all prices change by exactly the same proportion from one period to the next).
4. Levell (2015) provides a detailed comparison of the Carli and Jevons price index formulae. Carli takes an arithmetic mean of the price relatives while Jevons takes a geometric mean. While Levell ends up rightly favoring Jevons, he is at times too kind to Carli, which could cause some confusion among users. Indeed, there seems to be a perception in some circles that there are trade-offs between Carli and Jevons. For example, Leyland (2011) states: "The RSS does not have a view on whether the arithmetic or geometric mean is the better approach but it does consider the issue a major concern."
5. In my opinion the use of the Carli index is indefensible. To see why, *I* will revisit some of the points made by Levell. Levell assesses the Carli index from three perspectives, referred to in the literature as the test, statistical, and economic approaches. From the test perspective, Jevons is unambiguously better than Carli. Jevons is the only elementary price index formula that satisfies all the 14 tests considered by Levell. Up to this point *I* am in complete agreement with Levell. Turning to the statistical approach discussed in page 316, Levell states that "Ultimately our object of interest here is $E(p_1^i/p_0^i)$." He then goes on to show that Jevons is a downward-biased measure of $E(p_1^i/p_0^i)$. My problem here is that *I* disagree that $E(p_1^i/p_0^i)$ should be our object of interest since it treats price rises and falls asymmetrically. A better approach is to focus on the natural logarithm of the price indices with the following object of interest: $E[ln(p_1^i/p_0^i)]$. In this setting Jevons unambiguously outperforms Carli under the statistical approach. Turning finally to the economic approach, Levell notes that in the case of Leontief preferences – where there is no substitution effect – a case can be made for Carli. This argument dates back at least to the ILO *CPI Manual* of 2004, which on page 16 contains the following statement: "With Leontief preferences, a Laspeyres index provides an exact measure of the cost of living index. In this case, the Carli calculated for a random sample would provide an estimate of the cost of living provided that the items were selected with probabilities proportional to the population expenditure shares." This statement has caused a lot of confusion in the literature. *I* agree that in this case Laspeyres is an exact measure. But what follows regarding Carli is misleading. First, the whole point with elementary indices is that there are no expenditure shares. Second, if we assume the items are sampled proportionally to expenditure shares, then what we have is not Carli but a weighted arithmetic mean of the price relatives. If we assume further that the reference expenditure shares are those of the earlier of the two periods being compared, then instead of Carli we have Laspeyres. So what this statement is really saying is that if we have Leontief preferences and we replace Carli with Laspeyres, then we will get the right answer. This is not very helpful. It is not true that Carli performs well when preferences are Leontief or close to Leontief. The only situation when Carli is free of upward bias is when all prices change at the same rate (which is the Hicks, not Leontief, aggregation case). In conclusion, whichever way you look at it, the Carli index is flawed and should not be used. Jevons has much better properties (Robert Hill (2018)).

Hill makes two important conclusions in his point 5:

- The econometric or statistical approach to index number theory frequently assumes that the goal of the exercise is to measure the average relative price increase—that is, to measure some average over n of the price ratios, p_{tn}/p_{1n}. Using this perspective, econometricians may assert that, for example, the Törnqvist–Theil index is a biased estimator for the target index. But this "bias" vanishes if we make the goal the measurement of the average $\log(p_{tn}/p_{1n})$. As Hill notes, the first approach treats price rises and falls more asymmetrically than the second approach. In any case, the more important economic and basket approaches to consumer index number theory do not take the statistical approach to index number theory.
- Hill's second main point has to do with justifications for the use of the Carli index under special assumptions about the nature of consumer preferences. His dismissal of this type of argument seems to be on target.

References

Abdirahman, Mo, Diane Coyle, Richard Heys, and Will Stewart. 2020. "A Comparison of Deflators for Telecommunications Services Output." *Economie et Statistique / Economics and Statistics* 517-518-519: 103–22.

Abdirahman, Mo, Diane Coyle, Richard Heys, and Will Stewart. 2022. "Telecoms Deflators: A Story of Volume and Revenue

Weights." *Economie et Statistique / Economics and Statistics* 530–531: 43–59.

Balk, Bert M. 1980. "A Method for Constructing Price Indexes for Seasonal Commodities." *Journal of the Royal Statistical Society A* 143: 68–75.

Balk, Bert M. 1994. "On the First Step in the Calculation of a Consumer Price Index." Paper presented at the first Ottawa Group Meeting, Ottawa, October 31–November 4.

Balk, Bert M. 2000. "On Curing the CPI's Substitution and New Goods Bias." Research Paper 0005, Department of Statistical Methods, Statistics Netherlands, Voorburg.

Balk, Bert M. 2002. "Price Indexes for Elementary Aggregates: The Sampling Approach." Research Report, Voorburg: Statistics Netherlands.

Balk, Bert M. 2008. *Price and Quantity Index Numbers*. New York: Cambridge University Press.

Carli, Gian-Rinaldo. 1804. "Del valore e della proporzione de' metalli monetati." In *Scrittori classici italiani di economia politica*, Volume 13, Milano: G.G. Destefanis (originally published in 1764), pp. 297-366.

Carruthers, A.G., D.J. Sellwood, and P.W. Ward. 1980. "Recent Developments in the Retail Prices Index." *The Statistician* 29: 1–32.

Coggeshall, F. 1887. "The Arithmetic, geometric and Harmonic Means." *Quarterly Journal of Economics* 1: 83–86.

Dalén, Jörgen. 1992. "Computing Elementary Aggregates in the Swedish Consumer Price Index." *Journal of Official Statistics* 8: 129–47.

Dalén, Jörgen. 1994. "Sensitivity Analyses for Harmonizing European Consumer Price Indexes." In *International Conference on Price Indexes: Papers and Final Report*, First Meeting of the International Working Group on Price Indexes, November, Ottawa: Statistics Canada, pp. 147–71.

Davies, George R. 1924. "The Problem of a Standard Index Number Formula." *Journal of the American Statistical Association* 19: 180–88.

Davies, George R. 1932. "Index Numbers in Mathematical Economics." *Journal of the American Statistical Association* 27: 58–64.

de Haan, Jan, and Frances Krsinich. 2014. "Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes". *Journal of Business and Economic Statistics* 32: 341–58.

de Haan, Jan, and Heymerik van der Grient. 2011. "Eliminating Chain Drift in Price Indexes Based on Scanner Data." *Journal of Econometrics* 161: 36–46.

Diewert, W. Erwin. 1978. "Superlative Index Numbers and Consistency in Aggregation." *Econometrica* 46: 883–900.

Diewert, W. Erwin. 1993. "The Early History of Price Index Research." In *Essays in Index Number Theory*, Volume 1, edited by W. Erwin Diewert and Alice O. Nakamura (pp. 33–65). Amsterdam: North-Holland.

Diewert, W. Erwin. 1995. "Axiomatic and Economic Approaches to Elementary Price Indexes." Discussion Paper No. 95–01, Department of Economics, University of British Columbia, Vancouver, Canada.

Diewert, W. Erwin. 1998. "Index Number Issues in the Consumer Price Index." *Journal of Economic Perspectives* 12 (1): 47–58.

Diewert, W. Erwin. 2002. "Harmonized Indexes of Consumer Prices: Their Conceptual Foundations." *Swiss Journal of Economics and Statistics* 138 (4): 547–637.

Diewert, W. Erwin. 2013. "Answers to Questions Arising from the RPI Consultation." Discussion Paper 13–04, Vancouver School of Economics, University of British Columbia, Vancouver, Canada, V6T 1L4.

Diewert, W. Erwin, Jan de Haan, and Kevin J. Fox. 2016. "A Newly Identified Source of Potential CPI Bias: Weekly versus Monthly Unit Value Price Indexes." *Economics Letters* 141: 169–72.

Diewert, W. Erwin, and Peter von der Lippe. 2010. "Notes on Unit Value Index Bias." *Journal of Economics and Statistics* 230: 690–708.

Dutot, Charles. 1738. *Réflexions politiques sur les finances et le commerce*, Volume 1. La Haye: Les frères Vaillant et N. Prevost.

Eichhorn, Wolfgang. 1978. *Functional Equations in Economics*. London: Addison-Wesley.

Evans, Bruce. 2012. "International Comparison of the Formula Effect between the CPI and the RPI." Office for National Statistics (ONS), Newport, UK.

Feenstra, Robert C. 1994. "New Product varieties and the Measurement of International Prices." *American Economic Review* 34: 157–77.

Feenstra, Robert C., and Matthew D. Shapiro. 2003. "High Frequency Substitution and the Measurement of Price Indexes." In *Scanner Data and Price Indexes*, Studies in Income and Wealth Volume 64, edited by Robert C. Feenstra and Matthew D. Shapiro (pp. 123–46). Chicago: The University of Chicago Press.

Fisher, Irving. 1921. "The Best Form of Index Number." *Quarterly Publication of the American Statistical Association* 17: 533–37.

Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton-Mifflin.

Fox, Kevin J., and Iqbal A. Syed. 2016. "Price Discounts and the Measurement of Inflation." *Journal of Econometrics* 191: 398–406.

Gudnason, Rosmundur. 2003. "How Do We Measure Inflation? Some Measurement Problems." 7th Ottawa Group Meeting on Price Indexes, May 27–29, Paris.

Hardy, Godfrey H., John E. Littlewood, and György Pólya. 1934. *Inequalities*. Cambridge: Cambridge University Press.

Hawkes, William J. 1997. "Reconciliation of Consumer Price Index Trends in Average Prices for Quasi-Homogeneous Goods Using Scanning Data." Third Meeting of the International Working Group on Price Indexes, Statistics Netherlands, Voorburg, April 16–18.

Hawkes, William J., and Frank W. Piotrowski. 2003, "Using Scanner Data to Improve the Quality of Measurement in the Consumer Price Index." In *Scanner Data and Price Indexes*, Studies in Income and Wealth Volume 64, edited by Robert C. Feenstra and Matthew D. Shapiro (pp. 17–38). Chicago: The University of Chicago Press.

Hill, Robert. 2018. Statement Submitted to the House of Lords Economic Affairs Committee Inquiry into the Use of the Retail Price Index, July 21, 2018.

ILO, IMF, OECD, UNECE, Eurostat, and World Bank. 2004. *Consumer Price Index Manual: Concepts and Practice 2020*, Peter Hill (ed.). Geneva: International Labour Office.

IMF, ILO, Eurostat, UNECE, OECD, and World Bank. 2020. *Consumer Price Index Manual: Theory and Practice*, Brian Graf (ed.). Washington, DC: International Monetary Fund.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2011. "Scanner Data, Time Aggregation and the Construction of Price Indexes." *Journal of Econometrics* 161: 24–35.

Ivancic, Lorraine, and Kevin J. Fox. 2013. "Understanding Price variation Across Stores and Supermarket Chains: Some Implications for CPI Aggregation Methods." *Review of Income and Wealth* 59 (4): 629–47.

Jevons, William S. 1865. "The variation of Prices and the Value of the Currency since 1782." *Journal of the Statistical Society of London* 28: 294–320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119–50.

Koskimäki, Timo, and Mari Ylä-Jarkko. 2003. "Segmented Markets and CPI Elementary Classifications." 7th Ottawa Group Meeting on Price Indexes, Paris, May 27–29.

Koskimäki, Timo, and Yrjö vartia. 2001. "Beyond Matched Pairs and Griliches Type Hedonic Methods for Controlling Quality Changes in CPI Subindexes." 6th Ottawa Group Meeting on Price Indexes, Canberra, April 2–6.

Lawrence, Denis, and W. Erwin Diewert. 2006. "Regulating Electricity Networks: The ABC of Setting X in New Zealand." In *Performance Measurement and Regulation of Network Utilities*, edited by T. Coelli and D. Lawrence (pp. 207–41). Cheltenham: Edward Elgar Publishing.

Levell, Peter. 2015. "Is the Carli Index Flawed?: Assessing the Case for the New Retail Price Index RPIJ." *Journal of the Royal Statistical Society Series A* 178(Part 2): 303–36.

Ley, Eduardo. 2003. "Comment." In *Scanner Data and Price Indexes*, Studies in Income and Wealth Volume 64, edited by Robert C. Feenstra and Matthew D. Shapiro (pp. 379–82). Chicago: The University of Chicago Press.

Leyland, Jill. 2011. "RPI versus CPI—the Definitive Account." *Significance Magazine*. Web address: https://www.significancemagazine.com/business/159-rpi-versus-cpi-the-definitive-account

Reinsdorf, Marshall. 1994. "The Effect of Price Dispersion on Cost of Living Indexes." *International Economic Review* 35: 137–49.

Reinsdorf, Marshall. 1996. "Constructing Basic Component Indexes for the U.S. CPI from Scanner Data: A Test Using Data on Coffee." BLS Working Paper 277, Bureau of Labor Statistics, Washington, DC, April.

Reinsdorf, Marshall, and Brent R. Moulton. 1997. "The Construction of Basic Components of Cost of Living Indexes." In *The Economics of New Goods*, edited by Timothy F. Bresnahan and Robert J. Gordon (pp. 397–436). Chicago: University of Chicago Press.

Reinsdorf, Marshall, and Jack Triplett. 2009. "A Review of Reviews: Ninety Years of Professional Thinking About the Consumer Price Index." In *Price Index Concepts and Measurement*, edited by W. Erwin Diewert, John Greenlees and Charles Hulten (pp. 17–84). Studies in Income and Wealth, Volume 70. Chicago: University of Chicago Press.

Richardson, David H. 2003. "Scanner Indexes for the Consumer Price Index." In *Scanner Data and Price Indexes*, Studies in Income and Wealth Volume 64, edited by Robert C. Feenstra and Matthew D. Shapiro (pp. 39–65). Chicago: The University of Chicago Press.

Silver, Mick. 1995. "Elementary Aggregates, Micro-Indexes and Scanner Data: Some Issues in the Compilation of Consumer Price Indexes." *Review of Income and Wealth* 41: 427–38.

Silver, Mick. 2010. "The Wrong and Rights of Unit Value Indexes." *Review of Income and Wealth* 56 (S1): 206–223

Silver, Mick. 2011. "An Index Number Formula Problem: The Aggregation of Broadly Comparable Items." *Journal of Official Statistics* 27 (4): 1–17.

Silver, Mick, and Saeed Heravi. 2001. "Scanner Data and the Measurement of Inflation." *The Economic Journal* 111: F384–F405.

Silver, Mick, and Saeed Heravi. 2003. "The Measurement of Quality Adjusted Price Changes" In *Scanner Data and Price Indexes*, Studies in Income and Wealth Volume 64, edited by Robert C. Feenstra and Matthew D. Shapiro (pp. 277–316). Chicago: The University of Chicago Press.

Silver, Mick, and Saeed Heravi. 2005. "A Failure in the Measurement of Inflation: Results from a Hedonic and Matched Experiment using Scanner Data." *Journal of Business and Economic Statistics* 23 (3): 269–81.

Szulc, Bohdan J. 1983. "Linking Price Index Numbers". In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette (pp. 537–66). Ottawa: Statistics Canada.

Szulc, Bohdan J. 1987. "Price Iindexes below the Basic Aggregation Level." *Bulletin of Labour Statistics* 2: 9–16.

Theil, Henri. 1954. *Linear Aggregation of Economic Relations*. Amsterdam: North-Holland.

Triplett, Jack E. 2003. "Using Scanner Data in Consumer Price Indexes: Some Neglected Conceptual Considerations." In *Scanner Data and Price Indexes*, Studies in Income and Wealth Volume 64, edited by Robert C. Feenstra and Matthew D. Shapiro (pp. 151–62). Chicago: The University of Chicago Press.

Triplett, Jack E. 2004. *Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes*, Directorate for Science, Technology and Industry, DSTI/DOC(2004)9. Paris: OECD.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Walsh, C. Moylan. 1921a. *The Problem of Estimation*. London: P.S. King & Son.

Walsh, C. Moylan. 1921b. "Discussion." *Journal of the American Statistical Association* 17: 537–44.

White, Alan G. 1999. "Measurement Biases in Consumer Price Indexes." *International Statistical Review* 67 (3): 301–25.

White, Alan G. 2000. "Outlet Types and the Canadian Consumer Price Index." *Canadian Journal of Economics* 33: 488–505.

THE CHAIN DRIFT PROBLEM AND MULTILATERAL INDICES

7

1. Introduction

The *Consumer Price Index Manual*[1] recommended that the Fisher, Walsh, or Törnqvist–Theil price index be used as a *target month-to-month index* in a CPI, provided that monthly price and expenditure data for the class of expenditures in scope were available. In recent years, retail chains in several countries (for example, Australia, Canada, Japan, the Netherlands, Norway, and Switzerland) have been willing to donate their sales value and quantity sold information by detailed product to their national statistical agencies, so it has become possible to calculate month-to-month superlative indices for at least some strata of the country's CPI.[2] However, the following issue arises: Should the indices fix a base month (for 12 or 13 months) and calculate Fisher fixed-base indices, or should they calculate chained month-to-month Fisher indices? The 2004 *CPI Manual* offered the following advice on this choice in the chapter on seasonal commodities:[3]

- Determine the set of commodities that are present in the marketplace in both months of the comparison of prices between the two periods.
- For this maximum overlap set of commodities, calculate one of the three indices recommended in previous chapters using the chain principle; that is, calculate the chained Fisher, Walsh, or Törnqvist–Theil index.

The *CPI Manual* suggested the use of chained superlative indices as a target index for the following three reasons:[4]

- The set of seasonal commodities that overlaps during two consecutive months is likely to be much larger than the set obtained by comparing the prices of any given month with a fixed-base month (like January of a base year). Hence, the comparisons made using chained indices will be more comprehensive and accurate than those made using a fixed-base.
- In many economies, on average 2 or 3 percent of price quotes disappear each month due to the introduction of new commodities and the disappearance of older ones. This rapid sample attrition means that fixed-base indices rapidly become unrepresentative, and hence it seems preferable to use chained indices that can more closely follow marketplace developments.
- If prices and quantities are trending relatively smoothly over time, chaining will reduce the spread between the Paasche and Laspeyres indices.[5] Since these indices provide reasonable bounds for true cost of living indices, reducing the spread between these indices will narrow the zone of uncertainty about the cost of living.

Thus, the 2004 *Manual* recommended the use of chained Fisher, Walsh, or Törnqvist–Theil indices as a target index concept. But, as will be seen in the subsequent text, this advice does not always work out too well.

The problem with this advice is the assumption of smooth trends in prices and quantities. Hill (1993, 388), drawing on the earlier research of Szulc (1983, 1987) and Hill (1988, 136–37), noted that it is not appropriate to use the chain system when prices oscillate or "bounce" to use Szulc's (1983, 548) term. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of sales. The extent of the *price bouncing problem* or the problem of *chain drift* can be measured if we make use of the following test due to Walsh (1901, 389; 1921b, 540):[6]

multiperiod identity test:

$$P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)P(p^2,p^0,q^2,q^0) = 1,$$

where $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ are the period t price and quantity vectors and p_{tn} and q_{tn} are the period t price and quantity for commodity n for $n = 1, \ldots, N$ in the class of commodities under consideration. $P(p^0,p^1,q^0,q^1)$ is a bilateral index number formula that is a function of the prices and quantities of periods 0 and 1. Thus, price change

[1] See paragraph 22.63 in the ILO, Eurostat, IMF, OECD, UN, and the World Bank (2004).

[2] Some countries may be able to obtain price and quantity data for individual products from third-party data aggregators. This can be a cost-effective strategy for a statistical agency. In other cases, price and quantity data for regulated industries can be obtained from regulators.

[3] For more on the economic approach and the assumptions on consumer preferences that can justify month-to-month maximum overlap indices, see Diewert (1999a, 51–56).

[4] See the ILO, Eurostat, IMF, OECD, UN, and the World Bank (2004, 407).

[5] See Diewert (1978, 895) and Hill (1988, 1993, 387–388). Chaining under these conditions will also reduce the spread between fixed-base and chained indices using p_F, p_W or p_T as the basic bilateral formula.

[6] Fisher (1922, 293) realized that the chained Carli, Laspeyres, and Young indices could be subject to upward chain drift, but for his empirical example, there was no evidence of chain drift for the Fisher formula. However, Persons (1921, 110) came up with an empirical example where the Fisher index exhibited substantial downward chain drift. Frisch (1936, 9) seems to have been the first to use the term "chain drift." Both Frisch (1936, 8–9) and Persons (1928, 100–5) discussed and analyzed the chain drift problem. These indices will be formally defined later in the chapter.

is calculated over consecutive periods, but an artificial final period is introduced as the final period where the prices and quantities revert back to the prices and quantities in the very first period. The test asks that the product of all of these price changes should equal unity. If prices have no definite trends but are simply bouncing up and down in a range, then this test can be used to evaluate the amount of chain drift that occurs if chained indices are used under these conditions. *Chain drift* occurs when an index does not return to unity when prices in the current period return to their levels in the base period.[7] Fixed-base indices that satisfy the time reversal test will satisfy Walsh's test and hence will not be subject to chain drift as long as the base period is not changed.

The *Manual* did not take into account how severe the chain drift problem could be in practice.[8] The problem is mostly caused by *sales* (that is, highly discounted prices) of products.[9] An example will illustrate the problem.

Suppose that we are given the price and quantity data for two commodities for four periods. The data are listed in Table 7.1.[10]

The first commodity is subject to periodic sales (in period 2), when the price drops to ½ of its normal level of 1. In period 1, we have "normal" off-sale demand for commodity 1, which is equal to 10 units. In period 2, the sale takes place and demand explodes to 5,000 units.[11] In period 3, the commodity is off sale and the price is back to 1, but many shoppers have stocked up in the previous period, so demand falls to only 1 unit. Finally in period 4, the commodity is off sale and we are back to the "normal" demand of 10 units. Commodity 2 exhibits no price or quantity change across periods: Its price is 1 in all periods and the quantity sold is 100 units in each period. Note that the only thing that has happened going from period 3 to 4 is that the demand for commodity one has picked up from 1 unit to the "normal" level of 10 units. Also note that, conveniently, the period 4 data are exactly equal to the period 1 data so that for Walsh's test to be satisfied, the product of the period-to-period chain links must equal one.[12]

Table 7.2 lists the fixed-base Fisher, Laspeyres, and Paasche price indices, $P_{F(FB)}$, $P_{L(FB)}$, and $P_{P(FB)}$, and as expected, they behave perfectly in period 4, returning to the period 1 level of 1. Then the chained Fisher, Törnqvist–Theil, Laspeyres, and Paasche price indices, $P_{F(CH)}$, $P_{T(CH)}$, $P_{L(CH)}$, and $P_{P(CH)}$, are listed. Obviously, the chained Laspeyres and Paasche indices have chain drift bias that is extraordinary, but what is interesting is that the chained Fisher has a 2 percent downward bias and the chained Törnqvist has a close to 3 percent downward bias.

Table 7.1 Price and Quantity Data for Two Products for Four Periods

Period t	p_1^t	p_2^t	q_1^t	q_2^t
1	1.0	1.0	10	100
2	0.5	1.0	5000	100
3	1.0	1.0	1	100
4	1.0	1.0	10	100

Table 7.2 Fixed-Base and Chained Fisher, Törnqvist–Theil, Laspeyres, and Paasche Indices

Period	$P_{F(FB)}$	$P_{L(FB)}$	$P_{P(FB)}$	$P_{F(CH)}$	$P_{T(CH)}$	$P_{L(CH)}$	$P_{P(CH)}$
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.698	0.955	0.510	0.698	0.694	0.955	0.510
3	1.000	1.000	1.000	0.979	0.972	1.872	0.512
4	1.000	1.000	1.000	0.979	0.972	1.872	0.512

What explains the results in Table 7.2? The problem is this: When commodity one comes off sale and goes back to its regular price in period 3, *the corresponding quantity does not return to the level it had in period 1*—the period 3 demand is only 1 unit, whereas the "normal" period 1 demand for commodity 1 was 10 units. It is only in period 4 that demand for commodity one recovers to the period 1 level. However, since prices are the same in periods 3 and 4, all of the chain links show no change (even though quantities are changing), and this is what causes the difficulties. If demand for commodity one in period 3 had immediately recovered to its "normal" period 1 level of 10, then there would be no chain drift problem.[13]

There are at least four possible real-time solutions to the chain drift problem:

- Use a fixed-base index
- Use a multilateral index[14]

[7] See the ILO, Eurostat, IMF, OECD, UN and the World Bank (2004, 445).
[8] Szulc (1983, 1987) demonstrated how big the chain drift problem could be using chained Laspeyres indices, but the authors of the 2004 *Manual* did not realize that chain drift could also be a problem with chained *superlative* indices.
[9] Pronounced fluctuations in the prices and quantities of seasonal commodities can also cause chain drift.
[10] This example is taken from Diewert (2012).
[11] This example is based on an actual example that used Dutch scanner data. When the price of a detergent product went on sale in the Netherlands at approximately one half of the regular price, the volume sold shot up approximately one thousand fold; see de Haan (2008, 15) and de Haan and van der Grient (2011). These papers brought home the magnitude of volume fluctuations due to sales and led Ivancic, Diewert, and Fox (2009, 2011) to propose the use of rolling window multilateral indices to mitigate the chain drift problem.
[12] The traditional economic approach to index number theory does not take into account the stockpiling problem. It is possible to modify the traditional economic approach, but the resulting theory is difficult to implement; see Section 10 of Diewert and Shimizu (2022).

[13] If the economic approach to index number theory is adopted, what causes chain drift in the above example is *inventory stocking behavior* on the part of households. The standard theory for the cost of living index implicitly assumes that all purchased goods are nondurable and used up in the period of purchase. In real life, households can stockpile goods when they go on sale, and it is this stockpiling phenomenon that leads to a downward chain drift for a superlative index. For an example where a chained superlative index has upward chain drift, see Section 7. Feenstra and Shapiro (2003) also looked at the chain drift problem that was caused by sales and restocking dynamics. Their suggested solution to the chain drift problem was to use a fixed-base index, which was also the advice of Persons (1921, 112).
[14] A multilateral price index compares the average price levels over multiple periods. A bilateral price index compares the price levels over two periods. Multilateral price indices were originally applied in making *cross-country* comparisons of prices. The use of multilateral indices in the time series context dates back to Persons (1921), Fisher (1922, 297–308), Gini (1931), and Balk (1980, 1981). Fisher (1922, 305) suggested taking the arithmetic average of the Fisher "star" indices, whereas Gini suggested taking the geometric mean of the star indices. For additional material on

- Use annual weights for a past year or
- Give up on the use of weights at the first stage of aggregation, and simply use the Jevons index, which does not rely on representative weights.

There are two problems with the first solution: (i) The results depend asymmetrically on the choice of the base period, and (ii) with new and disappearing products,[15] the base period prices and quantities may lose their representativeness; that is, over long periods of time, matching products becomes very difficult.[16]

A problem with the second solution is that as an extra period of data becomes available, the indices may have to be recomputed. This is not a major problem. A solution to this problem is to use a rolling window of observations and the results of the current window to update the index to the current period. This methodology was suggested by Ivancic, Diewert, and Fox (2009, 2011) and is being used by the Australian Bureau of Statistics (2016). There is the problem of deciding exactly how to link the results of the current rolling window to the indices generated by the previous rolling window, but again, this is not a major problem.[17] However, it is possible to solve these linking problems by making use of a different class of multilateral methods, namely methods that rely on linking the data of the current period with a prior period that has the most similar structure of relative prices. This new class of multilateral methods will be explained in Sections 18 and 20.

The problem with the third possible solution is that the use of annual weights will inevitably result in some substitution bias, usually in the range of 0.15–0.40 percentage points per year.[18]

The problem with the fourth possible solution is that the use of an index that does not use quantity or expenditure weights will give equal weight to the prices of products that may be unimportant in household budgets, which can lead to a biased CPI.

There is a possible fifth method to avoid chain drift within a year when using a superlative index, and that is to simply compute a sequence of 12 year-over-year monthly indices so that say January prices in the previous year would be compared with January prices in the current year and so on. Handbury, Watanabe, and Weinstein (2013) used this methodological approach for the construction of year–over-year monthly superlative Japanese CPIs using the Nikkei point-of-sale database. This database has monthly price and expenditure data covering the years 1988 to 2010 and contains 4.82 billion price and quantity observations. This type of index number was recommended in Chapter 22 of the 2004 *Consumer Price Index Manual* as a valid year-over-year index that would avoid seasonality problems. However, central banks and other users require month-to-month CPIs in addition to year-over-year monthly CPIs, and so the approach of Handbury, Watanabe, and Weinstein does not solve the problems associated with the construction of superlative month-to-month indices.

Many national statistical agencies are using web-scraping to collect large numbers of prices as a substitute for selective sampling of prices at the first stage of aggregation. Thus, it is of interest to look at elementary indices that depend only on prices, such as the Carli (1804), Dutot (1738), and Jevons (1865) indices, and compare these indices to superlative indices; that is, under what conditions will these indices adequately approximate a superlative index.[19]

The two superlative indices that we will consider in this chapter are the Fisher (1922) and the Törnqvist[20] indices. The reasons for singling out these two indices as preferred bilateral index number formulae are as follows: (i) both indices can be given a strong justification from the viewpoint of the economic approach to index number theory; (ii) the Fisher index emerges as probably being the "best" index from the viewpoint of the axiomatic or test approach to index number theory;[21] (iii) the Törnqvist index has a strong justification from the viewpoint of the stochastic approach to index number theory.[22] Thus, there are strong cases for the use of these two indices when making comparisons of prices between two periods when detailed price and quantity data are available.

multilateral indices, see Diewert (1988, 1999b), Balk (1996, 2008), and Diewert and Fox (2020).

[15] We use the term "products" as meaning "goods and services."

[16] Persons (1928, 99–100) has an excellent discussion on the difficulties of matching products over time.

[17] Ivancic, Diewert, and Fox (2009, 2011) suggested that the movement of the rolling window indices for the last two periods in the new window be linked to the last index value generated by the previous window. However, Krsinich (2016) suggested that the movement of the indices generated by the new window be linked to the previous window index value for the second period in the previous window. Krsinich called this a *window splice* as opposed to the IDF *movement splice*. de Haan (2015, 27) suggested that perhaps the linking period should be in the middle of the old window, which the Australian Bureau of Statistics (2016, 12) termed a *half splice*. Ivancic, Diewert, and Fox (2010) suggested that the *average* of all links for the last period in the new window to the observations in the old window could be used as the linking factor. Diewert and Fox (2021) looked at these alternative methods for linking. *Average* or *mean linking* seems to be the safest strategy. Of course, another strategy would be to use *relative price similarity linking* and an expanding window length. This method is explained in Section 18.

[18] For retrospective studies on upper-level substitution bias for national CPIs, see Diewert, Huwiler, and Kohli (2009); Huang, Wimalaratne, and Pollard (2015); and Armknecht and Silver (2014). For studies of lower-level substitution bias for a Lowe index, see Diewert, Finkel, and Artsev (2009) and Diewert (2014).

[19] We will also look at the approximation properties of the CES price index with equal weights.

[20] The usual reference is Törnqvist (1936), but the index formula did not actually appear in this paper. It did appear explicitly in Törnqvist and Törnqvist (1937). It was listed as one of Fisher's (1922) many indices: namely number 123. It was explicitly recommended as one of his top five ideal indices by Warren Persons (1928, 86), so it probably should be called the *Persons index*. Theil (1967) developed a compelling descriptive statistics justification for the index. Superlative indices are explained in Diewert (1976, 2021a).

[21] See Diewert (1992).

[22] See Theil (1967, 136–37) or Chapter 4.

When comparing two indices, two methods for making the comparisons will be used: (i) second-order Taylor series approximations to the index differences; (ii) the difference between two indices can frequently be written as a covariance, and it is possible in many cases to determine the likely sign of the covariance.[23]

When looking at scanner data from a retail outlet (or price and quantity data from a firm that uses dynamic pricing to price its products or services[24]), a fact emerges: If a product or a service is offered at a highly discounted price (that is, it goes on sale), then the quantity sold of the product can increase by a very large amount. This empirical observation will allow us to make reasonable guesses about the signs of various covariances that express the difference between two indices. If we are aggregating products that are close substitutes for each other, then a heavily discounted price may not only increase the *quantities sold* of the product but also increase the *expenditure share* of the sales in the list of products or services that are in scope for the index.[25] It turns out that the behavior of shares in response to discounted prices does make a difference in analyzing the differences between various indices: In the context of highly substitutable products, a heavily discounted price will probably increase the market share of the product, but if the products are weak substitutes (which is typically the case at higher levels of aggregation), then a discounted price will typically increase sales of the product but not increase its market share. These two cases (strong or weak substitutes) will play an important role in our analysis.

Sections 2 and 3 look at relationships between the fixed-base and chained Carli, Dutot, Jevons, and constant elasticity of substitution (CES) elementary indices that do not use expenditure share or quantity information. These indices are used by national statistical agencies at the first stage of aggregation when they calculate price indices for components of their CPIs in the case when quantity or value information is not available. It should be noted that we will start our analysis of various index number formulae by first developing the concept of a *price level*, which is an average of prices pertaining to a given period of time. A bilateral *price index* calculates *price change* between two periods. A price index could be a ratio of two price levels, or it could be an average of price ratios, where the price of a good or service in the comparison period is in the numerator and the corresponding price in the base period is in the denominator. Comparing price levels for two periods is quite different from undertaking price comparisons over multiple periods. In the multiple period case, it turns out to be easier to compare price *levels* across periods rather than taking averages of price ratios as is done in the case of bilateral comparisons. Thus, from the viewpoint of the economic approach to index number theory, it is simpler to target the estimation of unit cost functions rather than target the estimation of a ratio of unit cost functions. Once we have estimates for period-by-period price levels, we can easily form ratios of these estimates, which will give us "normal" index numbers.

Section 4 looks at the relationships between the Laspeyres, Paasche, geometric Laspeyres, geometric Paasche, Fisher, and Törnqvist bilateral price indices. Section 5 investigates how close the unweighted Jevons index is to the geometric Laspeyres $P_{GL}{}^{t}$, geometric Paasche $P_{GP}{}^{t}$, and Törnqvist $P_{T}{}^{t}$ price indices.

Section 6 develops some relationships between the Törnqvist index and geometric indices that use average *annual* shares as weights.

Section 7 looks at the differences between fixed-base and chained Törnqvist indices.

Multilateral indices finally make their appearance in Section 8: the fixed-base Törnqvist index is compared to the Gini, Eltetö, Köves, and Szulc (GEKS) and GEKS-Törnqvist or Caves, Christensen, Diewert, and Inklaar (CCDI) multilateral indices.

Sections 9 and 10 compare unit value and quality-adjusted unit value indices to the Fisher index. It turns out that some multilateral indices are actually quality-adjusted unit value indices, as will be seen in Section 12. Section 11 compares the Lowe index to the Fisher index.

Section 12 examines the Geary–Khamis (GK) multilateral index and shows that it is actually a special case of a quality-adjusted unit value index.

Sections 13 and 14 introduce time product dummy multilateral indices. Section 13 assumes that there are no missing products in the window of time periods under consideration, while Section 14 deals with the case of missing products. Sections 15 and 16 introduce weighted time product dummy indices for the case of two periods; the missing products case is considered in Section 16. Finally, the weighted TPD multilateral indices for T periods with missing products are discussed in Section 17. Readers who are only interested in the general case can skip Sections 13–16 and just consider the general case in Section 17.

Section 18 introduces a less familiar multilateral method that is based on linking observations that have the most similar structure of relative prices. This *similarity method* for linking observations has for the most part been used in the context of making cross-country comparisons. This class of methods depends on the choice of a measure of *dissimilarity* between the prices of two observations. The dissimilarity measure used in Section 18 is Diewert's (2009) asymptotic linear measure of relative price dissimilarity.

A problem with the dissimilarity measure used in Section 18 is that it requires positive prices for all products.[26] Thus, in Section 19, a simple method for constructing imputed prices for missing products is described.

In Section 20, a new measure of relative price dissimilarity, the predicted share measure of relative price dissimilarity, is defined as the measure that does not require positive prices for all products in the two periods that are compared. This new measure can be adapted to measures of dissimilarity between relative quantities. Section 20 also introduces another method for constructing bilateral index number links between pairs

[23] This second method for making comparisons can be traced back to Bortkiewicz (1923).

[24] Airlines and hotels are increasingly using dynamic pricing; that is, they change prices frequently.

[25] In the remainder of this chapter, we will speak of products, but the same analysis applies to services.

[26] Products that are absent in both periods that are being compared can be ignored. However, for products that are present in only one of the two comparison periods, the dissimilarity measure defined in Section 18 requires that an imputed price for the missing products be constructed.

of observations that have either proportional price vectors or proportional quantity vectors. This new method has some good axiomatic properties, as will be seen in Section 21.

Section 21 introduces an axiomatic or test approach to evaluate the properties of alternative multilateral methods for generating price and quantity levels cross multiple time periods. However, this section makes only a start on the axiomatic approach to evaluating alternative price levels for many time periods.

Section 22 summarizes some of the more important results in this chapter.

The Annex evaluates all of these indices for a grocery store scanner data set that is publicly available. This data set had a number of missing prices and quantities. Some of these missing prices may be due to lack of sales or shortages of inventory. A general problem is how should the introduction of new products and the disappearance of (possibly) obsolete products be treated in the context of forming a CPI? Hicks (1940, 140) suggested a general approach to this measurement problem in the context of the economic approach to index number theory. His approach was to apply normal index number theory but estimate (or guess at) hypothetical prices that would induce utility-maximizing purchasers of a related group of products to demand 0 units of unavailable products. With these virtual (or reservation or imputed) prices in hand, one can just apply normal index number theory using the augmented price data and the observed quantity data. The empirical example discussed in the Annex uses the scanner data that was used in Diewert and Feenstra (2017, 2022) for frozen juice products for a Dominick's store in Chicago for three years. This data set had 20 observations where $q_{tn} = 0$. For these 0 quantity observations, Diewert and Feenstra estimated positive Hicksian reservation prices for these missing price observations, and these imputed prices are used in the empirical example in the Annex. The Annex lists the Dominick's data along with the estimated reservation prices. The Annex also has tables and charts of the various index number formulae that are discussed in the main text of the study.

2. Comparing CES Price Levels and Price Indices

In this section, we will begin our analysis by considering alternative methods by which the prices for N related products could be aggregated into an *aggregate price level* for the products for a given period.

We introduce some notation that will be used in the rest of the chapter. It is supposed that price and quantity data for N closely related products has been collected for T time periods.[27] Typically, a time period is a month. Denote the price of product n in period t as p_{tn} and the corresponding quantity during period t as q_{tn} for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. Usually, p_{tn} will be the period t *unit value price* for product n in period t; that is, $p_{tn} \equiv v_{tn}/q_{tn}$, where v_{tn} is the total value of product n that is sold or purchased during period t and q_{tn} is the total quantity of product n that is sold or purchased during period t. We assume that $q_{tn} \geq 0$ and $p_{tn} > 0$ for all t and n.[28] The restriction that all products have positive prices associated with them is a necessary one for much of our analysis since many popular index numbers are constructed using logarithms of prices, and the logarithm of a zero price is not well defined. However, our analysis does allow for possible 0 quantities and values for some products for some time periods. Denote the period t strictly positive *price vectors* as $p^t \equiv [p_{t1}, \ldots, p_{tN}] \gg 0_N$ and nonnegative (and nonzero) *quantity vectors* as $q^t \equiv [q_{t1}, \ldots, q_{tN}] > 0_N$, respectively, for $t = 1, \ldots, T$, where 0_N is an N-dimensional vector of zeros. As usual, the inner product of the vectors p^t and q^t is denoted by $p^t \cdot q^t \equiv \Sigma_{n=1}^N p_{tn} q_{tn} > 0$. Define the period t sales (or expenditure) share for product n as $s_{tn} \equiv p_{tn} q_{tn}/p^t \cdot q^t$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. The *period t sales or expenditure share vector* is defined as $s^t \equiv [s_{t1}, \ldots, s_{tN}] > 0_N$ for $t = 1, \ldots, T$.

In many applications, the N products will be closely related, and they will have common units of measurement (by weight, or by volume, or by "standard" package size). In this context, it is useful to define the period t "real" share for product n of total product sales or purchases, $S_{tn} \equiv q_{tn}/1_N \cdot q^t$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$, where 1_N is an N-dimensional vector of ones. Denote the *period t real share vector* as $S^t \equiv [S_{t1}, \ldots, S_{tN}]$ for $t = 1, \ldots, T$.

Define a generic *product weighting vector* as $\alpha \equiv [\alpha_1, \ldots, a_N]$. We assume that α has strictly positive components that sum to one; that is, we assume that α satisfies

$$\alpha \cdot 1_N = 1; \; a \gg 0_N. \tag{1}$$

Let $p \equiv [p_1, \ldots, p_N] \gg 0_N$ be a strictly positive price vector. The corresponding *mean of order r of the prices p* (*with weights* α) or *CES price level* $m_{r,\alpha}(p)$ is defined as follows:[29]

$$\begin{aligned} m_{r,\alpha}(p) &\equiv [\Sigma_{n=1}^N \alpha_n p_n^{\,r}]^{1/r}; \; r \neq 0; \\ &\equiv \textstyle\prod_{n=1}^N (p_n); \; r = 0. \end{aligned} \tag{2}$$

It is useful to have a special notation for $m_{r,\alpha}(p)$ when $r = 1$:

$$p_\alpha \equiv \Sigma_{n=1}^N \alpha_n p_n = \alpha \cdot p. \tag{3}$$

Thus, p_α is an α-weighted arithmetic mean of the prices $p_1, p_2, \ldots, p_N$, and it can be interpreted as a *weighted Dutot price level*.[30]

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[27] The T periods can be regarded as a window of observations, followed by another window of length T that has dropped the first period from the window and added the data of period $T + 1$ to the window. The literature on how to link the results of one window to the next window was briefly discussed in the Introduction section and is discussed at length by Diewert and Fox (2021).

[28] In the case where $q_{tn} = 0$, $v_{tn} = 0$ as well and hence $p_{tn} \equiv v_{tn}/q_{tn}$ is not well defined in this case. In the case where $q_{tn} = 0$, we will assume that p_{tn} is a positive imputed price. Imputed prices will be discussed in Section 19.

[29] Hardy, Littlewood, and Pólya (1934, 12–13) refer to this family of means or averages as *elementary weighted mean values* and study their properties in great detail. The function $m_{r,\alpha}(p)$ can also be interpreted as a *Constant Elasticity of Substitution (CES) unit cost function* if $r \leq 1$. The corresponding utility or production function was introduced into the economics literature by Arrow et al. (1961). For additional material on CES functions, see Diewert (2022a), Feenstra (1994), and Diewert and Feenstra (2017, 2022).

[30] The ordinary Dutot (1738) price level for the period t prices p^t is defined as $p_D^{\,t} \equiv (1/N)\Sigma_{n=1}^N p_{tn}$. Thus, it is equal to $m_{1,\alpha}(p^t)$ when $\alpha = (1/N)1_N$.

From Schlömilch's (1858) Inequality,[31] we know that $m_{r,\alpha}(p) \geq m_{s,\alpha}(p)$ if $r \geq s$ and $m_{r,\alpha}(p) \leq m_{s,\alpha}(p)$ if $r \leq s$. However, we do not know how big the gaps are between these price levels for different r and s. When $r = 0$, $m_{0,\alpha}(p)$ becomes a weighted geometric mean or a *weighted Jevons* (1865) or *Cobb–Douglas price level*, and it is of interest to know how much higher the weighted Dutot price level is than the corresponding weighted Jevons price level. Proposition 1 provides an approximation to the gap between $m_{r,\alpha}(p)$ and $m_{1,\alpha}(p)$ for any r, including $r = 0$.

Define the *α-weighted variance* of $p/p_a \equiv [p_1/p_\alpha, \ldots, p_N/p_\alpha]$, where p_α is defined by (3) as follows:[32]

$$\mathrm{var}_\alpha(p/p_\alpha) \equiv \Sigma_{n=1}^{N} \alpha_n[(p_n/p_\alpha) - 1]^2. \quad (4)$$

Proposition 1: Let $p >> 0_N$, $\alpha >> 0_N$, and $\alpha \cdot 1_N = 1$. Then $m_{r,\alpha}(p)/m_{1,\alpha}(p)$ is approximately equal to the following expression for any r:

$$m_{r,\alpha}(p)/m_{1,\alpha}(p) \approx 1 + (½)(r - 1)\mathrm{var}_\alpha(p/p_\alpha), \quad (5)$$

where $\mathrm{var}_\alpha(p/p_\alpha)$ is defined by (4). The expression on the right-hand side of (5) uses a second-order Taylor series approximation to $m_{r,\alpha}(p)$ around the equal price point $p = p_\alpha 1_N$, where p_α is defined by (3).[33]

Proof: Straightforward calculations show that the level, vector of first-order partial derivatives and matrix of second-order partial derivatives of $m_{r,\alpha}(p)$ evaluated at the equal price point $p = p_\alpha 1_N$ are equal to the following expressions: $m_{r,\alpha}(p_\alpha 1_N) = p_\alpha \equiv \alpha \cdot p$; $\nabla_p m_{r,\alpha}(p_\alpha 1_N) = \alpha$; $\nabla^2_{pp} m_{r,\alpha}(p_\alpha 1_N) = (p_\alpha)^{-1}(r - 1)(\hat{\alpha} - \alpha\alpha^T)$, where $\hat{\alpha}$ is a diagonal N by N matrix with the elements of the column vector α running down the main diagonal and α^T is the transpose of the column vector α. Thus, $\alpha\alpha^T$ is a rank one N by N matrix.

Thus, the second-order Taylor series approximation to $m_{r,\alpha}(p)$ around the point $p = p_\alpha 1_N$ is given by the following expression:

$$\begin{aligned} m_{r,\alpha}(p) &\approx p_\alpha + \alpha \cdot (p - p_\alpha 1_N) + (½)(p - p_\alpha 1_N)^T (p_\alpha)^{-1}(r - 1)(\hat{\alpha} - \alpha\alpha^T)(p - p_\alpha 1_N) \quad (6)\\ &= p_\alpha + (½)(p_\alpha)^{-1}(r - 1)(p - p_\alpha 1_N)^T(p_\alpha)^{-1}(\hat{\alpha} - \alpha\alpha^T)(p - p_\alpha 1_N) \text{ using (1) and (3)}\\ &= p_\alpha[1 + (½)(r - 1)(p_\alpha)^{-2}(p - p_\alpha 1_N)^T(\hat{\alpha} - \alpha\alpha^T)(p - p_\alpha 1_N)]\\ &= m_{1,\alpha}(p)[1 + (½)(r - 1)\mathrm{var}_\alpha(p/p_a)] \text{ using (2), (3) and (4).} \end{aligned}$$

Q.E.D.

The approximation (6) also holds if $r = 0$. In this case, (6) becomes the following approximation:[34]

$$\begin{aligned} m_{0,\alpha}(p) &\equiv \textstyle\prod_{n=1}^{N} (p_n)^{\alpha_n} \quad (7)\\ &\approx m_{1,\alpha}(p)[1 - (½)\mathrm{var}_\alpha(p/p_\alpha)]\\ &= m_{1,\alpha}(p)\{1 - (½)\Sigma_{n=1}^{N} \alpha_n[(p_n/p_\alpha) - 1]^2\} \text{ using (4)}\\ &= [\Sigma_{n=1}^{N} \alpha_n p_n]\{1 - (½)\Sigma_{n=1}^{N} \alpha_n[(p_n/p_\alpha) - 1]^2\} \text{ using (2) for } r = 1\\ &\leq \Sigma_{n=1}^{N} \alpha_n p_n. \end{aligned}$$

Thus, the bigger is the variation in the N prices $p_1, \ldots, p_N$, the bigger will be $\mathrm{var}_\alpha(p/p_\alpha)$ and the more the weighted arithmetic mean of the prices, $\Sigma_{n=1}^{N} \alpha_n p_n$ will be greater than the corresponding weighted geometric mean of the prices $\prod_{n=1}^{N} (p_n)^{\alpha_n}$. Note that if all of the p_n are equal, then $\mathrm{var}_\alpha(p/p_\alpha)$ will be equal to 0 and the approximations in (6) and (7) become exact equalities.

At this point, it is useful to define the Jevons (1865) and Dutot (1738) period *t price levels* for the prices in our window of observations, $p_J^{\,t}$ and $p_D^{\,t}$, and the corresponding Jevons and Dutot *price indices*, $P_J^{\,t}$ and $P_D^{\,t}$, for $t = 1, \ldots, T$:

$$p_D^{\,t} \equiv \Sigma_{n=1}^{N} (1/N)p_{tn}; \quad (8)$$

$$p_J^{\,t} \equiv \textstyle\prod_{n=1}^{N} p_{tn}^{\;1/N}; \quad (9)$$

$$P_D^{\,t} \equiv p_D^{\,t}/p_D^{\,1}; \quad (10)$$

$$P_J^{\,t} \equiv p_J^{\,t}/p_J^{\,1} = \textstyle\prod_{n=1}^{N} (p_{tn}/p_{1n})^{1/N}. \quad (11)$$

Thus, the period t price index is simply the period t price level divided by the corresponding period 1 price level. Note that the Jevons price index can also be written as the geometric mean of the long-term price ratios (p_{tn}/p_{1n}) between the period t prices relative to the corresponding period 1 prices.

The *weighted Dutot and Jevons period t price levels* using a weight vector α that satisfies the restrictions (1), $p_{D\alpha}^{\,t}$ and $p_{J\alpha}^{\,t}$, are defined by (12) and (13) and the corresponding *weighted Dutot and Jevons period t price indices*, $P_{D\alpha}^{\,t}$[35] and $P_{J\alpha}^{\,t}$,[36] are defined by (14) and (15) for $t = 1, \ldots, T$:

$$p_{D\alpha}^{\,t} \equiv \Sigma_{n=1}^{N} \alpha_n p_{tn} = m_{1,\alpha}(p^t); \quad (12)$$

$$p_{J\alpha}^{\,t} \equiv \textstyle\prod_{n=1}^{N} (p_{tn})^{\alpha_n} = m_{0,\alpha}(p^t); \quad (13)$$

$$P_{D\alpha}^{\,t} \equiv p_{D\alpha}^{\,t}/p_{D\alpha}^{\,1} = \alpha \cdot p^t/\alpha \cdot p^1; \quad (14)$$

$$P_{J\alpha}^{\,t} \equiv p_{J\alpha}^{\,t}/p_{J\alpha}^{\,1} = \textstyle\prod_{n=1}^{N} (p_{tn}/p_{1n})^{\alpha_n}. \quad (15)$$

Obviously, (12)–(15) reduce to definitions (8)–(11) if $\alpha = (1/N)1_N$. We can use the approximation (7) for $p = p^1$ and $p = p^t$ in order to obtain the following approximate relationship between the weighted Dutot price index for period t, $P_{D\alpha}^{\,t}$, and the corresponding weighted Jevons index $P_{J\alpha}^{\,t}$:

$$\begin{aligned} P_{J\alpha}^{\,t} &\equiv p_{J\alpha}^{\,t}/p_{J\alpha}^{\,1};\ t = 1, \ldots, T \quad (16)\\ &= m_{0,\alpha}(p^t)/m_{0,\alpha}(p^1) \text{ using (2) and (13)} \end{aligned}$$

[31] See Hardy, Littlewood, and Pólya (1934, 26) for a proof of this result.
[32] Note that the α-weighted mean of p/p_α is equal to $\Sigma_{n=1}^{N} \alpha_n p_n/p_\alpha = 1$. Thus, (4) defines the corresponding weighted variance.
[33] For alternative approximations for the differences between mean of order r averages, see vartia (1978, 278–79). vartia's approximations involve variances of logarithms of prices, whereas our approximations involve variances of deflated prices. Our analysis is a variation on his pioneering analysis.
[34] Note that $m_{0,\alpha}(p)$ can be regarded as a weighted Jevons (1865) price level or a Cobb and Douglas (1928) price level. Similarly, $p_\alpha \equiv m_{1,\alpha}(p)$ can be regarded as a weighted Dutot (1738) price level or a Leontief (1936) price level.
[35] A weighted Dutot index can also be interpreted as a Lowe (1823) index.
[36] This type of index is frequently called a *geometric Young index*; see Armknecht and Silver (2014, 4–5).

$$\approx m_{1,\alpha}(p^t)\{1 - (½)\Sigma_{n=1}^N \alpha_n[(p_{tn}/p_\alpha^{\ t}) - 1]^2\}/m_{1,\alpha}(p^1)\{1 - (½)\Sigma_{n=1}^N \alpha_n[(p_{1n}/p_\alpha^{\ 1}) - 1]^2\} \text{ using (7) for } p = p^t \text{ and } p = p^1 \text{ where } p_\alpha^{\ t} \equiv \alpha\cdot p^t \text{ and } p_\alpha^{\ 1} \equiv \alpha\cdot p^1$$

$$= P_{D\alpha}^{\ t}\{1 - (½)\Sigma_{n=1}^N \alpha_n[(p_{tn}/p_\alpha^{\ t}) - 1]^2\}/\{1 - (½)\Sigma_{n=1}^N \alpha_n[(p_{1n}/p_\alpha^{\ 1}) - 1]^2\}$$

$$= P_{D\alpha}^{\ t}\{1 - (½)\mathrm{var}_\alpha(p^t/p_\alpha^{\ t})\}/\{1 - (½)\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})\}.$$

In the elementary index context where there are no trends in prices in diverging directions, it is likely that $\mathrm{var}_\alpha(p^t/p_\alpha^{\ t})$ will be approximately equal to $\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})$.[37] Under this condition, *the weighted Jevons price index $P_{J\alpha}^{\ t}$ is likely to be approximately equal to the corresponding weighted Dutot price index*, $P_{D\alpha}^{\ t}$. Of course, this approximate equality result extends to the case where $\alpha = (1/N)1_N$, and so it is likely that the Dutot price indices $P_D^{\ t}$ are approximately equal to their Jevons price index counterparts $P_J^{\ t}$.[38] However, if the variance of the deflated period 1 prices is unusually large (small), then there will be a tendency for $P_J^{\ t}$ to exceed (to be less than) $P_D^{\ t}$ for $t > 1$.[39]

At higher levels of aggregation where the products may not be very similar,[40] it is likely that there will be *divergent trends in prices* over time. In this case, we can expect $\mathrm{var}_\alpha(p^t/p_\alpha^{\ t})$ to exceed $\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})$. Thus, using (16) under these circumstances leads to the likelihood that the weighted index $P_{J\alpha}^{\ t}$ will be significantly lower than $P_{D\alpha}^{\ t}$. Similarly, under the *diverging trends in prices hypothesis*, we can expect the ordinary Jevons index $P_J^{\ t}$ to be lower than the ordinary Dutot index $P_D^{\ t}$.[41]

We conclude this section by finding an approximate relationship between a CES price index and the corresponding weighted Dutot price index $P_{D\alpha}^{\ t}$. This approximation result assumes that econometric estimates for the parameters of the CES unit cost function $m_{r,\alpha}(p)$ defined by (2) are available so that we have estimates for the weighting vector α (which we assume satisfies the restrictions (1)) and the parameter r, which we assume satisfies $r \le 1$.[42] The *CES period t price levels* using a weight vector α that satisfies the restrictions (1) and an $r \le 1$, $p_{CES\alpha,r}^{\ t}$, and the corresponding *CES period t price indices*, $P_{CES\alpha,r}^{\ t}$, are defined as follows for $t = 1, \ldots, T$:

$$p_{CES\alpha,r}^{\ t} \equiv [\Sigma_{n=1}^N \alpha_n p_{tn}^{\ r}]^{1/r} = m_{r,\alpha}(p^t); \quad (17)$$
$$P_{CES\alpha,r}^{\ t} \equiv p_{CES\alpha,r}^{\ t}/p_{CES\alpha,r}^{\ 1} = m_{r,\alpha}(p^t)/m_{r,\alpha}(p^1). \quad (18)$$

Now use the approximation (6) for $p = p^1$ and $p = p^t$ in order to obtain the following approximate relationship between the weighted Dutot price index for period t, $P_{D\alpha}^{\ t}$, and the corresponding period t CES index, $P_{CES\alpha,r}^{\ t}$ for $t = 1, \ldots, T$:

$$P_{CES\alpha,r}^{\ t} \equiv p_{CES\alpha,r}^{\ t}/p_{CES\alpha,r}^{\ 1}; \quad (19)$$
$$= m_{r,\alpha}(p^t)/m_{r,\alpha}(p^1) \text{ using (18)}$$
$$\approx [m_{1,\alpha}(p^t)/m_{1,\alpha}(p^1)][1 + (½)(r-1)\mathrm{var}_\alpha(p^t/p_\alpha^{\ t})]/[1 + (½)(r-1)\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})]$$
$$= P_{D\alpha}^{\ t}\{1 + (½)(r-1)\Sigma_{n=1}^N \alpha_n[(p_{tn}/p_\alpha^{\ t}) - 1]^2\}/\{1 + (½)(r-1)\Sigma_{n=1}^N \alpha_n[(p_{1n}/p_\alpha^{\ 1}) - 1]^2\},$$

where we used definitions (4), (12), and (14) to establish the last equality in (19). Again, in the elementary index context with no diverging trends in prices, we could expect $\mathrm{var}_\alpha(p^t/p_\alpha^{\ t}) \approx \mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})$ for $t = 2, \ldots, T$. Using this assumption about the approximate constancy of the (weighted) variance of the deflated prices over time and using (16) and (19), we obtain the following approximations for $t = 2,3, \ldots, T$:

$$P_{CES\alpha,r}^{\ t} \approx P_{J\alpha}^{\ t} \approx P_{D\alpha}^{\ t}. \quad (20)$$

Thus, under the assumption of approximately *constant variances* for deflated prices, the CES, weighted Jevons, and weighted Dutot price indices should approximate each other fairly closely, provided that the same weighting vector α is used in the construction of these indices.[43]

The parameter r that appears in the definition of the CES unit cost function is related to the *elasticity of substitution* Σ; that is, it turns out that $\Sigma = 1 - r$.[44] Thus, as r takes on values from 1 to $-\infty$, Σ will take on values from 0 to $+\infty$. In the case where the products are closely related, typical estimates for Σ range from 1 to 10. If we substitute $\Sigma = 1 - r$ into the approximations (19), we obtain the following approximations for $t = 1, \ldots, T$:

$$P_{CES\alpha,r}^{\ t} \approx P_{D\alpha}^{\ t}\,[1 - (½)\Sigma\mathrm{var}_\alpha(p^t/p_\alpha^{\ t})]/[1 - (½)\Sigma\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})]. \quad (21)$$

The approximations in (21) break down for large and positive Σ (or equivalently, for very negative r); that is, the expressions in square brackets on the right-hand sides of (21)

[37] Note that the vectors $p^t/p_\alpha^{\ t}$ and $p^1/p_\alpha^{\ 1}$ are price vectors that are divided by their α-weighted arithmetic means. Thus, these vectors have eliminated general inflation between periods 1 and t.

[38] The same approximate inequalities hold for the weighted case. An approximation result similar to (16) for the equal weights case where $a = (1/N)1_N$ was first obtained by Carruthers, Sellwood, and Ward (1980, 25). See Diewert (2022b), equation (16).

[39] If we are allowed to change the units of measurement for the N products, then if we choose units of measurement that divide the price of product n in each period t by its price in period 1, then the transformed Dutot index becomes $P_D^{\ t*} \equiv \Sigma_{n=1}^N (p_{tn}/p_{1n})/\Sigma_{n=1}^N (p_{1n}/p_{1n}) = \Sigma_{n=1}^N (p_{tn}/p_{1n})/N$, which equals the Carli index, $P_C^{\ t}$. On the other hand, the transformed Jevons index becomes $P_J^{\ t*} \equiv \Pi_{n=1}^N (p_{tn}/p_{1n})^{1/N}/\Pi_{n=1}^N (p_{1n}/p_{1n})^{1/N} = \Pi_{n=1}^N (p_{tn}/p_{1n})^{1/N} = P_J^{\ t}$. Thus the Jevons index remains unchanged by the change of units but the Dutot index has become the Carli index $P_C^{\ t}$ which is always greater than the corresponding transformed Dutot index $P_D^{\ t*}$ using Schlomilch's inequality unless prices are proportional over the two periods, in which case, we have equality. Thus in general, we expect the transformed Dutot index to have an upward bias relative to the transformed Jevons index which is equal to the original Jevons index.

[40] If the products are not very similar, then the Dutot index should not be used since it is not invariant to changes in the units of measurement.

[41] Furthermore, as we shall see later, the Dutot index can be viewed as a fixed basket index where the basket is a vector of ones. Thus it is subject to substitution bias that will show up under the divergent price trends hypothesis.

[42] These restrictions imply that $m_{r,\alpha}(p)$ is a linearly homogeneous, nondecreasing, and concave function of the price vector p. These restrictions must be satisfied if we apply the economic approach to price index theory.

[43] Again, the approximate relationship $P_{CES\alpha,r}^{\ t} \approx P_{D\alpha}^{\ t}$ may not hold if the variance of the prices in the base period, $\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})$, is unusually large or small. Also under the diverging trends in price assumption, $\mathrm{var}_\alpha(p^t/p_\alpha^{\ t})$ will tend to increase relative to $\mathrm{var}_\alpha(p^1/p_\alpha^{\ 1})$ and the approximate equalities in (20) will become inequalities.

[44] See Feenstra (1994, 158) or equation (115) in Diewert (2022a).

will pass through 0 and become meaningless as s becomes very large. The approximations become increasingly accurate as Σ approaches 0 (or as r approaches 1). Of course, the approximations also become more accurate as the dispersion of prices within a period becomes smaller. For Σ between 0 and 1 and with "normal" dispersion of prices, the approximations in (21) should be reasonably good. However, as Σ becomes larger, the expressions in square brackets will become closer to 0, and the approximations in (21) will become more volatile and less accurate as Σ increases from an initial 0 value.

If the products in the aggregate are not very similar, it is more likely that there will be *divergent trends in prices* over time, and in this case, we can expect $\mathrm{var}_\alpha(p^t/p_\alpha{}^t)$ to exceed $\mathrm{var}_\alpha(p^1/p_\alpha{}^1)$. In this case, the approximate *equalities* (20) will no longer hold. In the case where the elasticity of substitution Σ is greater than 1 (so $r < 0$) and $\mathrm{var}_a(p^t/p_\alpha{}^t) > \mathrm{var}_\alpha(p^1/p_\alpha{}^1)$, we can expect that $P_{CES\alpha,r}{}^t < P_{D\alpha}{}^t$ and the gaps between these two indices will grow bigger over time as $\mathrm{var}_\alpha(p^t/p_\alpha{}^t)$ grows larger than $\mathrm{var}_\alpha(p^1/p_\alpha{}^1)$.

In the following section, we will use the mean of order r function to aggregate the price ratios p_{tn}/p_{1n} into an aggregate price index for period t directly; that is, we will not construct *price levels* as a preliminary step in the construction of a *price index*.

3. Using Means of Order *r* to Aggregate Price Ratios

In the previous section, we compared various elementary indices using approximate relationships between price levels constructed by using means of order r to construct the aggregate *price levels*. In this section, we will develop approximate relationships between price indices constructed by using means of order r to aggregate over *price ratios*.

In what follows, it is assumed that the weight vector α satisfies conditions (1); that is, $\alpha >> 0_N$ and $\alpha\cdot 1_N = 1$. Define the *mean of order r price index for period t* (relative to period 1), $P_{r,\alpha}{}^t$, as follows for $t = 1, \ldots, T$:

$$P_{r,\alpha}{}^t \equiv [\Sigma_{n=1}{}^N \alpha_n(p_{tn}/p_{1n})^r]^{1/r};\ r \neq 0; \qquad (22)$$
$$\equiv \prod_{n=1}{}^N (p_{tn}/p_{1n})\alpha_n;\ r = 0.$$

When $r = 1$ and $\alpha = (1/N)1_N$, then $P_{r,\alpha}{}^t$ becomes the *fixed-base Carli* (1804) *price index* (for period t relative to period 1), $P_C{}^t$, defined as follows for $t = 1, \ldots, T$:

$$P_C{}^t \equiv \Sigma_{n=1}{}^N (1/N)(p_{tn}/p_{1n}). \qquad (23)$$

With a general α and $r = 1$, $P_{r,a}{}^t$ becomes the *fixed-base weighted Carli price index*, $P_{C\alpha}{}^t$,[45] defined as follows for $t = 1, \ldots, T$:

$$P_{C\alpha}{}^t \equiv \Sigma_{n=1}{}^N \alpha_n(p_{tn}/p_{1n}). \qquad (24)$$

Using (24), it can be seen that the α-weighted mean of the period t long-term price ratios p_{tn}/p_{1n} divided by $P_{Ca}{}^t$ is equal to 1; that is, we have for $t = 1, \ldots, T$:

$$\Sigma_{n=1}{}^N \alpha_n(p_{tn}/p_{1n}P_{C\alpha}{}^t) = 1. \qquad (25)$$

Denote the α-weighted variance of the deflated period t price ratios $p_{tn}/p_{1n}P_{C\alpha}{}^t$ as $\mathrm{var}_\alpha(p^t/p^tP_{C\alpha}{}^t)$ and define it as follows for $t = 1, \ldots, T$:

$$\mathrm{var}_\alpha(p^t/p^1P_{C\alpha}{}^t) \equiv \Sigma_{n=1}{}^N \alpha_n[(p_{tn}/p_{1n}P_{C\alpha}{}^t) - 1]^2. \qquad (26)$$

Proposition 2: Let $p >> 0_N$, $\alpha >> 0_N$, and $\alpha\cdot 1_N = 1$. Then, $P_{r,\alpha}{}^t/P_{1,a}{}^t = P_{r,a}{}^t/P_{C\alpha}{}^t$ is approximately equal to the following expression for any r and $t = 1, \ldots, T$:

$$P_{r,\alpha}{}^t/P_{C\alpha}{}^t \approx 1 + (½)(r - 1)\mathrm{var}_\alpha(p^t/p^1P_{C\alpha}{}^t), \qquad (27)$$

where $P_{r,\alpha}{}^t$ is the mean of order r price index (with weights α) defined by (22), $P_{C\alpha}{}^t$ is the α-weighted Carli index defined by (24) and $\mathrm{var}_a(p^t/p^1P_{C\alpha}{}^t)$ is the α-weighted variance of the deflated long-term price ratios $(p_{tn}/p_{1n})/P_{C\alpha}{}^t$ defined by (26).

Proof: Replace the vector p in Proposition 1 by the vector $[p_{t1}/p_{11}, p_{t2}/p_{12}, \ldots, p_{tN}/p_{1N}]$.[46] Then the ratio $m_{r,\alpha}(p)/m_{1,\alpha}(p)$ which appears on the left-hand side of (5) becomes the ratio $P_{r,\alpha}{}^t/P_{1,a}{}^t = P_{r,a}{}^t/P_{C\alpha}{}^t$ using definitions (22) and (24). The terms p_α and $\mathrm{var}_\alpha(p/p_\alpha)$ which appear on the right-hand side of (5) become $P_{C\alpha}{}^t$ and $\mathrm{var}_\alpha(p^t/p^1P_{C\alpha}{}^t)$, respectively. With these substitutions, (5) becomes (27) and we have established Proposition 2. Q.E.D.

It is useful to look at the special case of (27) when $r = 0$. In this case, using definitions (22) and (15), we can establish the following equalities for $t = 1, \ldots, T$:

$$P_{0,\alpha}{}^t \equiv \prod_{n=1}{}^N (p_{tn}/p_{1n})^{\alpha_n} = P_{J\alpha}{}^t, \qquad (28)$$

where $P_{J\alpha}{}^t$ is the period *t weighted Jevons* or *Cobb–Douglas price index* defined by (15) in the previous section.[47] Thus, when $r = 0$, the approximations defined by (27) become the following approximations for $t = 1, \ldots, T$:

$$P_{J\alpha}{}^t/P_{C\alpha}{}^t \approx 1 - (½)\mathrm{var}_\alpha(p^t/p^1P_{C\alpha}{}^t). \qquad (29)$$

Thus, the bigger is the α-weighted variance of the deflated period t long-term price ratios, $(p_{t1}/p_{11})/P_{C\alpha}{}^t, \ldots, (p_{tn}/p_{1n})/P_{C\alpha}{}^t$, the more the period t-weighted Carli index $P_{C\alpha}{}^t$ will exceed the corresponding period t-weighted Jevons index $P_{J\alpha}{}^t$.

[45] This type of index was developed by Arthur Young (1812, 72), and so we could call this index the *Young index*, $P_{Y\alpha}{}^t$.

[46] In Proposition 1, some prices in either period could be 0. However, Proposition 2 requires that all period 1 prices be positive.

[47] Again, recall that Armknecht and Silver (2014, 4) call this index the geometric Young index.

When $\alpha = (1/N)1_N$, the approximations (29) become the following approximate relationships between the period t *Carli index* $P_C{}^t$ defined by (23) and the period t *Jevons index* $P_J{}^t$ defined by (11) for $t = 1, \ldots, T$:[48]

$$P_J{}^t/P_C{}^t \approx 1 - (½)\text{var}_{(1/N)1}(p^t/p^1P_C{}^t) = 1 - (½)\Sigma_{n=1}{}^N (1/N)[(p_{tn}/p_{1n}P_C{}^t) - 1]^2. \quad (30)$$

Thus, the Carli price indices $P_C{}^t$ will exceed their Jevons counterparts $P_J{}^t$ (unless $p^t = \lambda_t p^1$ in which case prices in period t are proportional to prices in period 1 and in this case, $P_C{}^t = P_J{}^t$).[49] This is an important result, since from an axiomatic perspective, the Jevons price index has much better properties than the corresponding Carli indices[50] and in particular, typically *chaining Carli indices will lead to large upward biases as compared to their Jevons counterparts*.

The results in this section can be summarized as follows: holding the weight vector a constant, the weighted Jevons price index for period t, $P_{J\alpha}{}^t$ will lie below the corresponding weighted Carli index, $P_{C\alpha}{}^t$, (unless all prices move in a proportional manner, in which case $P_{J\alpha}{}^t$ will equal $P_{C\alpha}{}^t$) with the gap growing as the α-weighted variance of the deflated price ratios, $(p_{t1}/p_{11})/P_{C\alpha}{}^t, \ldots, (p_{tn}/p_{1n})/P_{C\alpha}{}^t$, increases.[51]

In the following section, we turn our attention to weighted price indices where the weights are not exogenous constants but depend on observed sales or expenditure shares.

4. Relationships between Some Share-Weighted Price Indices

In this section (and in subsequent sections), we will look at comparisons between price indices that use information on the observed expenditure or sales shares of products in addition to price information. Recall that $s_{tn} \equiv p_{tn}q_{tn}/p^t \cdot q^t$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$.

The *fixed-base Laspeyres* (1871) *price index* for period t, $P_L{}^t$, is defined as the following base period share-weighted *arithmetic* average of the price ratios, p_{tn}/p_{1n}, for $t = 1, \ldots, T$:

$$P_L{}^t \equiv \Sigma_{n=1}{}^N s_{1n}(p_{tn}/p_{1n}). \quad (31)$$

It can be seen that $P_L{}^t$ is a weighted Carli index $P_{C\alpha}{}^t$ of the type defined by (24) in the previous section where $\alpha \equiv s^1 \equiv [s_{11}, s_{12}, \ldots, s_{1N}]$. We will compare $P_L{}^t$ with its weighted geometric mean counterpart $P_{GL}{}^t$, which is a weighted Jevons index $P_{J\alpha}{}^t$ where the weight vector is $\alpha = s^1$. Thus, the logarithm of the *fixed-base geometric Laspeyres price index* for $t = 1, \ldots, T$ is defined as follows:[52]

$$\ln P_{GL}{}^t \equiv \Sigma_{n=1}{}^N s_{1n} \ln(p_{tn}/p_{1n}). \quad (32)$$

Since $P_{GL}{}^t$ and $P_L{}^t$ are weighted geometric and arithmetic means of the price ratios p_{tn}/p_{1n} (using the weights in the period 1 share vector s^1), Schlömilch's inequality implies that $P_{GL}{}^t \le P_L{}^t$ for $t = 1, \ldots, T$. The inequalities (29), with $\alpha = s^1$, give us approximations to the gaps between $P_{GL}{}^t = P_{J\alpha}{}^t$ and $P_{C\alpha}{}^t = P_L{}^t$. Thus, we have following approximate equalities for $\alpha = s^1$ and $t = 1, \ldots, T$:

$$P_{GL}{}^t/P_L{}^t \approx 1 - (½)\text{var}_\alpha(p^t/p^1P_L{}^t) = 1 - (½)\Sigma_{n=1}{}^N s_{1n}[(p_{tn}/p_{1n}P_L{}^t) - 1]^2. \quad (33)$$

The *fixed-base Paasche* (1874) *price index* for period t, $P_P{}^t$, is defined as the following period t share-weighted *harmonic* average of the price ratios, p_{tn}/p_{1n}, for $t = 1, \ldots, T$:

$$P_P{}^t \equiv [\Sigma_{n=1}{}^N s_{tn}(p_{tn}/p_{1n})^{-1}]^{-1}. \quad (34)$$

We will compare $P_P{}^t$ with its weighted geometric mean counterpart $P_{GP}{}^t$, which is a weighted Jevons index $P_{J\alpha}{}^t$ where the weight vector is $\alpha = s^t$. The logarithm of the *fixed-base geometric Paasche price index* for $t = 1, \ldots, T$ *is defined as follows*:

$$\ln P_{GP}{}^t \equiv \Sigma_{n=1}{}^N s_{tn} \ln(p_{tn}/p_{1n}). \quad (35)$$

Since $P_{GP}{}^t$ and $P_P{}^t$ are weighted geometric and harmonic means of the price ratios p_{tn}/p_{1n} (using the weights in the period t share vector s^t), Schlömilch's inequality implies that $P_P{}^t \le P_{GP}{}^t$ for $t = 1, \ldots, T$. However, we cannot apply the inequalities (29) directly to give us an approximation to the *size* of the gap between $P_{GP}{}^t$ and $P_P{}^t$. According to definition (34), the reciprocal of $P_P{}^t$ is a period t share-weighted average of the reciprocals of the long-term price ratios, $p_{11}/p_{t1}, p_{12}/p_{t2}, \ldots, p_{1N}/p_{tN}$. Thus, using definition (34), we have the following equations and inequalities for $\alpha = s^t$ and $t = 1, \ldots, T$:

$$\begin{aligned}[P_P{}^t]^{-1} &= \Sigma_{n=1}{}^N s_{tn}(p_{1n}/p_{tn}) \\ &\ge \Pi_{n=1}{}^N (p_{1n}/p_{tn})^{s_{tn}} \\ &= [P_{GP}{}^t]^{-1} \text{ using definition (35),}\end{aligned} \quad (36)$$

where the inequalities follow from Schlömilch's inequality; that is, a weighted arithmetic mean is always equal to or greater than the corresponding weighted geometric mean. Note that the first equation in (36) implies that the period t share-weighted mean of the reciprocal price

[48] Results that are essentially equivalent to (30) were first obtained by Dalén (1992) and Diewert (1995). The approximations in (27) and (29) for weighted indices are new. vartia and Suoperä (2018, 5) derived alternative approximations. The analysis in this section is similar to vartia's (1978, 276–89) analysis of Fisher's (1922) five-tined fork.

[49] From Schlömilch's inequality, we know that P_C is always equal to or greater than P_J; the approximate result (30) provides an indication of the size of the gap between the two indices.

[50] See Diewert (1995, 2022b) and Reinsdorf (2007) on the axiomatic approach to equally weighted elementary indices. The Jevons index emerges as the best index from the viewpoint of the axiomatic approach.

[51] Since the Jevons price index has the best axiomatic properties, this result implies that CPI compilers should avoid the use of the Carli index in the construction of a CPI. This advice goes back to Fisher (1922, 29–30). Since the Dutot index will approximate the corresponding Jevons index provided that the products are similar and there are no systematic divergent trends in prices, Dutot indices can be satisfactory at the elementary level. If the products are not closely related, Dutot indices become problematic since they are not invariant to changes in the units of measurement. Moreover, in the case of nonsimilar products, divergent trends in prices become more probable and, using (16), the Dutot index will tend to be above the corresponding Jevons index.

[52] vartia (1978, 272) used the terms "geometric Laspeyres" and "geometric Paasche" to describe the indices defined by (32) and (35).

ratios, p_{1n}/p_{tn}, is equal to the reciprocal of $P_P^{\,t}$. Now adapt the approximate equalities (29) in order to establish the following approximate equalities for $t = 1, \ldots, T$:

$$[P_{GP}^{\,t}]^{-1}/[P_P^{\,t}]^{-1} \approx 1 - (\tfrac{1}{2})\Sigma_{n=1}^{N} s_{tn}[(p_{1n}/p_{tn}[P_P^{\,t}]^{-1}) - 1]^2. \quad (37)$$

The approximate equalities (37) for $t = 1, \ldots, T$ may be rewritten as follows:

$$P_{GP}^{\,t} \approx P_P^{\,t}/\{1 - (\tfrac{1}{2})\Sigma_{n=1}^{N} s_{tn}[(p_{1n}P_P^{\,t}/p_{tn}) - 1]^2\}. \quad (38)$$

Thus, for $t = 1, \ldots, T$, we have $P_{GP}^{\,t} \geq P_P^{\,t}$ (and the approximate equalities (38) measure the gaps between these indices) and $P_{GL}^{\,t} \leq P_L^{\,t}$ (and the approximate equalities (33) measure the gaps between these indices). Later we will show that the inequalities $P_{GP}^{\,t} \leq P_{GL}^{\,t}$ are likely if the N products are close substitutes for each other.

Suppose that prices in period t are proportional to the corresponding prices in period 1 so that $p^t = \lambda_t p^1$ where λ_t is a positive scalar. Then it is straightforward to show that $P_P^{\,t} = P_{GP}^{\,t} = P_{GL}^{\,t} = P_L^{\,t} = \lambda_t$ and the implicit error terms for equation t in (33) and (38) are equal to 0.

Define the period t fixed-base Fisher (1922) and Törnqvist–Theil price indices, $P_F^{\,t}$ and $P_T^{\,t}$, as the following geometric means for $t = 1, \ldots, T$:

$$P_F^{\,t} \equiv [P_L^{\,t}\, P_P^{\,t}]^{1/2}; \quad (39)$$
$$P_T^{\,t} \equiv [P_{GL}^{\,t}\, P_{GP}^{\,t}]^{1/2}. \quad (40)$$

Thus, $P_F^{\,t}$ is the geometric mean of the period t fixed-base Laspeyres and Paasche price indices while $P_T^{\,t}$ is the geometric mean of the period t fixed-base geometric Laspeyres and geometric Paasche price indices. Now use the approximate equalities in (33) and (38) and substitute these equalities into (40) in order to obtain the following approximate equalities between $P_T^{\,t}$ and $P_F^{\,t}$ for $t = 1, \ldots, T$:

$$\begin{aligned} P_T^{\,t} &\equiv [P_{GL}^{\,t}\, P_{GP}^{\,t}]^{1/2} \quad (41)\\ &\approx [P_L^{\,t}\, P_P^{\,t}]^{1/2}\, \varepsilon(p^1,p^t,s^1,s^t)\\ &= P_F^{\,t}\, \varepsilon(p^1,p^t,s^1,s^t), \end{aligned}$$

where the approximation error function $\varepsilon(p^1,p^t,s^1,s^t)$ for $t = 1, \ldots, T$ *is defined as follows*:

$$\varepsilon(p^1,p^t,s^1,s^t) \equiv \{1 - (\tfrac{1}{2})\Sigma_{n=1}^{N} s_{1n}[(p_{tn}/p_{1n}P_L^{\,t}) - 1]^2\}^{1/2} / \{1 - (\tfrac{1}{2})\Sigma_{n=1}^{N} s_{tn}[(p_{1n}P_P^{\,t}/p_{tn}) - 1]^2\}^{1/2}. \quad (42)$$

Thus, $P_T^{\,t}$ is approximately equal to $P_F^{\,t}$ for $t = 1, \ldots, T$. But how good are these approximations? We know from Diewert (1978) that $P_T^{\,t} = P_T(p^1,p^t,s^1,s^t)$ approximates $P_F^{\,t} = P_F(p^1,p^t,s^1,s^t)$ to the second order around any point where $p^t = p^1$ and $s^t = s^1$.[53] Since the approximations in (33) and (38) are also second order approximations, it is likely that the approximation given by (41) is fairly good.[54]

In general, if the products are highly substitutable and if prices and shares trend in opposite directions, then we expect that the base period share-weighted variance $\Sigma_{n=1}^{N} s_{1n}[(p_{tn}/p_{1n}P_L^{\,t}) - 1]^2$ and the current period share-weighted variance $\Sigma_{n=1}^{N} s_{tn}[(p_{1n}P_P^{\,t}/p_{tn}) - 1]^2$ will increase as t increases. It appears that the second variance expression increases more than the first one because the change in expenditure shares from s_{1n} to s_{tn} tends to magnify the squared differences $[(p_{1n}P_P^{\,t}/p_{tn}) - 1]^2$. Thus, as say p_{tn} increases and the difference $(p_{1n}P_P^{\,t}/p_{tn}) - 1$ decreases, the share s_{tn} will become smaller, and this decreasing share weight s_{tn} will lead to a further shrinkage of the term $s_{tn}[(p_{1n}P_P^{\,t}/p_{tn}) - 1]^2$. On the other hand, if p_{tn} decreases substantially, the difference $(p_{1n}P_P^{\,t}/p_{tn}) - 1$ will substantially increase and the share s_{tn} will become larger, and this increasing share weight s_{tn} will further magnify the term $s_{tn}[(p_{1n}P_P^{\,t}/p_{tn}) - 1]^2$. For large changes in prices, the magnification effects will tend to be more important than the shrinkage effects of changing expenditure shares. This overall *share magnification effect* does not occur for the base period share-weighted variance $\Sigma_{n=1}^{N} s_{1n}[(p_{tn}/p_{1n}P_L^{\,t}) - 1]^2$. Thus, if the products are highly substitutable and there are large divergent trends in prices, P_T will tend to increase relative to P_F as time increases under these conditions. The more substitutable the products are, the greater will be this tendency.

Our tentative conclusion at this point is that the approximations defined by (33), (38) and (41) are good enough to provide rough estimates of the differences in the six price indices involved in these approximate equalities. In an empirical example using scanner data, Diewert (2018) found that the variance terms on the right-hand sides of (38) tended to be larger than the corresponding variances on the right-hand sides of (33) and these differences led to a tendency for the fixed-base Fisher price indices $P_F^{\,t}$ to be slightly smaller than the corresponding fixed-base Törnqvist–Theil price indices $P_T^{\,t}$.[55]

We conclude this section by developing an exact relationship between the geometric Laspeyres and Paasche price indices. Using definitions (32) and (35) for the logarithms of these indices, we have the following exact decomposition for the logarithmic difference between these indices for $t = 1, \ldots, T$:[56]

$$\begin{aligned} lnP_{GP}^{\,t} - lnP_{GL}^{\,t} &= \Sigma_{n=1}^{N} s_{tn}\, ln(p_{tn}/p_{1n}) - \Sigma_{n=1}^{N} s_{1n}\, ln(p_{tn}/p_{1n})\\ &= \Sigma_{n=1}^{N} [s_{tn} - s_{1n}][lnp_{tn} - lnp_{1n}]. \quad (43) \end{aligned}$$

[53] This result can be generalized to the case where $p^t = \lambda p^1$ and $s^t = s^1$.

[54] However, the Diewert (1978) second-order approximation is different from the present second-order approximations that are derived from Proposition 2. Thus, the closeness of $e(p^1,p^t,s^1,s^t)$ to 1 depends on the closeness of the Diewert second-order approximation of $P_T^{\,t}$ to $P_F^{\,t}$ and the closeness of the second-order approximations that were used in (33) and (38), which use different Taylor series approximations. vartia and Suoperä (2018) used alternative Taylor series approximations to obtain relationships between various indices.

[55] vartia and Suoperä (2018) also found a tendency for the Fisher price index to lie slightly below their Törnqvist counterparts in their empirical work.

[56] vartia and Suoperä (2018, 26) derived this result and noticed that the right-hand side of (43) could be interpreted as a covariance. They also developed several alternative exact decompositions for the difference $lnP_{GP}^{\,t} - lnP_{GL}^{\,t}$. Their paper also developed a new theory of "excellent" index numbers.

Define the vectors $lnp^t \equiv [lnp_{t1}, lnp_{t2}, \ldots, lnp_{tN}]$ for $t = 1, \ldots, T$. It can be seen that the right-hand side of equation t in (43) is equal to $[s^t - s^1]\cdot[lnp^t - lnp^1]$, the inner product of the vectors $x \equiv s^t - s^1$ and $y \equiv lnp^t - lnp^1$. Let x^* and y^* denote the arithmetic means of the components of the vectors x and y. Note that $x^* \equiv (1/N)1_N\cdot x = (1/N)1_N\cdot[s^t - s^1] = (1/N)[1 - 1] = 0$. The covariance between x and y is defined as $\mathrm{Cov}(x,y) \equiv (1/N)[x - x^*1_N]\cdot[y - y^*1_N] = (1/N)\, x\cdot y - x^*y^* = (1/N)\, x\cdot y$[57] since x^* is equal to 0. Thus, the right-hand side of (43) is equal to $N\,\mathrm{Cov}(x,y) = N\,\mathrm{Cov}(s^t - s^1, lnp^t - lnp^1)$; that is, the right-hand side of (43) is equal to N times the covariance of the long-term share difference vector, $s^t - s^1$, with the long-term log price difference vector, $lnp^t - lnp^1$. Hence if this covariance is positive, then $lnP_{GP}{}^t - lnP_{GL}{}^t > 0$ and $P_{GP}{}^t > P_{GL}{}^t$. If this covariance is negative, then $P_{GP}{}^t < P_{GL}{}^t$. We argue here that for the case where the N products are close substitutes, it is likely that the covariances on the right-hand side of equations (43) are negative for $t > 1$.

Suppose that the observed price and quantity data are approximately consistent with purchasers having identical Constant Elasticity of Substitution preferences. CES preferences are dual to the CES unit cost function $m_{r,\alpha}(p)$, which is defined by (2), where α satisfies (1) and $r \le 1$. It can be shown[58] that the sales share for product n in a period where purchasers face the strictly positive price vector $p \equiv [p_1, \ldots, p_N]$ is the following share:

$$s_n(p) \equiv \alpha_n p_n{}^r/\Sigma_{i=1}{}^N \alpha_i p_i{}^r;\ n = 1, \ldots, N. \tag{44}$$

Upon differentiating $s_n(p)$ with respect to p_n, we find that the following relations hold:

$$\partial lns_n(p)/\partial lnp_n = r[1 - s_n(p)];\ n = 1, \ldots, N. \tag{45}$$

Thus, $\partial \ln s_n(p)/\partial lnp_n < 0$ if $r < 0$ (or equivalently, if the elasticity of substitution $\Sigma \equiv 1 - r$ is greater than 1) and $\partial \ln s_n(p)/\partial lnp_n > 0$ if r satisfies $0 < r < 1$ (or equivalently, if the elasticity of substitution satisfies $0 < \Sigma < 1$).[59] If we are aggregating prices at the first stage of aggregation where the products are close substitutes and purchasers have common CES preferences, then it is likely that the elasticity of substitution is greater than 1 and hence as the price of product n decreases, it is likely that the share of that product will increase. Hence we expect the terms $[s_{tn} - s_{1n}][lnp_{tn} - lnp_{1n}]$ to be predominantly negative; that is, if p_{1n} is unusually low, then $lnp_{tn} - lnp_{1n}$ is likely to be positive and $s_{tn} - s_{1n}$ is likely to be negative. On the other hand, if p_{tn} is unusually low, then $lnp_{tn} - lnp_{1n}$ is likely to be negative and $s_{tn} - s_{1n}$ is likely to be positive. Thus, for closely related products, we expect the covariances on the right-hand sides of (43) to be negative and for $P_{GP}{}^t$ to be less than $P_{GL}{}^t$. We can combine this inequality with our previously established inequalities to conclude that for closely related products, it is likely that $P_P{}^t < P_{GP}{}^t < P_T{}^t < P_{GL}{}^t < P_L{}^t$. On the other hand, if we are aggregating at higher levels of aggregation, then it is likely that the elasticity of substitution is in the range $0 < \Sigma < 1$,[60] and in this case, the covariances on the right-hand sides of (43) will tend to be positive and hence in this case, it is likely that $P_{GP}{}^t > P_{GL}{}^t$. We also have the inequalities $P_P{}^t < P_{GP}{}^t$ and $P_{GL}{}^t < P_L{}^t$ in this case.[61]

We turn now to some relationships between weighted and unweighted (that is, equally weighted) geometric price indices.

5. Relationships between the Jevons, geometric Laspeyres, geometric Paasche, and Törnqvist Price Indices

In this section, we will investigate how close the unweighted Jevons index $P_J{}^t$ is to the geometric Laspeyres $P_{GL}{}^t$, geometric Paasche $P_{GP}{}^t$, and Törnqvist $P_T{}^t$ price indices.

We first investigate the difference between the logarithms of $P_{GL}{}^t$ and $P_J{}^t$. Using the definitions for these indices, we have the following log differences for $t = 1, \ldots, T$:

$$\begin{aligned} lnP_{GL}{}^t - lnP_J{}^t &= \Sigma_{n=1}{}^N [s_{1n} - (1/N)][lnp_{tn} - lnp_{1n}] \\ &= N\mathrm{Cov}(s^1 - (1/N)1_N,\ lnp^t - lnp^1) \equiv \varepsilon_t. \end{aligned} \tag{46}$$

In the elementary index context where the N products are close substitutes and product shares in period 1 are close to being equal, it is likely that ε_t is positive; that is, if $\ln p_{1n}$ is unusually low, then s_{1n} is likely to be unusually high and thus it is likely that $s_{1n} - (1/N) > 0$ and $lnp_{tn} - lnp_{1n}$ minus the mean of the log ratios $\ln(p_{tn}/p_{1n})$ is likely to be greater than 0, and hence ε_t is likely to be greater than 0, implying that $P_{GL}{}^t > P_J{}^t$. However, if N is small and the shares have a high variance and if product n goes on sale in period 1, then we cannot assert that s_{1n} is likely to be greater than $1/N$, and hence we cannot be confident that ε_t is likely to be greater than 0 and hence we cannot predict with certainty that $P_{GL}{}^t$ will be greater than $P_J{}^t$.

There are three simple sets of conditions that will imply that $P_{GL}{}^t = P_J{}^t$: (i) the covariance on the right-hand side of (46) equals 0; that is, $\mathrm{Cov}(s^1 - (1/N)1_N,\ lnp^t - lnp^1) = 0$; (ii) period t price proportionality; that is, $p^t = \lambda_t p^1$ for some $l_t > 0$; (iii) equal sales shares in period 1; that is, $s^1 = (1/N)1_N$.

Now look at the difference between the logarithms of $P_{GP}{}^t$ and $P_J{}^t$. Using the definitions for these indices, for $t = 1, \ldots, T$, we have:

$$\begin{aligned} lnP_{GP}{}^t - lnP_J{}^t &= \Sigma_{n=1}{}^N [s_{tn} - (1/N)][lnp_{tn} - lnp_{1n}] \\ &= N\mathrm{Cov}(s^t - (1/N)1_N,\ lnp^t - lnp^1) \equiv c_t. \end{aligned} \tag{47}$$

[57] This equation is the *covariance identity* that was first used by Bortkiewicz (1923) to show that normally the Paasche price index is less than the corresponding Laspeyres index.

[58] See equations (110) in Diewert (2022a) or Diewert and Feenstra (2017).

[59] Thus, define product n as a *strong substitute* compared with all other products if $\partial \ln s_n(p)/\partial lnp_n < 0$ and as a *weak substitute* if $\partial \ln s_n(p)/\partial lnp_n > 0$.

[60] See Shapiro and Wilcox (1997), who found that $\Sigma = 0.7$ fit the US data well at higher levels of aggregation. See also Armknecht and Silver (2014, 9), who noted that estimates for Σ tend to be greater than 1 at the lowest level of aggregation and less than 1 at higher levels of aggregation.

[61] See vartia (1978, 276–90) for a similar discussion about the relationships between $P_L{}^t$, $P_P{}^t$, $P_F{}^t$, $P_{GL}{}^t$, $P_{GP}{}^t$, and $P_T{}^t$. vartia extended the discussion to include period 1 and period t share-weighted harmonic averages of the price ratios, p_{tn}/p_{1n}. See also Armknecht and Silver (2014, 10) for a discussion on how weighted averages of the above indices could approximate a superlative index at higher levels of aggregation.

In the elementary index context where the N products are close substitutes and the shares s^t are close to being equal, then it is likely that η_t is negative; that is, if $\ln p_{tn}$ is unusually low, then s_{tn} is likely to be unusually high and thus it is likely that $s_{tn} - (1/N) > 0$ and $lnp_{tn} - lnp_{1n}$ minus the mean of the log ratios $\ln(p_{tn}/p_{1n})$ is likely to be less than 0 and hence h_t is likely to be less than 0 implying that $P_{GP}{}^t < P_J{}^t$. However if N is small and the period t shares s^t are not close to being equal, then again, we cannot confidently predict the sign of the covariance in (47).

Again, there are three simple sets of conditions that will imply that $P_{GP}{}^t = P_J{}^t$: (i) the covariance on the right-hand side of (47) equals 0; that is, $\mathrm{Cov}(s^t - (1/N)1_N, lnp^t - lnp^1) = 0$; (ii) period t price proportionality; that is, $p^t = \lambda_t p^1$ for some $1_t > 0$; and (iii) equal sales shares in period t; that is, $s^t = (1/N)1_N$.

Using the definitions for $P_T{}^t$ and $P_J{}^t$, the log difference between these indices for $t = 1, \ldots, T$ is as follows:

$$\begin{aligned} lnP_T{}^t - lnP_J{}^t &= \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][lnp_{tn} - lnp_{1n}] \\ &= N\mathrm{Cov}[(½)s^t + (½)s^1 - (1/N)1_N, lnp^t - lnp^1] \\ &= (N/2)\mathrm{Cov}(s^t - (1/N)1_N, lnp^t - lnp^1) + (N/2) \\ &\quad \mathrm{Cov}(s^1 - (1/N)1_N, lnp^t - lnp^1) \\ &= (½)\varepsilon_t + (½)\eta_t. \end{aligned} \tag{48}$$

As usual, there are three simple sets of conditions that will imply that $P_T{}^t = P_J{}^t$: (i) the covariance on the right-hand side of (48) equals 0; that is, $\mathrm{Cov}[(½)s^t + (½)s^1 - (1/N)1_N, lnp^t - lnp^1] = 0 = (½)\varepsilon_t + (½)\eta_t$ or equivalently, $\mathrm{Cov}(s^t - (1/N)1_N, lnp^t - lnp^1) = -\mathrm{Cov}(s^1 - (1/N)1_N, lnp^t - lnp^1)$; (ii) period t price proportionality; that is, $p^t = 1_t p^1$ for some $\lambda_t > 0$; and (iii) the arithmetic average of the period 1 and t sales shares are all equal to $1/N$; that is, $(½)s^t + (½)s^1 = (1/N)1_N$.

If the trend deflated prices $p_{tn}/1_t$ are distributed *independently* across time and *independently* of the sales shares s_{tn}, then it can be seen that the expected values of the ε_t and η_t will be 0 and hence $P_T{}^t \approx P_J{}^t$ for $t = 1, \ldots, T$. Thus, it can be the case that the ordinary Jevons price index is able to provide an adequate approximation to the superlative Törnqvist price index in the elementary price index context. However, if the shares are trending and if prices are trending in divergent directions, then $P_J{}^t$ will not be able to approximate $P_T{}^t$.

In the general case, we expect $P_T{}^t$ to be less than $P_J{}^t$. The mean of the average shares for product n in periods 1 and t, $(½)s_{tn} + (½)s_{1n}$, is $1/N$. Define the means of the log prices in period t as $lnp_t{}^* \equiv (1/N)\Sigma_{n=1}{}^N lnp_{tn}$ for $t = 1, \ldots, T$. Note that $p_t{}^*$ is the geometric mean of the period t prices. Thus, using the first line of (48) and the covariance identity, we have:

$$\begin{aligned} lnP_T{}^t - lnP_J{}^t &= \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][lnp_{tn} - lnp_{1n}] \\ &= \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][lnp_{tn} - lnp_{1n} - lnp_t{}^* + lnp_1{}^*] \\ &= \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)][ln(p_{tn}/p_t{}^*) \\ &\quad - ln(p_{1n}/p_1{}^*)]. \end{aligned} \tag{49}$$

The second line in (49) follows from the first line because $\Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - (1/N)] = 0$ and so if these N terms are multiplied by a constant, the resulting sum of terms will still equal 0. Define the *deflated price* for product n in period t as $p_{tn}/p_t{}^*$ for $t = 1, \ldots, T$. Assume that the products are highly substitutable. Suppose that the deflated price of product n goes down between periods 1 and t so that $\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)$ is negative. Under these conditions, there will be a tendency for the average expenditure share for product n, $(½)s_{tn} + (½)s_{1n}$, to be greater than the average of these shares, which is $1/N$. Thus, the term $[(½)s_{tn} + (½)s_{1n} - (1/N)][\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)]$ is *likely* to be negative. Now suppose that the deflated price of product n goes up between periods 1 and t so that $\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)$ is positive. Under these conditions, there will be a tendency for the average expenditure share for product n, $(½)s_{tn} + (½)s_{1n}$, to be less than the average of these shares. Again, the term $[(½)s_{tn} + (½)s_{1n} - (1/N)][\ln(p_{tn}/p_t{}^*) - \ln(p_{1n}/p_1{}^*)]$ is *likely* to be negative. Thus, if the products under consideration are highly substitutable, we expect $P_T{}^t$ to be less than $P_J{}^t$.[62] If the products are not highly substitutable, we expect $P_T{}^t$ to be greater than $P_J{}^t$.

The results in this section can be summarized as follows: The unweighted Jevons index, $P_J{}^t$, can provide a reasonable approximation to a fixed-base superlative index like $P_T{}^t$ *provided* that the expenditure shares do not systematically trend with time and prices do not systematically grow at diverging rates. If these assumptions are not satisfied, then it is likely that the Jevons index will have some bias relative to a superlative index; $P_J{}^t$ is likely to exceed $P_T{}^t$ as t becomes large if the products are close substitutes and $P_J{}^t$ is likely to be less than $P_T{}^t$ if the products are not close substitutes.

6. Relationships between Superlative Fixed-Base Indices and Geometric Indices That Use Average Annual Shares as Weights

We consider the properties of weighted Jevons indices where the weight vector is an *annual average* of the observed monthly shares in a previous year. Recall that the weighted Jevons (or Cobb Douglas) price index $P_{J\alpha}{}^t$ was defined by (15) in Section 2 as $P_{J\alpha}{}^t \equiv \prod_{n=1}{}^N (p_{tn}/p_{1n})\alpha_n$ where the product weighting vector α satisfied the conditions $\alpha >> 0_N$ and $\alpha \cdot 1_N = 1$. The following counterparts to the covariance identities (46)–(48) hold for $t = 1, \ldots, T$ where the geometric Young index or weighted Jevons index $P_{J\alpha}{}^t$ has replaced $P_J{}^t$:[63]

$$\begin{aligned} lnP_{GL}{}^t - lnP_{J\alpha}{}^t &= \Sigma_{n=1}{}^N [s_{1n} - \alpha_n][lnp_{tn} - lnp_{1n}] \\ &= N\mathrm{Cov}(s^1 - \alpha, lnp^t - lnp^1); \end{aligned} \tag{50}$$

$$\begin{aligned} lnP_{GP}{}^t - lnP_{J\alpha}{}^t &= \Sigma_{n=1}{}^N [s_{tn} - \alpha_n][lnp_{tn} - lnp_{1n}] \\ &= N\mathrm{Cov}(s^t - \alpha, lnp^t - lnp^1); \end{aligned} \tag{51}$$

$$lnP_T{}^t - lnP_{J\alpha}{}^t = \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{1n} - \alpha_n][lnp_{tn} - lnp_{1n}]$$

[62] This is perhaps an important result in the context where a statistical agency is collecting web scraped prices for very similar products and using an equally weighted geometric mean of these scraped prices as an estimated elementary price level. The resulting Jevons price index may have an upward bias relative to its superlative counterpart.

[63] The relationship (52) was obtained by Armknecht and Silver (2014, 9); that is, by taking logarithms on both sides of their equation (12), we obtain the first equation in equations (52).

$$= N\text{Cov}[(½)s^t + (½)s^1 - \alpha, lnp^t - lnp^1]$$
$$= (½)[lnP_{GL}{}^t - lnP_{J\alpha}{}^t] + (½)[lnP_{GP}{}^t - lnP_{J\alpha}{}^t]. \quad (52)$$

Define α as the *arithmetic average of the first T^* observed share vectors s^t*:

$$\alpha \equiv \Sigma_{t=1}{}^{T^*} (1/T^*)s^t. \quad (53)$$

In the context where the data consist of monthly periods, T^* will typically be equal to 12; that is, the elementary index under consideration is the weighted Jevons index $P_{J\alpha}{}^t$ where the weight vector a is the average of the observed expenditure shares for the first 12 months in the sample.

The decompositions (50)–(52) will hold for α defined by (53). If the N products are highly substitutable, it is likely that $\text{Cov}(s^1 - a, lnp^t - lnp^1) > 0$ and $\text{Cov}(s^t - \alpha, lnp^t - lnp^1) < 0$ and hence it is likely that $P_{GL}{}^t > P_{J\alpha}{}^t$ and $P_{GP}{}^t < P_{J\alpha}{}^t$. If the products are not close substitutes, then it is likely that $P_{GL}{}^t < P_{J\alpha}{}^t$ and $P_{GP}{}^t > P_{J\alpha}{}^t$. If there are no divergent trends in prices, then it is possible that the *average share price index* $P_{J\alpha}{}^t$ could provide an adequate approximation to the superlative Törnqvist index $P_T{}^t$.

Note $t = 1, \ldots, T$ in equations (50)–(52). However, annual share indices that are implemented by statistical agencies are not constructed in exactly this manner. The practical month-to-month indices that are constructed by statistical agencies using annual shares of the type defined by (53) do not choose the reference month for prices to be month 1; rather they chose the reference month for prices to be $T^* + 1$, the month that follows the first year.[64] Thus, the reference *year* for share weights precedes the reference *month* for prices. In this case, the logarithm of the month $t \geq T^* + 1$ annual share-weighted Jevons index, $lnP_{J\alpha}{}^t$, is defined as follows:

$$lnP_{J\alpha}{}^t \equiv \Sigma_{n=1}{}^N \alpha_n[lnp_{tn} - lnp_{T^*+1,n}];$$
$$t = T^* + 1, T^* + 2, \ldots, T, \quad (54)$$

where α is the vector of annual average share weights defined by (53). The following counterparts to the identities (50)–(52) hold for $t = T^* + 1, T^* + 2, \ldots, T$, where α is defined by (53) and $P_{J\alpha}{}^t$ is defined by (54):

$$lnP_{GL}{}^t - lnP_{J\alpha}{}^t = \Sigma_{n=1}{}^N [s_{T^*+1,n} - \alpha_n][lnp_{tn} - lnp_{T^*+1,n}]$$
$$= N\text{Cov}(s^{T^*+1} - \alpha, lnp^t - lnp^{T^*+1}); \quad (55)$$

$$lnP_{GP}{}^t - lnP_{J\alpha}{}^t = \Sigma_{n=1}{}^N [s_{tn} - \alpha_n][lnp_{tn} - lnp_{T^*+1,n}]$$
$$= N\text{Cov}(s^t - \alpha, lnp^t - lnp^{T^*+1}); \quad (56)$$

$$lnP_T{}^t - lnP_{J\alpha}{}^t = \Sigma_{n=1}{}^N [(½)s_{tn} + (½)s_{T^*+1,n} - \alpha_n][lnp_{tn} - lnp_{T^*+1,n}]$$
$$= N\text{Cov}[(½)s^t + (½)s^{T^*+1} - \alpha, lnp^t - lnp^{T^*+1}]$$
$$= (½)[lnP_{GL}{}^t - lnP_{Ja}{}^t] + (½)[lnP_{GP}{}^t - lnP_{Ja}{}^t]. \quad (57)$$

If the N products are highly substitutable, it is likely that $\text{Cov}(s^{T^*+1} - \alpha, lnp^t - lnp^{T^*+1}) > 0$ so that $P_{GL}{}^t > P_{J\alpha}{}^t$. It is also likely that $\text{Cov}(s^t - \alpha, lnp^t - lnp^{T^*+1}) < 0$ and hence it is likely that $P_{GP}{}^t < P_{J\alpha}{}^t$ in the highly substitutable case. If the products are not close substitutes, then it is likely that $P_{GL}{}^t < P_{J\alpha}{}^t$ and $P_{GP}{}^t > P_{J\alpha}{}^t$. If there are no divergent trends in prices, then it is possible that the *average share price index* $P_{J\alpha}{}^t$ could provide an adequate approximation to the superlative Törnqvist index $P_T{}^t$. However, if there are divergent trends in prices and shares and the products are highly substitutable with each other, then we expect the covariance in (56) to be more negative than the covariance in (55) is positive so that $P_T{}^t$ will tend to be less than the annual shares geometric index $P_{J\alpha}{}^t$. Thus, $P_{J\alpha}{}^t$ will tend to have a slight substitution bias if the products are highly substitutable, which is an intuitively plausible result.

As usual, there are three simple sets of conditions that will imply that $P_T{}^t = P_{J\alpha}{}^t$: (i) the covariance on the right-hand side of (57) equals 0; that is, $\text{Cov}[(½)s^t + (½)s^{T^*+1} - \alpha, lnp^t - lnp^{T^*+1}] = 0$ or equivalently, $\text{Cov}(s^{T^*+1} - \alpha, lnp^t - lnp^{T^*+1}) = -\text{Cov}(s^t - \alpha, lnp^t - lnp^{T^*+1})$; (ii) period t price proportionality (to the prices of the price reference period); that is, $p^t = l_t p^{T^*+1}$ for some $\lambda_t > 0$; (iii) the arithmetic average of the period $T^* + 1$ and t sales shares are all equal to α defined by (53); that is, $(½)s^t + (½)s^{T^*+1} = \alpha$. This last condition will hold if the shares s^t are constant over all time periods and a is defined by (53).

Suppose that there are *linear trends in shares* and *divergent linear trends in log prices*; that is, suppose that the following assumptions hold for $t = 2,3, \ldots, T$:

$$s^t = s^1 + \beta(t-1); \quad (58)$$
$$lnp^t = lnp^1 + \gamma(t-1); \quad (59)$$

where $\beta \equiv [\beta_1, \ldots, b_N]$ and $\gamma \equiv [\gamma_1, \ldots, g_N]$ are constant vectors and b satisfies the additional restriction:[65]

$$\beta \cdot 1_N = 0. \quad (60)$$

In the case where the products are highly substitutable, if the price of product n, p_{tn}, is trending upward so that γ_n is positive, then we could expect that the corresponding share s_{tn} is trending downward so that β_n is negative. Similarly, if γ_n is negative, then we expect that the corresponding β_n is positive. Thus, we expect that $\Sigma_{n=1}{}^N \beta_n\gamma_n = \beta \cdot \gamma < 0$.

Substituting (58) into definition (53) gives us the following equation for the annual share weight vector under the linear trends assumption:

$$\alpha \equiv \Sigma_{t=1}{}^{T^*} (1/T^*)s^t$$
$$= \Sigma_{t=1}{}^{T^*} (1/T^*)[s^1 + \beta(t-1)]$$
$$= s^1 + (½)\beta(T^*-1). \quad (61)$$

Using (57)–(59) and (61), we have the following equations for $t = T^* + 1, T^* + 2, \ldots, T$:

[64] In actual practice, the reference month for prices can be many months after T^*.

[65] Since expenditure shares must be nonnegative, if $\beta \neq 0_N$ then some components of β will be negative, and thus the linear trends in shares assumption (58) cannot hold forever. Assumptions (58) and (59) will generally be only approximately true, and they cannot hold indefinitely.

$$lnP_T{}^t - lnP_{J\alpha}{}^t = [(½)s^t + (½)s^1 - \alpha]\cdot[lnp^t - lnp^1] = (½)\beta\cdot\gamma t(t-T^*-1). \quad (62)$$

Thus, if the inner product of the vectors β and γ is not equal to 0, $lnP_T{}^t$ and $lnP_{J\alpha}{}^t$ will diverge at a *quadratic rate* as t increases. Under these trend assumptions, the average share geometric index $P_{J\alpha}{}^t$ will be subject to some substitution bias (as compared to $P_T{}^t$ which controls for substitution bias[66]), which will grow over time.[67] As indicated earlier, it is likely that $\beta\cdot\gamma < 0$ so that it is likely that $P_T{}^t$ will be below $P_{J\alpha}{}^t$ under the assumption of strong substitutability and diverging trends in prices and shares.

Note that in real life, new products appear and existing products disappear. The analysis presented in this section and in previous sections can take this fact into account *in theory* if the price statistician has somehow calculated approximate reservation prices for products that are not available in the current period. Note that product churn means that shares are not constant over time; that is, *product churn will lead to nonsmooth trends in product shares*. However, superlative indices like $P_F{}^t$ and $P_T{}^t$ can deal with new and disappearing products in a way that is consistent with consumer theory, provided that suitable reservation prices have been either estimated or approximated by suitable rules of thumb.

7. To Chain or Not to Chain

In the discussions here, attention has been focused on direct indices that compare the prices of period t with the prices of period 1. But it is also possible to move from period 1 prices to period t prices by moving from one period to the next and cumulating the jumps. If the second method is used, the resulting period t price index is called a *chained index*. In this section, we will examine the possible differences between direct and chained Törnqvist price indices.

It is convenient to introduce some new notation. Denote the Törnqvist price index that compares the prices of period j to the prices of period i (the base period for the comparison) by $P_T(i,j)$. The logarithm of $P_T(i,j)$ is defined as follows for $i,j = 1, \ldots, N$:

$$lnP_T(i,j) \equiv (½)\Sigma_{n=1}{}^N (s_{in} + s_{jn})(lnp_{jn} - lnp_{in}) = (½)(s^i + s^j)\cdot(lnp^j - lnp^i). \quad (63)$$

The chained Törnqvist price index going from period 1 to T will coincide with the corresponding direct index if the indices $P_T(i,j)$ satisfy the following *multiperiod identity test*, which was developed by Walsh (1901, 389; 1921b, 540):

$$P_T(1,2)P_T(2,3)\ldots P_T(T-1,T)P_T(T,1) = 1. \quad (64)$$

This test can be used to measure the amount that the chained indices between periods 1 and T differ from the corresponding direct index that compares the prices of period 1 and T; that is, if the product of indices on the left-hand side of (64) is different from unity, then we say that the index number formula is subject to *chain drift* and the difference between the left-hand and right-hand sides of (64) is used to measure the magnitude of the chain drift problem.[68] In order to determine whether the Törnqvist price index formula satisfies the multiperiod identity test (64), take the logarithm of the left-hand side of (64) and check whether it is equal to the logarithm of 1 which is 0. Thus, substituting definitions (63) into the logarithm of the left-hand side of (64) leads to the following expressions:[69]

$$\begin{aligned}
&lnP_T(1,2) + lnP_T(2,3) + \ldots + lnP_T(T-1,T) + lnP_T(T,1)\\
&= ½\Sigma_{n=1}{}^N (s_{1n} + s_{2n})(lnp_{2n} - lnp_{1n}) + ½\Sigma_{n=1}{}^N (s_{2n} + s_{3n})(lnp_{3n} - lnp_{2n}) + \ldots\\
&\quad + ½\Sigma_{n=1}{}^N (s_{T-1,n} + s_{Tn})(lnp_{Tn} - lnp_{T-1,n}) + ½\Sigma_{n=1}{}^N (s_{Tn} + s_{1n})(lnp_{1n} - lnp_{Tn})\\
&= ½\Sigma_{n=1}{}^N (s_{1n} - s_{3n})lnp_{2n} + ½\Sigma_{n=1}{}^N (s_{2n} - s_{4n})lnp_{3n} + \ldots + ½\Sigma_{n=1}{}^N (s_{T-2,n} - s_{Tn})lnp_{T-1,n}\\
&\quad + ½\Sigma_{n=1}{}^N (s_{Tn} - s_{2n})lnp_{1n} + ½\Sigma_{n=1}{}^N (s_{T-1,n} - s_{1n})lnp_{Tn}. \quad (65)
\end{aligned}$$

In general, it can be seen that the Törnqvist price index formula will be subject to some chain drift; that is, the sums of terms on the right-hand side of (65) will not equal 0 in general. However, there are four sets of conditions where these terms will sum to 0.

The first set of conditions makes use of the first equality on the right-hand side of (65). If the prices vary in strict proportion over time, so that $p^t = \lambda_t p^1$ for $t = 2,3,\ldots,T$, then it is straightforward to show that (64) is satisfied.

The second set of conditions makes use of the second equality in equations (65). If the shares s^t are constant over time,[70] then it is obvious that (64) is satisfied.

The third set of conditions also makes use of the second equality in (65). The sum of terms $\Sigma_{n=1}{}^N (s_{1n} - s_{3n})lnp_{2n}$ is equal to $(s^1-s^3)\cdot lnp^2$, which in turn is equal to $(s^1-s^3)\cdot(lnp^2 - lnp^{2*}) = N\text{Cov}(s^1-s^3, lnp^2)$, where $lnp^{2*} \equiv (1/N)\,\Sigma_{n=1}{}^N lnp_{2n}$, the mean of the components of lnp^2. Thus, the N sets of summations on the right-hand side of the second equation in (65) can be interpreted as constant times the covariances of a difference in shares (separated by one or more time periods) with the logarithm of a price vector for a time period that is not equal to either of the time periods involved in the difference in shares. Thus, if the covariance equalities $\text{Cov}(s^1-s^3, lnp^2) = \text{Cov}(s^2-s^4, lnp^3) = \ldots = \text{Cov}(s^{T-2}-s^T, lnp^{T-1}) = \text{Cov}(s^T-s^2, lnp^1) = \text{Cov}(s^{T-1}-s^1, lnp^T) = 0$, then (64) will be satisfied. These zero

[66] We regard an index as having some substitution bias if it diverges from a superlative index which controls for substitution bias. See Diewert (1976) for the formal definition of a superlative index.

[67] If all prices grow at the same geometric rate, then it can be verified that $P_{J\alpha}{}^t = P_{GL}{}^t = P_{GP}{}^t = P_T{}^t$. If in addition, assumptions (58)–(60) hold, then $\gamma = \lambda 1_N$ for some scalar $\lambda > 0$ and using assumption (60), we have $\beta\cdot\gamma = 0$, and thus $P_T{}^t = P_{J\alpha}{}^t$ under our assumptions.

[68] Walsh (1901, 401) was the first to propose this methodology to measure chain drift. It was independently proposed later by Persons (1921, 110) and Szulc (1983, 540). Fisher's (1922, 284) circular gap test could also be interpreted as a test for chain drift.

[69] Persons (1928, 101) developed a similar decomposition using the bilateral Fisher formula instead of the Törnqvist formula. See also de Haan and Krsinich (2014) for an alternative decomposition.

[70] If purchasers of the products have Cobb–Douglas preferences, then the sales shares will be constant.

covariance conditions will be satisfied if the log prices of one period are uncorrelated with the shares of all other periods. If the time period is long enough and there are no trends in log prices and shares, so that prices are merely bouncing around in a random fashion,[71] then these zero covariance conditions are likely to be satisfied to a high degree of approximation, and thus under these conditions, the Törnqvist–Theil price index is likely to be largely free of chain drift. However, in the elementary index context where retailers have periodic highly discounted prices, the zero correlation conditions are unlikely to hold. Suppose that product n goes on sale during period 2 so that lnp_{2n} is well below the average price for period 2. Suppose product n is not on sale during periods 1 and 3. If purchasers have stocked up on product n during period 2, it is likely that s_{3n} will be less than s_{1n}, and thus it is likely that $\mathrm{Cov}(s^1–s^3,lnp^2) < 0$. Now suppose that product n is not on sale during period 2. In this case, it is likely that lnp_{2n} is greater than the average log price during period 2. If product n was on sale during period 1 but not period 3, then s_{1n} will tend to be greater than s_{3n}, and thus $\mathrm{Cov}(s^1–s^3,lnp^2) > 0$. However, if product n was on sale during period 3 but not period 1, then s_{1n} will tend to be less than s_{3n}, and thus $\mathrm{Cov}(s^1–s^3,lnp^2) < 0$. These last two cases should largely offset each other, and so we are left with the likelihood that $\mathrm{Cov}(s^1–s^3,lnp^2) < 0$. Similar arguments apply to the other covariances, and so we are left with the expectation that the chained Törnqvist index used in the elementary index context is likely to drift downward relative to its fixed-base counterpart.[72]

Since the Fisher index normally approximates the Törnqvist fairly closely, we expect both the chained Fisher and Törnqvist indices to exhibit downward chain drift. However, it is not always the case that a superlative index is subject to downward chain drift. Feenstra and Shapiro (2003) found upward chain drift in the Törnqvist formula using a scanner data set. Persons (1928, 100–5) had an extensive discussion of the chain drift problem with the Fisher index, and he gave a numerical example on page 102 of his article that showed how upward chain drift could occur. We have adapted his example in Table 7.3.

Product 1 is on sale in period 1 and goes back to a relatively high price in periods 2 and 3 and then goes on sale again, but the discount is not as steep as the period 1 discount. Product 2 is at its "regular" price for periods 1–3 and then rises steeply in period 4. Products 1 and 2 are close substitutes, so when product 1 is steeply discounted, only 1 unit of product 2 is sold in period 1, while 100 units of product 1 are sold. When the price of product 1 increases fivefold in period 2, demand for the product falls and purchasers switch to product 2, but the adjustment to the new higher price of product 1 is not complete in period 2: in period 3 (where prices are unchanged from period 2), purchasers continue to

Table 7.3 Prices and Quantities for Two Products and the Fisher Fixed-Base and Chained Price Indices

t	p_1^t	p_2^t	q_1^t	q_2^t	P_F^t	P_{FCh}^t
1	**2**	**1**	**100**	**1**	**1.00000**	**1.00000**
2	**10**	**1**	**40**	**40**	**4.27321**	**4.27321**
3	**10**	**1**	**25**	**80**	**3.55553**	**4.27321**
4	**5**	**2**	**50**	**20**	**2.45676**	**2.96563**

move away from product 1 and toward product 2. It is this *incomplete adjustment* that causes the chained index to climb above the fixed-base index in period 3.[73] Thus, it is not always the case that the Fisher index is subject to downward chain drift, but we do expect that "normally," this would be the case.

The fourth set of conditions that ensure that there is no chain drift are assumptions (58) and (59); that is, the assumption that *shares and log prices have linear trends*. To prove this assertion, substitute these equations into either one of the two right-hand side equations in (65), and we find that the resulting sum of terms is 0.[74] This result is of some importance at higher levels of aggregation where aggregate prices and quantities are more likely to have smooth trends. If the trends are actually linear, then this result shows that there will be no chain drift if the Törnqvist–Theil index number formula is used to aggregate the data.[75] However, when this formula is used at the elementary level when there are frequent fluctuations in prices and quantities, chain drift is likely to occur, and thus the use of a fixed-base index or a multilateral index is preferred under these conditions.

As was mentioned in the introduction, a main advantage of the chain system is that under conditions where prices and quantities are trending smoothly, chaining will reduce the spread between the Paasche and Laspeyres indices.[76] These two indices each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration, and it could be expected that a single-point estimate of the aggregate price change should lie between these two estimates. Thus, at higher levels of aggregation, the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth." However, at lower levels of aggregation, smooth changes in prices and quantities are unlikely to occur.

An alternative to the use of a fixed-base index is the use of a *multilateral index*. A problem with the use of a fixed-base index is that it depends asymmetrically on the choice of the base period. If the structure of prices and quantities

[71] Szulc (1983) introduced the term "price bouncing" to describe the behavior of soft drink prices in Canada at the elementary level.

[72] Fisher (1922, 284) found little difference in the fixed-base and chained Fisher indices for his particular data set which he used to compare 119 different index number formulae. Fisher noted that the Carli, Laspeyres, and share-weighted Carli chained indices showed upward chain drift. However, Persons (1921, 110) showed that the Fisher chained index ended up about 4 percent lower than its fixed-base counterpart for his agricultural data set covering 10 years. This is an early example of the downward chain drift associated with the use of the Fisher index.

[73] Persons (1928, 102) explained that it was *incomplete adjustment* that caused the Fisher chained index to climb above the corresponding fixed-base index in his example. Ludwig von Auer (2019) proposed a similar theory.

[74] This result was first established by Alterman, Diewert, and Feenstra (1999, 61–65).

[75] This transitivity property carries over to an *approximate* transitivity property for the Fisher and Walsh index number formulae using the fact that these indices approximate the Törnqvist–Theil index to the second order around an equal price and quantity point; see Diewert (1978) on these approximations.

[76] See Diewert (1978, 895) and Hill (1988) for additional discussion on the benefits and costs of chaining.

for the base period is unusual and fixed-base index numbers are used, then the choice of the base period could lead to "unusual" results. Multilateral indices treat each period symmetrically and thus avoid this problem. In the following section, we will introduce some possible multilateral indices that are free of chain drift (within our window of T observations).[77]

8. Relationships between the Törnqvist Index and the GEKS and CCDI Multilateral Indices

It is useful to introduce some additional notation at this point. Denote the Laspeyres, Paasche, and Fisher price indices that compare the prices of period j to the prices of period i (the base period for the comparison) by $P_L(i,j)$, $P_P(i,j)$, and $P_F(i,j)$, respectively. These indices for $r,t = 1,\ldots,N$ are defined as follows:

$$P_L(r,t) \equiv p^t \cdot q^r / p^r \cdot q^r; \tag{66}$$

$$P_P(r,t) \equiv p^t \cdot q^t / p^r \cdot q^t; \tag{67}$$

$$P_F(r,t) \equiv [P_L(r,t)P_P(r,t)]^{1/2}. \tag{68}$$

The Fisher indices have very good axiomatic properties and hence are preferred indices from the viewpoint of the test or axiomatic approach.[78]

Obviously, one could choose period 1 as the base period and form the following sequence of price levels relative to period 1: $P_F(1,1) = 1$, $P_F(1,2)$, $P_F(1,3)$, . . . , $P_F(1,T)$. But one could also use period 2 as the base period and use the following sequence of price levels: $P_F(2,1)$, $P_F(2,2) = 1$, $P_F(2,3)$, . . . , $P_F(2,T)$. Each period could be chosen as the base period, and thus we end up with T alternative series of Fisher price levels. Since each of these sequences of price levels is equally plausible, Gini (1931) suggested that it would be appropriate to take the geometric average of these alternative price levels in order to determine the final set of price levels. Thus, the *GEKS price levels*[79] for periods $t = 1,2,\ldots,T$ are defined as follows:

$$p_{GEKS}^{\,t} \equiv [\textstyle\prod_{r=1}^{T} P_F(r,t)]^{1/T}. \tag{69}$$

Note that all time periods are treated in a symmetric manner in these definitions. The GEKS price indices $P_{GEKS}^{\,t}$ are obtained by normalizing the aforementioned price levels so that the period 1 index is equal to 1. Thus, we have the following definitions for $P_{GEKS}^{\,t}$ for $t = 1,\ldots,T$:

$$P_{GEKS}^{\,t} \equiv p_{GEKS}^{\,t} / p_{GEKS}^{\,1}. \tag{70}$$

It is straightforward to verify that the GEKS price indices satisfy Walsh's multiperiod identity test, which becomes the following test in the present context:

$$[P_{GEKS}^{\,2}/P_{GEKS}^{\,1}][P_{GEKS}^{\,3}/P_{GEKS}^{\,2}] \ldots [P_{GEKS}^{\,T}/P_{GEKS}^{\,T-1}][P_{GEKS}^{\,1}/P_{GEKS}^{\,T}] = 1. \tag{71}$$

Thus, the GEKS indices are not subject to chain drift within the window of T periods under consideration.

Recall definition (63), which defined the logarithm of the Törnqvist price index, $lnP_T(i,j)$, that compared the prices of period j to the prices of period i. The GEKS methodology can be applied using $P_T(r,t)$ in place of the Fisher $P_F(r,t)$ as the basic bilateral index building block. Thus, define the *period t GEKS Törnqvist price level*, $p_{GEKST}^{\,t}$, for $t = 1,\ldots,T$ as follows:

$$p_{GEKST}^{\,t} \equiv [\textstyle\prod_{r=1}^{T} P_T(r,t)]^{1/T}. \tag{72}$$

The *GEKST price indices* $P_{GEKST}^{\,t}$ are obtained by normalizing these price levels so that the period 1 index is equal to 1. Thus, we have the following definitions for $P_{GEKST}^{\,t}$ for $t = 1,\ldots,T$:

$$P_{GEKST}^{\,t} \equiv p_{GEKST}^{\,t} / p_{GEKST}^{\,1}. \tag{73}$$

Since $P_T(r,t)$ approximates $P_F(r,t)$ to the second order around an equal price and quantity point, $P_{GEKST}^{\,t}$ will usually be quite close to the corresponding $P_{GEKS}^{\,t}$ indices.

It is possible to provide a very simple alternative approach to the derivation of the GEKS Törnqvist price indices.[80] Define the *sample average sales share* for product n, $s_{\bullet n}$, and the *sample average log price* for product n, $lnp_{\bullet n}$, for $n = 1,\ldots,N$ as follows:

$$s_{\bullet n} \equiv \textstyle\sum_{t=1}^{T} (1/T) s_{tn}; \tag{74}$$

$$lnp_{\bullet n} \equiv \textstyle\sum_{t=1}^{T} (1/T) lnp_{tn}. \tag{75}$$

The logarithm of the *CCDI price level for period t*, $lnp_{CCDI}^{\,t}$, is defined by comparing the prices of period t with the sample average prices using the bilateral Törnqvist formula; that is, for $t = 1,\ldots,T$, we have the following definitions:

$$lnp_{CCDI}^{\,t} \equiv \textstyle\sum_{n=1}^{N} \tfrac{1}{2}(s_{tn} + s_{\bullet n})(lnp_{tn} - lnp_{\bullet n}). \tag{76}$$

The *CCDI price index for period t*, $P_{CCDI}^{\,t}$, is defined as the following normalized CCDI price level for $t = 1,\ldots,T$:

$$P_{CCDI}^{\,t} \equiv p_{CCDI}^{\,t} / p_{CCDI}^{\,1}. \tag{77}$$

Using the aforementioned definitions, the logarithm of the CCDI price index for period t is equal to the following expression for $t = 1,\ldots,T$:

$$lnP_{CCDI}^{\,t} = lnp_{CCDI}^{\,t} - lnp_{CCDI}^{\,1}$$

[77] Ivancic, Diewert, and Fox (2009, 2011) advocated the use of multilateral indices adapted to the time series context in order to control chain drift. Balk (1980, 1981) also advocated the use of multilateral indices in order to address the problem of seasonal commodities.

[78] See Diewert (1992) for details on the axiomatic properties of the Fisher index.

[79] Eltetö and Köves (1964) and Szulc (1964) independently derived the GEKS price indices using an alternative route. Thus, the name GEKS has the initials of all four primary authors of the method. Ivancic, Diewert, and Fox (2009, 2011) suggested the use of the GEKS index in the time series context.

[80] This approach was developed by Inklaar and Diewert (2016). It is an adaptation of the distance function approach used by Caves, Christensen, and Diewert (1982) to the price index context.

$$= \Sigma_{n=1}^{N} (½)(s_{tn} + s_{\bullet n})(lnp_{tn} - lnp_{\bullet n}) - \Sigma_{n=1}^{N} (½)(s_{1n} + s_{\bullet n})(lnp_{1n} - lnp_{\bullet n})$$
$$= lnP_T^{\,t} + \Sigma_{n=1}^{N} (½)(s_{tn} - s_{\bullet n})(lnp_{1n} - lnp_{\bullet n}) - \Sigma_{n=1}^{N} (½)(s_{1n} - s_{\bullet n})(lnp_{tn} - lnp_{\bullet n})$$
$$= lnP_{GEKST}^{\,t}, \quad (78)$$

where the last equality follows from direct computation or from the computations of Inklaar and Diewert (2016).[81] Thus, the CCDI multilateral price indices are equal to the GEKS Törnqvist multilateral indices defined by (73). Define $s^{\bullet} \equiv [s_{\bullet 1}, \ldots, s_{\bullet N}]$ as the vector of sample average shares and $lnp^{\bullet} \equiv [lnp_{\bullet 1}, \ldots, lnp_{\bullet N}]$ as the vector of sample average log prices. Then the last two terms on the right-hand side of the penultimate equality in (78) can be written as $(½)N\text{Cov}(s^t - s^{\bullet}, lnp^1 - lnp^{\bullet}) - (½)N\text{Cov}(s^1 - s^{\bullet}, lnp^t - lnp^{\bullet})$. If the fluctuations in shares and prices are not too high, it is likely that both covariances are close to 0, and thus $lnP_{CCDI}^{\,t} \approx lnP_T^{\,t}$ for each t.[82] Thus, under these conditions, it is likely that $lnP_{CCDI}^{\,t} \approx lnP_T^{\,t}$ for each t. Moreover, under the assumptions of linear trends in log prices and linear trends in shares, assumptions (58) and (59), it was seen in the previous section that the period t bilateral Törnqvist price index, $P_T^{\,t}$, was equal to its chained counterpart for any t.[83] This result implies that $P_T^{\,t} = P_{CCDI}^{\,t} = P_{GEKST}^{\,t}$ for $t = 1, \ldots, T$ under the linear trends assumption. Thus, we expect the period t multilateral index, $P_{GEKST}^{\,t} = P_{CCDI}^{\,t}$, to approximate the corresponding fixed-base period t Törnqvist price index, $P_T^{\,t}$, provided that prices and quantities have smooth trends.

Since $P_F^{\,t}$ approximates $P_T^{\,t}$, we expect that the following approximate equalities will hold under the smooth trends assumption for $t = 1, \ldots, T$:

$$P_F^{\,t} \approx P_T^{\,t} \approx P_{GEKS}^{\,t} \approx P_{GEKST}^{\,t} = P_{CCDI}^{\,t}. \quad (79)$$

These indices will be free from chain drift within the window of T periods;[84] that is, if prices and quantities for any two periods in the sample are equal, then the price index will register the same value for these two periods.

Unit values taken over heterogeneous products are often used at the first stage of aggregation. In the following section, bias estimates for unit value price levels will be derived, and in the subsequent section, quality-adjusted unit value price levels will be studied.

[81] The second from last equality was derived by Diewert and Fox (2021).

[82] For Diewert's (2018) empirical example, the sample average of these two sets of covariance terms turned out to be 0 with variances equal to 0.00024 and 0.00036, respectively.

[83] See the discussion below (65) in the previous section. Note that the assumption of linear trends in shares is not consistent with the existence of new and disappearing products.

[84] See de Haan (2015) and Diewert and Fox (2021) for discussions of the problems associated with linking the results from one rolling window multilateral comparison to a subsequent window of observations. Empirically, there does not appear to be much chain drift when the indices generated by subsequent windows are linked.

9. Unit Value Price and Quantity Indices

As was mentioned in Section 2, there was a preliminary aggregation over time problem that needed to be addressed; that is, exactly how should the period t prices and quantities for commodity n, $p_n^{\,t}$, and $q_n^{\,t}$ that are used in an index number formula be defined? During any time period t, there will typically be many transactions in a specific commodity n at a number of different prices. Hence, there is a need to provide a more precise definition for the "average" or "representative" price for commodity n in period t, $p_n^{\,t}$. Starting with Drobisch (1871), many measurement economists and statisticians advocated the use of the *unit value* (total value transacted divided by the total quantity) as the appropriate price $p_n^{\,t}$ for commodity n and the total quantity transacted during period t as the appropriate quantity, $q_n^{\,t}$; for example, see Walsh (1901, 96; 1921a, 88), Fisher (1922, 318), and Davies (1924, 183; 1932, 59). If it is desirable to have $q_n^{\,t}$ be equal to the total quantity of commodity n transacted during period t and also desirable to have the product of the price $p_n^{\,t}$ times quantity $q_n^{\,t}$ to be equal to the value of period t transactions in commodity n, then one is *forced* to define the aggregate period t price for commodity n, $p_n^{\,t}$, to be the total value transacted during the period divided by the total quantity transacted, which is the unit value for commodity n.[85]

There is general agreement that a unit value price is an appropriate price concept to be used in an index number formula if the transactions refer to a narrowly defined homogeneous commodity. Our task in this section is to look at the properties of a unit value price index when aggregating over commodities that are *not* completely homogeneous. We will also look at the properties of the companion unit value quantity index in this section.

The period *t unit value price level*, $p_{UV}^{\,t}$, and the corresponding period *t unit value price index*, which compares the price level in period t to that in period 1, $P_{UV}^{\,t}$, for $t = 1, \ldots, T$ are defined as follows:

$$p_{UV}^{\,t} \equiv p^t \cdot q^t / 1_N \cdot q^t; \quad (80)$$
$$P_{UV}^{\,t} \equiv p_{UV}^{\,t} / p_{UV}^{\,1} = [p^t \cdot q^t / 1_N \cdot q^t]/[p^1 \cdot q^1 / 1_N \cdot q^1] = [p^t \cdot q^t / p^1 \cdot q^1]/Q_{UV}^{\,t}, \quad (81)$$

where the period *t unit value quantity index*, $Q_{UV}^{\,t}$, for $t = 1, \ldots, T$ is defined as follows:

$$Q_{UV}^{\,t} \equiv 1_N \cdot q^t / 1_N \cdot q^1. \quad (82)$$

It can be seen that the unit value price index satisfies Walsh's multiperiod identity test, and thus $P_{UV}^{\,t}$ is free from chain drift.

However, there is a big problem in using the unit value price index when the commodities in scope are not homogeneous: *The unit value price index is not invariant to changes*

[85] For additional discussion on unit value price indices, see Balk (2008, 72–74), Diewert, and von der Lippe (2010), Silver (2010, 2011), and de Haan and Krsinich (2018).

in the units of measurement of the individual products in the aggregate.

We will look at the relationship of the *unit value quantity indices*, $Q_{UV}{}^t$, with the corresponding *Laspeyres*, *Paasche*, and *Fisher fixed-base quantity indices*, $Q_L{}^t$, $Q_P{}^t$, and $Q_F{}^t$, defined here for $t = 1, \ldots, T$:

$$Q_L{}^t \equiv p^1 \cdot q^t / p^1 \cdot q^1 = \Sigma_{n=1}{}^N s_{1n}(q_{tn}/q_{1n}); \quad (83)$$

$$Q_P{}^t \equiv p^t \cdot q^t / p^t \cdot q^1 = [\Sigma_{n=1}{}^N s_{tn}(q_{tn}/q_{1n})^{-1}]^{-1}; \quad (84)$$

$$Q_F{}^t \equiv [Q_L{}^t Q_P{}^t]^{1/2}. \quad (85)$$

For the second set of equations in (83), we require that $q_{1n} > 0$ for all n, and for the second set of equations in (84), we require that all $q_{tn} > 0$. Recall that the period t *sales* or *expenditure share* vector $s^t \equiv [s_{t1}, \ldots, s_{tN}]$ was defined at the beginning of Section 2. The period t *quantity* share vector $S^t \equiv [S_{t1}, \ldots, S_{tN}]$ was also defined in Section 2 for $t = 1, \ldots, T$ as follows:

$$S^t \equiv q^t / 1_N \cdot q^t. \quad (86)$$

Here, we will make use of the following identities (87), which hold for $t = 1, \ldots, T$:

$$\Sigma_{n=1}{}^N [p_{UV}{}^t - p_{tn}]q_{tn} = \Sigma_{n=1}{}^N [(p^t \cdot q^t / 1_N \cdot q^t) - p_{tn}]q_{tn} \text{ using definitions (80)}$$
$$= (p^t \cdot q^t / 1_N \cdot q^t) 1_N \cdot q^t - p^t \cdot q^t = 0. \quad (87)$$

The following relationships between $Q_{UV}{}^t$ and $Q_L{}^t$ hold for $t = 1, \ldots, T$:

$$Q_{UV}{}^t - Q_L{}^t = [1_N \cdot q^t / 1_N \cdot q^1] - [p^1 \cdot q^t / p^1 \cdot q^1] \text{ using (82) and (83)}$$
$$= \Sigma_{n=1}{}^N S_{1n}(q_{tn}/q_{1n}) - \Sigma_{n=1}{}^N s_{1n}(q_{tn}/q_{1n}) \text{ using (86) and (83)}$$
$$= \Sigma_{n=1}{}^N [S_{1n} - s_{1n}](q_{tn}/q_{1n})$$
$$= N\text{Cov}(S^1 - s^1, q^t/q^1), \quad (88)$$

where the vector of period t to period 1 relative quantities is defined as $q^t/q^1 \equiv [q_{t1}/q_{11}, q_{t2}/q_{12} \ldots, q_{tN}/q_{1N}]$. As usual, there are three special cases of (88), which will imply that $Q_{UV}{}^t = Q_L{}^t$. (i) $S^1 = s^1$ so that the vector of period 1 real quantity shares S^1 is equal to the period 1 sales share vector s^1. This condition is equivalent to $p^1 = \lambda_1 1_N$ so that all period 1 prices are equal.[86] (ii) $q^t = \lambda_t q^1$ for $t = 2, 3, \ldots, T$ so that quantities vary in strict proportion over time and (iii) $\text{Cov}(S^1 - s^1, q^t/q^1) = 0$.[87]

There are two problems with this bias formula: (i) It is difficult to form a judgment on the sign of the covariance $\text{Cov}(S^1 - s^1, q^t/q^1)$ and (ii) the decomposition given by (88) requires that all components of the period 1 quantity vector be positive.[88] It would be useful to have a decomposition that allowed some quantities (and sales shares) to be equal to 0. Consider the following alternative decomposition to (88) for $t = 1, \ldots, T$:

$$Q_{UV}{}^t - Q_L{}^t = [1_N \cdot q^t / 1_N \cdot q^1] - [p^1 \cdot q^t / p^1 \cdot q^1] \text{ using (82) and (83)}$$
$$= \Sigma_{n=1}{}^N [(q_{tn}/1_N \cdot q^1) - (p_{1n} q_{tn} / p^1 \cdot q^1)]$$
$$= \Sigma_{n=1}{}^N [(1/1_N \cdot q^1) - (p_{1n}/p^1 \cdot q^1)]q_{tn}$$
$$= \Sigma_{n=1}{}^N [(p^1 \cdot q^1 / 1_N \cdot q^1) - p_{1n}][q_{tn}/p^1 \cdot q^1]$$
$$= \Sigma_{n=1}{}^N [p_{UV}{}^1 - p_{1n}][q_{tn}/p^1 \cdot q^1] \text{ using (80) for } t = 1$$
$$= \Sigma_{n=1}{}^N [p_{UV}{}^1 - p_{1n}][q_{tn} - q_{1n} Q_{UV}{}^t]/p^1 \cdot q^1 \text{ using (87) for } t = 1$$
$$= Q_{UV}{}^t \Sigma_{n=1}{}^N [p_{UV}{}^1 - p_{1n}][(q_{tn}/Q_{UV}{}^t) - q_{1n}]/p^1 \cdot q^1$$
$$= Q_{UV}{}^t \Sigma_{n=1}{}^N s_{1n}[(p_{UV}{}^1/p_{1n}) - 1][(q_{tn}/q_{1n} Q_{UV}{}^t) - 1] \text{ if } q_{1n} > 0 \text{ for all } n$$
$$= Q_{UV}{}^t \varepsilon_L{}^t, \quad (89)$$

where the *period t error term* $\varepsilon_L{}^t$ for $t = 1, \ldots, T$ is defined as

$$\varepsilon_L{}^t \equiv \Sigma_{n=1}{}^N [p_{UV}{}^1 - p_{1n}][(q_{tn}/Q_{UV}{}^t) - q_{1n}]/p^1 \cdot q^1.\text{[89]} \quad (90)$$

If $q_{1n} > 0$ for $n = 1, \ldots, N$, then $\varepsilon_L{}^t$ is equal to $\Sigma_{n=1}{}^N s_{1n}[(p_{UV}{}^1/p_{1n}) - 1][(q_{tn}/q_{1n} Q_{UV}{}^t) - 1]$.

Note that the terms on the right-hand side of (90) can be interpreted as $(N/p^1 \cdot q^1)$ times the covariance $\text{Cov}(p_{UV}{}^1 1_N - p^1, q^t - Q_{UV}{}^t q^1)$ since $1_N \cdot (q^t - Q_{UV}{}^t q^1) = 0$. If the products are substitutes, it is likely that this covariance is *negative*, since if p_{1n} is unusually low, we would expect that it would be less than the period 1 unit value price level $p_{UV}{}^1$ so that $p_{UV}{}^1 - p_{1n} > 0$. Furthermore, if p_{1n} is unusually low, then we would expect that the corresponding q_{1n} is unusually high, and thus it is likely that q_{1n} is greater than $q_{tn}/Q_{UV}{}^t$, and so $q_{tn} - q_{1n} Q_{UV}{}^t < 0$. Thus, the N terms in the covariance will tend to be negative provided that there is some degree of substitutability between the products.[90] Looking at formula (90) for $\varepsilon_L{}^t$, it can be seen that all terms on the right-hand side of (90) do not depend on t, except for the N period t deflated product quantity terms, $q_{tn}/Q_{UV}{}^t$ for $n = 1, \ldots, N$. Hence, if there is a great deal of variation in the period t quantities q_{tn}, then $q_{tn}/Q_{UV}{}^t - q_{1n}$ could be positive or negative, and thus the tendency for $\varepsilon_L{}^t$ to be negative will be a weak one. Thus, our expectation is that the error term $\varepsilon_L{}^t$ is likely to be negative, and hence $Q_{UV}{}^t < Q_L{}^t$ for $t \geq 2$, but this expectation is a weak one.

[86] Consider the case where $p^1 = \lambda 1_N$. Units of measurement for the N commodities can always be chosen so that all prices are equal in period 1. Then $Q_{UV}{}^t = Q_L{}^t$, and hence $P_{UV}{}^t = P_P{}^t$, where $P_{UV}{}^t$ is defined by (81) and $P_P{}^t$ is the fixed-base Paasche price index defined by (34). Thus, for this particular choice for units of measurement, the unit value price index $P_{UV}{}^t$ is equal to a fixed-base Paasche price index, which will typically have a downward bias relative to a superlative index.

[87] For similar bias formulae, see Balk (2008, 73–74) and Diewert and von der Lippe (2010).

[88] We are assuming that all prices are positive in all periods (so if there are missing prices they must be replaced by positive imputed prices), but we are not assuming that all quantities (and expenditure shares) are positive.

[89] Note that this error term is homogeneous of degree 0 in the components of p^1, q^1, and q^t. Hence, it is invariant to proportional changes in the components of these vectors.

[90] The results in previous sections looked at responses of product *shares* to changes in prices and with data that are consistent with CES preferences, and the results depended on whether the elasticity of substitution was greater or less than unity. In the present section, the results depend on whether the elasticity of substitution is equal to or greater than 0; that is, it is the response of *quantities* (rather than *shares*) to lower prices that matters.

It should be noted that $P_{UV}{}^t$ and $Q_{UV}{}^t$ *do not depend on the estimated reservation prices* for the missing products; that is, the definitions of $P_{UV}{}^t$ and $Q_{UV}{}^t$ zero out the estimated reservation prices.

As usual, there are three special cases of (89) that will imply that $Q_{UV}{}^t = Q_L{}^t$: (i) $p^1 = \lambda_1 1_N$ so that all period 1 prices are equal; (ii) $q^t = \lambda_t q^1$ for $t = 2,3, \ldots, T$ so that the quantities vary in strict proportion over time; (iii) $\mathrm{Cov}(p_{UV}{}^1 1_N - p^1, q^t - Q_{UV}{}^t q^1) = 0$. These conditions are equivalent to our earlier conditions of (88).

If we divide both sides of equation t in equations (89) by $Q_{UV}{}^t$, we obtain the following system of identities for $t = 1, \ldots, T$:

$$Q_L{}^t/Q_{UV}{}^t = 1 - \varepsilon_L{}^t, \qquad (91)$$

where we expect $\varepsilon_L{}^t$ to be a small negative number in the elementary index context.

The identities in (89) and (91) are valid if we interchange prices and quantities. The quantity counterparts to $p_{UV}{}^t$ and $P_{UV}{}^t$ defined by (80) and (81) are the period t *Dutot quantity level* $q_D{}^t$ and *quantity index* $Q_D{}^t$[91] defined as $q_D{}^t \equiv p^t \cdot q^t/1_N \cdot p^t = \alpha^t \cdot q^t$ (where $\alpha^t \equiv p^t/1_N \cdot p^t$ is a vector of period t price weights for q^t) and $Q_D{}^t \equiv q_{UV}{}^t/q_{UV}{}^1 = [p^t \cdot q^t/p^1 \cdot q^1]/P_D{}^t$, where we redefine the period t Dutot price level as $p_D{}^t \equiv 1_N \cdot p^t$ and the period t Dutot price index as $P_D{}^t \equiv p_D{}^t/p_D{}^1 = 1_N \cdot p^t/1_N \cdot p^1$, which coincides with our earlier definition (10) for $P_D{}^t$. Using these definitions and interchanging prices and quantities, equation (91) becomes the following equations for $t = 1, \ldots, T$:

$$P_L{}^t/P_D{}^t = 1 - \varepsilon_L{}^{t*}, \qquad (92)$$

where the period t error term $\varepsilon_L{}^{t*}$ for $t = 1, \ldots, T$ is defined as

$$\varepsilon_L{}^{t*} \equiv \Sigma_{n=1}{}^N [q_D{}^1 - q_{1n}][(p_{tn}/P_D{}^t) - p_{1n}]/p^1 \cdot q^1. \qquad (93)$$

If p_{1n} is unusually low, then it is likely that it will be less than $p_{tn}/P_D{}^t$, and it is also likely that q_{1n} will be unusually high and hence greater than the average period 1 Dutot quantity level, $q_D{}^1$. Thus, the N terms in the definition of $\varepsilon_L{}^{t*}$ will tend to be negative, and thus $1 - \varepsilon_L{}^{t*}$ will tend to be greater than 1. Thus, there will be a tendency for $P_D{}^t < P_L{}^t$ for $t \geq 2$, but again, this expectation is a weak one if there are large fluctuations in the deflated period t prices, $p_{tn}/P_D{}^t$, for $n = 1, \ldots, N$.

It can be verified that the following identities hold for the period t Laspeyres, Paasche, and unit value price and quantity indices for $t = 1, \ldots, T$:

$$p^t \cdot q^t/p^1 \cdot q^1 = P_{UV}{}^t Q_{UV}{}^t = P_P{}^t Q_L{}^t = P_L{}^t Q_P{}^t. \qquad (94)$$

Equation (94) implies the following identities for $t = 1, \ldots, T$:

$$P_{UV}{}^t/P_P{}^t = Q_L{}^t/Q_{UV}{}^t = 1 - \varepsilon_L{}^t, \qquad (95)$$

which follow from equations (91). Thus, we expect that $P_{UV}{}^t > P_P{}^t$ for $t = 2,3, \ldots, T$ if the products are substitutes and $e_L{}^t$ is negative.[92]

We now turn our attention to developing an exact relationship between $Q_{UV}{}^t$ and the Paasche quantity index $Q_P{}^t$. Using definitions (82) and (84), we have for $t = 1, \ldots, T$:

$$\begin{aligned}[Q_{UV}{}^t]^{-1} - [Q_P{}^t]^{-1} &= [1_N \cdot q^1/1_N \cdot q^t] - [p^t \cdot q^1/p^t \cdot q^t] \\ &\quad \text{using (82) and (84)} \\ &= \Sigma_{n=1}{}^N [S_{tn} - s_{tn}][q_{1n}/q_{tn}] \\ &= N\mathrm{Cov}(S^t - s^t, q^1/q^t), \end{aligned} \qquad (96)$$

where the second set of equalities follows from (88) and (86), assuming that $q_{tn} > 0$ for $n = 1, \ldots, N$.

As usual, there are three special cases of (96) that will imply that $Q_{UV}{}^t = Q_P{}^t$: (i) $S^t = s^t$ so that the vector of period t real quantity shares S^t is equal to the period t sales share vector s^t. This condition is equivalent to $p^t = \lambda_t 1_N$, which implies that all period t prices are equal.[93] (ii) $q^t = 1_t q^1$ for $t = 2,3, \ldots, T$ so that quantities vary in strict proportion over time. (iii) $N\mathrm{Cov}(S^t - s^t, q^1/q^t) = 0$.

Again, there are two problems with this bias formula: (i) it is difficult to form a judgment on the sign of the covariance $N\mathrm{Cov}(S^t - s^t, q^1/q^t)$ and (ii) the decomposition given by (96) requires that all components of the period t quantity vector be positive. We will proceed to develop a decomposition that does not require the positivity of q^t. The following exact decomposition holds for $t = 1, \ldots, T$:

$$\begin{aligned}[Q_{UV}{}^t]^{-1} - [Q_P{}^t]^{-1} &= [1_N \cdot q^1/1_N \cdot q^t] - [p^t \cdot q^1/p^t \cdot q^t] \\ &= \Sigma_{n=1}{}^N [(q_{1n}/1_N \cdot q^t) - (p_{tn} q_{1n}/p^t \cdot q^t)] \\ &= \Sigma_{n=1}{}^N [(1/1_N \cdot q^t) - (p_{tn}/p^t \cdot q^t)] q_{1n} \\ &= \Sigma_{n=1}{}^N [(p^t \cdot q^t/1_N \cdot q^t) - p_{tn}][q_{1n}/p^t \cdot q^t] \\ &= \Sigma_{n=1}{}^N [p_{UV}{}^t - p_{tn}][q_{1n}/p^t \cdot q^t] \text{ using (80) } \textit{for } t = t \\ &= \Sigma_{n=1}{}^N [p_{UV}{}^t - p_{tn}][q_{1n} - (q_{tn}/Q_{UV}{}^t)]/p^t \cdot q^t \text{ using (87) for } t = t \\ &= [Q_{UV}{}^t]^{-1} \Sigma_{n=1}{}^N [p_{UV}{}^t - p_{tn}][(q_{1n} Q_{UV}{}^t) - q_{tn}]/p^t \cdot q^t \\ &= [Q_{UV}{}^t]^{-1} \Sigma_{n=1}{}^N s_{tn}[(p_{UV}{}^t/p_{tn}) - 1][(q_{1n} Q_{UV}{}^t/q_{tn}) - 1] \text{ if } q_{tn} > 0 \text{ for all } n \\ &= [Q_{UV}{}^t]^{-1} \varepsilon_P{}^t, \end{aligned} \qquad (97)$$

where the period t error term $\varepsilon_P{}^t$ for $t = 1, \ldots, T$ is defined as follows:

$$\varepsilon_P{}^t \equiv \Sigma_{n=1}{}^N [p_{UV}{}^t - p_{tn}][(q_{1n} Q_{UV}{}^t) - q_{tn}]/p^t \cdot q^t.^{94} \qquad (98)$$

[91] Balk (2008, 7) called $Q_{UV}{}^t$ a Dutot-type quantity index.

[92] As was discussed earlier, if all prices are equal in the base period, then $\varepsilon_L{}^t = 0$ and $P_{UV}{}^t/P_P{}^t = Q_L{}^t/Q_{UV}{}^t = 0$.

[93] If $p^t = \lambda 1_N$, so that all prices are equal in period t, then it can be shown directly that $P_{UV}{}^t = P_L{}^t$. Thus, for the particular choice for units of measurement that makes all prices equal in period t, the unit value price index $P_{UV}{}^t$ is equal to a fixed-base Laspeyres price index which will typically have an upward bias relative to a superlative index.

[94] Note that this error term is homogeneous of degree 0 in the components of p^t, q^1, and q^t. Thus, for $\lambda > 0$, we have $\varepsilon_P(p^t,q^1,q^t) = \varepsilon_P(\lambda p^t,q^1,q^t) = \varepsilon_P(p^t,\lambda q^1,q^t) = \varepsilon_P(p^t,q^1,\lambda q^t)$. Note also that $\varepsilon_P{}^t$ is well defined if some quantities are equal to 0 and $\varepsilon_P{}^t$ does depend on the reservation prices p_{tn} for products n that are not present in period t. If product n is missing in period t, then it is likely that the reservation price p_{tn} is greater than the unit value price level for period t, $p_{UV}{}^t$, and since $q_{tn} = 0$, it can be seen that the nth term on the right-hand side of (98) will be negative; that is, the greater the number of missing products in period t, the greater is the likelihood that $\varepsilon_P{}^t$ is negative.

If $q_{tn} > 0$ for $n = 1, \ldots, N$, then $\varepsilon_P^{\,t}$ is equal to $\Sigma_{n=1}^{N}$ $s_{tn}[(p_{UV}^{\,t}/p_{tn}) - 1][(q_{1n}Q_{UV}^{\,t}/q_{tn}) - 1]$.

Note that the terms on the right-hand side of (97) can be interpreted as $(N/p^t \cdot q^t)$ times $\mathrm{Cov}(p_{UV}^{\,t}1_N - p^t, q^1 - [Q_{UV}^{\,t}]^{-1}q^t)$ since $1_N \cdot (q^1 - [Q_{UV}^{\,t}]^{-1}q^t) = 0$. If the products are substitutable, it is likely that this covariance is *negative*, since if p_{tn} is unusually low, we would expect that it would be less than the period t unit value price $p_{UV}^{\,t}$ so that $p_{UV}^{\,t} - p_{tn} > 0$. If p_{tn} is unusually low, then we also expect that the corresponding q_{tn} is unusually high, and thus it is likely that q_{tn} is greater than $q_{1n}Q_{UV}^{\,t}$, and so $q_{1n}Q_{UV}^{\,t} - q_{tn} < 0$. Thus, the N terms in the covariance will tend to be negative. Thus, our expectation is that the error term $\varepsilon_P^{\,t} < 0$ and $[Q_{UV}^{\,t}]^{-1} < [Q_P^{\,t}]^{-1}$ or $Q_{UV}^{\,t} > Q_P^{\,t}$ for $t \geq 2$.[95]

There are three special cases of (97) that will imply that $Q_{UV}^{\,t} = Q_P^{\,t}$: (i) $p^t = \lambda_t 1_N$ so that all period t prices are equal; (ii) $q^t = \lambda_t q^1$ for $t = 2,3, \ldots, T$ so that quantities vary in strict proportion over time; and (iii) $\mathrm{Cov}(p_{UV}^{\,t}1_N - p^t, q^1 - [Q_{UV}^{\,t}]^{-1}q^t) = 0$.

If we divide both sides of equation t in equations (97) by $[Q_{UV}^{\,t}]^{-1}$, we obtain the following system of identities for $t = 1, \ldots, T$:

$$Q_P^{\,t}/Q_{UV}^{\,t} = [1 - \varepsilon_P^{\,t}]^{-1}, \qquad (99)$$

where we expect $\varepsilon_P^{\,t}$ to be a small negative number if the products are substitutable. Thus, we expect $Q_P^{\,t} < Q_{UV}^{\,t} < Q_L^{\,t}$ for $t = 2,3, \ldots, T$.

Equations (97) and (99) are valid if we interchange prices and quantities. Using the definitions for the Dutot price and quantity levels and indices t and interchanging prices and quantities, equation (99) becomes $P_P^{\,t}/P_D^{\,t} = [1 - \varepsilon_P^{\,t*}]^{-1}$, where $\varepsilon_P^{\,t*} \equiv \Sigma_{n=1}^{N}[q_D^{\,t} - q_{tn}][(p_{1n}P_D^{\,t}) - p_{tn}]/p^t \cdot q^t$ for $t = 1, \ldots, T$. If p_{tn} is unusually low, then it is likely that it will be less than $p_{tn}/P_D^{\,t}$, and it is also likely that q_{tn} will be unusually high and hence greater than the average period t Dutot quantity level $q_D^{\,t}$. Thus, the N terms in the definition of $\varepsilon_P^{\,t*}$ will tend to be negative and, there is hence a tendency for $[1 - \varepsilon_P^{\,t*}]^{-1}$ to be less than 1. Thus, there will be a tendency for $P_P^{\,t} < P_D^{\,t}$ for $t \geq 2$.

Equations (94) imply the following identities for $t = 1, \ldots, T$:

$$P_{UV}^{\,t}/P_L^{\,t} = Q_P^{\,t}/Q_{UV}^{\,t} = [1 - \varepsilon_P^{\,t}]^{-1}, \qquad (100)$$

which follow from equations (99). Thus, we expect that $P_P^{\,t} < P_{UV}^{\,t} < P_L^{\,t}$ for $t = 2,3, \ldots, T$ if the products are substitutes.[96]

Equations (95) and (100) develop exact relationships for the unit value price index $P_{UV}^{\,t}$, with the corresponding fixed-base Laspeyres and Paasche price indices, $P_L^{\,t}$ and $P_P^{\,t}$. Taking the square root of the product of these two sets of equations leads to the following exact relationships between the fixed-base Fisher price index, $P_F^{\,t}$, and its unit value counterpart period t index, $P_{UV}^{\,t}$, for $t = 1, \ldots, T$:

$$P_{UV}^{\,t} = P_F^{\,t}\{(1 - \varepsilon_L^{\,t})/(1 - \varepsilon_P^{\,t})\}^{1/2}, \qquad (101)$$

where $\varepsilon_L^{\,t}$ and $\varepsilon_P^{\,t}$ are defined by (90) and (98). If there are no strong (divergent) trends in prices and quantities, then it is likely that $e_L^{\,t}$ is approximately equal to $e_P^{\,t}$, and hence under these conditions, it is likely that $P_{UV}^{\,t} \approx P_F^{\,t}$; that is, the unit value price index will provide an adequate approximation to the fixed-base Fisher price index under these conditions. However, with diverging trends in prices and quantities (in opposite directions), we would expect the error term $\varepsilon_P^{\,t}$ defined by (98) to be more negative than the error term $\varepsilon_L^{\,t}$ defined by (90), and thus under these conditions, we expect the unit value price index $P_{UV}^{\,t}$ to have a *downward bias* relative to its Fisher price index counterpart $P_F^{\,t}$.[97]

However, if there are missing products in period 1 so that that some q_{1n} are equal to 0 and the corresponding imputed prices p_{1n} are greater than the unit value price for observation 1, $p_{UV}^{\,1}$, then the nth term in the sum of terms on the right-hand side of (90) can become negative and large in magnitude, which can make $e_L^{\,t}$ defined by (90) much more negative than $\varepsilon_P^{\,t}$, which in turn means that $P_{UV}^{\,t}$ will be greater than unit value price index $P_F^{\,t}$ using (101). Thus, under these circumstances, the unit value price index $P_{UV}^{\,t}$ will have an *upward bias* relative to its Fisher price index counterpart $P_F^{\,t}$.

It is possible that unit value price indices can approximate their Fisher counterparts to some degree in some circumstances, but these approximations are not likely to be very accurate. If the products are somewhat heterogeneous, and there are some divergent trends in price and quantities, then the approximations are likely to be poor.[98] They are also likely to be poor if there is substantial product turnover.

10. Quality-Adjusted Unit Value Price and Quantity Indices

In the previous section, the period t unit value quantity *level* was defined by $q_{UV}^{\,t} \equiv 1_N \cdot q^t = \Sigma_{n=1}^{N} q_{tn}$ for $t = 1, \ldots, T$. The corresponding period t unit value quantity *index* was defined by (82) for $t = 1, \ldots, T$; that is, $Q_{UV}^{\,t} \equiv 1_N \cdot q^t / 1_N \cdot q^1$. In the present section, we will consider *quality-adjusted unit value quantity levels*, $q_{UV\alpha}^{\,t}$, and the corresponding *quality-adjusted unit value quantity indices*, $Q_{UV\alpha}^{\,t}$, defined as follows for $t = 1, \ldots, T$:

[95] Our expectation that $e_P^{\,t}$ is negative is more strongly held than our expectation that $\varepsilon_L^{\,t}$ is negative.

[96] If $p^t = \lambda 1_N$, then $\varepsilon_P^{\,t} = 0$, $P_{UV}^{\,t} = P_L^{\,t}$ and $Q_{UV}^{\,t} = Q_P^{\,t}$. Thus, if prices in period t are all equal, the period t fixed-base unit value index will equal the fixed-base Laspeyres price index. Thus, the unit value index will tend to have an upward bias relative to a superlative index in this equal period t prices case.

[97] The Dutot price index counterparts to the exact relations (101) are $P_F^{\,t} = P_D^{\,t}\{(1 - \varepsilon_L^{\,t*})/(1 - \varepsilon_P^{\,t*})\}^{1/2}$ for $t = 1, \ldots, T$. Thus, with diverging trends in prices and quantities (in opposite directions), we would expect the error term $\varepsilon_P^{\,t*}$ to be more negative than the error term $\varepsilon_L^{\,t*}$ and hence we would expect $P_D^{\,t} > P_F^{\,t}$ for $t \geq 2$. Note that the Dutot price index can be interpreted as a *fixed basket price index* where the basket is proportional to a vector of ones. Thus, with divergent trends in prices and quantities in opposite directions, we would expect the Dutot index to exhibit substitution bias and hence we would expect $P_D^{\,t} > P_F^{\,t}$ for $t \geq 2$.

[98] The problem with unit value price indices is that they correspond to an additive quantity level. If one takes the economic approach to index number theory, then an additive quantity level corresponds to a linear utility function which implies an infinite elasticity of substitution between products, which is too high in general.

$$q_{UV\alpha}{}^{t} \equiv \alpha \cdot q^{t}; \qquad (102)$$
$$Q_{UV\alpha}{}^{t} \equiv q_{UV\alpha}{}^{t}/q_{UV\alpha}{}^{1} = \alpha \cdot q^{t}/\alpha \cdot q^{1}, \qquad (103)$$

where $\alpha \equiv [\alpha_1, \ldots, a_N]$ is a vector of positive *quality adjustment factors*. Note that if consumers value their purchases of the N products according to the linear utility function $f(q) \equiv \alpha \cdot q$, then the period t quality-adjusted aggregate quantity level $q_{UV\alpha}{}^{t} = \alpha \cdot q^{t}$ can be interpreted as the aggregate (sub) *utility* of consumers of the N products. Note that this utility function is linear, and thus the products are perfect substitutes, after adjusting for the relative quality of the products. The bigger a_n is, the more consumers will value a unit of product n over other products. The period *t quality-adjusted unit value price level* and *price index*, $p_{UV\alpha}{}^{t}$ and $P_{UV\alpha}{}^{t}$, are defined as follows for $t = 1, \ldots, T$:

$$p_{UV\alpha}{}^{t} \equiv p^{t} \cdot q^{t}/q_{UV\alpha}{}^{t} = p^{t} \cdot q^{t}/\alpha \cdot q^{t}; \qquad (104)$$
$$P_{UV\alpha}{}^{t} \equiv p_{UV\alpha}{}^{t}/p_{UV\alpha}{}^{1} = [p^{t} \cdot q^{t}/p^{1} \cdot q^{1}]/Q_{UV\alpha}{}^{t}. \qquad (105)$$

It is easy to check that the quality-adjusted unit value price index satisfies Walsh's multiperiod identity test and thus is free from chain drift.[99] Note that the $P_{UV\alpha}{}^{t}$ and $Q_{UV\alpha}{}^{t}$ *do not depend on any estimated reservation prices*; that is, the definitions of $P_{UV\alpha}{}^{t}$ and $Q_{UV\alpha}{}^{t}$ zero out any reservation prices that are applied to missing products.

Quality-adjusted unit value price indices are consistent with the economic approach to index number theory. If consumers of the N products under consideration all have linear utility functions of the form $f(q) \equiv \alpha \cdot q = \Sigma_{n=1}{}^{N} \alpha_n q_n$, then $Q_{UV\alpha}{}^{t}$ defined by (103) accurately represents real welfare growth going from period 1 to t, and $P_{UV\alpha}{}^{t}$ defined by (105) represents consumer inflation over this period. It does not matter if there are new or disappearing products over this period; aggregate welfare or utility for period t is well defined as $\Sigma_{n=1}{}^{N} \alpha_n q_{tn}$ even if some q_{tn} are equal to 0. If $q_{tn} = 0$, then the contribution of product n to utility in period t is $\alpha_n q_n = 0$. Furthermore, the quality-adjusted unit value price and quantity indices are invariant to changes in the units of measurement if we make the convention that if the units of measurement of q_n are changed to $\lambda_n q_n$ for some positive constant λ_n, then the corresponding α_n is changed to α_n/λ_n.[100] Note that regular unit value price indices are not invariant to changes in the units of measurement.

From the viewpoint of the economic approach to index number theory, the problem with quality-adjusted unit value price and quantity indices is that the underlying linear utility function assumes that the N products under consideration are perfect substitutes after quality adjustment. Linear preferences are a special case of Constant Elasticity of Substitution preferences, and the elasticity of substitution for a linear preference is equal to plus infinity. Empirical estimates for the elasticity of substitution are far less than plus infinity.[101]

We will start out by comparing $Q_{UV\alpha}{}^{t}$ to the corresponding Laspeyres, Paasche, and Fisher period t quantity indices, $Q_L{}^{t}$, $Q_P{}^{t}$ and $Q_F{}^{t}$. The algebra in this section follows the algebra in the preceding section. Thus, the counterparts to the identities (87) in the previous section are the following identities for $t = 1, \ldots, T$:

$$\Sigma_{n=1}{}^{N} [\alpha_n p_{UV\alpha}{}^{t} - p_{tn}] q_{tn} = \Sigma_{n=1}{}^{N} [\alpha_n (p^{t} \cdot q^{t}/\alpha \cdot q^{t}) - p_{tn}] q_{tn} \text{ using definitions (104)}$$
$$= (p^{t} \cdot q^{t}/\alpha \cdot q^{t}) \alpha \cdot q^{t} - p^{t} \cdot q^{t} = 0. \qquad (106)$$

The difference between the quality-adjusted unit value quantity index for period t, $Q_{UV\alpha}{}^{t}$, and the Laspeyres quantity index for period t, $Q_L{}^{t}$, can be written as follows for $t = 1, \ldots, T$:

$$Q_{UV\alpha}{}^{t} - Q_L{}^{t} = [\alpha \cdot q^{t}/\alpha \cdot q^{1}] - [p^{1} \cdot q^{t}/p^{1} \cdot q^{1}] \text{ using (83) and (103)}$$
$$= \Sigma_{n=1}{}^{N} [(\alpha_n q_{tn}/\alpha \cdot q^{1}) - (p_{1n} q_{tn}/p^{1} \cdot q^{1})]$$
$$= \Sigma_{n=1}{}^{N} [(\alpha_n/\alpha \cdot q^{1}) - (p_{1n}/p^{1} \cdot q^{1})] q_{tn}$$
$$= \Sigma_{n=1}{}^{N} [(\alpha_n p^{1} \cdot q^{1}/\alpha \cdot q^{1}) - p_{1n}][q_{tn}/p^{1} \cdot q^{1}]$$
$$= \Sigma_{n=1}{}^{N} [\alpha_n p_{UV\alpha}{}^{1} - p_{1n}][q_{tn}/p^{1} \cdot q^{1}] \text{ using (104) for } t = 1$$
$$= \Sigma_{n=1}{}^{N} [\alpha_n p_{UV\alpha}{}^{1} - p_{1n}][q_{tn} - q_{1n} Q_{UV\alpha}{}^{t}]/p^{1} \cdot q^{1} \text{ using (106) for } t = 1$$
$$= Q_{UV\alpha}{}^{t} \Sigma_{n=1}{}^{N} \alpha_n [p_{UV\alpha}{}^{1} - (p_{1n}/\alpha_n)][(q_{tn}/Q_{UV\alpha}{}^{t}) - q_{1n}]/p^{1} \cdot q^{1}$$
$$= Q_{UV\alpha}{}^{t} \varepsilon_{La}{}^{t}, \qquad (107)$$

where the period t error term $\varepsilon_{La}{}^{t}$ for $t = 1, \ldots, T$ is defined as

$$\varepsilon_{L\alpha}{}^{t} \equiv \Sigma_{n=1}{}^{N} \alpha_n [p_{UV\alpha}{}^{1} - (p_{1n}/\alpha_n)][(q_{tn}/Q_{UV\alpha}{}^{t}) - q_{1n}]/p^{1} \cdot q^{1}.^{102} \qquad (108)$$

Assuming that $\alpha_n > 0$ for $n = 1, \ldots, N$, the vector of period t *quality-adjusted prices* $p_\alpha{}^{t}$ for $t = 1, \ldots, T$ is defined as follows:

$$p_\alpha{}^{t} \equiv [p_{t1\alpha}, \ldots, p_{tN\alpha}] \equiv [p_{t1}/\alpha_1, p_{t2}/\alpha_2, \ldots, p_{tN}/\alpha_N]. \qquad (109)$$

It can be seen that $p_{UV\alpha}{}^{1} - (p_{1n}/\alpha_n)$ is the difference between the period 1 unit value price level, $p_{UV\alpha}{}^{1}$, and the period 1

[99] The term "quality-adjusted unit value price index" was introduced by Dalén (2001). Its properties were further studied by de Haan (2004b, 2010) and de Haan and Krsinich (2018). von Auer (2014) considered a wide variety of choices for the weight vector α (including $\alpha = p^{1}$ and $\alpha = p^{t}$), and he looked at the axiomatic properties of the resulting indices.

[100] Some methods for estimating α_n are suggested in Diewert and Feenstra (2017) and Diewert (2022c).

[101] Quality-adjusted unit value price and quantity levels are also consistent with Leontief (no substitution) preferences. In this case, the dual unit cost function is equal to $c(p) \equiv \Sigma_{n=1}{}^{N} \beta_n p_n$, where β_n are the positive preference parameters. The period t quantity vector that is consistent with these preferences is $q^{t} = u_t \beta$ for $t = 1, \ldots, T$, where $\beta \equiv [\beta_1, \ldots, \beta_N]$ and u_t is the period t utility level. Thus, the quantity vectors q^{t} will vary in strict proportion over time. This model of consumer behavior is inconsistent with situations where there are new and disappearing products over the T periods. Moreover, empirically, quantity vectors do not vary in a proportional manner over time.

[102] This error term is homogeneous of degree 0 in the components of p^{1}, q^{1}, and q^{t}. Hence, it is invariant to proportional changes in the components of these vectors. Definition (108) is only valid if all $\alpha_n > 0$. If this is not the case, redefine $\varepsilon_{L\alpha}{}^{t}$ as $\Sigma_{n=1}{}^{N} [\alpha_n p_{UV\alpha}{}^{1} - p_{1n}][q_{tn} - q_{1n} Q_{UV\alpha}{}^{t}]/p^{1} \cdot q^{1}$ and with this change, the decomposition defined by the last line of (107) will continue to hold. It should be noted that $\varepsilon_{L\alpha}{}^{t}$ does not have an interpretation as a *covariance* between a vector of price differences and a vector of quantity differences.

quality-adjusted price for product n, p_{1n}/α_n. Define the period t *quality-adjusted quantity share for product n* (using the vector a of quality adjustment factors) as follows for t = 1, . . .,T and n = 1, . . .,N:

$$S_{tn\alpha} \equiv \alpha_n q_{tn}/\alpha{\cdot}q^t. \qquad (110)$$

The vector of *period t quality-adjusted real product shares* (using the vector α of quality adjustment factors) is defined as $S_a^{\,t} \equiv [S_{t1\alpha}, S_{t2\alpha}, \ldots, S_{tN\alpha}]$ for t = 1, . . .,T. It can be seen that these vectors are share vectors in that their components sum to 1; that is, we have for t = 1, . . .,T:

$$1_N{\cdot}S_\alpha^{\,t} = 1. \qquad (111)$$

Using the above definitions, we can show that the period t quality-adjusted unit value price level, $p_{UV\alpha}^{\,t}$, defined by (104) is equal to a share-weighted average of the period t quality-adjusted prices $p_{tn\alpha} = p_{tn}/\alpha_n$ defined by (109); that is, for t = 1, . . .,T, we have the following equations:

$$\begin{aligned} p_{UV\alpha}^{\,t} &= p^t{\cdot}q^t/\alpha{\cdot}q^t \text{ using (104)} \\ &= \Sigma_{n=1}^{N} (p_{tn}/\alpha_n)(\alpha_n q_{tn})/\alpha{\cdot}q^t \\ &= \Sigma_{n=1}^{N} S_{tn\alpha} p_{tn\text{a}} \text{ using (109) and (110)} \\ &= S_\alpha^{\,t}{\cdot}p_\alpha^{\,t}. \end{aligned} \qquad (112)$$

Now we are in a position to determine the likely sign of $\varepsilon_{La}^{\,t}$ defined by (108). If the products are substitutable, it is likely that $\varepsilon_{L\alpha}^{\,t}$ is *negative*, since if p_{1n} is unusually low, then it is likely that the quality-adjusted price for product n, p_{1n}/α_n, is below the weighted average of the quality-adjusted prices for period 1, which is $p_{UV\alpha}^{\,1} = S_\alpha^{\,1}{\cdot}p_\alpha^{\,1}$ using (112) for t = 1. Thus, we expect that $p_{UV\text{a}}^{\,1} - (p_{1n}/\alpha_n) > 0$. If p_{1n} is unusually low, then we would expect that the corresponding q_{1n} is unusually high, and thus it is likely that q_{1n} is greater than $q_{tn}/Q_{UV\alpha}^{\,t}$, and so $q_{tn}/Q_{UV\alpha}^{\,t} - q_{1n} < 0$. Thus, the sum of the N terms on the right-hand side of (108) is likely to be negative. Our expectation[103] is that the error term $\text{e}_{L\alpha}^{\,t} < 0$, and hence $Q_{UV\alpha}^{\,t} < Q_L^{\,t}$ for $t \geq 2$.

As usual, there are three special cases of (108) that will imply that $Q_{UV\alpha}^{\,t} = Q_L^{\,t}$: (i) $p_\alpha^{\,1} = \lambda_1 1_N$ so that all period 1 quality-adjusted prices are equal;[104] (ii) $q^t = 1_t q^1$ for t = 2,3, . . .,T so that quantities vary in strict proportion over time; (iii) the following sum of price differences times quantity differences equals 0; that is, $\Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^{\,1} - p_{1n}][(q_{tn}/Q_{UV\alpha}^{\,t}) - q_{1n}] = 0$.

If we divide both sides of equation t in equations (108) by $Q_{UV\alpha}^{\,t}$, we obtain the following system of identities for t = 1, . . .,T:

$$Q_L^{\,t}/Q_{UV\alpha}^{\,t} = 1 - \varepsilon_{L\text{a}}^{\,t}, \qquad (113)$$

where we expect $\varepsilon_{L\alpha}^{\,t}$ to be a small negative number if the products are substitutes.[105]

The difference between the reciprocal of the quality-adjusted unit value quantity index for period t, $[Q_{UV\alpha}^{\,t}]^{-1}$, and the reciprocal of the Paasche quantity index for period t, $[Q_P^{\,t}]^{-1}$, can be written as follows for t = 1, . . .,T:

$$\begin{aligned} [Q_{UV\alpha}^{\,t}]^{-1} - [Q_P^{\,t}]^{-1} &= [\alpha{\cdot}q^1/\alpha{\cdot}q^t] - [p^t{\cdot}q^1/p^t{\cdot}q^t] \\ &\quad \text{using (84) and (103)} \\ &= \Sigma_{n=1}^{N} [(\alpha_n q_{1n}/\alpha{\cdot}q^t) - (p_{tn}q_{1n}/p^t{\cdot}q^t)] \\ &= \Sigma_{n=1}^{N} [(\alpha_n/\alpha{\cdot}q^t) - (p_{tn}/p^t{\cdot}q^t)]q_{1n} \\ &= \Sigma_{n=1}^{N} [(\alpha_n p^t{\cdot}q^t/\alpha{\cdot}q^t) - p_{tn}][q_{1n}/p^t{\cdot}q^t] \\ &= \Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^{\,t} - p_{tn}][q_{1n}/p^t{\cdot}q^t] \text{ using (104)} \\ &= \Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^{\,t} - p_{tn}][q_{1n} - (q_{tn}/Q_{UV\alpha}^{\,t})]/p^t{\cdot}q^t \text{ using (106)} \\ &= [Q_{UV\alpha}^{\,t}]^{-1} \Sigma_{n=1}^{N} \alpha_n[p_{UV\alpha}^{\,t} - (p_{tn}/\alpha_n)][(q_{1n}Q_{UV\alpha}^{\,t}) - q_{tn}]/p^t{\cdot}q^t \\ &= [Q_{UV\alpha}^{\,t}]^{-1}\, \varepsilon_{P\text{a}}^{\,t}, \end{aligned} \qquad (114)$$

where the period t error term $\varepsilon_{P\text{a}}^{\,t}$ for t = 1, . . .,T is defined as

$$\varepsilon_{P\alpha}^{\,t} \equiv \Sigma_{n=1}^{N} \alpha_n[p_{UV\alpha}^{\,t} - (p_{tn}/\alpha_n)][(q_{1n}Q_{UV\alpha}^{\,t}) - q_{tn}]/p^t{\cdot}q^t.\text{[106]} \qquad (115)$$

If the products are substitutable, it is likely that $\text{e}_{P\alpha}^{\,t}$ is *negative*, since if p_{tn} is unusually low, then it is likely that the period t quality-adjusted price for product n, p_{tn}/α_n, is below the weighted average of the quality-adjusted prices for period t, which is $p_{UV\alpha}^{\,t} = S_\alpha^{\,t}{\cdot}p_\alpha^{\,t}$ using (112). Thus, we expect that $p_{UV\text{a}}^{\,t} - (p_{tn}/\alpha_n) > 0$. If p_{tn} is unusually low, then we would expect that the corresponding q_{tn} is unusually high, and thus it is likely that q_{tn} is greater than $q_{1n}Q_{UV\alpha}^{\,t}$, and so $q_{1n}Q_{UV\alpha}^{\,t} - q_{tn} < 0$. Thus, the sum of the N terms on the right-hand side of (115) is likely to be negative. Thus, our expectation is that the error term $\varepsilon_{Pa}^{\,t} < 0$ and hence $[Q_{UV\alpha}^{\,t}]^{-1} < [Q_P^{\,t}]^{-1}$ for $t \geq 2$. Assuming that $\varepsilon_{La}^{\,t}$ is also negative, we have $Q_P^{\,t} < Q_{UV\alpha}^{\,t} < Q_L^{\,t}$ for t = 2, . . .,T as inequalities that are likely to hold.

As usual, there are three special cases of (114) that will imply that $Q_{UV\alpha}^{\,t} = Q_P^{\,t}$: (i) $p_\alpha^{\,t} = \lambda_t 1_N$ so that all period t quality-adjusted prices are equal; (ii) $q^t = \lambda_t q^1$ for t = 2, 3, . . .,T so that quantities vary in strict proportion over time; (iii) the following sum of price differences times quantity differences equals zero: that is, $\Sigma_{n=1}^{N} [\alpha_n p_{UV\text{a}}^{\,t} - p_{tn}][(q_{1n}Q_{UV\alpha}^{\,t}) - q_{tn}] = 0$.

If we divide both sides of equation t in equations (114) by $[Q_{UV\alpha}^{\,t}]^{-1}$, we obtain the following system of identities for t = 1, . . .,T:

$$Q_P^{\,t}/Q_{UV\alpha}^{\,t} = [1 - \varepsilon_{P\text{a}}^{\,t}]^{-1}, \qquad (116)$$

[103] As in the previous section, this expectation is not held with great conviction if the period t quantities have a large variance.

[104] The condition $p_\alpha^{\,1} = \lambda_1 1_N$ is equivalent to $p^1 = \lambda_1\alpha$. Thus, if we choose a to be proportional to the period 1 price vector p^1, then $Q_{UV\alpha}^{\,t} = Q_L^{\,t}$ and $P_{UV\alpha}^{\,t} = P_P^{\,t}$, the fixed-base Paasche price index. Thus, with this choice of a, the quality-adjusted unit value index will usually have a downward bias relative to a superlative index. This result requires that p^1 be strictly positive.

[105] If $q_{1n} = 0$ and the period 1 quality-adjusted reservation price p_{1n}/α_n is greater than the period 1 unit value price $p_{UV\alpha}^{\,1}$, then $\varepsilon_{La}^{\,t}$ defined by (108) could be a large negative number.

[106] This error term is homogeneous of degree 0 in the components of p^t, q^1, and q^t. Hence, it is invariant to proportional changes in the components of these vectors. Definition (115) is only valid if all $\text{a}_n > 0$. If this is not the case, then redefine $\varepsilon_{P\alpha}^{\,t}$ as $\Sigma_{n=1}^{N} [\alpha_n p_{UV\alpha}^{\,t} - p_{tn}][(q_{1n}Q_{UV\alpha}^{\,t}) - q_{tn}]/p^t{\cdot}q^t$, and with this change, the decomposition defined by the last line of (114) will continue to hold..

where we expect $\varepsilon_{P\alpha}{}^t$ to be a small negative number if the products are substitutes.

Equations (113) and (116) develop exact relationships for the quality-adjusted unit value quantity index $Q_{UV\alpha}{}^t$ with the corresponding fixed-base Laspeyres and Paasche quantity indices, $Q_L{}^t$ and $Q_P{}^t$. Taking the square root of the product of these two sets of equations leads to the following exact relationships between the fixed-base Fisher quantity index, $Q_F{}^t$, and its quality-adjusted unit value counterpart period t quantity index, $Q_{UV\alpha}{}^t$, for $t = 1, \ldots, T$:

$$Q_F{}^t = Q_{UV\alpha}{}^t\{(1 - \varepsilon_{L\alpha}{}^t)/(1 - \varepsilon_{P\alpha}{}^t)\}^{1/2}, \tag{117}$$

where $\varepsilon_{L\alpha}{}^t$ and $\varepsilon_{P\alpha}{}^t$ are defined by (108) and (115). If there are no strong (divergent) trends in prices and quantities, then it is likely that $\varepsilon_{L\alpha}{}^t$ is approximately equal to $e_{P\alpha}{}^t$, and hence under these conditions, it is likely that $Q_{UV\alpha}{}^t \approx Q_F{}^t$; that is, the quality-adjusted unit value quantity index will provide an adequate approximation to the fixed-base Fisher price index under these conditions. However, if there are divergent trends in prices and quantities (in opposite directions), then it is likely that $e_{P\alpha}{}^t$ will be more negative than $\varepsilon_{L\alpha}{}^t$, and hence it is likely that $Q_F{}^t < Q_{UV\alpha}{}^t$ for $t = 2, \ldots, T$; that is, *with divergent trends in prices and quantities, the quality-adjusted unit value quantity index is likely to have an upward bias* relative to its Fisher quantity index counterparts.[107]

Using equations (105), we have the following counterparts to equations (94) for $t = 1, \ldots, T$:

$$p^t{\cdot}q^t/p^1{\cdot}q^1 = P_{UV\alpha}{}^tQ_{UV\alpha}{}^t = P_P{}^tQ_L{}^t = P_L{}^tQ_P{}^t. \tag{118}$$

Equations (113), (116), and (118) imply the following identities for $t = 1, \ldots, T$:

$$P_{UV\alpha}{}^t/P_P{}^t = Q_L{}^t/Q_{UV\alpha}{}^t = 1 - \varepsilon_{L\alpha}{}^t; \tag{119}$$
$$P_{UV\alpha}{}^t/P_L{}^t = Q_P{}^t/Q_{UV\alpha}{}^t = [1 - \varepsilon_{P\alpha}{}^t]^{-1}. \tag{120}$$

We expect that $\varepsilon_L{}^t$ and $\varepsilon_{P\alpha}{}^t$ will be predominantly negative if the products are highly substitutable, and thus in this case, the quality-adjusted unit value indices $P_{UV\alpha}{}^t$ should satisfy the inequalities $P_P{}^t < P_{UV\alpha}{}^t < P_L{}^t$ for $t = 2,3, \ldots, T$.

Taking the square root of the product of equations (119) and (120) leads to the following exact relationships between the fixed-base Fisher price index, $P_F{}^t$, and its quality-adjusted unit value counterpart period t index, $P_{UV\alpha}{}^t$, for $t = 1, \ldots, T$:

$$P_{UV\alpha}{}^t = P_F{}^t\{(1 - \varepsilon_{L\alpha}{}^t)/(1 - \varepsilon_{P\alpha}{}^t)\}^{1/2}, \tag{121}$$

where $\varepsilon_{L\alpha}{}^t$ and $\varepsilon_{P\alpha}{}^t$ are defined by (108) and (115). If there are no strong (divergent) trends in prices and quantities, then it is likely that $\varepsilon_{L\alpha}{}^t$ is approximately equal to $e_{P\alpha}{}^t$, and hence under these conditions, it is likely that $P_{UV\alpha}{}^t \approx P_F{}^t$; that is, the quality-adjusted unit value price index will provide an adequate approximation to the fixed-base Fisher price index under these conditions. However, if there are divergent trends in prices and quantities, then we expect $\varepsilon_{P\alpha}{}^t$ to be more negative than $e_{L\alpha}{}^t$, and hence there is an expectation that $P_{UV\alpha}{}^t < P_F{}^t$ for $t = 2, \ldots, T$; that is, we expect that normally $P_{UV\alpha}{}^t$ will have a *downward bias* relative to $P_F{}^t$.[108] However, if there are missing products in period 1, then the bias of $P_{UV\alpha}{}^t$ relative to $P_F{}^t$ is uncertain.

11. Relationships between Lowe and Fisher Indices

We now consider how a Lowe (1823) price index is related to a fixed-base Fisher price index. The framework that we consider is similar to the framework developed in Section 6 for the annual share-weighted Jevons index, $P_{J\alpha}{}^t$. In the present section, instead of using the average sales shares for the first year in the sample as weights for a weighted Jevons index, we use annual average quantities sold (or purchased) in the first year as a vector of quantity weights for subsequent periods. Define the *annual average quantity vector* $q^* \equiv [q_1{}^*, \ldots, q_N{}^*]$ for the first T^* periods in the sample that make up a year, q^*, as follows:[109]

$$q^* \equiv (1/T^*)\Sigma_{t=1}{}^{T^*}\, q^t. \tag{122}$$

As was the case in Section 6, the reference year for the weights precedes the reference month for the product prices. Define the *period t Lowe* (1823) *price level* and *price index*, $p_{Lo}{}^t$ and $P_{Lo}{}^t$ by (123) and (124), respectively, for $t = T^* + 1, T^* + 2, \ldots, T$:

$$p_{Lo}{}^t \equiv p^t{\cdot}\alpha; \tag{123}$$
$$P_{Lo}{}^t \equiv p_{Lo}{}^t/p_{Lo}{}^{T^*+1} = p^t{\cdot}\alpha/p^{T^*+1}{\cdot}\alpha, \tag{124}$$

where the constant price weights vector α is the annual average weights vector q^* defined by (122); that is, we have

$$\alpha \equiv q^*. \tag{125}$$

The *period t Lowe quantity level*, $q_{Lo}{}^t$, and the corresponding *period t Lowe quantity index*, $Q_{Lo}{}^t$, for $t = T^* + 1, T^* + 2, \ldots, T$ are defined as follows:

[107] As was the case in the previous section, if there are missing products in period 1, the expected inequality $Q_F{}^t < Q_{UV\alpha}{}^t$ may be reversed because $\varepsilon_{L\alpha}{}^t$ defined by (108) may become significantly negative if some q_{1n} equal 0, while their corresponding reservation prices p_{1n} are positive.

[108] Recall that the weighted unit value quantity level, $q_{UV\alpha}{}^t$ is defined as the linear function of the period t quantity data, $\alpha{\cdot}q^t$. If T ≥ 3 and the price and quantity data are consistent with purchasers maximizing a utility function that generates data that is exact for the Fisher price index $Q_F{}^t$, then $Q_{UV\alpha}{}^t$ will tend to be greater than $Q_F{}^t$ (and hence $P_{UV\alpha}{}^t$ will tend to be less than $P_F{}^t$) for $t \geq 2$. See Marris (1984, 52), Diewert (1999b, 49), and Diewert and Fox (2021) on this point.

[109] If product n was not available in the first year of the sample, then the nth component of q^*, $q_n{}^*$, will equal 0 and hence the nth component of the weight vector α defined by (125) will also equal 0. If product n was also not available in periods $t \geq T^* + 1$, then looking at definitions (123) and (124), it can be seen that $P_{LO}{}^t$ will not depend on the reservation prices p_{nt} for these subsequent periods where product n is not available. Thus, under these circumstances, the Lowe index cannot be consistent with the (Hicksian) economic approach to index number theory since Konüs (1924) true cost of living price indices will depend on the reservation prices. However, if the products in the elementary aggregate are indeed highly substitutable, then the assumption of a linear utility function will provide an adequate approximation to the "truth" and the estimation of reservation prices becomes unimportant.

$$q_{Lo}{}^{t} \equiv p^{t}\cdot q^{t}/p_{Lo}{}^{t} = p^{t}\cdot q^{t}/p^{t}\cdot\alpha = \Sigma_{n=1}{}^{N}\,(p_{tn}\alpha_{n}/p^{t}\cdot\alpha)(q_{tn}/\alpha_{n});^{110} \quad (126)$$

$$Q_{Lo}{}^{t} \equiv q_{Lo}{}^{t}/q_{Lo}{}^{T^*+1} = [p^{t}\cdot q^{t}/p^{T^*+1}\cdot q^{T^*+1}]/P_{Lo}{}^{t}. \quad (127)$$

It can be seen that the Lowe price index defined by (124) is equal to a *weighted Dutot price index*; see definition (14). It is also structurally identical to the quality-adjusted unit value quantity index $Q_{UV\alpha}{}^{t}$ defined in the previous section, except that the role of prices and quantities has been reversed. Thus, the identity (107) in the previous section will be valid if we replace $Q_{UV\alpha}{}^{t}$ by $P_{Lo}{}^{t}$, replace $Q_{L}{}^{t}$ by $P_{L}{}^{t}$, and interchange prices and quantities on the right-hand side of (107).[111] The resulting identities for $t = T^* + 1, T^* + 2, \ldots, T$ are as follows:

$$\begin{aligned} P_{Lo}{}^{t} - P_{L}{}^{t} &= \Sigma_{n=1}{}^{N}\,[(\alpha_{n}p_{tn}/\alpha\cdot p^{T^*+1}) - (p_{tn}q_{T^*+1,n}/p^{T^*+1}\cdot q^{T^*+1})] \\ &= \Sigma_{n=1}{}^{N}\,[(\alpha_{n}/\alpha\cdot p^{T^*+1}) - (q_{T^*+1,n}/p^{T^*+1}\cdot q^{T^*+1})]p_{tn} \\ &= \Sigma_{n=1}{}^{N}\,[(\alpha_{n}p^{T^*+1}\cdot q^{T^*+1}/\alpha\cdot p^{T^*+1}) - q_{T^*+1,n}][p_{tn}/p^{T^*+1}\cdot q^{T^*+1}] \\ &= \Sigma_{n=1}{}^{N}\,[\alpha_{n}q_{Lo}{}^{T^*+1} - q_{T^*+1,n}][p_{tn}/p^{T^*+1}\cdot q^{T^*+1}] \text{ using (126) for } t = T^*+1 \\ &= \Sigma_{n=1}{}^{N}\,[\alpha_{n}q_{Lo}{}^{T^*+1} - q_{T^*+1,n}][p_{tn} - p_{T^*+1,n}P_{Lo}{}^{t}]/p^{T^*+1}\cdot q^{T^*+1}\ ^{112} \\ &= P_{Lo}{}^{t}\,\Sigma_{n=1}{}^{N}\,[\alpha_{n}q_{Lo}{}^{T^*+1} - q_{T^*+1,n}][(p_{tn}/P_{Lo}{}^{t}) - p_{T^*+1,n}]/p^{T^*+1}\cdot q^{T^*+1} \\ &= P_{Lo}{}^{t}\,\Sigma_{n=1}{}^{N}\,\alpha_{n}[q_{Lo}{}^{T^*+1} - (q_{T^*+1,n}/\alpha_{n})][(p_{tn}/P_{Lo}{}^{t}) - p_{T^*+1,n}]/p^{T^*+1}\cdot q^{T^*+1} \\ &= P_{Lo}{}^{t}\,\varepsilon_{L\alpha}{}^{t}, \end{aligned} \quad (128)$$

where the period t error term $\varepsilon_{La}{}^{t}$ is now defined for $t = T^* + 1, \ldots, T$ as follows:

$$\varepsilon_{L\alpha}{}^{t} \equiv \Sigma_{n=1}{}^{N}\,\alpha_{n}[q_{Lo}{}^{T^*+1} - (q_{T^*+1,n}/\alpha_{n})][(p_{tn}/P_{Lo}{}^{t}) - p_{T^*+1,n}]/p^{T^*+1}\cdot q^{T^*+1}.^{113} \quad (129)$$

If the products are substitutable, it is likely that $e_{L\alpha}{}^{t}$ is *negative*, since if $p_{T^*+1,n}$ is unusually low, then it is likely that $(p_{tn}/P_{Lo}{}^{t}) - p_{T^*+1,n} > 0$ and that $q_{T^*+1,n}/\alpha_{n}$ is unusually large and hence is greater than $q_{Lo}{}^{T^*+1}$, which is a weighted average of the period $T^* + 1$ quantity ratios, $q_{T^*+1,1}/\alpha_{1}, q_{T^*+1,2}/\alpha_{2}, \ldots, q_{T^*+1,N}/\alpha_{N}$ using definition (126) for $t = T^* + 1$. Thus, the sum of the N terms on the right-hand side of (129) is likely to be negative. Thus, our expectation[114] is that the error term $e_{L\alpha}{}^{t} < 0$ and hence $P_{Lo}{}^{t} < P_{L}{}^{t}$ for $t > T^* + 1$.

α_{n} can be interpreted as *inverse quality indicators* of the utility provided by one unit of the nth product. Suppose purchasers of the N commodities have Leontief preferences with the utility function $f(q_1,q_2, \ldots, q_N) \equiv \min_n \{q_n/\alpha_n : n = 1,2, \ldots, N\}$. Then, the dual unit cost function that corresponds to this functional form is $c(p_1,p_2, \ldots, p_N) \equiv \Sigma_{n=1}{}^{N}\,p_n\alpha_n = p\cdot\alpha$. If we evaluate the unit cost function at the prices of period t, p^t, we obtain the Lowe price level for period t defined by (123); that is, $p_{Lo}{}^{t} \equiv p^{t}\cdot\alpha$. Thus, the bigger α_n is, the more units of q_n it will take for purchasers of the N commodities to attain one unit of utility. Thus, α_n can be interpreted as inverse indicators of the relative utility of each product.

As usual, there are three special cases of (128) that will imply that $P_{Lo}{}^{t} = P_{L}{}^{t}$: (i) $q^{T^*+1} = lq^*$ for some $\lambda > 0$ so that the period $T^* + 1$ quantity vector q^{T^*+1} is proportional to the annual average quantity vector q^* for the base year; (ii) $p^t = \lambda_t p^{T^*+1}$ for some $\lambda_t > 0$ and $t = T^* + 1, \ldots, T$ so that prices vary in strict proportion over time; and (iii) the sum of terms $\Sigma_{n=1}{}^{N}\,[\alpha_n q_{Lo}{}^{T^*+1} - q_{T^*+1,n}][(p_{tn}/P_{Lo}{}^{t}) - p_{T^*+1,n}] = 0$.

If we divide both sides of equation t in equations (128) by $P_{Lo}{}^{t}$, we obtain the following system of identities for $t = T^* + 1, \ldots, T$:

$$P_{L}{}^{t}/P_{Lo}{}^{t} = 1 - \varepsilon_{La}{}^{t}, \quad (130)$$

where we expect $\varepsilon_{L\alpha}{}^{t}$ to be a small negative number.

We turn now to developing a relationship between the Lowe and Paasche price indices. The difference between the reciprocal of the Lowe price index for period t, $[P_{Lo}{}^{t}]^{-1}$, and the reciprocal of the Paasche price index for period t, $[P_{P}{}^{t}]^{-1}$, can be written as follows for $t = T^* + 1, \ldots, T$:

$$\begin{aligned} [P_{Lo}{}^{t}]^{-1} - [P_{P}{}^{t}]^{-1} &= [\alpha\cdot p^{T^*+1}/\alpha\cdot p^{t}] - [q^{t}\cdot p^{T^*+1}/q^{t}\cdot p^{t}] \\ &= \Sigma_{n=1}{}^{N}\,[(\alpha_{n}p_{T^*+1,n}/\alpha\cdot p^{t}) - (q_{tn}p_{T^*+1,n}/p^{t}\cdot q^{t})] \\ &= \Sigma_{n=1}{}^{N}\,[(\alpha_{n}/\alpha\cdot p^{t}) - (q_{tn}/p^{t}\cdot q^{t})]p_{T^*+1,n} \\ &= \Sigma_{n=1}{}^{N}\,[(\alpha_{n}p^{t}\cdot q^{t}/\alpha\cdot p^{t}) - q_{tn}][p_{T^*+1,n}/p^{t}\cdot q^{t}] \\ &= \Sigma_{n=1}{}^{N}\,[\alpha_{n}q_{Lo}{}^{t} - q_{tn}][p_{T^*+1,n}/p^{t}\cdot q^{t}] \text{ using (126)} \\ &= \Sigma_{n=1}{}^{N}\,[\alpha_{n}q_{Lo}{}^{t} - q_{tn}][p_{T^*+1,n} - (p_{tn}/P_{Lo}{}^{t})]/p^{t}\cdot q^{t\,115} \\ &= [P_{Lo}{}^{t}]^{-1}\,\Sigma_{n=1}{}^{N}\,[\alpha_{n}q_{Lo}{}^{t} - q_{tn}][p_{T^*+1,n}P_{Lo}{}^{t} - p_{tn}]/p^{t}\cdot q^{t} \\ &= [P_{Lo}{}^{t}]^{-1}\,\Sigma_{n=1}{}^{N}\,\alpha_{n}[q_{Lo}{}^{t} - (q_{tn}/\alpha_{n})][p_{T^*+1,n}P_{Lo}{}^{t} - p_{tn}]/p^{t}\cdot q^{t} \text{ if all } \alpha_{n} > 0 \\ &= [P_{Lo}{}^{t}]^{-1}\,\varepsilon_{Pa}{}^{t}, \end{aligned} \quad (131)$$

where the period t error term $\varepsilon_{Pa}{}^{t}$ for $t = T^* + 1, \ldots, T$ is defined as

$$\varepsilon_{P\alpha}{}^{t} \equiv \Sigma_{n=1}{}^{N}\,\alpha_{n}[q_{Lo}{}^{t} - (q_{tn}/\alpha_{n})][p_{T^*+1,n}P_{Lo}{}^{t} - p_{tn}]/p^{t}\cdot q^{t}.^{116} \quad (132)$$

[110] This last inequality is only valid if all $\alpha_n > 0$. It can be seen that the Lowe quantity level for period t, $q_{Lo}{}^{t}$, is a share-weighted sum of the period t quality-adjusted quantities, q_{tn}/α_n.

[111] We also replace period 1 by period $T^* + 1$.

[112] This step follows from the following counterpart to (106): $\Sigma_{n=1}{}^{N}\,[\alpha_n q_{Lo}{}^{T^*+1} - q_{T^*+1,n}]p_{T^*+1,n} = 0$.

[113] Note that this error term is homogeneous of degree 0 in the components of p^{T^*+1}, q^{T^*+1}, and p^t. Hence, it is invariant to proportional changes in the components of these vectors. Definition (129) is valid only if all $\alpha_n > 0$. If this is not the case, redefine $\varepsilon_{L\alpha}{}^{t}$ as $\Sigma_{n=1}{}^{N}\,[\alpha_n q_{Lo}{}^{T^*+1} - q_{T^*+1,n}][(p_{tn}/P_{Lo}{}^{t}) - p_{T^*+1,n}]/p^{T^*+1}\cdot q^{T^*+1}$, and with this change, the decomposition defined by the last line of (128) will continue to hold.

[114] This expectation is not held with great conviction if the period t prices have a large variance.

[115] This step follows from the following counterpart to (106): $\Sigma_{n=1}{}^{N}\,[\alpha_n q_{Lo}{}^{t} - q_{tn}]p_{tn} = 0$.

[116] This error term is homogeneous of degree 0 in the components of q^t, p^{T^*+1}, and p^t. Hence, it is invariant to proportional changes in the components of these vectors. Definition (132) is only valid if all $\alpha_n > 0$. If this

If the products are substitutable, it is likely that $e_{P\alpha}{}^t$ is *negative*, since if p_{tn} is unusually low, then it is likely that it will be less than the inflation-adjusted *n*th component of the period $T^* + 1$ price, $p_{T^*+1,n}P_{LO}{}^t$. If p_{tn} is unusually low, then it is also likely that the period *t* quality-adjusted quantity for product *n*, q_{tn}/α_n, is above the weighted average of the quality-adjusted quantities for period *t*, which is $q_{Lo}{}^t$. Thus, the sum of the *N* terms on the right-hand side of (132) is likely to be negative. Moreover, our expectation is that the error term $\varepsilon_{P\alpha}{}^t < 0$ and hence $[P_{Lo}{}^t]^{-1} < [P_P{}^t]^{-1}$ for $T^* + 2, \ldots, T$. Assuming that $\varepsilon_{La}{}^t$ is also negative, we have $P_P{}^t < P_{Lo}{}^t < P_L{}^t$ for $t = T^* + 2, T^* + 3, \ldots, T$ as inequalities that are likely to hold.

As usual, there are three special cases of (131) that will imply that $P_{Lo}{}^t = P_P{}^t$: (i) $q^t = \lambda q^*$ for some $\lambda > 0$ so that the period *t* quantity vector q^t is proportional to the annual average quantity vector q^* for the reference year prior to the reference month; (ii) $p^t = \lambda_t p^{T^*+1}$ for $t = T^* + 2, T^* + 3, \ldots, T$ so that prices vary in strict proportion over time; and (iii) the sum of terms $\Sigma_{n=1}{}^N [\alpha_n q_{Lo}{}^t - q_{tn}][p_{T^*+1,n}P_{Lo}{}^t - p_{tn}] = 0$.

If we divide both sides of equation *t* in equations (131) by $[P_{Lo}{}^t]^{-1}$, we obtain the following system of identities for $t = T^* + 1, \ldots, T$:

$$P_P{}^t/P_{Lo}{}^t = [1 - \varepsilon_{Pa}{}^t]^{-1}, \qquad (133)$$

where we expect $\varepsilon_{P\alpha}{}^t$ to be a negative number.

Equations (130) and (133) develop exact relationships for the Lowe price index $P_{Lo}{}^t$ with the corresponding fixed-base Laspeyres and Paasche price indices, $P_L{}^t$ and $P_P{}^t$. Taking the square root of the product of these two sets of equations leads to the following exact relationships between the fixed-base Fisher price index, $P_F{}^t$, and the corresponding Lowe period *t* price index, $P_{Lo}{}^t$, for $t = T^* + 1, \ldots, T$:

$$P_F{}^t = P_{Lo}{}^t\{(1 - \varepsilon_{L\alpha}{}^t)/(1 - \varepsilon_{Pa}{}^t)\}^{1/2}, \qquad (134)$$

where $\varepsilon_{L\alpha}{}^t$ and $\varepsilon_{Pa}{}^t$ are defined by (129) and (132). If there are no strong (divergent) trends in prices and quantities, then it is likely that $\varepsilon_{L\alpha}{}^t$ is approximately equal to $e_{P\alpha}{}^t$, and hence under these conditions, it is likely that $P_{Lo}{}^t \approx P_F{}^t$; that is, the Lowe price index will provide an adequate approximation to the fixed-base Fisher price index under these conditions. However, if there are divergent trends in prices and quantities (in diverging directions), then it is likely that $e_{P\alpha}{}^t$ will be more negative than $\varepsilon_{L\alpha}{}^t$, and hence it is likely that $P_F{}^t < P_{Lo}{}^t$ for $t = T^* + 2, \ldots, T$; that is, *with divergent trends in prices and quantities, the Lowe price index is likely to have an upward bias* relative to its Fisher Price index counterpart. This is an intuitively plausible result since the Lowe index is a fixed basket type index and hence will be subject to some upward substitution bias relative to the Fisher index, which is able to control the substitution bias.

In the following section, we show that the GK multilateral indices can be regarded as quality-adjusted unit value price indices, and hence the analysis in Section 10 on quality-adjusted unit value price indices can be applied to GK multilateral indices.

is not the case, redefine $\varepsilon_{Pa}{}^t$ as $\Sigma_{n=1}{}^N [\alpha_n q_{Lo}{}^t - q_{tn}][p_{T^*+1,n}P_{Lo}{}^t - p_{tn}]/p^t \cdot q^t$, and with this change, the decomposition defined by the last line of (131) will continue to hold.

12. Geary–Khamis Multilateral Indices

The GK multilateral method was introduced by Geary (1958) in the context of making international comparisons of prices. Khamis (1970) showed that the equations that define the method have a positive solution under certain conditions. A modification of this method has been adapted to the time series context and is being used to construct some components of the Dutch CPI; see Chessa (2016). The GK index was the multilateral index chosen by the Dutch to avoid the chain drift problem for the segments of their CPI that use scanner data.

The GK system of equations for *T* time periods involves *T price levels* ${}_{GK}{}^1, \ldots, p_{GK}{}^T$ and *N quality adjustment factors* $\alpha_1, \ldots, \alpha_N$.[117] Let p^t and q^t denote the *N*-dimensional price and quantity vectors for period *t* (with components p_{tn} and q_{tn} as usual). Define the total consumption (or sales) vector *q* over the entire window of observations as the following simple sum of the period-by-period consumption vectors:

$$q \equiv \Sigma_{t=1}{}^T q^t, \qquad (135)$$

where $q \equiv [q_1, q_2, \ldots, q_N]$. The equations that determine the *GK price levels* $p_{GK}{}^1, \ldots, p_{GK}{}^T$ and *quality adjustment factors* $\alpha_1, \ldots, \alpha_N$ (up to a scalar multiple) are as follows:

$$\alpha_n = \Sigma_{t=1}{}^T [q_{tn}/q_n][p_{tn}/p_{GK}{}^t];\ n = 1, \ldots, N; \qquad (136)$$

$$p_{GK}{}^t = p^t \cdot q^t/\alpha \cdot q^t = \Sigma_{n=1}{}^N [\alpha_n q_{tn}/\alpha \cdot q^t][p_{tn}/\alpha_n];\quad t = 1, \ldots, T, \qquad (137)$$

where $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$ is the vector of GK quality adjustment factors. The sample share of period *t*'s purchases of commodity *n* in total sales of commodity *n* over all *T* periods can be defined as $S_{tn} \equiv q_{tn}/q_n$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. Thus, $a_n \equiv \Sigma_{t=1}{}^T S_{tn}[p_{tn}/p_{GK}{}^t]$ is a (real) share-weighted average of the period *t* inflation-adjusted prices $p_{tn}/p_{GK}{}^t$ for product *n* over all *T* periods. The period *t* quality-adjusted sum of quantities sold is defined as the *period t GK quantity level*, $q_{GK}{}^t \equiv \alpha \cdot q^t = S_{n=1}{}^N \alpha_n q_{tn}$.[118] This period *t* quantity level is divided into the value of period *t* sales, $p^t \cdot q^t = \Sigma_{n=1}{}^N p_{tn}q_{tn}$, in order to obtain the period *t* GK price level, $p_{GK}{}^t$. Thus, the GK price level for period *t* can be interpreted as a *quality-adjusted unit value index*, where α_n act as the quality adjustment factors.

Note that the GK price level, $p_{GK}{}^t$, defined by (137) *does not depend on the estimated reservation prices*; that is, the definition of $p_{GK}{}^t$ zeros out any reservation prices that are applied to missing products, and thus $P_{GK}{}^t \equiv p_{GK}{}^t/p_{GK}{}^1$ does

[117] In the international context, α_n are interpreted as international commodity reference prices.
[118] Khamis (1972, 101) also derived this equation in the time series context.

not depend on reservation prices.[119] A related property of the GK price levels is the following one: If a product n^* is only available in a single period t^*, then the GK price levels $p_{GK}{}^t$ do not depend on $p_{n^*t^*}$ or $q_{n^*t^*}$.[120]

It can be seen that if a solution to equations (136) and (137) exists, then if all of the period price levels $p_{GK}{}^t$ are multiplied by a positive scalar λ, say, and all of the quality adjustment factors α_n are divided by the same λ, then another solution to (136) and (137) is obtained. Hence, α_n and $p_{GK}{}^t$ are only determined up to a scalar multiple, and an additional normalization such as $p_{GK}{}^1 = 1$ or $\alpha_1 = 1$ is required to determine a unique solution to the system of equations defined by (136) and (137).[121] It can also be shown that only $N + T - 1$ of the $N + T$ equations in (136) and (137) are independent.

Using the normalization $p_{GK}{}^1 = 1$, it is straightforward to show that the GK price levels, $p_{GK}{}^t$, are invariant to changes in the units of measurement. Suppose we have a solution $p_{GK}{}^t$ and α_n for $t = 1, \ldots, T$ and $n = 1, \ldots, N$ with $p_{GK}{}^1 \equiv 1$. Let $\lambda_n > 0$ for $n = 1, \ldots, N$. Use these λ_n to measure prices and quantities in new units of measurement; that is, define $p_{tn}{}^* \equiv \lambda_n p_{tn}$ and $q_{tn}{}^* \equiv (\lambda_n)^{-1} q_{tn}$ for $t = 1, \ldots, T$ and $n = 1, \ldots, N$. Now substitute these transformed prices and quantities into equations (135)–(137). It is straightforward to show that the initial solution GK price levels, $p_{GK}{}^t$, along with new $\alpha_n{}^* \equiv \lambda_n \alpha_n$ also satisfy the new GK equations (135)–(137).

A traditional method for obtaining a solution to (136) and (137) is to iterate between these equations. Thus, set $a = 1_N$, a vector of ones, and use equations (137) to obtain an initial sequence for the $p_{GK}{}^t$. Substitute these $p_{GK}{}^t$ estimates into equation (136) and obtain α_n estimates. Substitute these α_n estimates into equation (137) and obtain a new sequence of $p_{GK}{}^t$ estimates. Continue iterating between the two systems until convergence is achieved.

An alternative method is more efficient. Following Diewert (1999b, 26),[122] substitute equations (137) into equations (136), and after some simplification, obtain the following system of equations that will determine the components of the α vector:

$$[\mathrm{I}_N - C]\alpha = 0_N, \qquad (138)$$

where I_N is the N by N identity matrix, 0_N is a vector of zeros of dimension N, and the C matrix is defined as follows:

$$C \equiv \hat{q}^{-1} \Sigma_{t=1}{}^T s^t q^{tT}, \qquad (139)$$

where $\hat{q}$ is an N by N diagonal matrix with the elements of the total window purchase vector q running down the main diagonal, and $\hat{q}^{-1}$ denotes the inverse of this matrix, s^t is the period t expenditure share column vector, q^t is the column vector of quantities purchased during period t, and q_n is the nth element of the sample total q defined by (135).

The matrix $\mathrm{I}_N - C$ is singular, which implies that the N equations in (138) are not all independent. In particular, if the first N–1 equations in (138) are satisfied, then the last equation in (138) will also be satisfied. It can also be seen that the N equations in (138) are homogeneous of degree one in the components of the vector α. Thus, to obtain a unique b solution to (138), set α_N equal to 1, drop the last equation in (138), and solve the remaining N–1 equations for $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$. Once the α_n are known, equations (137) can be used to determine the GK price levels, $p_{GK}{}^t = p^t \cdot q^t / \alpha \cdot q^t$ for $t = 1, \ldots, T$.

Using equations (137), it can be seen that the *GK price index for period t* (relative to period 1) is equal to $P_{GK}{}^t \equiv p_{GK}{}^t / p_{GK}{}^1 = [p^t \cdot q^t / \alpha \cdot q^t] / [p^1 \cdot q^1 / \alpha \cdot q^1]$ for $t = 1, \ldots, T$, and thus these indices are *quality-adjusted unit value price indices* with a particular choice for the vector of quality adjustment factors α. Thus, these indices lead to corresponding *additive quantity levels* $q_{GK}{}^t$ that correspond to the linear utility function, $f(q) \equiv \alpha \cdot q$.[123] As we saw in Section 10, this type of index can approximate the corresponding fixed-base Fisher price index provided that there are no systematic divergent trends in prices and quantities. However, if there are diverging trends in prices and quantities (in opposite directions), then we expect the GK price indices to be subject to some *substitution bias* with the expectation that the GK price index for period $t \geq 2$ be somewhat *below* the corresponding Fisher fixed-base price index. Thus, we expect GK and quality-adjusted unit value price indices to *normally* have a downward bias relative to their Fisher and Törnqvist counterparts, provided that there are no missing products, the products are highly substitutable, and there are divergent trends in prices and quantities. However, if there are missing products in period 1, then it is quite possible for the GK price indices to have an upward bias relative to their Fisher fixed-base counterparts, which, in principle, use reservation prices for the missing products.[124]

In the following five sections, we will study in some detail another popular method for making price level comparisons over multiple periods: the weighted TPD multilateral indices. The general case with missing observations will be studied in

[119] In equations (136) and (137), each price p_{tn} always appears with the multiplicative factor q_{tn}. Thus, if p_{tn} is an imputed price, it will always be multiplied by $q_{tn} = 0$, and thus any imputed price will have no impact on α_n and $p_{GK}{}^t$. Thus, this method fails Test 9 in Section 21.

[120] Let product n^* be available only in period t^*. Using (136) for $n = n^*$, we have (i) $\alpha_{n^*} = p_{t^*n^*}/p_{GK}{}^{t^*}$. Equations (137) can be rewritten as follows: (ii) $p_{GK}{}^t \, \alpha \cdot q^t = p^t \cdot q^t$; $t = 1, \ldots, T$. Note that for $t \neq t^*$, these equations do not depend directly on $\alpha_{n^*}, p_{t^*n^*}$ or $q_{t^*n^*}$. For period $t = t^*$, equation t^* in (137) can be written as (iii) $p_{GK}{}^{t^*}(\Sigma_{n'n^*} a_n q_{t^*n} + \alpha_{n^*} q_{t^*n^*}) = (\Sigma_{n \neq n^*} p_{t^*n} q_{t^*n} + p_{t^*n^*} q_{t^*n^*})$. Substitute (i) into (iii) and after some simplification, we find that $p_{GK}{}^{t^*} = \Sigma_{n \neq n^*} p_{t^*n} q_{t^*n} / \Sigma_{n'n^*} a_n q_{t^*n}$. This proof was achieved by Claude Lamboray. Thus, this method fails Test 8 in Section 21.

[121] See Diewert and Fox (2021) for various solution methods.

[122] See also Diewert and Fox (2021) for additional discussion on this solution method.

[123] Using the economic approach to index number theory, it can be seen that the GK price indices will be exactly the correct price indices to use if purchasers maximize utility using a common linear utility function. Diewert (1999b, 27) and Diewert and Fox (2021) show that the GK price indices will also be exactly correct if purchasers maximize a Leontief no substitution utility function. These extreme cases are empirically unlikely. As was noted earlier in Section 10, Leontief preferences are not consistent with new and disappearing products.

[124] New products appear with some degree of regularity, and so it is likely that there will be missing products in period 1, and this may reverse the "normal" inequality, $P_{GK}{}^t < P_F{}^t$, as was the case for Diewert's (2018) scanner data set. This data set is used in the Annex to this chapter. The GK index, like all indices based on quality-adjusted unit values, zeros out the effects of reservation prices for the missing products, whereas Fisher indices can include the effects of reservation prices.

Section 17. It proves to be useful to consider simpler special cases of the method in Sections 13–16.

13. Time Product Dummy Regressions: The Case of No Missing Observations

In this section, it is assumed that price and quantity data for N products are available for T periods. As usual, let $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ denote the price and quantity vectors for time periods $t = 1, \ldots, T$. In this section, it is assumed that there are no missing prices or quantities, so that all NT prices and quantities are positive. We assume initially that purchasers of the N products maximize the linear utility function $f(q)$ defined as follows:

$$f(q) = f(q_1, q_2, \ldots, q_N) \equiv \Sigma_{n=1}^{N} \alpha_n q_n = \alpha \cdot q, \quad (140)$$

where α_n are the positive parameters, which can be interpreted as quality adjustment parameters. Under the assumption of maximizing behavior on the part of purchasers of the N commodities, Wold's Identity[125] applied to a linearly homogeneous utility function tells us that the purchasers' system of *inverse demand functions* should satisfy the following equations:

$$\begin{aligned} p^t &= v^t \nabla f(q^t)/f(q^t);\ t = 1, \ldots, T \\ &= [v^t/f(q^t)]\nabla f(q^t) \\ &= P^t \nabla f(q^t), \end{aligned} \quad (141)$$

where $v^t \equiv p^t \cdot q^t$ is period t expenditure on the N commodities, P^t is the *period t aggregate price level* defined as $v^t/f(q^t) = v^t/Q^t$, and $Q^t \equiv f(q^t)$ is the corresponding *period t aggregate quantity level* for $t = 1, \ldots, T$.

Since $f(q)$ is defined by (140), $\nabla f(q^t) = \alpha \equiv [\alpha_1, \ldots, \alpha_N]$ for $t = 1, \ldots, T$. By substituting these equations into equations (141), we obtain the following equations, which should hold exactly under our assumptions:

$$p_{tn} = \pi_t \alpha_n;\ n = 1, \ldots, N;\ t = 1, \ldots, T, \quad (142)$$

where we have redefined the period t price levels P^t in equations (141) as the parameters π_t for $t = 1, \ldots, T$.

Note that equation (142) forms the basis for the *time dummy hedonic regression model*, which was developed by Court (1939).[126]

At this point, it is necessary to point out that our consumer theory derivation of equation (142) is not accepted by all economists. Rosen (1974), Triplett (1987, 2004), and Pakes (2001)[127] have argued for a more general approach to the derivation of hedonic regression models that is based on supply conditions as well as on demand conditions. The present approach is obviously based on consumer demands and preferences only. This consumer-oriented approach was endorsed by Griliches (1971, 14–15), Muellbauer (1974, 988), and Diewert (2003a, 2003b).[128] Of course, the assumption that purchasers have the same linear utility function is quite restrictive but nevertheless, it is useful to imbed hedonic regression models in a traditional consumer demand setting.

Empirically, equation (142) is unlikely to hold exactly. Thus, we assume that the exact model defined by (142) holds only to some degree of approximation and so error terms, e_{tn}, are added to the right-hand sides of equation (142). The unknown price level parameters, $\pi \equiv [\pi_1, \ldots, \pi_T]$, and quality adjustment parameters, $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$, can be estimated as solutions to the following (nonlinear) least squares minimization problem:

$$\min_{\alpha,\pi} \{\Sigma_{n=1}^{N} \Sigma_{t=1}^{T} [p_{tn} - \pi_t \alpha_n]^2\}. \quad (143)$$

Our approach to the specification of the error terms will not be very precise. Throughout this chapter, we will obtain estimators for the aggregate price levels π_t and the quality adjustment parameters α_n as solutions to least squares minimization problems like those defined by (143) or as solutions to weighted least squares minimization problems that will be considered in subsequent sections. Our focus will not be on the distributional aspects of our estimators; rather, our focus will be on the *axiomatic* or *test properties* of the price levels that are solutions to the various least squares minimization problems.[129] Basically, the approach taken here is a descriptive statistics approach: We consider simple models that aggregate price and quantity information for a given period over a set of specified commodities into scalar measures of aggregate price and quantity that summarize

[125] See Section 4 in Diewert (2022a).

[126] This was Court's (1939, 109–11) hedonic suggestion number two. He transformed the underlying equations (142) by taking logarithms of both sides of these equations (which will be done below). He chose to transform the prices by the log transformation because the resulting regression model fit his data on automobiles better. Diewert (2003b) also recommended the log transformation on the grounds that multiplicative errors were more plausible than additive errors.

[127] "The derivatives of a hedonic price function should not be interpreted as either willingness to pay derivatives or cost derivatives; rather they are formed from a complex equilibrium process" (Ariel Pakes, 2001, 14).

[128] Diewert (2003b, 97) justified the consumer demand approach as follows: "After all, the purpose of the hedonic exercise is to find how demanders (and not suppliers) of the product value alternative models in a given period. Thus for the present purpose, it is the preferences of consumers that should be decisive, and not the technology and market power of producers. The situation is similar to ordinary general equilibrium theory where an equilibrium price and quantity for each commodity is determined by the interaction of consumer preferences and producer's technology sets and market power. However, there is a big branch of applied econometrics that ignores this complex interaction and simply uses information on the prices that consumers face, the quantities that they demand and perhaps demographic information in order to estimate systems of consumer demand functions. Then these estimated demand functions are used to form estimated consumer utility functions and these functions are often used in applied welfare economics. What producers are doing is entirely irrelevant to these exercises in applied econometrics with the exception of the prices that they are offering to sell at. In other words, we do not need information on producer marginal costs and markups in order to estimate consumer preferences: all we need are selling prices." Footnote 25 on page 82 of Diewert (2003b) explained how the present hedonic model can be derived from Diewert's (2003a) consumer-based model by strengthening the assumptions in the 2003a paper.

[129] For rigorous econometric approaches to the stochastic approach to index number theory, see Rao and Hajargasht (2016) and Gorajek (2018). These papers consider many transformations of the fundamental hedonic equations (143) and many methods for constructing averages of prices.

the detailed price and quantity information in a "sensible" way.[130]

The first-order necessary (and sufficient) conditions for $\pi \equiv [\pi_1, \ldots, \pi_T]$ and $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$ to solve the minimization problem defined by (143) are equivalent to the following $N + T$ equations:

$$\alpha_n = \Sigma_{t=1}^{T} \pi_t p_{tn} / \Sigma_{t=1}^{T} \pi_t^2 \quad n = 1, \ldots, N$$
$$= \Sigma_{t=1}^{T} \pi_t^2 (p_{tn}/\pi_t) / \Sigma_{t=1}^{T} \pi_t^2; \quad (144)$$

$$\pi_t = \Sigma_{n=1}^{N} \alpha_n p_{tn} / \Sigma_{n=1}^{N} \alpha_n^2 \quad t = 1, \ldots, T$$
$$= \Sigma_{n=1}^{N} \alpha_n^2 (p_{tn}/\alpha_n) / \Sigma_{n=1}^{N} \alpha_n^2. \quad (145)$$

Solutions to the two sets of equations can readily be obtained by iterating between the two sets of equations. Thus, set $\alpha^{(1)} = 1_N$ (a vector of ones of dimension N) in equations (145) and calculate the resulting $\pi^{(1)} = [\pi_1^{(1)}, \ldots, p_T^{(1)}]$. Then substitute $\pi^{(1)}$ into the right-hand sides of equations (144) to calculate $\alpha^{(2)} \equiv [\alpha_1^{(2)}, \ldots, \alpha_N^{(2)}]$. This is continued until convergence is achieved.

If $p^* \equiv [\pi_1^*, \ldots, \pi_T^*]$ and $\alpha^* \equiv [\alpha_1^*, \ldots, \alpha_N^*]$ is a solution to (144) and (145), then $\lambda\pi^*$ and $\lambda^{-1}\alpha^*$ is also a solution for any $\lambda > 0$. Thus to obtain a unique solution we impose the normalization $\pi_1^* = 1$. Then, $1, \pi_2^*, \ldots, \pi_T^*$ is the sequence of fixed-base aggregate price levels that is generated by the least squares minimization problem defined by (143).

If quantity data are available, then aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \alpha^* \cdot q^t = \Sigma_{n=1}^{N} \alpha_n^* q_{tn}$ for $t = 1, \ldots, T$. Estimated aggregate price levels can be obtained directly from the solution to (143); that is, set $P^{t*} = \pi_t^*$ for $t = 1, \ldots, T$. Alternative price levels can be *indirectly* obtained as $P^{t**} \equiv p^t \cdot q^t / Q^{t*} = p^t \cdot q^t / \alpha^* \cdot q^t$ for $t = 1, \ldots, T$. If the optimized objective function in (143) is 0 (so that all errors $e_{tn}^* \equiv p_{tn} - \pi_t^* \alpha_n^*$ equal 0 for $t = 1, \ldots, T$ and $n = 1, \ldots, N$), then P^{t*} will equal P^{t**} for all t. However, usually nonzero errors will occur and so a choice between the two sets of estimators must be made.[131]

From (144), it can be seen that α_n^*, the quality adjustment parameter for product n, is a weighted average of the T inflation-adjusted prices for product n, the p_{tn}/π_t^*, where the weight for p_{tn}/π_t^* is $\pi_t^{*2} / \Sigma_{\tau=1}^{T} \pi_\tau^{*2}$. This means that the weight for p_{tn}/π_t^* will be very high for periods t where general inflation is high, which seems rather arbitrary. From (145), it can be seen that π_t^*, the period t price level (and fixed-base price index), is weighted average of the N quality-adjusted prices for period t, the p_{tn}/α_n^*, where the weight for p_{tn}/α_n^* is $\alpha_n^{*2} / \Sigma_{i=1}^{N} \alpha_i^{*2}$. It is a positive feature of the method that π_t^* is a weighted average of the quality-adjusted prices for period t but the quadratic nature of the weights is not an attractive feature.

In addition to having unattractive weighting properties, the estimates generated by solving the least squares minimization problem (143) suffer from a fatal flaw: *the estimates are not invariant to changes in the units of measurement*. In order to remedy this defect, we turn to an alternative error specification.

Instead of adding approximation errors to the exact equations (142), we could append multiplicative approximation errors. Thus the exact equations become $p_{tn} = \pi_t \alpha_n e_{tn}$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. By taking logarithms of both sides of these equations, we obtain the following system of estimating equations:

$$lnp_{tn} = ln\pi_t + ln\alpha_n + lne_{tn}; \; n = 1, \ldots, N; \; t = 1, \ldots, T$$
$$= \rho_t + \beta_n + \varepsilon_{tn}, \quad (146)$$

where $\rho_t \equiv \ln\pi_t$ for $t = 1, \ldots, T$ and $\beta_n \equiv \ln\alpha_n$ for $n = 1, \ldots, N$. The model defined by (146) is the basic *TPD regression model* with no missing observations.[132] Now choose the ρ_t and β_n to minimize the sum of squared residuals, $\Sigma_{n=1}^{N} \Sigma_{t=1}^{T} \varepsilon_{tn}^2$. Thus let $\rho \equiv [\rho_1, \ldots, r_T]$ and $\beta \equiv [\beta_1, \ldots, b_N]$ be a solution to the following least squares minimization problem:

$$\min_{\rho, \beta} \{\Sigma_{n=1}^{N} \Sigma_{t=1}^{T} [lnp_{tn} - \rho_t - \beta_n]^2\}. \quad (147)$$

The first-order necessary conditions for $\rho_1, \ldots, \rho_T$ and $\beta_1, \ldots, \beta_N$ to solve (147) are given by the following $T + N$ equations:

$$N\rho_t + \Sigma_{n=1}^{N} \beta_n = \Sigma_{n=1}^{N} lnp_{tn}; \; t = 1, \ldots, T; \quad (148)$$
$$\Sigma_{t=1}^{T} \rho_t + T\beta_n = \Sigma_{t=1}^{T} lnp_{tn}; \; n = 1, \ldots, N. \quad (149)$$

Replace ρ_t and β_n in equations (148) and (149) by $\ln\pi_t$ and $\ln\alpha_n$, respectively, for $t = 1, \ldots, T$ and $n = 1, \ldots, N$. After some rearrangement, the resulting equations become

$$\pi_t = \prod_{n=1}^{N} (p_{tn}/\alpha_n)^{1/N}; \; t = 1, \ldots, T; \quad (150)$$
$$\alpha_n = \prod_{t=1}^{T} (p_{tn}/\pi_t)^{1/T}; \; n = 1, \ldots, N. \quad (151)$$

Thus, the period t aggregate price level, π_t, is equal to the geometric average of the N quality-adjusted prices for period t, $p_{t1}/\alpha_1, \ldots, p_{tN}/\alpha_N$, while the quality adjustment factor for product n, α_n, is equal to the geometric average of the T inflation-adjusted prices for product n, $p_{1n}/\pi_1, \ldots, p_{Tn}/\pi_T$. These estimators look very reasonable (if quantity weights are not available).

Solutions to (150) and (151) can readily be obtained by iterating between the two sets of equations. Thus set $\alpha^{(1)} = 1_N$ (a vector of ones of dimension N) in equations (150) and calculate the resulting $\pi^{(1)} = [\pi_1^{(1)}, \ldots, p_T^{(1)}]$. Then substitute $\pi^{(1)}$ into the right-hand side of equation (151) to calculate $\alpha^{(2)} \equiv [\alpha_1^{(2)}, \ldots, \alpha_N^{(2)}]$. This is continued until convergence is achieved. Alternatively, equations (148) and (149) are linear in the

[130] Our approach here is broadly similar to Theil's (1967, 136–37) descriptive statistics approach to index number theory.

[131] Usually, the direct estimates for the price levels will be used in hedonic regression studies or in applications of the TPD method; that is, the $P^{t*} = \pi_t^*$ estimates will be used. For statistical agencies, an advantage of the direct estimates is that they can be calculated without the use of quantity information. However, later in this chapter, we will note some advantages of the indirect method if quantity information is available.

[132] In the statistics literature, this type of model is known as a fixed effects model. A generalized version of this model (with missing observations) was proposed by Summers (1973) in the international comparison context where it is known as the country product dummy regression model. A weighted version of this model (with missing observations) was proposed by Aizcorbe, Corrado, and Doms (2000).

unknown parameters and can be solved (after normalizing one parameter) by a simple matrix inversion. A final method of obtaining a solution to (148) and (149) is to apply a simple linear regression model to equations (146).[133]

If $\pi^* \equiv [\pi_1^*, \ldots, \pi_T^*]$ and $\alpha^* \equiv [\alpha_1^*, \ldots, \alpha_N^*]$ is a solution to (148) and (149), then $\lambda\pi^*$ and $\lambda^{-1}\alpha^*$ is also a solution for any $\lambda > 0$. Thus, to obtain a unique solution we impose the normalization $\pi_1^* = 1$ (which corresponds to $\rho_1 = 0$). Then $1, \pi_2^*, \ldots, \pi_T^*$ is the sequence of fixed-base index numbers that is generated by the least squares minimization problem defined by (147).

Once we have the unique solution $1, \pi_2^*, \ldots, \pi_T^*$ for the T price levels that are generated by solving (147) along with the normalization $\pi_1 = 1$, the *price index* between period t relative to period s can be defined as p_t^*/π_s^*. Using equations (150) for π_t^* and π_s^*, we have the following equation for these price indices:

$$\pi_t^*/\pi_s^* = \prod_{n=1}^{N} (p_{tn}/\alpha_n^*)^{1/N} / \prod_{n=1}^{N} (p_{sn}/\alpha_n^*)^{1/N} = \prod_{n=1}^{N} (p_{tn}/p_{sn})^{1/N}. \quad (152)$$

Thus, if there are no missing observations, the TPD price indices between any two periods in the window of T period under consideration is equal to the *Jevons index* between the two periods (the simple geometric mean of the price ratios, p_{tn}/p_{sn}).[134] This is a somewhat disappointing result since an equally weighted average of the price ratios is not necessarily a representative average of the prices; that is, unimportant products to purchasers (in the sense that they spend very little on these products) are given the same weight in the Jevons measure of inflation between the two periods as is given to high expenditure products.[135]

Since there are no missing observations, then it can be seen using equations (151) that the ratio of the quality adjustment factor for product n relative to product m is equal to the following sensible expression:

$$\alpha_n^*/\alpha_m^* = \prod_{t=1}^{T} (p_{tn}/\pi_t^*)^{1/T} / \prod_{t=1}^{T} (p_{tm}/\pi_t^*)^{1/T} = \prod_{t=1}^{T} (p_{tn}/p_{tm})^{1/T}. \quad (153)$$

If quantity data are available, then aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \alpha^* \cdot q^t = \Sigma_{n=1}^{N} \alpha_n^* q_{tn}$ for $t = 1, \ldots, T$. Estimated aggregate price levels can be obtained directly from the solution to (147); that is, set $P^{t*} = \pi_t^*$ for $t = 1, \ldots, T$. Alternative price levels can be obtained *indirectly* as $P^{t**} \equiv p^t \cdot q^t / Q^{t*} = p^t \cdot q^t / \alpha^* \cdot q^t$ for $t = 1, \ldots, T$.[136] If the optimized objective function in (147) is 0 (so that all errors $e_{tn}^* \equiv \ln p_{tn} - \rho_t^* - \beta_n^*$ equal 0 for $t = 1, \ldots, T$ and $n = 1, \ldots, N$), then P^{t*} will equal P^{t**} for all t. If the estimated residuals are not all equal to 0, then the two estimates for the period t price level P^t will differ in general. The two alternative estimates for P^t will generate different estimates for the companion aggregate quantity levels.

Note that the underlying exact model ($p_{tn} = \pi_t \alpha_n$ for all t and n) is the same for both least squares minimization problems, (143) and (147). However, different error specifications and different transformations of both sides of the equations $p_{tn} = \pi_t \alpha_n$ can lead to very different estimators for π_t and α_n. Our strategy in this section and in the following sections will be to choose specifications of the least squares minimization problem that lead to estimators for the price levels p_t that have good axiomatic properties.[137] From this perspective, it is clear that (147) leads to "better" estimates than (143).

In the following section, we allow for missing observations.

14. Time Product Dummy Regressions: The Case of Missing Observations

In this section, the least squares minimization problem defined by (147) is generalized to allow for missing observations. In order to make this generalization, it is first necessary to make some definitions. As in the previous section, there are N products and T time periods but not all products are purchased (or sold) in all time periods. For each period t, define the set of products n that are present in period t as $S(t) \equiv \{n: p_{tn} > 0\}$ for $t = 1, 2, \ldots, T$. It is assumed that these sets are not empty; that is, at least one product is purchased in each period. For each product n, define the set of periods t where product n is present as $S^*(n) \equiv \{t: p_{tn} > 0\}$. Again, assume that these sets are not empty; that is, each product is sold in at least one time period. Define the integers $N(t)$ and $T(n)$ as follows:

$$N(t) \equiv \Sigma_{n \in S(t)} 1; \; t = 1, \ldots, T; \quad (154)$$

$$T(n) \equiv \Sigma_{t \in S^*(n)} 1; \; n = 1, \ldots, N. \quad (155)$$

If all N products are present in period t, then $N(t) = N$; if product n is present in all T periods, then $T(n) = T$.

The multilateral methods studied in previous sections assumed that reservation prices were available for missing products in any period. Thus, the methods discussed up until the present section assumed that there were no missing product prices: p_{tn} was either an actual period t price for product n or an estimated price for the product if it was missing in period t. When discussing the TPD multilateral price levels and indices, we do *not* assume that reservation prices for missing products have been estimated. Instead, the method generates estimated prices for the missing products.

[133] Again we require one normalization on the parameters such as $\rho_1 = 0$.

[134] This result is a special case of a more general result obtained by Triplett and McDonald (1977, 150).

[135] However, if quantity data are not available, the Jevons index has the strongest axiomatic properties; see Diewert (2021b).

[136] The fact that a time dummy hedonic regression model generates two alternative decompositions of the value aggregate into price and quantity aggregates was first noted by de Haan and Krsinich (2018).

[137] From the perspective of the economic approach to index number theory, the minimization problems (143) and (147) have exactly the same justification; that is, they are based on the same economic model of consumer behavior.

Using the above notation for missing products, the counterpart to (147) when there are missing products is the following least squares minimization problem:

$$\min_{\rho,\beta} \{\Sigma_{t=1}^{T} \Sigma_{n\in S(t)} [lnp_{tn} - \rho_t - \beta_n]^2\}$$
$$= min_{\rho,\beta} \{\Sigma_{n=1}^{N} \Sigma_{t\in S^*(n)} [lnp_{tn} - \rho_t - \beta_n]^2\}. \qquad (156)$$

Note that there are two equivalent ways of writing the least squares minimization problem.[138] The first-order necessary conditions for $\rho_1, \ldots, r_T$ and $\beta_1, \ldots, b_N$ to solve (156) are the following counterparts to (148) and (149):

$$\Sigma_{n\in S(t)} [\rho_t + \beta_n] = \Sigma_{n\in S(t)}\, lnp_{tn};\ t = 1, \ldots, T; \qquad (157)$$
$$\Sigma_{t\in S^*(n)} [\rho_t + \beta_n] = \Sigma_{t\in S^*(n)}\, lnp_{tn};\ n = 1, \ldots, N. \qquad (158)$$

As in the previous section, let $\rho_t \equiv \ln\pi_t$ for $t = 1, \ldots, T$ and let $\beta_n \equiv \ln\alpha_n$ for $n = 1, \ldots, N$. Substitute these definitions into equations (157) and (158). After some rearrangement and using definitions (154) and (155), equations (157) and (158) become the following ones:

$$\pi_t = \prod_{n\in S(t)} [p_{tn}/\alpha_n]^{1/N(t)};\ t = 1, \ldots, T; \qquad (159)$$
$$\alpha_n = \prod_{t\in S^*(n)} [p_{tn}/\pi_t]^{1/T(n)};\ n = 1, \ldots, N. \qquad (160)$$

The same iterative procedure that was explained in the previous section will work to generate a solution to equations (159) and (160).[139] As was the case in the previous section, solutions to (159) and (160) are not unique; if π^*, α^* is a solution to (159) and (160), then $\lambda\pi^*$ and $\lambda^{-1}\alpha^*$ is also a solution for any $\lambda > 0$. Thus, to obtain a unique solution we impose the normalization $\pi_1^* = 1$ (which corresponds to $\rho_1 = 0$). Then $1, \pi_2^*, \ldots, \pi_T^*$ is the sequence of (normalized) price levels that is generated by the least squares minimization problem defined by (156).[140] In this case, $\pi_t^* = \prod_{n\in S(t)}[p_{tn}/\alpha_n^*]^{1/N(t)}$ is the equally weighted geometric mean of all of the quality-adjusted prices for the products that are available in period t for $t = 2,3, \ldots, T$ and the quality adjustment factors are normalized so that $\pi_1^* = \prod_{n\in S(1)} [p_{1n}/\alpha_n^*]^{1/N(1)} = 1$. From (160), we can deduce that α_n^* will be larger for products that are relatively expensive and will be smaller for cheaper products.

Once we have the unique solution $1, \pi_2^*, \ldots, \pi_T^*$ for the T price levels that are generated by solving (156), the *price index* between period t relative to period r can be defined as π_t^*/π_r^*. Using equations (159) and (160), we have the following expressions for p_t^*/π_r^* and α_n^*/α_m^*:

$$\pi_t^*/\pi_r^* = \prod_{n\in S(t)}[p_{tn}/\alpha_n^*]^{1/N(t)} / \prod_{n\in S(r)}[p_{rn}/\alpha_n^*]^{1/N(r)};$$
$$1 \le t, r \le T; \qquad (161)$$
$$\alpha_n^*/\alpha_m^* = \prod_{t\in S^*(n)}[p_{tn}/\pi_t^*]^{1/T(n)} / \prod_{t\in S^*(m)}[p_{tm}/\pi_t^*]^{1/T(m)};$$
$$1 \le n, m \le N. \qquad (162)$$

Note that, in general, the quality adjustment factors α_n^* do not cancel out for the indices π_t^*/π_r^* defined by (161) as they did in the previous section. However, these price indices do have some good axiomatic properties.[141] If the set of available products is the same in periods r and t, then the quality adjustment factors do cancel and the price index for period t relative to period r is $\pi_t^*/\pi_r^* = \prod_{n\in S(t)}[p_{tn}/p_{rn}]^{1/N(t)}$, which is the Jevons index between periods r and t. Again, while this index is an excellent one if quantity information is not available, it is not satisfactory when quantity information is available due to its equal weighting of economically important and unimportant price ratios.[142]

There is another problematic property of the estimated price levels that are generated by solving the TPD hedonic model that is defined by (156): A product that is available only in one period out of the T periods has no influence on the aggregate price levels π_t^*.[143] To see this, consider equations (157) and (158) and suppose that product n^* was available only in period t^*.[144] Equation n^* in the N equations in (158) becomes the equation $[\rho_{t^*} + \beta_{n^*}] = lnp_{t^*n^*}$. Thus, once ρ_{t^*} has been determined, b_{n^*} can be defined as $\beta_{n^*} \equiv lnp_{t^*n^*} - \rho_{t^*}$. Subtract the equation $[\rho_{t^*} + \beta_{n^*}] = lnp_{t^*n^*}$ from equation t^* and the resulting equations in (157) can be written as equations (163). Dropping equation n^* in equations (158) leads to equations (164):

$$\Sigma_{n\in S(t),\, n\neq n^*}[\rho_t + \beta_n] = \Sigma_{n\in S(t),\, n\neq n^*}\, lnp_{tn};\ t = 1, \ldots, T; \qquad (163)$$
$$\Sigma_{t\in S^*(n)} [\rho_t + \beta_n] = \Sigma_{t\in S^*(n)}\, lnp_{tn};$$
$$n = 1, \ldots, n^*-1, n^* + 1, \ldots, N. \qquad (164)$$

Equations (163) and (164) are $T + N - 1$ equations that do not involve $p_{t^*n^*}$. After making the normalization $\rho_1^* = 0$,

[138] The first expression is used when (156) is differentiated with respect to ρ_t, and the second expression is used when differentiating (156) with respect to β_n.

[139] Of course, it is not necessary to use the iterative procedure to find a solution to equations (157) and (158). After setting $\rho_1 = 0$ and dropping the first equation in (157), matrix algebra can be used to find a solution to the remaining equations. Alternatively, after setting $\rho_1 = 0$, use the equations $lnp_{tn} = \rho_t + \beta_n + \varepsilon_{tn}$ for $t = 1, \ldots, T$ and $n\in S(t)$ to set up a linear regression model with time and product dummy variables and use a standard ordinary least squares econometric software package to obtain the solution $\rho_2^*, \ldots, \rho_T^*, \beta_1^*, \ldots, \beta_N^*$ to the linear regression model $lnp_{tn} = \rho_t + \beta_n + \varepsilon_{tn}$ for $t = 1, \ldots, T$ and $n\in S(t)$. We need to assume that the X matrix for this linear regression model has full column rank.

[140] We need enough observations on products that are present so that a full rank condition is satisfied for equations (157) and (158) after dropping one equation and setting $r_1 = 0$. If there is a rapid proliferation of new and disappearing products, then it may not be possible to invert the coefficient matrix that is associated with the modified equations (157) and (158). In subsequent models with missing observations, we will assume that a similar full rank condition is satisfied.

[141] The index π_t^*/π_r^* satisfies the identity test (if prices are the same in periods r and t, then the index is equal to 1), and it is invariant to changes in the units of measurement. It is also homogeneous of degree one in the prices of period t and homogeneous of degree minus one in the prices of period r.

[142] However, if the estimated squared residuals are small in magnitude for periods τ and t, then the index p_t^*/π_r^* defined by (161) will be satisfactory, since in this case $p^\tau \approx \pi_\tau^*\alpha^*$ and p^t » $p_t^*\alpha^*$ so that prices are approximately proportional for these two periods and π_t^*/π_r^* defined by (161) will be approximately correct. Any missing prices for any period t and product n are defined as $p_{tn}^* \equiv \pi_t^*\alpha_n^*$.

[143] This property of the TPD model was first noticed by Diewert (2004) (in the context of the country product dummy model).

[144] We assume that products other than product n^* are available in period t^*.

these equations can be solved for $\rho_2^*, \ldots, \rho_T^*, \beta_1^*, \ldots, \beta_{n^*-1}^*, \beta_{n^*+1}^*, \ldots, b_N^*$. Now define $\beta_{n^*}^* \equiv lnp_{t^*n^*} - \rho_{t^*}$, and we have the (normalized) solution for (156). Since ρ_t^* do not involve $p_{t^*n^*}$, the resulting $\pi_t^* \equiv \exp[\rho_t^*]$ for $t = 1, \ldots, T$ also do not depend on the isolated price $p_{t^*n^*}$. This proof can be repeated for any number of isolated prices. This property of the TPD model is unfortunate because it means that when a new product enters the marketplace in period T, it has no influence on the price levels $1, \pi_2^*, \ldots, \pi_T^*$ that are generated by solving the least squares minimization problem defined by (156). In other words, an expansion in the choice of products available to consumers will have no effect on price levels.

If quantity data are available, then aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \Sigma_{n\in S(t)} \alpha_n^* q_{tn}$ for $t = 1, \ldots, T$.[145] Estimated aggregate price levels can be obtained directly from the solution to (42); that is, set $P^{t*} = \pi_t^*$ for $t = 1, \ldots, T$. Alternative price levels can be obtained indirectly as $P^{t**} \equiv \Sigma_{n\in S(t)} p_n q_{tn}/Q^{t*} = \Sigma_{n\in S(t)} p_n q_{tn}/\Sigma_{n\in S(t)} \alpha_n^* q_{tn}$ for $t = 1, \ldots, T$.[146] If the optimized objective function in (156) is 0, so that all errors $\varepsilon_{tn}^* \equiv \ln p_{tn} - \rho_t^* - \beta_n^*$ equal 0 for $t = 1, \ldots, T$ and $n\in S(t)$, then P^{t*} will equal P^{t**} for all t. If the estimated residuals are not all equal to 0, then the two estimates for the period t price level P^t will differ. The two estimates for P^t will generate different estimates for the companion aggregate quantity levels.

15. Weighted Time Product Dummy Regressions: The Bilateral Case

A major problem with the indices discussed in the previous two sections is the fact that they do not weight the individual product prices by their economic importance. The first serious index number economist to stress the importance of weighting was Walsh (1901).[147] Keynes was quick to follow up on the importance of weighting,[148] and Fisher emphatically endorsed weighting.[149] Griliches also endorsed weighting in the hedonic regression context.[150]

In this section, we will discuss some alternative methods for weighting by economic importance in the context of a bilateral time product dummy regression model.[151] We also assume that there are no missing observations in this section.

Recall the least squares minimization problem defined by (147) in Section 13. The squared residuals, $[lnp_{tn} - \rho_t - \beta_n]^2$, appear in this problem without any weighting. Thus, products, which have a high volume of sales in any period, are given the same weight in the least squares minimization problem as products that have very few sales. In order to take economic importance into account, for the case of two time periods, replace (147) by the following *weighted least squares minimization problem*:

$$\min_{\rho,\beta} \{\Sigma_{n=1}^N q_{1n}[lnp_{1n} - \beta_n]^2 + \Sigma_{n=1}^N q_{2n}[lnp_{2n} - \rho_2 - \beta_n]^2\}, \qquad (165)$$

where we have set $\rho_1 = 0$. The squared error for product n in period t is repeated q_{tn} times to reflect the sales of the product in period t. Thus, the new problem (165) takes into account the popularity of each product.[152]

The first-order necessary conditions for the minimization problem defined by (165) are the following $N + 1$ equations:

$$(q_{1n} + q_{2n})\beta_n = q_{1n}lnp_{1n} + q_{2n}(lnp_{2n} - \rho_2); \quad n = 1, \ldots, N; \qquad (166)$$

$$(\Sigma_{n=1}^N q_{2n})\rho_2 = \Sigma_{n=1}^N q_{2n}(lnp_{2n} - \beta_n). \qquad (167)$$

[145] Note that each $a_n^* > 0$ since $\alpha_n^* \equiv \exp[\beta_n^*]$ for $n = 1, \ldots, N$.

[146] Note that $P^{t**} \equiv \Sigma_{n\in S(t)} p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^* q_{tn}$ is a period t *quality-adjusted unit value price level*; see Section 10. The corresponding quantity level is $Q^{t**} \equiv \Sigma_{n\in S(t)} p_{tn}q_{tn}/P^{t**} = \Sigma_{n\in S(t)} \alpha_n^* q_{tn}$, which is the level generated by a *linear aggregator function*. By looking at (156), it can be seen that if prices are identical in periods t and r so that $p^t = p^r$, then $P^{t*} = P^{r*}$; that is, an identity test for the direct hedonic price levels will be satisfied. However, the corresponding Q^{t*} will not satisfy the identity test for quantity levels; that is, if quantities q_{tn} and q_{rn} are equal in periods t and r for all n, it is not the case that $Q^{t*} \equiv \Sigma_{n=1}^N p_{tn}q_{tn}/p_t^*$ will equal $Q^{r*} \equiv \Sigma_{n=1}^N p_{rn}q_{rn}/p_r^*$ for $r \neq t$ unless prices are also equal for the two periods. On the other hand, it can be seen that $Q^{t**} = \Sigma_{n\in S(t)} \alpha_n^* q_{tn} = \Sigma_{n\in S(t)} \alpha_n^* q_{rn} = Q^{r*}$ if $q_{tn} = q_{rn}$ for all n even if prices are not identical for the two periods. Thus, the choice between using P^{t*} and P^{t**} could be made on the basis of choosing which identity test is more important to satisfy. The analysis here follows that of de Haan and Krsinich (2018, 763–64).

[147] See Walsh (1901). This book laid the groundwork for the test or axiomatic approach to index number theory that was further developed by Fisher (1922). In his second book on index number theory, Walsh made the case for weighting by economic importance as follows: "It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. . . . A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worth, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth" (Correa Moylan Walsh, 1921a, 82–83).

[148] "It is also clear that the so-called unweighted index numbers, usually employed by practical statisticians, are the worst of all and are liable to large errors which could have been easily avoided" (J.M. Keynes, 1909, 79). This paper won the Cambridge University Adam Smith Prize for that year. Keynes (1930, 76–77) again stressed the importance of weighting in a later paper which drew heavily on his 1909 paper.

[149] "It has already been observed that the purpose of any index number is to strike a fair average of the price movements or movements of other groups of magnitudes. At first a simple average seemed fair, just because it treated all terms alike. And, in the absence of any knowledge of the relative importance of the various commodities included in the average, the simple average is fair. But it was early recognized that there are enormous differences in importance. Everyone knows that pork is more important than coffee and wheat than quinine. Thus the quest for fairness led to the introduction of weighting" (Irving Fisher, 1922, 43).

[150] "But even here, we should use a weighted regression approach, since we are interested in an estimate of a weighted average of the pure price change, rather than just an unweighted average over all possible models, no matter how peculiar or rare" (Zvi Griliches, 1971, 8).

[151] The approach taken in this section is based on Rao (1995, 2004, 2005) and Diewert (2003b, 2005a, 2005b). Diewert (2005a) considered all four forms of weighting that will be discussed in this section, while Rao (1995, 2005) discussed mainly the third form of weighting.

[152] One can think of repeating the term $[lnp_{1n} - \beta_n]^2$ for each unit of product n sold in period 1. The result is the term $q_{1n}[lnp_{1n} - \beta_n]^2$. A similar justification based on repeating the price according to its sales can also be made. This repetition methodology makes the stochastic specification of the error terms somewhat complicated. However, as indicated in the Introduction section, we leave these difficult distributional problems to other more capable econometricians.

The solution to (166) and (167) is the following one:[153]

$$\rho_2^* \equiv S_{n=1}^{N} q_{1n}q_{2n}(q_{1n} + q_{2n})^{-1}\, ln(p_{2n}/p_{1n})/ \Sigma_{i=1}^{N} q_{1i}q_{2i}(q_{1i} + q_{2i})^{-1}; \quad (168)$$

$$\beta_n^* \equiv q_{1n}(q_{1n} + q_{2n})^{-1}\, ln(p_{1n}) + q_{2n}(q_{1n} + q_{2n})^{-1}\, ln(p_{2n}/\pi_2^*); \quad n = 1, \ldots, N, \quad (169)$$

where $\pi_2^* \equiv \exp[\rho_2^*]$. Note that the weight for the term $\ln(p_{2n}/p_{1n})$ in (168) can be written as follows:

$$q_n^* \equiv \Sigma_{n=1}^{N} q_{1n}q_{2n}(q_{1n} + q_{2n})^{-1} / \Sigma_{i=1}^{N} q_{1i}q_{2i}(q_{1i} + q_{2i})^{-1}; \quad n = 1, \ldots, N$$
$$= h(q_{1n}, q_{2n}) / \Sigma_{i=1}^{N} h(q_{1i}, q_{2i}), \quad (170)$$

where $h(a,b) \equiv 2ab/(a + b) = [\tfrac{1}{2} a^{-1} + \tfrac{1}{2} b^{-1}]^{-1}$ is the *harmonic mean* of a and b.[154]

Note that q_n^* sum to 1 and thus ρ_2^* is a weighted average of the logarithmic price ratios $\ln(p_{2n}/p_{1n})$. Using $\pi_2^* = \exp[\rho_2^*]$ and $\pi_1^* = \exp[\rho_1^*] = \exp[0] = 1$, the bilateral price index that is generated by the solution to (165) is

$$\pi_2^*/\pi_1^* = \exp[\rho_2^*] = \exp[\Sigma_{n=1}^{N} q_n^*\, ln(p_{2n}/p_{1n})]. \quad (171)$$

Thus, π_2^*/π_1^* is a weighted geometric mean of the price ratios p_{2n}/p_{1n} with weights q_n^* defined by (170). Although this seems to be a reasonable bilateral index number formula, it must be rejected for practical use on the grounds that *the index is not invariant to changes in the units of measurement*.

Since values are invariant to changes in the units of measurement, the lack of invariance problem can be solved if we replace the quantity weights in (165) with expenditure or sales weights.[155] This leads to the following weighted least squares minimization problem where the weights v_{tn} are defined as $p_{tn}q_{tn}$ for $t = 1,2$ and $n = 1, \ldots, N$:

$$\min_{\rho,\beta} \{\Sigma_{n=1}^{N} v_{1n}[lnp_{1n} - \beta_n]^2 + \Sigma_{n=1}^{N} v_{2n}[lnp_{2n} - \rho_2 - \beta_n]^2\}. \quad (172)$$

It can be seen that problem (172) has exactly the same mathematical form as problem (165) except that v_{tn} has replaced q_{tn}, and so the solutions (168) and (169) will be valid in the present context if v_{tn} replaces q_{tn} in these formulae. Thus, the solution to (172) is

$$\rho_2^* \equiv \Sigma_{n=1}^{N} v_{1n}v_{2n}(v_{1n} + v_{2n})^{-1}\, ln(p_{2n}/p_{1n})/ \Sigma_{i=1}^{N} v_{1i}v_{2i}(v_{1i} + v_{2i})^{-1}; \quad (173)$$

$$\beta_n^* \equiv v_{1n}(v_{1n} + v_{2n})^{-1}\, ln(p_{1n}) + v_{2n}(v_{1n} + v_{2n})^{-1}\, ln(p_{2n}/\pi_2^*); \quad n = 1, \ldots, N, \quad (174)$$

where $\pi_2^* \equiv \exp[\rho_2^*]$.

The resulting price index, $\pi_2^*/\pi_1^* = \pi_2^* = \exp[\rho_2^*]$, is indeed invariant to changes in the units of measurement. However, if we regard π_2^* as a function of the price and quantity vectors for the two periods, say $P(p^1,p^2,q^1,q^2)$, then another problem emerges for the price index defined by the solution to (172): $P(p^1,p^2,q^1,q^2)$ is not homogeneous of degree 0 in the components of q^1 or in the components of q^2. These properties are important because it is desirable that the companion implicit quantity index defined as $Q(p^1,p^2,q^1,q^2) \equiv [p^2 \cdot q^2/p^1 \cdot q^1]/P(p^1,p^2,q^1,q^2)$ be homogeneous of degree 1 in the components of q^2 and homogeneous of degree minus 1 in the components of q^1.[156] We also want $P(p^1,p^2,q^1,q^2)$ to be homogeneous of degree 1 in the components of p^2 and homogeneous of degree minus 1 in the components of p^1 and these properties are also not satisfied. Thus, we conclude that the solution to the weighted least squares problem defined by (172) does not generate a satisfactory price index formula.

These deficiencies can be remedied if the *expenditure amounts* v_{tn} in (172) are replaced by *expenditure shares*, s_{tn}, where $v_t \equiv \Sigma_{n=1}^{N} v_{tn}$ for $t = 1,2$ and $s_{tn} \equiv v_{tn}/v_t$ for $t = 1,2$ and $n = 1, \ldots, N$. This replacement leads to the following weighted least squares minimization problem:[157]

$$\min_{\rho,\beta} \{\Sigma_{n=1}^{N} s_{1n}[lnp_{1n} - \beta_n]^2 + \Sigma_{n=1}^{N} s_{2n}[lnp_{2n} - \rho_2 - \beta_n]^2\}. \quad (175)$$

Again, it can be seen that problem (175) has exactly the same mathematical form as problem (165) except that s_{tn} has replaced q_{tn}, and so the solutions (168) and (169) will be valid in the present context if s_{tn} replaces q_{tn} in these formulae. Thus, the solution to (175) is

$$\rho_2^* \equiv \Sigma_{n=1}^{N} s_{1n}s_{2n}(s_{1n} + s_{2n})^{-1}\, ln(p_{2n}/p_{1n})/ \Sigma_{i=1}^{N} s_{1i}s_{2i}(s_{1i} + s_{2i})^{-1}; \quad (176)$$

$$\beta_n^* \equiv s_{1n}(s_{1n} + s_{2n})^{-1}\, ln(p_{1n}) + s_{2n}(s_{1n} + s_{2n})^{-1}\, ln(p_{2n}/\pi_2^*); \quad n = 1, \ldots, N, \quad (177)$$

[153] See Diewert (2005a).

[154] $h(a,b)$ is well defined by $ab/(a + b)$ if a and b are nonnegative and at least one of these numbers is positive. In order to write $h(a,b)$ as $[\tfrac{1}{2} a^{-1} + \tfrac{1}{2} b^{-1}]^{-1}$, we require $a > 0$ and $b > 0$.

[155] "But on what principle shall we weight the terms? Arthur Young's guess and other guesses at weighting represent, consciously or unconsciously, the idea that relative money values of the various commodities should determine their weights. A value is, of course, the product of a price per unit, multiplied by the number of units taken. Such values afford the only common measure for comparing the streams of commodities produced, exchanged, or consumed, and afford almost the only basis of weighting which has ever been seriously proposed" (Irving Fisher, 1922, 45).

[156] Thus, we want Q to have the following properties: $Q(p^1,p^2,q^1,\lambda q^2) = \lambda Q(p^1,p^2,q^1,q^2)$ and $Q(p^1,p^2,\lambda q^1,q^2) = \lambda^{-1}Q(p^1,p^2,q^1,q^2)$ for all $\lambda > 0$.

[157] Note that the minimization problem defined by (175) is equivalent to the problem of minimizing $\Sigma_{n=1}^{N} e_{1n}^{\ 2} + \Sigma_{n=1}^{N} e_{2n}^{\ 2}$ with respect to $\rho_2, \beta_1, \ldots, \beta_N$, where the error terms e_{tn} are defined by the equations $s_{1n}^{1/2}lnp_{1n} = s_{1n}^{1/2}\beta_n + e_{1n}$ for $n = 1, \ldots, N$ and $s_{2n}^{1/2}lnp_{2n} = s_{2n}^{1/2}\rho_2 + s_{2n}^{1/2}\beta_n + e_{2n}$ for $n = 1, \ldots, N$. Thus, the solution to (175) can be found by running a linear regression using the above two sets of estimating equations. The numerical equivalence of the least squares estimates obtained by repeating multiple observations or by using the square root of the weight transformation was noticed long ago as the following quotation indicates: "It is evident that an observation of weight w enters into the equations exactly as if it were w separate observations each of weight unity. The best practical method of accounting for the weight is, however, to prepare the equations of condition by multiplying each equation throughout by the square root of its weight" (E. T. Whittaker and G. Robinson, 1940, 224).

where $\pi_2^* \equiv \exp[\rho_2^*]$. Define the *normalized harmonic mean share weights* as $s_n^* \equiv h(s_{1n},s_{2n})/\Sigma_{i=1}^N h(s_{1i},s_{2i})$ for $n = 1, \ldots, N$. Then the weighted time product dummy bilateral price index, $P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \pi_2^*/\pi_1^* = \pi_2^*$, has the following logarithm:

$$ln\ P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \Sigma_{n=1}^N s_n^*\ ln(p_{2n}/p_{1n}). \qquad (178)$$

Thus, $P_{WTPD}(p^1,p^2,q^1,q^2)$ is equal to a share-weighted geometric mean of the price ratios, p_{2n}/p_{1n}.[158] This index is a satisfactory one from the viewpoint of the test approach to index number theory. It can be shown that $P_{WTPD}(p^1,p^2,q^1,q^2)$ satisfies the following tests:

(i) the *identity test*; that is, $P_{WTPD}(p^1,p^2,q^1,q^2) = 1$ if $p^1 = p^2$;

(ii) the *time reversal test*; that is, $P_{WTPD}(p^2,p^1,q^2,q^1) = 1/P_{WTPD}(p^1,p^2,q^1,q^2)$;[159]

(iii) *homogeneity of degree 1 in period 2 prices*; that is, $P_{WTPD}(p^1,\lambda p^2,q^1,q^2) = \lambda P_{WTPD}(p^1,p^2,q^1,q^2)$; (iv) *homogeneity of degree –1 in period 1 prices*; that is, $P_{WTPD}(\lambda p^1,p^2,q^1,q^2) = \lambda^{-1}P_{WTPD}(p^1,p^2,q^1,q^2)$; (v) *homogeneity of degree 0 in period 1 quantities*; that is, $P_{WTPD}(p^1,p^2,\lambda q^1,q^2) = P_{WTPD}(p^1,p^2,q^1,q^2)$; (vi) *homogeneity of degree 0 in period 2 quantities*; that is, $P_{WTPD}(p^1,p^2,q^1,\lambda q^2) = P_{WTPD}(p^1,p^2,q^1,q^2)$; (vii) *invariance to changes in the units of measurement*;

(viii) the *min-max test*; that is,

$$\min_n\{p_{2n}/p_{1n} : n = 1, \ldots, N\} \le P_{WTPD}(p^1,p^2,q^1,q^2) \le \max_n\{p_{2n}/p_{1n} : n = 1, \ldots, N\};\ \text{and}$$

(ix) the *invariance to the ordering of the products test*.

Moreover, it can be shown that $P_{WTPD}(p^1,p^2,q^1,q^2)$ approximates the superlative Törnqvist–Theil index to the second order around an equal price and quantity point where $p^1 = p^2$ and $q^1 = q^2$.[160] Thus, if changes in prices and quantities going from one period to the next are not too large and there are no missing products, P_{WTPD} should be close to the superlative Fisher (1922) and Törnqvist–Theil indices.[161]

Recall the results from Section 13 above for the unweighted time product dummy model. From equation (152), it can be seen that the unweighted bilateral time product dummy regression model generated the Jevons index as the solution to the unweighted least squares minimization problem that is a counterpart to the weighted problem defined by (175). Thus, appropriate weighting of the squared errors has changed the solution index dramatically: The index defined by (178) weights products by their economic importance and has good test properties, whereas the Jevons index can generate very problematic results due to its lack of weighting according to economic importance. Note that both models have the same underlying structure; that is, they assume that p_{tn} is approximately equal to $\pi_t\alpha_n$ for $t = 1,2$ and $n = 1, \ldots, N$. *Thus, weighting by economic importance has converted a least squares minimization problem that generates a rather poor price index into a problem that generates a rather good index*.

There is one more weighting scheme that generates an even better index in the bilateral context where we are running a TPD hedonic regression using the price and quantity data for only two periods. Consider the following weighted least squares minimization problem:

$$\min_{\rho,\beta} \{\Sigma_{n=1}^N (½)(s_{1n} + s_{2n})[lnp_{1n} - \beta_n]^2 + \Sigma_{n=1}^N (½)(s_{1n} + s_{2n})[lnp_{2n} - \rho_2 - \beta_n]^2\}. \qquad (179)$$

As usual, it can be seen that problem (179) has exactly the same mathematical form as problem (165) except that $(½)(s_{1n} + s_{2n})$ has replaced q_{tn}, and so the solutions (168) and (169) will be valid in the present context if $(½)(s_{1n} + s_{2n})$ replaces q_{tn} in these formulae. Thus, the solution to (179) simplifies to the following solution:

$$\rho_2^* \equiv \Sigma_{n=1}^N (½)(s_{1n} + s_{2n})ln(p_{2n}/p_{1n}); \qquad (180)$$

$$\beta_n^* \equiv (½)ln(p_{1n}) + (½)ln(p_{2n}/\pi_2^*);\ n = 1, \ldots, N, \qquad (181)$$

where $\pi_2^* \equiv \exp[\rho_2^*]$ and $\pi_1^* \equiv \exp[\rho_1^*] = \exp[0] = 1$ since we have set $\rho_1^* = 0$. Thus, the bilateral index number formula that emerges from the solution to (179) is $\pi_2^*/\pi_1^* = \exp[\Sigma_{n=1}^N (½)(s_{1n} + s_{2n})\ln(p_{2n}/p_{1n})] \equiv P_T(p^1,p^2,q^1,q^2)$, which is the Törnqvist–Theil (1967, 137–38) bilateral index number formula. Thus, the use of the weights in (179) has generated an even better bilateral index number formula than the formula that resulted from the use of the weights in (175). This result reinforces the case for using appropriately weighted versions of the basic TPD hedonic regression model.[162] However, if the implied residuals in the original unweighted minimization problem (147) are small (or equivalently, if the fit in the linear regression model that can be associated with (147) is high so that predicted values for log prices are close to actual log prices), then *weighting will not matter very much*, and the unweighted model (147) will give results that are similar to the results generated by the weighted model defined by (179). This comment applies to all of the weighted hedonic regression models that are considered in this paper.[163]

The aggregate quantity levels for the t periods can be obtained as $Q^{t*} \equiv \alpha^* \cdot q^t = \Sigma_{n=1}^N \alpha_n^* q_{tn}$ for $t = 1,2$, where α_n^* are defined as the exponentials of β_n^* defined by (181). Estimated aggregate price levels can be obtained directly

[158] See Diewert (2002, 2005a).

[159] See Diewert (2003b, 2005b).

[160] Diewert (2005a, 564) noted this result. Thus, P_{WTPD} is a pseudo-superlative index. For the definition of a superlative index, see Diewert (1976, 2021a). A pseudo-superlative index approximates a superlative index to the second order around any point where $p^1 = p^2$ and $q^1 = q^2$; see Diewert (1978).

[161] However, with large changes in price and quantities going from period 1 to 2, P_{WTPD} will tend to lie below its superlative counterparts; see Diewert (2018, 53) and an example provided by Diewert and Fox (2021).

[162] Note that the bilateral regression model defined by the minimization problem (175) is readily generalized to the case of T periods, whereas the bilateral regression model defined by the minimization problem (179) cannot be generalized to the case of T periods. These facts were noted by de Haan and Krsinich (2014).

[163] If the residuals are small for (147), then prices will vary almost proportionally over time, and all reasonable index number formulae will register price levels that are close to the estimated π_t^*; that is, we will have $p^t \approx \pi_t^* p^1$ for $t = 2,3, \ldots, T$ if the residuals are small for (147).

from the solution to (179); that is, set $P^{t*} = \pi_t^*$ for $t = 1,2$.[164] Alternative price levels can be obtained indirectly as $P^{t**} \equiv p^t{\cdot}q^t/Q^{t*} = p^t{\cdot}q^t/\alpha^*{\cdot}q^t$ for $t = 1,2$. If the optimized objective function in (179) is 0, so that all errors equal 0, then P^{t*} will equal P^{t**} for $t = 1,2$. If the estimated residuals are not all equal to 0, then the two estimates for the period t price level P^t will differ, and the alternative estimates for P^t will generate different estimates for the companion aggregate quantity levels.

It should be noted that we have not made any bias corrections due to the fact that our model estimates the logarithm of π_t instead of π_t itself. This is due to our perspective that simply tries to fit an exact model by transforming it in a way that leads to solutions π_t^* to a least squares minimization problem, where π_t^* have good axiomatic properties.[165] There is more work to be done in working out the distributional properties of the above estimators for the price levels.

16. Weighted Time Product Dummy Regressions: The Bilateral Case with Missing Observations

In this section, we will generalize the last two models in the previous section to cover the case where there are missing observations.[166] Thus, we assume that there are products that are missing in period 2 that were present in period 1 and some new products that appear in period 2. As in Section 14, $S(t)$ denotes the set of products n that are present in period t for $t = 1,2$. It is assumed that $S(1)\cap S(2)$ is not the empty set; that is, there are one or more products that are present in both periods. We need some new notation to deal with missing prices and quantities. For the present, if product n is not present in period t, define p_{tn} and q_{tn} to equal 0. This enables us to define the N-dimensional period t price and quantity vectors as $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ for $t = 1,2$. Thus, the missing prices and quantities are simply set equal to 0. The period t share of sales or expenditures for product n is defined in the usual case as $s_{tn} \equiv p_{tn}q_{tn}/p^t{\cdot}q^t$ for $n = 1, \ldots, N$ and $t = 1,2$. With these notational conventions, the new weighted least squares minimization problem that generalizes (175) is the following minimization problem:[167]

$$\min_{\rho,\,\beta} \{\Sigma_{n\in S(1)}\, s_{1n}[lnp_{1n} - \beta_n]^2 + \Sigma_{n\in S(2)}\, s_{2n}[lnp_{2n} - \rho_2 - \beta_n]^2\}. \quad (182)$$

The first-order conditions for $\rho_2^*, \beta_1^*, \ldots, \beta_N^*$ to solve (182) are equivalent to the following equations:

$$\Sigma_{n\in S(2)}\, s_{2n}\rho_2^* + \Sigma_{n\in S(2)}\, s_{2n}\beta_n^* = \Sigma_{n\in S(2)}\, s_{2n} lnp_{2n}; \quad (183)$$

$$s_{2n}\rho_2^* + (s_{1n} + s_{2n})\beta_n^* = s_{1n} lnp_{1n} + s_{2n} lnp_{1n}; \quad n\in S(1)\cap S(2); \quad (184)$$

$$\beta_n^* = lnp_{1n}; \quad n\in S(1),\ n\notin S(2); \quad (185)$$

$$\rho_2^* + \beta_n^* = lnp_{2n}; \quad n\in S(2),\ n\notin S(1). \quad (186)$$

Define the intersection set of products S^* as follows:

$$S^* \equiv S(1)\cap S(2). \quad (187)$$

Substituting equations (186) into equation (183) leads to the following equation:

$$\Sigma_{n\in S^*}\, s_{2n}[lnp_{2n} - \rho_2^* - \beta_n^*] = 0. \quad (188)$$

Consider the following least squares minimization problem that is defined over the set of products that are present in both periods:

$$\min_{\rho,\,\beta} \{\Sigma_{n\in S^*}\, s_{1n}[lnp_{1n} - \beta_n]^2 + \Sigma_{n\in S^*}\, s_{2n}[lnp_{2n} - \rho_2 - \beta_n]^2\}. \quad (189)$$

The first-order conditions for this problem are (188) and (184). Once we find the solution to this problem, define β_n^* for the products that are not present in both periods by equations (185) and (186). This augmented solution will solve problem (182). The solution to (189) can be found by adapting the solution to (175) to the current situation. Recall equations (176) and (177) from the previous section. Replacing the entire set of product indices $n = 1, \ldots, N$ by the intersection set S^* defined by (187) leads to the following solution to (189):

$$\rho_2^* \equiv [\Sigma_{n\in S^*}\, s_{1n}s_{2n}(s_{1n} + s_{2n})^{-1}\, ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*}\, s_{1i}s_{2i}(s_{1i} + s_{2i})^{-1}]; \quad (190)$$

$$\beta_n^* \equiv s_{1n}(s_{1n} + s_{2n})^{-1}\, ln(p_{1n}) + s_{2n}(s_{1n} + s_{2n})^{-1}\, ln(p_{2n}/\pi_2^*); \quad n\in S^*, \quad (191)$$

where $\pi_2^* \equiv \exp[\rho_2^*]$. Define the *normalized harmonic mean share weights* for the always present products as $s_n^* \equiv h(s_{1n},s_{2n})/\Sigma_{i\in S^*}\, h(s_{1i},s_{2i})$ for $n\in S^*$. Using these definitions for the shares s_n^*, the *weighted TPD bilateral price index with missing observations*, $P_{WTPD}(p^1,p^2,q^1,q^2) \equiv \pi_2^*/\pi_1^* = \pi_2^*$, has the following logarithm:

[164] In this case, alternative period t quantity levels are defined as $Q^{1**} \equiv p^1{\cdot}q^1$ and $Q^{2**} \equiv p^2{\cdot}q^2/\pi_2^* = [v_2/v_1]/P_T(p^1,p^2,q^1,q^2)$. If the squared errors in (179) are all 0, then the alternative quantity estimates are equal to each other and the model $lnp_{tn} = \rho_t + \beta_n$ holds exactly for each t and n, which means that prices are proportional across the two periods; that is, we have $p^t = \pi_t^*\alpha^*$ for $t = 1,2$ and $\alpha^* \equiv [\alpha_1^*, \ldots, \alpha_N^*]$. In the case where the squared errors are nonzero, the π_t^*, Q^{t**} aggregates are preferred since $P_T(p^1,p^2,q^1,q^2)$ is a superlative index and thus has a strong economic justification.

[165] We note that de Haan and Krsinich (2018, 769–70) make the following comments on possible biases that result from the use of a weighted least squares model to generate price indices: "Finally, we will elaborate on a few econometric issues. The estimated quality adjusted prices . . . are biased as taking exponentials is a nonlinear transformation. The time dummy index is similarly biased. It is questionable whether bias adjustments would be appropriate, though, at least from an index number point of view. For instance, recall the two-period case with only matched items, where Diewert's (2004) choice of regression weights ensures that the time dummy index is equal to the superlative Törnqvist price index. Correcting for the 'bias' would mean that this useful property does no longer hold, and so there is a tension between econometrics and index number theory."

[166] The results in this section are closely related to the results derived by de Haan (2004a), Silver and Heravi (2005), and de Haan and Krsinich (2014, 2018). However, our method of derivation is somewhat different.

[167] This form of weighting was suggested by Rao (1995, 2004, 2005), Diewert (2002, 2004, 2005a), and de Haan (2004a).

$$lnP_{WTPD}(p^1,p^2,q^1,q^2) \equiv \Sigma_{n\in S^*} s_n^* ln(p_{2n}/p_{1n}). \qquad (192)$$

Note that $P_{WTPD} \equiv \pi_2^*/\pi_1^*$ depends directly on the price ratios for the products *that are present in both periods*. However, it also depends on the shares s_{tn}, which in turn depend on all of the price and quantity information for both periods. It can be seen that $P_{WTPD}(p^1,p^2,q^1,q^2)$ is a weighted geometric mean of the matched prices p_{2n}/p_{1n} for products n that are present in both periods. Thus, if matched product prices are equal in the two periods, then $P_{WTPD}(p^1,p^2,q^1,q^2)$ will equal unity even if there is an expanding or contracting choice set over the two periods; that is, alternative reservation prices for any missing products will not affect the estimated price levels and price indices.

However, the hedonic regression model that is generated by solving (189) can be used to impute (neutral) reservation prices for missing observations. Thus, define $\alpha_n^* \equiv \exp[\beta_n^*]$ for $n = 1, \ldots, N$. Then the missing prices p_{tn}^* can be defined as follows:

$$p_{2n}^* \equiv \pi_2^*\alpha_n^* = \pi_2^* p_{1n} \quad n\in S(1),\ n\notin S(2); \qquad (193)$$
$$p_{1n}^* \equiv \pi_1^*\alpha_n^* = p_{2n}/\pi_2^* \quad n\in S(2),\ n\notin S(1). \qquad (194)$$

Thus, the missing prices for period 2, p_{2n}^*, are the corresponding *inflation-adjusted carry-forward prices* from period 1, p_{1n} times π_2^*, and the missing prices for period 1, p_{1n}^*, are the corresponding *inflation-adjusted carry-backward prices* from period 2, p_{2n}, deflated by π_2^*, where π_2^* is the weighted TPD price index $P_{WTPDM}(p^1,p^2,q^1,q^2)$ defined as $\pi_2^* \equiv \exp[\rho_2^*]$, where ρ_2^* is defined by (190).[168] As noted earlier, these reservation prices are neutral in the sense that they do not affect the definition of ρ_2^*, and hence they do not affect the definition of $P_{WTPDM}(p^1,p^2,q^1,q^2)$.

Estimated aggregate price levels can be obtained directly from the solution to (189); that is, set $P^{1*} = 1$ and $P^{2*} = \pi_2^*$. The corresponding quantity levels are defined as $Q^{1*} \equiv p^1\cdot q^1$ and $Q^{2*} \equiv p^2\cdot q^2/\pi_2^*$. Alternative price and quantity levels can be obtained as $Q^{t**} \equiv \alpha^*\cdot q^t$ and $P^{t**} \equiv p^t\cdot q^t/Q^{t**}$ for $t = 1,2$. If the optimized objective function in (189) is 0, so that all errors equal 0, then P^{t*} will equal P^{t**} for all t. If the estimated residuals are not all equal to 0, then the two estimates for the period 2 price level P^2 will differ, and, as usual, the alternative estimates for P^2 will generate different estimates for the companion aggregate quantity levels.

This analysis is not quite the end of the story. The expenditure shares s_{1n} and s_{2n} that appear in (182) are not the expenditure shares that characterize the always present products; they are the original expenditure shares defined over all N products. It is of interest to compare $P_{WTPD}(p^1, p^2,q^1,q^2)$ defined implicitly by (192) with the weighted TPD index, $P_{WTPDM}(p^{1*},p^{2*},q^{1*},q^{2*})$, that is defined over the common set of products, S^*;[169] that is, P_{WTPDM} is the weighted TPD regression model that is defined over the set of *matched products* for the two periods under consideration.

[168] The corresponding imputed values for the missing quantities in each period are set equal to 0.

[169] Define p^{t*} and q^{t*} as the period t price and quantity vectors that include only products that are present in both periods.

Define $v_t^* \equiv \Sigma_{n\in S^*} v_{tn}$ as the total expenditure on always present products for $t = 1,2$ and define the corresponding *restricted expenditure shares* as[170]

$$s_{tn}^* \equiv v_{tn}/v_t^*; \quad t = 1,2;\ n\in S^*. \qquad (195)$$

The matched model version of (189) is the following weighted least squares minimization problem:

$$\min_{\rho,\beta} \{\Sigma_{n\in S^*} s_{1n}^*[lnp_{1n} - \beta_n]^2 + \Sigma_{n\in S^*} s_{2n}^*[lnp_{2n} - \rho_2 - \beta_n]^2\}. \qquad (196)$$

The ρ_2 solution to (196) is the following one:

$$\rho_2^{**} \equiv [\Sigma_{n\in S^*} s_{1n}^* s_{2n}^* (s_{1n}^* + s_{2n}^*)^{-1} ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} s_{1i}^* s_{2i}^* (s_{1i}^* + s_{2i}^*)^{-1}]$$
$$= [\Sigma_{n\in S^*} h(s_{1n}^*,s_{2n}^*)\, ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} h(s_{1i}^*,s_{2i}^*)], \qquad (197)$$

where $h(s_{1n}^*,s_{2n}^*)$ is the harmonic mean of the restricted shares s_{1n}^* and s_{2n}^*. Thus, $P_{WTPDM}(p^{1*},p^{2*},q^{1*},q^{2*}) \equiv \exp[\rho_2^{**}]$, where ρ_2^{**} is defined by (197).

The relationship between the *true shares*, s_{tn}, and the *restricted shares*, s_{tn}^*, for the always present products is given by the following equations:

$$s_{tn} \equiv v_{tn}/v_t = [v_{tn}/v_t^*][v_t^*/v_t] = s_{tn}^* f_t; \quad t = 1,2;\ n\in S^*, \qquad (198)$$

where the *fraction* of expenditures on always available commodities compared to expenditures on all commodities during period t is $f_t \equiv v_t^*/v_t$ for $t = 1,2$. Using definitions (190) and (198), it can be seen that the logarithm of $P_{WTPD}(p^1,p^2,q^1,q^2)$ defined by (192) is given by

$$\rho_2^* \equiv [\Sigma_{n\in S^*} h(s_{1n},s_{2n})\, ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} h(s_{1i},s_{2i})]$$
$$= [\Sigma_{n\in S^*} h(f_1 s_{1n}^*, f_2 s_{2n}^*)\, ln(p_{2n}/p_{1n})]/[\Sigma_{i\in S^*} h(f_1 s_{1i}^*, f_2 s_{2i}^*)]. \qquad (199)$$

Now compare (197) and (199). If either (i) $p_{2n} = \lambda p_{1n}$ for all $n\in S^*$ so that we have price proportionality for the always present products or (ii) $f_1 = f_2$ so that the ratio of expenditures on always present products to total expenditure in each period is constant across the two periods, then $\rho_2^{**} = \rho_2^*$. However, if these conditions are not satisfied and there is considerable variation in prices and quantities across periods, then ρ_2^{**} could differ substantially from ρ_2^*. Since neither index is superlative, it is difficult to recommend one of these indices over the other as the "optimal" carry-forward and carry-backward inflation rate that could be used to construct the inflation-adjusted carry-forward and carry-backward estimates for the missing prices.[171]

[170] The matched product expenditure shares defined by (195), $s_{tn}^* \equiv v_{tn}/v_t^*$, differ from the original "true" expenditure shares defined as $s_{tn} \equiv v_{tn}/v_t$ because the true period t expenditures v_t include expenditures on "isolated" products that are present in only one of the two periods under consideration. Thus, if there are isolated products in both periods, v^t will be greater than v^{t*} for $t = 1,2$, and thus the two sets of shares will be different.

[171] For another alternative weighting scheme for a bilateral TPD model in the case of two periods that generalizes the model defined by (179) to the case of missing observations, see de Haan (2004a).

In the following section, we define weighted time dummy regression models for the general case of T periods and missing observations.

17. Weighted Time Product Dummy Regressions: The General Case

We first consider the case of T periods and no missing observations. The generalization of the two-period weighted least squares minimization problem that was defined by (175) in Section 15 to the case of $T > 2$ periods is (200):[172]

$$\min_{\rho,\beta} \{\Sigma_{n=1}^{N} \Sigma_{t=1}^{T} s_{tn}[lnp_{tn} - \rho_t - \beta_n]^2\}. \quad (200)$$

The first-order necessary conditions for $\rho^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\beta^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ to solve (200) are the following T equations (201) and N equations (202):

$$\rho_t^* = \Sigma_{n=1}^{N} s_{tn}[lnp_{tn}^* - \beta_n^*];\ t = 1, \ldots, T; \quad (201)$$
$$\beta_n^* = \Sigma_{t=1}^{T} s_{tn}[lnp_{tn}^* - \rho_t^*]/(\Sigma_{t=1}^{T} s_{tn});\ n = 1, \ldots, N. \quad (202)$$

As usual, the solution to (200) given by (201) and (202) is not unique: If $\rho^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\beta^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ solve (201) and (202), then so do $[\rho_1^* + \lambda, \ldots, \rho_T^* + \lambda]$ and $[\beta_1^* - \lambda, \ldots, \beta_N^* - \lambda]$ for all λ. Thus, we can set $\rho_1^* = 0$ in equations (201) and drop the first equation in (201) and use linear algebra to find a unique solution for the resulting equations.[173] Once the solution is found, define the estimated *price levels* π_t^* and *quality adjustment factors* α_n^* as follows:

$$\pi_t^* \equiv \exp[\rho_t^*];\ t = 2,3, \ldots, T;\ \alpha_n^* \equiv \exp[\beta_n^*];$$
$$n = 1, \ldots, N. \quad (203)$$

Note that the resulting *price index* between periods t and τ is equal to the following expression:

$$\pi_t^*/\pi_\tau^* = \prod_{n=1}^{N} \exp[s_{tn}ln(p_{tn}/\alpha_n^*)]/\prod_{n=1}^{N} exp[s_{\tau n}ln(p_{\tau n}/\alpha_n^*)];$$
$$1 \le t, \tau \le T. \quad (204)$$

If $s_{tn} = s_{\tau n}$ for $n = 1, \ldots, N$, then π_t^*/π_r^* will equal a weighted geometric mean of the price ratios $p_{tn}/p_{\tau n}$, where the weight for $p_{tn}/p_{\tau n}$ is the common expenditure share $s_{tn} = s_{\tau n}$. Thus, π_t^*/π_τ^* will not depend on a_n^* in this case.[174]

The price levels π_t^* defined by (203) are functions of the T price vectors, $p^1, \ldots, p^T$, and the T quantity vectors, $q^1, \ldots, q^T$. These price level functions have some good axiomatic properties: (i) π_t^* are invariant to changes in the units of measurement; (ii) π_t^* regarded as a function of the period t price vector p^t is linearly homogeneous in the components of p^t; that is, $\pi_t^*(\lambda p^t) = l\pi_t^*(p^t)$ for all $p^t >> 0_N$ and $\lambda > 0$; (iii) π_t^* regarded as a function of the period t quantity vector q^t is homogeneous of degree 0 in the components of q^t; that is, $\pi_t^*(\lambda q^t) = \pi_t^*(q^t)$ for all $q^t >> 0_N$ and $\lambda > 0$;[175] (iv) π_t^* satisfy a version of Walsh's (1901, 389; 1921b, 540) *multiperiod identity test*; that is, if $p^t = p^\tau$ and $q^t = q^\tau$, then $\pi_t^* = \pi_\tau^*$.[176]

Once the estimates for π_t and α_n have been computed, we have the usual two methods for constructing period-by-period price and quantity levels, P^t and Q^t for $t = 1, \ldots, T$. The π_t^* estimates can be used to form the aggregates using equations (205), or the α_n^* estimates can be used to form the aggregates using equations (206):[177]

$$P^{t*} \equiv \pi_t^*;\ Q^{t*} \equiv p^t \cdot q^t/\pi_t^*;\ t = 1, \ldots, T; \quad (205)$$
$$Q^{t**} \equiv \alpha^* \cdot q^t;\quad P^{t**} \equiv p^t \cdot q^t/\alpha^* \cdot q^t;\quad t = 1, \ldots, T. \quad (206)$$

Define the error terms $e_{tn} \equiv lnp_{tn} - \ln\pi_t^* - \ln\alpha_n^*$ for $t = 1, \ldots, T$ and $n = 1, \ldots, N$. If all $e_{tn} = 0$, then P^{t*} will equal P^{t**} and Q^{t*} will equal Q^{t**} for $t = 1, \ldots, T$. However, if the error terms are not all equal to zero, then the statistical agency will have to decide on pragmatic grounds on which option to choose.

It is straightforward to generalize the weighted least squares minimization problem (200) to the case where there are missing prices and quantities. As in Section 14 we assume that there are N products and T time periods, but not all products are purchased (or sold) in all time periods. For each period t, define the set of products n that are present in period t as $S(t) \equiv \{n\colon p_{tn} > 0\}$ for $t = 1,2, \ldots, T$. It is assumed that these sets are not empty; that is, at least one product is purchased in each period. For each product n, define the set of periods t where product n is present as $S^*(n) \equiv \{t\colon p_{tn} > 0\}$. Again, assume that these sets are not empty; that is, each product is sold in at least one time period. The generalization of (200) to the case of missing products is the following weighted least squares minimization problem:

$$\min_{\rho,\beta} \Sigma_{t=1}^{T} \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \beta_n]^2 = \min_{\rho,\beta}$$
$$\Sigma_{n=1}^{N} \Sigma_{t\in S^*(n)} s_{tn}[lnp_{tn} - \rho_t - \beta_n]^2. \quad (207)$$

Note that there are two equivalent ways of writing the least squares minimization problem. The first-order necessary conditions for $\rho_1, \ldots, \rho_T$ and $\beta_1, \ldots, \beta_N$ to solve (207) are the following counterparts to (201) and (202):[178]

$$\Sigma_{n\in S(t)} s_{tn}[\rho_t^* + \beta_n^*] = \Sigma_{n\in S(t)} s_{tn}lnp_{tn};\ t = 1, \ldots, T; \quad (208)$$
$$\Sigma_{t\in S^*(n)} s_{tn}[\rho_t^* + \beta_n^*] = \Sigma_{t\in S^*(n)} s_{tn}lnp_{tn};$$
$$n = 1, \ldots, N. \quad (209)$$

[172] Rao (1995, 2004, 2005, 574) was the first to consider this model using expenditure share weights. However, Balk (1980, 70) suggested this class of models much earlier using somewhat different weights.

[173] Alternatively, one can set up the linear regression model defined by $(s_{tn})^{1/2}lnp_{tn} = (s_{tn})^{1/2}\rho_t + (s_{tn})^{1/2}\beta_n + e_{tn}$ for $t = 1, \ldots, T$ and $n = 1, \ldots, N$, where we set $\rho_1 = 0$ to avoid exact multicollinearity. Iterating between equations (201) and (202) will also generate a solution to these equations and the solution can be normalized so that $\rho_1 = 0$.

[174] This case is consistent with utility-maximizing purchasers having common Cobb Douglas preferences.

[175] By looking at the minimization problem defined by (200), it is also straightforward to show that $\pi_t^*(\lambda q^\tau) = \pi_t^*(q^\tau)$ for all $q^t >> 0_N$ and $\lambda > 0$ for $\tau = 1, \ldots, T$.

[176] We would like the π_t^* to satisfy the usual (strong) identity test, which is: if $p^t = p^\tau$, then $\pi_t^* = \pi_\tau^*$. However, if the share weights for the two periods are different, then this test no longer holds. However, if we define the period t price and quantity levels using definitions (206), it can be seen that the resulting Q^{t**} will satisfy the usual (strong) identity test for quantities. If our perspective is one of measuring economic welfare, then we may want to choose (206) over (205).

[177] Note that the price level P^{t**} defined in (206) is a quality-adjusted unit value index of the type studied by de Haan (2004b).

[178] Equations (208) and (209) show that the solution to (207) does not depend on any independently determined reservation prices p_{tn} for products n that are missing in period t.

As usual, the solution to (208) and (209) is not unique: if $\rho^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\beta^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ solve (208) and (209), then so do $[\rho_1^* + \lambda, \ldots, \rho_T^* + \lambda]$ and $[\beta_1^* - \lambda, \ldots, \beta_N^* - \lambda]$ for all λ. Thus, we can set $\rho_1^* = 0$ in equations (208) and drop the first equation in (208) and use linear algebra to find a unique solution for the resulting equations.

Define the estimated *price levels* π_t^* and *quality adjustment factors* α_n^* by definitions (203). The weighted TPD price level for period t is defined as $p_{WTPD}{}^t \equiv \pi_t^*$ for $t = 1, \ldots, T$. Substitute these definitions into equations (208) and (209). After some rearrangement, equations (208) and (209) become the following ones:

$$\pi_t^* = \exp[\Sigma_{n\in S(t)} s_{tn} ln(p_{tn}/\alpha_n^*)] \equiv p_{WTPD}{}^t; \quad t = 1, \ldots, T; \tag{210}$$

$$\alpha_n^* = \exp[\Sigma_{t\in S^*(n)} s_{tn} ln(p_{tn}/\pi_t^*)/\Sigma_{t\in S^*(n)} s_{tn}]; \quad n = 1, \ldots, N. \tag{211}$$

Once the estimates for π_t and α_n have been computed, we have the usual two methods for constructing period-by-period price and quantity levels, P^t and Q^t for $t = 1, \ldots, T$; see (205) and (206).[179]

The new price levels π_t^* defined by (210) are functions of the T price vectors, $p^1, \ldots, p^T$, and the T quantity vectors $q^1, \ldots, q^T$. If there are missing products, the corresponding prices and quantities, p_{tn} and q_{tn}, are temporarily set equal to 0. The new price level functions defined by (210) have the same axiomatic properties (i)–(iv), which were noted earlier in this section.[180] The present price level functions take the economic importance of the products into account and thus are a clear improvement over their unweighted counterparts, which were discussed in Section 14. If the estimated errors $e_{tn}^* \equiv lnp_{tn} - \rho_t^* - \beta_n^*$ that implicitly appear in the weighted least squares minimization problem (207) turn out to be small, then the underlying exact model, $p_{tn} = \pi_t\alpha_n$ for $t = 1, \ldots, T$, $n\in S(t)$, provides a good approximation to reality, and thus this weighted TPD regression model can be used with some confidence.

The solution to the weighted least squares minimization problem defined by (207), π_t^* for $t = 1, \ldots, T$ and α_n^* for $n = 1, \ldots, N$ can be used to define (neutral) *reservation prices* for missing observations. For any missing price for product n in period t, define p_{tn}^* as follows:

$$p_{tn}^* \equiv \pi_t^*\alpha_n^*; \quad n\notin S^*(t). \tag{212}$$

In what follows, we will use the prices defined by (212) to replace the 0 prices in the vectors p^t for $t = 1, \ldots, T$, so with the use of these imputed prices, all price vectors p^t have positive components. Of course, the quantities q_{tn} and the shares s_{tn} that correspond to the imputed prices defined by (212) are still equal to 0.

The weighted TPD price level functions $p_{WTPD}{}^t$ defined by (210) have the same unsatisfactory property that their unweighted counterparts had in previous sections: a product that is available only in one period out of the T periods has no influence on the aggregate price levels $p_{WTPD}{}^t \equiv \pi_t^*$.[181] This means that the price of a new product that appears in period T has no influence on the price levels, and thus the benefits of an expanding consumption set are not measured by this multilateral method. This is a significant shortcoming of this method. However, on the positive side of the ledger, this method does satisfy the strong identity test for the companion quantity index, a property that it shares with the GK multilateral method.[182]

Once the WTPD price levels $p_{WTPD}{}^t$ have been defined,[183] the *weighted TPD price index* for period t (relative to period 1) is defined as $P_{WTPD}{}^t \equiv p_{WTPD}{}^t/p_{WTPD}{}^1$ and the logarithm of $P_{WTPD}{}^t$ is equal to the following expression:

$$lnP_{WTPD}{}^t = \Sigma_{n=1}^N s_{tn}(lnp_{tn} - \beta_n^*) - \Sigma_{n=1}^N s_{1n}(lnp_{1n} - \beta_n^*); \quad t = 1, \ldots, T. \tag{213}$$

With this expression for $lnP_{WTPD}{}^t$ in hand, we can compare $lnP_{WTPD}{}^t$ to $lnP_T{}^t$. Using (213) and definition (40),[184] we can derive the following expressions for $t = 1,2, \ldots, T$:

$$lnP_{WTPD}{}^t - lnP_T{}^t = \tfrac{1}{2}\Sigma_{n=1}^N (s_{tn} - s_{1n})(lnp_{tn} - \beta_n^*) + \tfrac{1}{2}\Sigma_{n=1}^N (s_{tn} - s_{1n})(lnp_{1n} - \beta_n^*). \tag{214}$$

179 The counterparts to definitions (205) are now: $P^{t*} \equiv \pi_t^* = \prod_{n\in S(t)} \exp[s_{tn}\ln(p_{tn}/\alpha_n^*)]$, a share-weighted geometric mean of the quality-adjusted prices present in period t, and $Q^{t*} \equiv \Sigma_{n\in S(t)} p_{tn}q_{tn}/P^{t*}$ for $t = 1, \ldots, T$. The counterparts to equations (206) are now: $Q^{t**} \equiv \Sigma_{n\in S(t)} \alpha_n^* q_{tn}$ and $P^{t**} \equiv \Sigma_{n\in S(t)} p_{tn}q_{tn}/Q^{t**} = \Sigma_{n\in S(t)} p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^* q_{tn} = \Sigma_{n\in S(t)} p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^* (p_{tn})^{-1} p_{tn}q_{tn} = [\Sigma_{n\in S(t)} s_{tn}(p_{tn}/\alpha_n^*)^{-1}]^{-1}$, a share-weighted harmonic mean of the quality-adjusted prices present in period t. Thus, using Schlömilch's inequality (see Hardy, Littlewood, and Polyá (1934, 26)), we see that $P^{t**} \leq P^{t*}$, which in turn implies that $Q^{t**} \geq Q^{t*}$ for $t = 1, \ldots, T$. This algebra is was developed by de Haan (2004b, 2010) and de Haan and Krsinich (2018, 763). If the variance of prices increases over time, it is likely that P^{t**}/P^{1**} will be less than P^{t*}/P^{1*}, and vice versa, if the variance of prices decreases; see de Haan and Krsinich (2018, 771) and Diewert (2018, 10) on this last point. Note that the work of de Haan and Krsinich provides us with a concrete formula for the difference between P^{t*} and P^{t**}. The model used by de Haan and Krsinich is a more general hedonic regression model which includes the time dummy model used in the present section as a special case.

180 However, we would like the P^{t*} to satisfy a strong identity test as noted above; that is, we would like P^{t*} to equal P^{r*} if the prices in periods t and r are identical. The P^{t*} equal to the π_t^* where π_t^* are defined by (210) do not satisfy this strong identity test for price levels. However, the Q^{t**} defined as $\Sigma_{n\in S(t)} \alpha_n^* q_{tn}$ do satisfy the strong identity test for quantities and this suggests that the P^{t**}, Q^{t**} decomposition of period t sales may be a better choice than the P^{t*}, Q^{t*} decomposition.

181 See Diewert (2004) for a proof or modify the proof in Section 16.

182 Both methods are basically quality-adjusted unit value methods. Thus, if the products under consideration are highly substitutable, then both methods may give satisfactory results. From the viewpoint of the economic approach to index number theory, the GK method is consistent with utility-maximizing behavior if purchasers have either Leontief (no substitution) preferences or linear preferences (perfect substitution preferences after quality adjustment). The weighted TPD method is consistent with utility-maximizing behavior if purchasers have either Cobb–Douglas preferences or linear preferences. Note that Cobb–Douglas preferences are not consistent with situations where there are new and disappearing products.

183 See (210).

184 If product n in period t is missing, we use the imputed price p_{tn}^* defined by (212) as the positive reservation price for this observation in the definitions for both $P_{WTPD}{}^t$ and $P_T{}^t$, which appear in equations (213) and (214). Thus, the summations in (213) and (214) are over all N products.

Since $\Sigma_{n=1}^{N} (s_{tn} - s_{1n}) = 0$ for each t, the two sets of terms on the right-hand side of equation t in (214) can be interpreted as normalizations of the covariances between the vectors $s^t - s^1$ and $lnp^t - \beta^*$ for the first set of terms and between $s^t - s^1$ and $lnp^1 - \beta^*$ for the second set of terms. If the products are highly substitutable with each other, then a low p_{tn} will usually imply that lnp_{tn} is less than the average log price for product n, β_n^*, and it is also likely that s_{tn} is greater than s_{1n} so that $(s_{tn} - s_{1n})(lnp_{tn} - \beta_n^*)$ is likely to be negative. Hence, the covariance between $s^t - s^1$ and $lnp^t - \beta^*$ will tend to be negative. On the other hand, if p_{1n} is unusually low, then lnp_{1n} will be less than the average log price β_n^*, and it is likely that s_{1n} is greater than s_{tn} so that $(s_{tn} - s_{1n})(lnp_{1n} - \beta_n^*)$ is likely to be positive. Hence, the covariance between $s^t - s^1$ and $lnp^1 - \beta^*$ will tend to be positive. Thus, the first set of terms on the right-hand side of (214) will tend to be negative, while the second set will tend to be positive. If there are no divergent trends in log prices and sales shares, then it is likely that these two terms will largely offset each other and under these conditions P_{WTPD}^t is likely to approximate P_T^t reasonably well. However, with divergent trends and highly substitutable products, it is likely that the first set of negative terms will be larger in magnitude than the second set of terms, and thus P_{WTPD}^t is likely to be below P_T^t under these conditions.[185] But if some product n is not available in period 1 so that $s_{1n} = 0$ and if the logarithm of the imputed price for this product p_{1n}^* defined by (212) is greater than β_n^*, then it can happen that the second covariance term on the right-hand side of (214) becomes very large and positive so that it overwhelms the first negative covariance term, and thus P_{WTPD}^t ends up above P_T^t rather than below it.

To sum up, the weighted time product indices can be problematic in the elementary index context when price and quantity data are available as compared to a fixed-base superlative index (that uses reservation prices):

- If there are no missing products and the products are strong substitutes, the WTPD indices will tend to have a downward bias.
- If there are no missing products and the products are weak substitutes, the WTPD indices will tend to have an upward bias.
- If there are missing products in period 1, the relationship between the WTPD indices and the corresponding Törnqvist–Theil indices is uncertain.
- If there are missing products, the weighted TPD price levels and price indices do not depend on reservation prices (which could be regarded as an advantage of the WTPD indices for price statisticians who want to avoid making imputations).

18. Linking Based on Relative Price Similarity

The GEKS multilateral method treats each set of price indices using the prices of one period as the base period as being equally valid, and hence an averaging of the resulting parities seems to be appropriate under this hypothesis. Thus, the method is "democratic" in that each bilateral index number comparison between any two periods gets the same weight in the overall method. However, it is not the case that all bilateral comparisons of price between two periods are equally accurate: If the relative prices in periods r and t are very similar, then the Laspeyres and Paasche price indices will be very close to each other, and hence it is likely that the "true" price comparison between these two periods (using the economic approach to index number theory) will be very close to the bilateral Fisher index that compares prices between the two periods under consideration. In particular, if the two price vectors are exactly proportional, then we want the price index between these two periods to be equal to the factor of proportionality and the direct Fisher index between these two periods satisfies this proportionality test. On the other hand, the GEKS index comparison between the two periods would not in general satisfy this proportionality test.[186] Also if prices are identical between two periods but the quantity vectors are different, then GEKS price index between the two periods would not equal unity in general.[187] These considerations suggest that a more accurate set of price indices could be constructed if initially a bilateral comparison was made between the two periods that have the most *similar relative price structures*. At the next stage of comparison, look for a third period that had the most similar relative price structure to the first two periods and link in this third country to the comparisons of volume between the first two countries, and so on. At the end of this procedure, a pathway through the periods in the window would be constructed that minimized the sum of the relative price dissimilarity measures. In the context of making comparisons of prices across countries, this method of linking countries with the most similar structure of relative prices has been pursued by Hill (1997, 1999a, 1999b, 2009), Hill and Timmer (2006), Diewert (2009, 2013, 2018) and Hill et al. (2017). Hill (2001, 2004) also pursued this similarity of relative prices approach in the time series context. Our conclusion is that similarity linking using Fisher ideal price indices as the bilateral links is an attractive alternative to GEKS.

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two countries. various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981), Kravis, Heston, and

[185] If the products are not highly substitutable so that when a price goes up, the quantity purchased goes down but the expenditure share also goes up, then the inequalities are reversed; that is, if there are no missing products and long-term trends in prices and quantities, then P_{WTPD}^t is likely to be above P_T^t. If preferences of purchasers are Cobb and Douglas, then expenditure shares will remain constant over time, and P_{WTPD}^t will equal P_T^t for $t = 1, \ldots, T$.

[186] If both prices *and* quantities are proportional to each other for the two periods being compared, then the GEKS price index between the two periods will satisfy this (weak) proportionality test. However, we would like the GEKS price index between the two periods to satisfy the strong proportionality test; that is, if the two price vectors are proportional (and the two quantity vectors are not necessarily proportional to each other), then we would like the GEKS price index between the two periods to equal the factor of proportionality.

[187] See Zhang, Johansen, and Nygaard (2019, 689) on this point.

Summers (1982, 104–6), Hill (1997, 2009), Sergeev (2001, 2009), Hill and Timmer (2006), Aten and Heston (2009), and Diewert (2009, 2021a).

In this section, we will discuss the following *weighted asymptotic linear index of relative price dissimilarity*, Δ_{AL}, suggested by Diewert (2009):[188]

$$\Delta_{AL}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}^{N} \tfrac{1}{2}(s_{rn} + s_{tn})\{(p_{tn}/P_F(p^r,p^t,q^r,q^t)p_{rn}) + (P_F(p^r,p^t,q^r,q^t)p_{rn}/p_{tn}) - 2\}, \qquad (215)$$

where $P_F(p^r,p^t,q^r,q^t) \equiv [p^t{\cdot}q^r p^t{\cdot}q^t/p^r{\cdot}q^r\ p^r{\cdot}q^t]^{1/2}$ is the bilateral Fisher price index linking period t to period r, and p^r, q^r, s^r, and p^t, q^t, s^t are the price, quantity, and share vectors for periods r and t, respectively. This measure turns out to be nonnegative, and the bigger $\Delta_{AL}(p^r,p^t,q^r,q^t)$ is, the more dissimilar are the relative prices for periods r and t. Note that if $p^t = \lambda p^r$ for some positive scalar so that if prices are proportional for the two periods, then $\Delta_{AL}(p^r,p^t,q^r,q^t) = 0$. Note also that *all prices need to be positive* in order for $\Delta_{AL}(p^r,p^t,q^r,q^t)$ to be well defined. Thus, if there are missing products in one of the two periods being compared, reservation prices need to be estimated for the missing product prices in each period.[189] Alternatively, inflation-adjusted carry-forward or carry-backward prices can be used to fill in the missing prices.[190]

The method for constructing *similarity-linked Fisher* price indices in real time using the above measure of relative price similarity proceeds as follows. Set the similarity-linked price index for period 1, $P_{AL}^{1} \equiv 1$. The period 2 index is set equal to $P_F(p^1,p^2,q^1,q^2)$, and the Fisher index linking the period 2 prices to the period 1 prices. Thus, $P_{AL}^{2} \equiv P_F(p^1,p^2,q^1,q^2)P_{AL}^{1}$. For period 3, evaluate the dissimilarity indices $\Delta_{AL}(p^1,p^3,q^1,q^3)$ and $\Delta_{AL}(p^2,p^3,q^2,q^3)$ defined by (215). If $\Delta_{AL}(p^1,p^3,q^1,q^3)$ is the minimum of the two numbers, $\Delta_{AL}(p^1,p^3,q^1,q^3)$ and $\Delta_{AL}(p^2,p^3,q^2,q^3)$, define $P_{AL}^{3} \equiv P_F(p^1,p^3,q^1,q^3)P_{AL}^{1}$. If $\Delta_{AL}(p^2,p^3,q^2,q^3)$ is the minimum of these two numbers, define $P_{AL}^{3} \equiv P_F(p^2,p^3,q^2,q^3)P_{AL}^{2}$. For period 4, evaluate the dissimilarity indices $\Delta_{AL}(p^r,p^4,q^r,q^4)$ for $r = 1,2,3$. Let r^* be such that $\Delta_{AL}(p^{r*},p^4,q^{r*},q^4) = \min_r \{\Delta_{AL}(p^r,p^4,q^r,q^4);\ r = 1,2,3\}$.[191] Then, define $P_{AL}^{4} \equiv P_F(p^{r*},p^4,q^{r*},q^4)P_{AL}^{r*}$. Continue this process in the same manner; that is, for period t, let r^* be such that $\Delta_{AL}(p^{r*},p^t,q^{r*},q^t) = \min_r \{\Delta_{AL}(p^r,p^t,q^r,q^t);\ r = 1,2,\ldots,t{-}1\}$, and define $P_{AL}^{t} \equiv P_F(p^{r*},p^t,q^{r*},q^t)P_{AL}^{r*}$. This procedure allows for the construction of similarity-linked indices in real time.

Diewert (2018) implemented the above procedure with a retail outlet scanner data set and compared the resulting similarity-linked index, P_{AL}^{t}, to other indices that are based on the use of superlative indices and the economic approach to index number theory. The data set he used is listed in Section A.7.1 of the annex, and his results are listed in the annex along with some additional results. The comparison indices in his study were the fixed-base Fisher and Törnqvist indices, P_F^{t} and P_T^{t}, and the multilateral indices, P_{GEKS}^{t} and P_{CCDI}^{t}. The sample means for these five indices, P_{AL}^{t}, P_F^{t}, P_T^{t}, P_{GEKS}^{t}, and P_{CCDI}^{t}, were 0.97069, 0.97434, 0.97607, 0.97417, and 0.97602. Thus, on average, P_{AL}^{t} was about 0.5 percentage points below P_T^{t} and P_{CCDI}^{t} and about 0.35 percentage points below P_F^{t} and P_{GEKS}^{t}. These are fairly significant differences.[192]

What are some of the advantages and disadvantages of using P_{AL}^{t}, P_F^{t}, P_T^{t}, P_{GEKS}^{t}, or P_{CCDI}^{t} as target indices for an elementary index in a CPI? All of these indices are equally consistent with the economic approach to index number theory. The problem with the fixed-base Fisher and Törnqvist indices is that they depend too heavily on the base period. Moreover, sample attrition means that the base must be changed fairly frequently, leading to a potential chain drift problem. The GEKS and CCDI indices also suffer from problems associated with the existence of seasonal products: It makes little sense to include bilateral indices between all possible periods in a window of periods in the context of seasonal commodities. The similarity-linked indices address both the problem of sample attrition and the problem of seasonal commodities. Moreover, Walsh's multiperiod identity test is always satisfied using this methodology. Finally, there is no need to choose a window length and use a rolling window approach to construct the time series of indices if the price similarity linking method is used: The window length simply grows by one period as the data for an additional period becomes available.[193]

The procedure for constructing the time series of similarity-linked Fisher price indices, P_{AL}^{t}, is a *real-time procedure*; that is, there is no preliminary time period that is required in order to produce the final time series of aggregate price levels. However, the resulting pattern of bilateral links may not be "optimal" in the sense that the most similar sets of relative prices are linked to one another in the first year or so. This is apparent when the price level P_{AL}^{2} is constructed: It is simply equal to the Fisher index linking period 2 to 1; there are no other choices for a linking partner. A "better" set of bilateral links could potentially be obtained if a final set of bilateral links for the index could be obtained by forming a *spanning tree of comparisons*, say, for the first year of data.[194] Thus, a year of data on prices and quantities is used to form a set of bilateral links that minimizes the sum of the associated dissimilarity measures that link the observations for the first year. This leads to a *modified* set of price levels for the first year, say P_{ALM}^{t} for t in the first year. For months t that follow after the first "training" year, the bilateral links are the same as indicated earlier, but because the levels in the first year may have changed, the modified price levels P_{ALM}^{t} for months t that follow after the first year may differ from the real-time price levels P_{AL}^{t} described earlier. However, the trends in the two series will be similar.

[188] The discussion paper version of Diewert (2009) appeared in (2002).

[189] See Section 14 of Diewert (2022a) for additional information on reservation prices.

[190] See the discussion in the following section. Section A.7.6 of the annex compares P_{AL}^{t} computed using reservation prices and P_{ALC}^{t} which uses inflation-adjusted carry-forward/backward prices for the missing products. For our particular empirical example, there were small differences in the resulting indices.

[191] If the minimum occurs at more than one r, choose r^* to be the earliest of these minimizing periods.

[192] The final values for the five indices (P_{AL}^{t}, P_F^{t}, P_T^{t}, P_{GEKS}^{t}, and P_{CCDI}^{t}) were as follows: 0.92575, 0.95071, 0.95482, 0.94591, and 0.94834. Thus P_{AL}^{t} ended up significantly below the other indices. P_T^{t} is listed in Table 1.4 and the remaining indices are listed in Table 1.6 of the annex.

[193] In practice, as the number of periods grows and the structure of the economy evolves, it will become increasingly unlikely that a current observation will be linked to a distant observation. Thus eventually, it becomes practical to move to a rolling window framework with a large window length.

[194] See Hill (2001, 2004) for explanations of how this can be done.

In Section A.7.5 of the Annex, we calculate both $P_{AL}{}^{t}$ and $P_{ALM}{}^{t}$ for the data set listed in Section A1 of the Annex. There is little difference in these two series for our example data set, and in fact, both series end up at the same point.[195] Normally, we do not expect much difference between the original real-time method and the modified method, but the modified method is useful in the context of constructing price indices for strongly seasonal commodities because it will tend to reduce the magnitude of seasonal fluctuations.

Similarity-linked price indices suffer from at least two problems:

- A measure of relative price dissimilarity must be chosen, and there may be many "reasonable" choices for the measure of dissimilarity. These different choices can lead to different indices, which in turn can lead users to question the usefulness of the method.
- The measures of weighted price dissimilarity suggested by Diewert (2009) require that all prices in the comparison of prices between two periods be positive.

These problems will be addressed in Section 20, where an alternative measure of price dissimilarity that does not require strictly positive prices will be defined. Using the scanner data set listed in Section A1 of the Annex, this new measure of price (and quantity) dissimilarity generates indices $P_{SP}{}^{t}$ that are very similar to the $P_{AL}{}^{t}$ indices discussed in the present section.

It is a difficult econometric exercise to estimate reservation prices, and so a simpler method may be required in order to construct imputed prices for missing products in a scanner data set. In the following section, a standard method used by price statisticians is explained.

19. Inflation-Adjusted Carry-Forward and Carry-Backward Imputed Prices

When constructing elementary indices, statistical agencies often encounter situations where a product in an elementary index disappears. At the time of disappearance, it is unknown whether the product is temporarily unavailable, so the missing price could be set equal to the last available price; that is, the missing price could be replaced by a carry-*forward price*. Thus, carry-forward prices could be used in place of reservation prices, which are much more difficult to construct. This procedure is, in general, not a recommended one. A much better alternative to the use of a carry-forward price is an *inflation-adjusted carry-forward price*; that is, the last available price is escalated using the maximum overlap index between the period when the product was last available and the current period where an appropriate index number formula is used.[196] In this section, we use inflation-adjusted carry-forward and carry-backward prices in place of the reservation prices for our scanner data set and compare the resulting indices with our earlier indices that used the econometrically estimated reservation prices that were constructed by Diewert and Feenstra (2017) for the scanner data set listed in the annex.

Suppose we have price and quantity data for N products for T periods as usual. Let $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ denote the period t price and quantity vectors. If product n is not present in period t, define (for now) the corresponding p_{tn} and q_{tn} to be 0. Define $S(t)$ to be the set of products that are present in period t; that is, $S(t) \equiv \{n: p_{tn} > 0\}$.[197] Suppose that we want to make a Fisher index number comparison between periods r and t where $r < t$. The *maximum overlap set of products* that are present in periods r and t is the intersection set, $S(r) \cap S(t)$. We assume that this set is nonempty. Define the vectors p^{r*}, p^{t*}, q^{r*}, q^{t*} as the vectors that have only the products that are present in periods r and t. Define the *maximum overlap Fisher price index* for period t relative to period r as $P_{FM}(p^{r*},p^{t*},q^{r*},q^{t*})$. If there are products present in period r that are not present in period t, define the *inflation-adjusted carry-forward price* for such products as follows:

$$p_{tn} \equiv p_{rn} P_{FM}(p^{r*},p^{t*},q^{r*},q^{t*});\ n \in S(r);\ n \notin S(t). \qquad (216)$$

The corresponding quantities q_{tn} remains at their initially defined 0 levels. If there are products present in period t that are not present in period r, define the *inflation-adjusted carry-backward price* for such products as follows:

$$p_{rn} \equiv p_{tn} / P_{FM}(p^{r*},p^{t*},q^{r*},q^{t*});\ n \in S(t);\ n \notin S(r). \qquad (217)$$

The corresponding quantities q_{rn} remain at their initial 0 levels.

Using these definitions, we will have new price and quantity vectors that have well-defined price and quantity vectors p^{r**}, p^{t**}, q^{r**}, q^{t**} that have positive prices for products that belong to the union set of products that are present in both periods r and t, $S(r) \cup S(t)$. Denote the Fisher index for period t relative to period r over this union set of products as $P_F{}^{*}(p^{r**},p^{t**},q^{r**},q^{t**})$. This index can be used as the Fisher index linking periods r and t. Thus, the carry-forward and carry-backward prices defined by (216) and (217) can replace econometrically estimated reservation prices, and the similarity-linked price indices defined in the previous section can be calculated using the Fisher linking indices $P_F{}^{*}(p^{r**},p^{t**},q^{r**},q^{t**})$ in place of the $P_F(p^r,p^t,q^r,q^t)$ used in the previous section. Note that the components of the period t price vector p^{t**} will be equal to the components of the original period t price vector p^t except for components that correspond to missing products.

[195] See Table 1.7 and Figure A7.9 in the Annex. Although $P_{AL}{}^{t}$ and $P_{ALM}{}^{t}$ end up at the same level, the mean of the $P_{AL}{}^{t}$ was 0.97069 and the mean of the $P_{ALM}{}^{t}$ was 0.96437. The fluctuations in the $P_{ALM}{}^{t}$ series were somewhat smaller. This tendency for the modified series to be a bit smoother than the corresponding real-time series becomes important in the context of constructing indices for strongly seasonal commodities. In this context, the use of the modified similarity linking method is recommended in order to reduce seasonal fluctuations.

[196] Triplett (2004, 21–29) calls these two methods for replacing missing prices the *link to show no change method* and the *deletion method*. See section 14 in Diewert (2022a) and Diewert, Fox and Schreyer (2017) for a more extensive discussion on the problems associated with finding replacements for missing prices.

[197] Recall that this notation was used in previous sections.

It should be emphasized that, usually, it is important to make the index number adjustments to the carry-forward and carry-backward prices defined by (216) and (217) instead of simply carrying existing prices from one period to another period. Failure to make these index number adjustments could lead to substantial biases if substantial general inflation (or deflation) is present. From the perspective of the economic approach to index number theory, it is likely that the use of inflation-adjusted carry-backward prices in place of estimated reservation prices will generally lead to an upward bias in the linking index since the "true" reservation prices are likely to be higher than the adjusted prices in order to induce consumers to purchase zero units of the unavailable products in the prior period. Of course, the bias in using carry-forward prices for disappearing products works in the opposite direction.

In Section A.7.6 of the annex, we used our scanner data to compute the GEKS, Fisher, chained Fisher and the real-time similarity-linked index explained in the previous section which used the Δ_{AL} dissimilarity measure defined by (215). We also calculated the real-time predicted share similarity-linked indices that use the Δ_{SP} dissimilarity measure that will be defined by (218) in the following section. Denote the resulting period t index by $P_{SP}{}^t$. There were missing products in our scanner data set. As noted above, the missing prices were initially set equal to reservation prices calculated using econometrics. Denote these indices for period t (which used reservation prices) by $P_{GEKS}{}^t$, $P_F{}^t$, $P_{FCH}{}^t$, $P_{AL}{}^t$, and $P_{SP}{}^t$. The same five indices were recomputed using inflation-adjusted carry-forward and carry-backward prices for the missing product prices.[198] Denote the resulting period t indices by $P_{GEKSC}{}^t$, $P_{FC}{}^t$, $P_{FCHC}{}^t$, $P_{ALC}{}^t$, and $P_{SPC}{}^t$. As noted earlier, it turns out that the GK index ($P_{GK}{}^t$) and the weighted TPD index ($P_{WTPD}{}^t$) do not depend on the values of the missing prices and so these indices do not have to be recomputed using carry-forward prices in place of reservation prices. $P_{GK}{}^t$ and $P_{WTPD}{}^t$ are listed in Table 1.6 in Section A.7.5 of the annex. The series $P_{AL}{}^t$, $P_{ALC}{}^t$, $P_{SP}{}^t$, $P_{SPC}{}^t$, $P_{GEKS}{}^t$, $P_{GEKSC}{}^t$ is listed in Table 1.8 in Section A.7.6 of the annex along with the Fisher and chained Fisher indices using reservation prices, denoted by $P_F{}^t$ and $P_{FCH}{}^t$, and using carry-forward prices, denoted by $P_{FC}{}^t$ and $P_{FCHC}{}^t$.

A summary of the results using econometrically estimated reservation prices versus using carry-forward and carry-backward prices for the missing products is as follows: for our example, there was very little difference between the resulting index pairs using reservation prices versus using inflation-adjusted carry-forward prices. This is likely due to the fact that only 20 out of 741 prices were missing; that is, only 2.7 percent of the sample had missing products. (0.97542) and $P_{FCH}{}^A = 1.0589$ (1.0589). Our tentative conclusion here is that *for products that are highly substitutable, the use of inflation-adjusted carry-forward and carry-backward prices for missing products will probably generate weighted indices that are comparable to their counterparts that use econometrically estimated reservation prices*. For products which are not highly substitutable, it is likely that reservation prices will be higher than their inflation-adjusted carry-forward prices, and thus it is likely that the indices will differ in a more substantial manner. This conclusion is only tentative, and further research on the use of reservation prices is required.

20. Linking Based on Relative Price and Quantity Similarity

A problem with the measure of relative price dissimilarity $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215) is that it requires that all prices in the two periods being compared must be positive. Thus, if there are missing prices for some products present in one of the two periods but not in the other period, then the Δ_{AL} dissimilarity measure is not well defined.[199]

The following *predicted share measure of relative price dissimilarity*, $\Delta_{SP}(p^r,p^t,q^r,q^t)$, is well defined even if some product prices in the two periods being compared are equal to 0:

$$\Delta_{SP}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}{}^N [s_{tn} - (p_{rn}q_{tn}/p^r{\cdot}q^t)]^2 + \Sigma_{n=1}{}^N [s_{rn} - (p_{tn}q_{rn}/p^t{\cdot}q^r)]^2, \qquad (218)$$

where $s_{tn} \equiv p_{tn}q_{tn}/p^t{\cdot}q^t$ is the share of product n in period t expenditures on the N products for $t = 1,\ldots,T$ and $n = 1,\ldots,N$. We require that $p^r{\cdot}q^t > 0$ for $r = 1,\ldots,T$ and $t = 1,\ldots,T$ in order for $\Delta_{SP}(p^r,p^t,q^r,q^t)$ to be well defined for any pair of periods, r and t. Since the two summations on the right-hand side of (218) are sums of squared terms, we see that $\Delta_{SP}(p^r,p^t,q^r,q^t) \geq 0$.

The first set of N terms on the right-hand side of (218) is $\Sigma_{n=1}{}^N [s_{tn} - (p_{rn}q_{tn}/p^r{\cdot}q^t)]^2$. Note that the terms $p_{rn}q_{tn}/p^r{\cdot}q^t$ for $n = 1,\ldots,N$ are (hybrid) *shares*; that is, these terms are nonnegative and they sum to unity so that $\Sigma_{n=1}{}^N (p_{rn}q_{tn}/p^r{\cdot}q^t) = 1$. These shares use the prices of period r and the quantities of period t. They can be regarded as *predictions* for the actual period t shares, s_{tn}, using the prices of period r but using the quantities of period t. A similar interpretation applies to the second set of N terms on the right-hand side of (218); the hybrid shares that use the prices of period t and the quantities of period r, $p_{tn}q_{rn}/p^t{\cdot}q^r$, can be regarded as predictors for the actual period r shares, s_{rn}. Since each share s_{tn} in the first set of terms is already weighted by its economic importance, there is no need for any further weighting of the first set of N squared terms in the summation to account for economic importance. The same analysis applies to the second set of N sum of squared terms; each term in the summation is already weighted by its economic importance.

If prices in period t are proportional to prices in period r (so that $p^t = \lambda_t p^r$ for some scalar $\lambda_t > 0$ or $p^r = \lambda_r p^t$ for some $\lambda_r > 0$), then it is easy to verify that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is equal to 0.

[198] Inflation-adjusted carry-forward prices were used to compute prices for missing products except when a product was missing in period 1. In the latter case, inflation-adjusted carry backward prices were computed for the missing products.

[199] Diewert (2009, 205–6) recommended two other measures of price dissimilarity, but they also have the problem of not being well defined if some product prices are equal to 0. These alternative measures are the *weighted log quadratic measure of relative price dissimilarity*, $\Delta_{PLQ}(p^1,p^2,q^1,q^2) \equiv \Sigma_{n=1}{}^N (1/2)(s_n{}^1 + s_n{}^2)[\ln(p_n{}^2/p_n{}^1P(p^1,p^2,q^1,q^2))]^2$, and the *weighted asymptotically quadratic measure of relative price dissimilarity*, $\Sigma_{n=1}{}^N (1/2)(s_n{}^1 + s_n{}^2)\{[(p_n{}^2/p_n{}^1P(p^1,p^2,q^1,q^2) - 1]^2 + [(P(p^1,p^2,q^1,q^2)p_n{}^1/p_n{}^2) - 1]^2\} \equiv \Delta_{WAQ}(p^1,p^2,q^1,q^2)$, where $P(p^1,p^2,q^1,q^2)$ is any superlative bilateral price index formula. It can be shown that $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$ approximates $\Delta_{AL}(p^r,p^t,q^r,q^t)$ to the second order around any point where $p^1 = p^2 >> 0_N$ and $q^1 = q^2 >> 0_N$.

Now consider the implications on p^t and p^r if $\Delta_{SP}(p^r,p^t,q^r,q^t) = 0$. We need to consider a number of cases, depending on assumptions about the positivity of the prices and quantities in periods r and t. In all cases listed here, it is assumed that $p^r \cdot q^t > 0$ for $r = 1, \ldots, T$ and $t = 1, \ldots, T$.

Case (i): $\Delta_{SP}(p^r,p^t,q^r,q^t) = 0$ and $q^t >> 0_N$ *or* $q^r >> 0_N$; that is, assume that all components of the period t *or* period r quantity vectors are positive. If $q^t >> 0_N$ and $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is 0, then the first sum of squared terms, $\Sigma_{n=1}^{N} [s_{tn} - (p_{rn}q_{tn}/p^r \cdot q^t)]^2 = 0$, which implies that $p_{tn}q_{tn} = (p^t \cdot q^t / p^r \cdot q^t)p_{rn}q_{tn}$ which in turn implies that $p_{tn} = (p^t \cdot q^t/p^r \cdot q^t)p_{rn}$ since $q_{tn} > 0$ for $n = 1, \ldots, N$. Thus, $p^t = \lambda_{tr}p^r$, where $\lambda_{tr} \equiv p^t \cdot q^t/p^r \cdot q^t > 0$, which implies that the period t price vector is proportional to the period r price vector. If $q^r >> 0_N$ and $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is 0, then the second set of terms on the right-hand side of (218) is equal to zero. Thus, we must have $p_{rn} = (p^r \cdot q^r/p^t \cdot q^r)p_{tn}$ for $n = 1, \ldots, N$. Thus, $p^r = \lambda_{rt}p^t$, where $\lambda_{rt} \equiv p^r \cdot q^r/p^t \cdot q^r > 0$, which in turn implies that the period r price vector is proportional to the period t price vector.

Case (ii): $\Delta_{SP}(p^r,p^t,q^r,q^t) = 0$ and $q^r + q^t >> 0_N$ so that each product is present in at least one of the two periods, periods r and t, whose prices are being compared. We further assume that there is at least one product n^* that is present in both periods being compared; that is, there exists an n^* such that $q_{rn^*} > 0$ and $q_{tn^*} > 0$. Following the same type of argument that was pursued for Case (i), we find that our assumptions imply that $p_{tn} = \lambda_{tr}p_{rn}$ for n such that $q_{tn} > 0$ and $p_{rn} = \lambda_{rt}p_{tn}$ for n such that $q_{rn} > 0$. For products n^* that are present in both periods r and t, we have $p_{tn^*} = \lambda_{tr}p_{rn^*}$ and $p_{rn^*} = \lambda_{rt}p_{tn^*}$, and thus $\lambda_{tr} = 1/\lambda_{rt}$. These equalities imply that the period t price vector must be proportional to the period r price vector under our present assumptions.

Case (iii): Some products are not present in both periods r and t. This case can be reduced down to one of the previous cases for a new N^* that just includes the products that are present in in at least one of periods r and t.

Using the above analysis, it can be seen that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ equals 0 if and only if the period r and t price vectors are proportional. If the price vectors are not proportional, then $\Delta_{SP}(p^r,p^t,q^r,q^t)$ will be positive. A larger value for $\Delta_{SP}(p^r,p^t,q^r,q^t)$ indicates a bigger deviation from price proportionality. Thus, $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is a "reasonable" measure of bilateral relative price dissimilarity.

There are some aspects of the predicted price measure of relative price dissimilarity that require further discussion. When comparing the prices of periods r and t, suppose product 1 is present in period t but not present in period r. More precisely, suppose $q_{t1} > 0$ (and $p_{t1} > 0$) but $q_{r1} = 0$. What is the corresponding price for the missing product in period r; that is, what exactly is p_{r1}? Suppose we set $p_{r1} = 0$. For simplicity, suppose further that prices and quantities for products 2 to N are the same in periods r and t, so that $p_{rn} = p_{tn}$ and $q_{rn} = q_{tn}$ for $n = 2,3, \ldots, N$. Under these conditions, we find that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is equal to the following sum of squared terms:

$$\begin{aligned}\Delta_{SP}(p^r,p^t,q^r,q^t) &\equiv \Sigma_{n=1}^{N} [s_{tn} - (p_{rn}q_{tn}/p^r \cdot q^t)]^2 + \Sigma_{n=1}^{N} [s_{rn} - (p_{tn}q_{rn}/p^t \cdot q^r)]^2 \\ &= [s_{t1} - 0]^2 + \Sigma_{n=2}^{N} [s_{tn} - s_{rn}]^2 + \Sigma_{n=1}^{N} [s_{rn} - s_{rn}]^2 \\ &= s_{t1}^{\;2} + \Sigma_{n=2}^{N} [s_{tn} - s_{rn}]^2 > 0, \qquad (219)\end{aligned}$$

where the inequality follows from our assumptions, $s_{t1} > 0$. Thus, even if all prices and quantities are the same for products that are present in both periods r and t, the dissimilarity measure defined by (218) will be positive as long as there are some products that are present in only one of the two periods being compared. Thus, if we set the prices for missing products equal to 0, then the predicted share measure of relative price dissimilarity will automatically register a positive measure; that is, the measure will *penalize* a lack of matching of prices if we set the prices for missing products equal to 0.

Hill and Timmer were the first to point out the importance of having a measure of relative price dissimilarity that would penalize a lack of matching of the prices in the two periods being compared:

> In a survey of the literature on reliability measures, Rao and Timmer (2003) concluded that the main problem of existing measures, such as Hill's (1999) Paasche-Laspeyres spread and Diewert's (2002) class of relative price dissimilarity measures, is that they fail to make adjustments for gaps in the data. Rao and Timmer drew a distinction between statistical and index theoretic measures of reliability. The former take a sampling perspective; bilateral comparisons based on a small number of matched product headings or a low coverage of total expenditure or production (averaged across the two countries) are deemed less reliable. In addition to the standard statistical arguments regarding small samples and a low coverage not being representative, little overlap in the product headings priced by the two countries implies that they are very different and, by implication, inherently difficult to compare. Index theoretic measures, in contrast, focus on the sensitivity of a bilateral comparison to the choice of price index formula. Most of the reliability measures proposed in the literature, including Hill's (1999) Paasche-Laspeyres spread and Diewert's (2002) class of relative price dissimilarity measures, are of this type. Although these measures perform well when there are few gaps in the data, they can generate highly misleading results when there are many gaps. This is because they fail to penalize bilateral comparisons made over a small number of matched headings.
>
> Hill and Timmer (2006, 366)

These considerations suggest that the predicted share measure of relative price dissimilarity could be used under two different sets of circumstances when there are missing prices:

- Use carry-forward (or carry-backward) prices or reservation prices for the missing prices and use the measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) to link the observations. With a complete set of prices for each period in hand, the usual bilateral Fisher index could be used as the linking index. This approach is consistent with the economic approach to index number theory.
- Do not estimate carry-forward or reservation prices for the missing price observations (and set the prices of the

missing products equal to 0) but still use $\Delta_{SP}(p^r,p^t,q^r,q^t)$ to link the observations. In this case, the *maximum overlap* bilateral Fisher index is used as the linking index for each pair of links chosen by the similarity linking method. This approach is more consistent with the stochastic approach to index number theory used by Hill and Timmer (2006).

Both strategies are illustrated for our empirical example in the annex.

Some additional properties of $\Delta_{SP}(p^r,p^t,q^r,q^t)$ are as follows:

- *Symmetry*; that is, $\Delta_{SP}(p^r,p^t,q^r,q^t) = \Delta_{SP}(p^t,p^r,q^t,q^r)$.
- *Invariance to changes in the units of measurement*.
- *Homogeneity of degree 0 in the components of q^r and q^t*; that is, $\Delta_{SP}(p^r,p^t,\lambda_r q^r,\lambda_t q^t) = \Delta_{SP}(p^r,p^t,q^r,q^t)$ for all $\lambda_r > 0$ and $\lambda_t > 0$.
- *Homogeneity of degree 0 in the components of p^r and p^t*; that is, $\Delta_{SP}(\lambda_r p^r,\lambda_t p^t,q^r,q^t) = \Delta_{SP}(p^r,p^t,q^r,q^t)$ for all $\lambda_r > 0$ and $\lambda_t > 0$.

The relative price dissimilarity indices $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) can be used in place of the dissimilarity indices $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215) in Section 18 in order to link together bilateral Fisher indices. Thus, set the new relative price similarity-linked Fisher price index for period 1 equal to unity; that is, set $P_{SP}{}^1 \equiv 1$. The period 2 index is set equal to $P_F(p^1,p^2,q^1,q^2)$, the Fisher index linking the period 2 prices to the period 1 prices.[200] Thus, $P_{SP}{}^2 \equiv P_F(p^1,p^2,q^1,q^2)P_{SP}{}^1$. For period 3, evaluate the dissimilarity indices $\Delta_{SP}(p^1,p^3,q^1,q^3)$ and $\Delta_{SP}(p^2,p^3,q^2,q^3)$ defined by (218). If $\Delta_{SP}(p^1,p^3,q^1,q^3)$ is the minimum of these two numbers, define $P_{SP}{}^3 \equiv P_F(p^1,p^3,q^1,q^3)P_{PS}{}^1$. If $\Delta_{SP}(p^2,p^3,q^2,q^3)$ is the minimum of these two numbers, define $P_{SP}{}^3 \equiv P_F(p^2,p^3,q^2,q^3)P_{SP}{}^2$. For period 4, evaluate the dissimilarity indices $\Delta_{SP}(p^r,p^4,q^r,q^4)$ for $r = 1,2,3$. Let r^* be such that $\Delta_{SP}(p^{r*},p^4,q^{r*},q^4) = \min_r \{\Delta_{SP}(p^r,p^4,q^r,q^4);\ r = 1,2,3\}$.[201] Then define $P_{SP}{}^4 \equiv P_F(p^{r*},p^4,q^{r*},q^4)P_{SP}{}^{r*}$. Continue this process in the same manner; that is, for period t, let r^* be such that $D_{SP}(p^{r*},p^t,q^{r*},q^t) = \min_r \{\Delta_{SP}(p^r,p^t,q^r,q^t);\ r = 1,2,\ldots,t{-}1\}$ and define $P_{SP}{}^t \equiv P_F(p^{r*},p^t,q^{r*},q^t)P_{SP}{}^{r*}$. Again, as in Section 18, this procedure allows for the construction of similarity-linked indices in real time.

Using the scanner data listed in the annex, which included reservation prices for missing products, the new similarity-linked price indices $P_{SP}{}^t$ were calculated and compared to the price similarity-linked price indices $P_{AL}{}^t$ that were defined in Section 18. The new measure of relative price dissimilarity led to a different pattern of bilateral links: 7 of the 38 bilateral links changed when the dissimilarity measure was changed from $\Delta_{AL}(p^r,p^t,q^r,q^t)$ to $\Delta_{SP}(p^r,p^t,q^r,q^t)$. However, the price indices generated by these alternative methods for linking observations were very similar: The sample averages for $P_{AL}{}^t$ and $P_{SP}{}^t$ were 0.97069 and 0.97109, respectively, and the correlation coefficient between the two indices was 0.99681. Both indices ended up at 0.9275. Thus, even though the two measures of price dissimilarity generated a different pattern of bilateral links, the underlying indices $P_{Al}{}^t$ and $P_{SP}{}^t$ approximated each other very closely.

Both of the similarity-linked price indices $P_{AL}{}^t$ and $P_{SP}{}^t$ satisfy *a strong identity test*; that is, if $p^r = p^t$, then $P_{AL}{}^r = P_{AL}{}^t$ and $P_{SP}{}^r = P_{SP}{}^t$. It is not necessary for q^r to equal q^t for this strong identity test to be satisfied. Thus, these similarity-linked indices have an advantage over the corresponding GEKS and CCDI multilateral indices in that in order to ensure that $P_{GEKS}{}^r = P_{GEKS}{}^t$ and $P_{CCDI}{}^r = P_{CCDI}{}^t$, we require that $p^r = p^t$ *and* $q^r = q^t$; that is, we require that quantities be equal for the two periods as well as prices.

The preceding material can be adapted to measuring the *relative similarity of quantities* in place of prices. The incentive to use similarity of relative quantities is as follows: If the period r and t quantity vectors are proportional, then the Laspeyres, Paasche, and Fisher quantity indices will be equal to this factor of quantity proportionality. In particular, if $q^r = q^t$, then the Laspeyres, Paasche, Fisher, and any superlative quantity index will be equal to unity, without requiring p^t and p^r to be equal. Thus, when the quantity vectors are proportional, it makes sense to define the price indices residually using the Product Test. Thus, define the following measure of *relative quantity similarity* between the quantity vectors for periods r and t as follows:[202]

$$\Delta_{SQ}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}^N [s_{tn} - (p_{tn}q_{rn}/p^t{\cdot}q^r)]^2 + \Sigma_{n=1}^N [s_{rn} - (p_{rn}q_{tn}/p^r{\cdot}q^t)]^2. \qquad (220)$$

If the quantity vectors q^r and q^t are proportional to each other, then it is straightforward to verify that $\Delta_{SQ}(p^r,p^t,q^r,q^t) = 0$. On the other hand, if $\Delta_{SQ}(p^r,p^t,q^r,q^t) = 0$, then one can repeat Cases (i)–(iii), with prices and quantities interchanged, to show that q^r and q^t must be proportional to each other. Thus, $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ equals 0 if and only if the period r and t quantity vectors are proportional. If the quantity vectors are not proportional, then $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ will be positive. A larger value for $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ indicates a bigger deviation from quantity proportionality. An advantage of the measure of dissimilarity defined by (220) is that it can deal with q_{tn} that are equal to 0.[203]

The new dissimilarity measure $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ can be used in place of $\Delta_{SP}(p^r,p^t,q^r,q^t)$ in order to construct a new pattern of bilateral Fisher price index links,[204] leading to a new series of price indices, say $P_{SQ}{}^t$ for $t = 1,\ldots,T$. The advantage of computing this sequence of price indices is that they will satisfy the following *fixed basket test*: if $q^r = q^t \equiv q$ for $r < t$,

[200] In the present context, it is not necessary to have all prices positive in computing the Fisher indices. However, if the economic approach to index number theory is applied, then it is preferable to impute the missing prices. Missing quantities should be left at their 0 values using the economic approach.

[201] If the minimum occurs at more than one r, choose r^* to be the earliest of these minimizing periods.

[202] It can be seen that $\Delta_{SQ}(p^r,p^t,q^r,q^t) = \Delta_{SP}(q^r,q^t,p^r,p^t)$; that is, the role of prices and quantities is interchanged in the above measure of price dissimilarity $\Delta_{SP}(p^r,p^t,q^r,q^t)$.

[203] If one takes the economic approach to index number theory and adopts the reservation price methodology due to Hicks (1940), then 0 prices can be avoided by using reservation prices or approximations to them such as inflation-adjusted carry-forward or carry-backward prices. However, 0 quantities cannot be avoided, so we need measures of price and quantity dissimilarity that can accommodate 0 prices and quantities in a sensible way.

[204] The implicit Fisher price index that is defined residually using the product test turns out to be equal to the usual Fisher price index that is defined directly as the geometric mean of the Laspeyres and Paasche price indices.

then $P_{SQ}{}^t/P_{SQ}{}^r = p^t{\cdot}q/p^r{\cdot}q$. Note that this test does not require that $p^t = p^r$. Once the sequence of price indices $P_{SQ}{}^t$ has been constructed, the corresponding quantity levels can be defined as $Q_{SQ}{}^t \equiv p^t{\cdot}q^t/P_{SQ}{}^t$ for $t = 1, \ldots, T$. The fixed basket test for price indices translates into the following *strong identity test* for quantity indices: if $q^r = q^t \equiv q$ for $r < t$, then $Q_{SQ}{}^t/Q_{SQ}{}^r = 1$. Note that this test does not require that $p^r = p^t$. It can be seen that this is the advantage in using $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ as the dissimilarity measure in place of $\Delta_{SP}(p^r,p^t,q^r,q^t)$: if $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ is used, then the strong identity test for quantities will be satisfied by the resulting quantity indices, $Q_{SQ}{}^t$. On the other hand if $\Delta_{SP}(p^r,p^t,q^r,q^t)$ is used as the measure of relative price dissimilarity, then the resulting price indices $P_{SP}{}^t$ will satisfy the strong identity test for prices.

It is possible to design a measure that combines relative price dissimilarity with relative quantity dissimilarity such that the resulting dissimilarity measure when used with Fisher price index bilateral links in the usual manner gives rise to a sequence of price indices (relative to period 1) $P_{SPQ}{}^t$ that will satisfy both the fixed basket test and the strong identity test for prices. Define the following index for *relative price and quantity dissimilarity* between periods r and t, $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$, as follows:[205]

$$\Delta_{SPQ}(p^r,p^t,q^r,q^t) \equiv \min\{\Delta_{SP}(p^r,p^t,q^r,q^t), \Delta_{SQ}(p^r,p^t,q^r,q^t)\}. \qquad (221)$$

Thus, if prices are equal to each other for periods r and t, then $\Delta_{SP}(p^r,p^t,q^r,q^t)$ and $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will both equal 0, and our linking procedure will lead to equal price levels for periods r and t. On the other hand, if quantities are equal to each other for periods r and t, then $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ and $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will both equal 0, and our linking procedure will lead to equal quantity levels for periods r and t.[206] Denote the price indices relative to period 1 generated using $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ as the measure of dissimilarity by $P_{SPQ}{}^t$ for $t = 1, \ldots, T$. Call this method the *SPQ multilateral method*. Thus, the similarity-linked indices that are generated using the dissimilarity measure defined by (221) will lead to index levels that satisfy both a strong identity test for prices and a strong identity test for quantities. Thus, if prices are identical in the two periods being compared ($p^r = p^t$), then the similarity-linked price levels for periods r and t are equal *and* if quantities are identical in the two periods being compared ($q^r = q^t$), then the similarity-linked quantity levels for periods r and t are equal. No of the other multilateral methods studied in this chapter have this very strong property. *This property rules out chain drift both in the price and quantity levels.*

Using the scanner data listed in Annex 1, the new similarity-linked price indices that combine price and quantity similarity linking, $P_{SPQ}{}^t$, were calculated and compared to the price similarity-linked price indices $P_{SP}{}^t$ that were defined in the beginning of this section. For our sample data set, it turned out that predicted share quantity dissimilarity was always greater than the corresponding measure of predicted share price dissimilarity for each pair of observations in our sample. Under these conditions, it can be seen that $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will equal $\Delta_{SP}(p^r,p^t,q^r,q^t)$ for all periods r and t. Thus, the same set of bilateral Fisher index links that were generated using $\Delta_{SP}(p^r,p^t,q^r,q^t)$ were also generated using $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ defined by (221) as the measure of dissimilarity. It turns out that it was always the case that $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ was much bigger than the corresponding $\Delta_{SP}(p^r,p^t,q^r,q^t)$; that is, in all cases, *relative quantity dissimilarity was much bigger than the corresponding relative price dissimilarity*.[207]

In Section A.7.5 of the annex, some variations on the multilateral indices $P_{AL}{}^t$ and $P_{SP}{}^t$ are considered and evaluated using the price and quantity data for our empirical example. The indices $P_{ALM}{}^t$ and $P_{SPM}{}^t$ use the same tables of dissimilarity measures that were used to define the bilateral links for the indices $P_{AL}{}^t$ and $P_{SP}{}^t$ but instead of generating real-time indices, the new *modified* indices $P_{ALM}{}^t$ and $P_{SPM}{}^t$ use the observations for the first year of data in the sample to construct a *spanning tree* of comparisons; that is, the Robert Hill (2001) methodology is used to construct the set of bilateral comparisons for all months in the first year such that the resulting set of bilateral comparisons minimizes the sum of the dissimilarity measures for the chosen bilateral links. Once the set of bilateral links for the first year has been determined, subsequent months are linked to previous months in real time. Thus, the bilateral links for $P_{AL}{}^t$ and $P_{ALM}{}^t$ to the index levels of previous months are the same for all months t beyond the first year. Similar comments apply to $P_{SP}{}^t$ and $P_{SPM}{}^t$. It follows that the longer term trends in $P_{AL}{}^t$ and $P_{ALM}{}^t$ will be the same as those in $P_{SP}{}^t$ and $P_{SPM}{}^t$.[208]

The indices $P_{AL}{}^t$, $P_{SP}{}^t$, $P_{ALM}{}^t$, and $P_{SPM}{}^t$ use reservation prices for the prices of missing products. These reservation prices were estimated econometrically in an earlier study by Diewert and Feenstra (2017). It is not easy to estimate reservation prices. Moreover, reservation prices rely on the applicability of the economic approach to index number theory and many assumptions are required in order to implement this approach. Thus, many statistical agencies will want to avoid the use of estimated reservation prices when constructing their CPIs. As was indicated in the discussion following equation (219), the predicted share measure of relative price dissimilarity $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is well defined even if the prices for missing products are set equal to 0.[209] As was mentioned earlier in this section, it is possible to use $\Delta_{SP}(p^r,p^t,q^r,q^t)$ as a guide to linking

[205] This approach that combines measures of relative price dissimilarity with measures of relative quantity dissimilarity was developed by Allen and Diewert (1981), Hill (2004), and Hill and Timmer (2006, 277). Hill and Timmer also noted that, usually, the relative price dissimilarity measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ will be smaller than the relative quantity dissimilarity measure $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ in which case the combined measure $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ reduces to the price measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$. Diewert and Allen (1981) found this to be the case with their empirical example, and we find the same to be true for our empirical example in the annex.

[206] Thus, a strong version of Walsh's multiperiod identity test will hold using this procedure; that is, if $p^r = p^t$, then the period r and t price levels will coincide and if $q^r = q^t$, then the period r and t quantity levels will coincide. Note that these tests will hold no matter how large the number of observations T is.

[207] Allen and Diewert (1981) and Hill and Timmer (2006) found the same pattern for their empirical examples using their measures of price and quantity dissimilarity.

[208] For our empirical example, $P_{AL}{}^t$, $P_{SP}{}^t$, $P_{ALM}{}^t$, and $P_{SPM}{}^t$ all end up at the same level for the last month in our sample; see Table 1.7 and Figure A7.9 in the annex.

[209] This is not the case for the asymptotic linear measure of relative price dissimilarity $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215).

the observations even if the prices of missing products are set equal to 0. We explain how alternative versions of $P_{SP}{}^{t}$ and $P_{SPM}{}^{t}$ can be produced when the price vectors p^t have 0 components for missing products in period t in the following paragraph.

In order to explain how the alternative version of $P_{SP}{}^{t}$ (denoted by $P_{SP}{}^{t*}$), it is first necessary to calculate all possible *maximum overlap bilateral Fisher indices* for every pair of observations in the sample. Denote the maximum overlap Fisher price index for period t relative to the base period r as $P_{FMO}(p^r,p^t,q^r,q^t)$ for all observations r and t. When calculating $P_{FMO}(p^r,p^t,q^r,q^t)$, the usual inner products $p^r \cdot q^t = \Sigma_{n=1}{}^{N}\, p_{rn}q_{tn}$ that are used to construct the Fisher index between periods r and t are replaced by summations over n, where n is restricted to products that are present in both periods r and t. These four restricted inner products can be constructed very efficiently using matrix operations. As noted above, the dissimilarity measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218) is well defined even if the prices for missing products are set equal to zero. Set the maximum overlap similarity-linked price index $P_{SP}{}^{1*}$ for period 1 equal to unity; that is, set $P_{SP}{}^{1*} \equiv 1$. The period 2 index $P_{SP}{}^{2*}$ is set equal to $P_{FMO}(p^1,p^2,q^1,q^2)$, the maximum overlap Fisher index linking the period 2 prices to the period 1 prices. Thus, $P_{SP}{}^{2*} \equiv P_{FMO}(p^1,p^2,q^1,q^2)P_{SP}{}^{1*}$. For period 3, evaluate the dissimilarity indices $\Delta_{SP}(p^1,p^3,q^1,q^3)$ and $\Delta_{SP}(p^2,p^3,q^2,q^3)$ defined by (218). If $\Delta_{SP}(p^1,p^3,q^1,q^3)$ is the minimum of these two numbers, define $P_{SP}{}^{3*} \equiv P_{FMO}(p^1,p^3,q^1,q^3)P_{PS}{}^{1*}$. If $\Delta_{SP}(p^2,p^3,q^2,q^3)$ is the minimum of these two numbers, define $P_{SP}{}^{3*} \equiv P_{FMO}(p^2,p^3,q^2,q^3)P_{SP}{}^{2*}$. For period 4, evaluate the dissimilarity indices $\Delta_{SP}(p^r,p^4,q^r,q^4)$ for r = 1,2,3. Let r^* be such that $\Delta_{SP}(p^{r*},p^4,q^{r*},q^4) = \min_r \{\Delta_{SP}(p^r,p^4,q^r,q^4);\ r = 1,2,3\}$.[210] Then define $P_{SP}{}^{4*} \equiv P_{FMO}(p^{r*},p^4,q^{r*},q^4)P_{SP}{}^{r*}$. Continue this process in the same manner; that is, for period t, let r^* be such that $\Delta_{SP}(p^{r^{\circ}},p^t,q^{r*},q^t) = \min_r \{\Delta_{SP}(p^r,p^t,q^r,q^t);\ r = 1,2,\ldots,t{-}1\}$ and define $P_{SP}{}^{t*} \equiv P_{FMO}(p^{r*},p^t,q^{r*},q^t)P_{SP}{}^{r*}$. The procedure for constructing $P_{SP}{}^{t*}$ is exactly the same as the procedure for constructing $P_{SP}{}^{t}$ except that maximum overlap Fisher indices are used in place of regular Fisher indices defined over all products in order to implement the "best" set of bilateral links that are used to link all of the observations in the sample up to the current period t.[211]

Recall the definition for the modified set of price levels $P_{ALM}{}^{t}$ using the asypmtotic linear measure of relative price dissimilarity, which were similar to the $P_{AL}{}^{t}$ price levels except that a year of data on prices and quantities was used to form a set of bilateral links that minimizes the sum of the associated dissimilarity measures that link the observations for the first year. The same procedure can be used in the present context where $P_{SP}{}^{t*}$ can be replaced by the *modified predicted share indices*, $P_{SPM}{}^{t*}$.[212] For months t that follow after the first "training" year, the bilateral links are the same as the links used to calculate the predicted share indices $P_{SP}{}^{t*}$.[213]

The maximum overlap fixed-base Fisher indices, $P_{FMO}(p^1,p^t.q^1.p^t) \equiv P_F{}^{t*}$, and the GEKS indices $P_{GEKS}{}^{t*}$ using maximum overlap Fisher indices in place of regular Fisher indices are listed in the annex and can be compared to their counterparts $P_F{}^{t}$ and $P_{GEKS}{}^{t}$ that used reservation prices for the missing products. See Table 1.7 in Section A.7.5 of the annex for a listing of the following indices: $P_{AL}{}^{t}$, $P_{ALM}{}^{t}$, $P_{SP}{}^{t}$, $P_{SPM}{}^{t}$, $P_{SP}{}^{t*}$, $P_{SPM}{}^{t*}$, $P_{GEKS}{}^{t}$, $P_{GEKS}{}^{t*}$, $P_F{}^{t}$, and $P_F{}^{t*}$. The final level for these 10 indices after three years of data where the level in month 1 was 1.00000 was as follows: 0.92725, 0.92725, 0.92725, 0.92725, 0.92725, 0.92612, 0.92612, 0.94591, 0.94987, 0.95071, and 0.95610. Thus, the first four similarity-linked indices end up at the same price level, 0.92575, while the predicted share and modified predicted share indices that used maximum overlap prices, $P_{SP}{}^{t*}$ and $P_{SPM}{}^{t*}$, ended up at the same slightly higher price level, 0.92612. The two GEKS indices ($P_{GEKS}{}^{t}$ used reservation prices, while $P_{GEKS}{}^{t*}$ used maximum overlap Fisher links that did not depend on any imputed prices) ended up about 2 percentage points above the similarity-linked indices. Finally, the fixed-base Fisher index that used reservation prices and the fixed-base Fisher index that used maximum overlap bilateral links, $P_F{}^{t}$ and $P_F{}^{t*}$, ended up about 3 percentage points above the similarity-linked index levels. These results lead to two important (but tentative) conclusions:

- The similarity-linked indices considered in this section and the previous sections all generate approximately the same results.
- The similarity-linked indices appear to generate lower rates of overall price change than the fixed-base Fisher or the GEKS indices.

The first dot point is important if it is consistent with other empirical investigations. Some statistical agencies may prefer to use inflation-adjusted carry-forward prices to replace missing prices while other agencies may not wish to use any form of an imputed price in their indices. The results for our empirical example suggest that it may not matter very much which strategy is chosen, provided similarity linking of observations is used.

21. The Axiomatic Approach to Multilateral Price Levels

In this section, we look at the axiomatic or test properties of the five major multilateral methods studied in previous sections. The multilateral methods are the GEKS, CCDI, GK, WTPD, and SPQ (price and quantity similarity linking) methods. The price levels for period t for the five methods are

[210] If the minimum occurs at more than one r, choose r^* to be the earliest of these minimizing periods.

[211] In addition to using P_{FMO} in place of P_F, the other difference in the two procedures is the use of 0 prices for unavailable products in place of reservation or carry-forward prices when evaluating the dissimilarity measures $\Delta_{SP}(p^r,p^t,q^r,q^t)$. Thus, the set of optimal bilateral links can change as we move from the $P_{SP}{}^{t}$ indices to their maximum overlap counterpart $P_{SP}{}^{t*}$ indices.

[212] Note that we cannot construct $P_{AL}{}^{t*}$ or $P_{ALM}{}^{t*}$ in the present context where we have 0 prices for the missing products because $\Delta_{AL}(p^r,p^t,q^r,q^t)$ is not well defined when some prices are equal to zero.

[213] It is straightforward to apply the predicted share methodology when we have 0 prices and quantities for missing products to quantity indices. Apply definition (221); that is, define $\Delta_{SPQ}(p^r,p^t,q^r,q^t) \equiv \min\{\Delta_{SP}(p^r,p^t,q^r,q^t), \Delta_{SQ}(p^r,p^t,q^r,q^t)\}$as our new measure of relative price and quantity dissimilarity where *0* prices and quantities are allowed to appear in the price and quantity vectors. Using this measure of dissimilarity and maximum overlap Fisher price and quantity indices leads to the price levels $P_{SPQ}{}^{t*}$. For our empirical example, it was the case that $\Delta_{SP}(p^r,p^t,q^r,q^t)$ was always less than $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ so $P_{SPQ}{}^{t*}$ ended up being equal to $P_{SP}{}^{t*}$ for all t.

defined by (69) for $p_{GEKS}{}^t$, (76) for $p_{CCDI}{}^t$, (137) for $p_{GK}{}^t$, (210) for $p_{WTPD}{}^t$, and by (221) for $P_{SPQ}{}^t$.[214] We will look at the properties of these price level functions rather than at the corresponding price indices.[215] Denote the period t price level function for generic multilateral method M as $p_M{}^t(p^1, \ldots, p^T; q^1, \ldots, q^T)$ for $t = 1, \ldots, T$. We will follow the example of Dalén (2001, 2017) and Zhang, Johansen, and Nygaard (2019) in considering a *dynamic product universe*; that is, we will allow for new products and disappearing products in the tests that follow. N is the total number of products that are in the aggregate over all T periods. If a product n is not available in period t, we set q_{tn} equal to 0. We will assume that the corresponding price p_{tn} is a positive Hicksian reservation price or a positive inflation-adjusted carry-forward or carry-backward price. Thus, for each period t, the price vector $p^t >> 0_N$ but the corresponding period t quantity vector satisfies only $q^t > 0_N$; that is, the missing products in period t are assigned 0 values for the corresponding quantities.[216] It proves convenient to define the N by T matrices of prices and quantities as $P \equiv [p^1, \ldots, p^T]$ and $Q \equiv [q^1, \ldots, q^T]$. Thus, p^t and q^t are to be interpreted as column vectors of dimension N in the definitions of the matrices P and Q.

Consider the following nine tests for a system of generic multilateral price levels, $p_M{}^t(P,Q)$:

Test 1: *The strong identity test for prices.* If $p^r = p^t$, then $p_M{}^r(P,Q) = p_M{}^t(P,Q)$. Thus, if prices are equal in periods r and t, then the corresponding price levels are equal even if the corresponding quantity vectors q^r and q^t are not equal.

Test 2: *The fixed basket test for prices* or *the strong identity test for quantities.*[217] If $q^r = q^t \equiv q$, then the price index for period t relative to period r is $p_M{}^t(P,Q)/p_M{}^r(P,Q)$ which is equal to $p^t \cdot q / p^r \cdot q$.[218]

Test 3: *Linear homogeneity test for prices.* Let $r \neq t$ and $\lambda > 0$. Then $p_M{}^t(p^1, \ldots, p^{t-1}, \lambda p^t, p^{t+1}, \ldots, p^T, Q) / p_M{}^r(p^1, \ldots, p^{t-1}, \lambda p^t, p^{t+1}, \ldots, p^T, Q) = \lambda p_M{}^t(P,Q)/p_M{}^r(P,Q)$. Thus, if all prices in period t are multiplied by a common scalar factor λ, then the price level of period t relative to the price level of any other period r will increase by the multiplicative factor λ.

Test 4: *Homogeneity test for quantities.* Let $\lambda > 0$. Then $p_M{}^r(P, q^1, \ldots, q^{t-1}, \lambda q^t, q^{t+1}, \ldots, q^T) = p_M{}^r(P,Q)$ for $r = 1, \ldots, T$. Thus, if all quantities in period t are multiplied by a common scalar factor λ, then the price level of any period r remains unchanged. This property holds for all $t = 1, \ldots, T$.

Test 5: *Invariance to changes in the units of measurement*. The price level functions $p_M{}^t(P,Q)$ for $t = 1, \ldots, T$ remain unchanged if the N commodities are measured in different units of measurement.

Test 6: *Invariance to changes in the ordering of the commodities.* The price level functions $p_M{}^t(P,Q)$ for $t = 1, \ldots, T$ remain unchanged if the ordering of the N commodities is changed.

Test 7: *Invariance to changes in the ordering of the time periods.* If the T time periods are reordered by some permutation of the first T integers, then the new price level functions are equal to the same permutation of the initial price level functions. This test is considered to be an important one in the context of making cross sectional comparisons of price levels across countries. In the country context, if this test is satisfied, then all countries are treated in a symmetric manner. It is not so clear whether this test is important in the time series context.

Test 8: *Responsiveness to isolated products test*: If a product is available in only one period in the window of T periods, this test asks that the price level functions $p_M{}^t(P,Q)$ *respond to changes* in the prices of these isolated products; that is, the test asks that the price level functions *are not constant* as the prices for isolated products change. This test is a variation of Test 5 suggested by Zhang, Johansen, and Nygaard (2019), which was a bilateral version of this test.[219]

Test 9: *Responsiveness to changes in imputed prices for missing products test*: If there are missing products in one or more periods, then there will be imputed prices for these missing products according to our methodological framework. This test asks that the price level functions $p_M{}^t(P,Q)$ *respond to changes* in these imputed prices; that is, the test asks that the price level functions *are not constant* as the imputed prices change. This test is essentially an extension of the previous Test 8. This test allows a price level to decline if new products enter the market place during the period and for consumer utility to increase as the number of available products increases. If this test is not satisfied, then the price levels will be subject to *new products bias*.[220] This is an important source of bias in a dynamic product universe.

It can be shown that GEKS and CCDI fail Tests 1 and 2, GK fails Tests 1,4, 8, and 9, WTPD fails Tests 1, 2,[221] 8, and 9, and SPQ fails Test 7. The above five multilateral methods pass the remaining Tests. Since Test 7 may not be so important in the time series context, it appears that the price and quantity

[214] The price and quantity similarity-linked price levels $P_{SPQ}{}^t$ have been normalized to equal 1 in period 1. The other four sets of price levels have not been normalized.

[215] For earlier work on the axiomatic properties of multilateral price and quantity indices, see Diewert (1988, 1999b) and Balk (2008). These earlier studies did not look at the properties of standalone price level functions.

[216] It is necessary to have strictly positive prices in order to calculate the CCDI price levels. The remaining multilateral methods do not require strictly positive prices for all products and all periods to be well defined, but our last test involves imputed prices for missing products. Thus, we need to introduce these imputed prices at the outset of our axiomatic framework.

[217] The period t quantity level that matches up with the period t price level is $q_M{}^t(P,Q) \equiv p^t \cdot q^t / p_M{}^t(P,Q)$ for $t = 1, \ldots, T$. Test 2 translates into the *strong identity test for quantity levels*; that is, if $q^r = q^t$, then $q_M{}^r(P,Q) = q_M{}^t(P,Q)$ even if the price vectors p^r and p^t for the two periods are not equal.

[218] Tests 1 and 2 are essentially versions of Tests 1 and 2 suggested by Zhang, Johansen, and Nygaard (2019).

[219] This test was explicitly suggested by Claude Lamboray. Some care is needed in interpreting this test since the test framework assumes that there are imputed prices for the missing products.

[220] On new goods bias, see Boskin et al. (1996), Nordhaus (1997), Diewert (1998), and the references in Section 14 of Diewert (2022a).

[221] The weighted TPD price levels fail Test 2 if definition (205) is used to define the period t price levels. This is the option that statistical agencies are using at present. However, the WTPD price levels P^{t**} and the corresponding quantity levels Q^{t**} defined by (206) will satisfy Test 2. If all errors are equal to 0, equations (205) and (206) will generate the same estimated price and quantity levels.

similarity method of linking, the SPQ method, is the "best" for the above tests. However, other reasonable tests could be considered in a more systematic exploration of the test approach to multilateral comparisons so our endorsement of the SPQ method is tentative at this point. Furthermore, the method needs to be tested on alternative data sets to see if "reasonable" indices are generated by the method.

22. Summary of Results

Some of the more important results in each section of the chapter will be summarized here.

- If there are divergent trends in product prices, the Dutot index is likely to have an upward bias relative to the Jevons index; see Section 2.
- The Carli index has an upward bias relative to the Jevons index (unless all prices move proportionally over time in which case both indices will capture the common trend). The same result holds for the weighted Carli (or Young) index relative to the corresponding weighted Jevons index; see Section 3.
- The useful relationship (41) implies that the Fisher index $P_F^{\,t}$ will be slightly less than the corresponding fixed-base Törnqvist index $P_T^{\,t}$, provided that the products in scope for the index are highly substitutable and there are divergent trends in prices; see Section 4. Under these circumstances, the following inequalities between the Paasche, geometric Paasche, Törnqvist, geometric Laspeyres, and Laspeyres indices are likely to hold: $P_P^{\,t} < P_{GP}^{\;\;t} < P_T^{\,t} < P_{GL}^{\;\;t} < P_L^{\,t}$.
- The covariance identity (48) provides an exact relationship between the Jevons and Törnqvist indices. Some conditions for equality and for divergence between these two indices are provided at the end of Section 5.
- In Section 6, a geometric index that uses annual expenditure sales of a previous year as weights, $P_{J\alpha}^{\;\;t}$, is defined and compared to the Törnqvist index, $P_T^{\,t}$. Equation (62) provides an exact covariance decomposition of the difference between these two indices. If the products are highly substitutable and there are divergent trends in prices, then it is likely that $P_T^{\,t} < P_{J\alpha}^{\;\;t}$.
- Section 7 derives an exact relationship (65) between the fixed-base Törnqvist index, $P_T^{\,t}$, and its chained counterpart, $P_{TCh}^{\;\;t}$. This identity is used to show that it is likely that the chained index will "drift" below its fixed-base counterpart if the products in scope are highly substitutable and prices are frequently heavily discounted. However, a numerical example shows that if quantities are slow to adjust to the lower prices, then upward chain drift can occur.
- Section 8 introduces two multilateral indices $P_{GEKS}^{\;\;t}$ and $P_{CCDI}^{\;\;t}$. The exact identity (78) for the difference between $P_{CCDI}^{\;\;t}$ and $P_T^{\,t}$ is derived. This identity and the fact that $P_F^{\,t}$ usually closely approximates $P_T^{\,t}$ lead to the conclusion (79) that typically $P_F^{\,t}$, $P_T^{\,t}$, $P_{GEKS}^{\;\;t}$, and $P_{CCDI}^{\;\;t}$ will approximate each other fairly closely.
- Section 9 introduces the unit value price index $P_{UV}^{\;\;t}$ and shows that if there are divergent trends in prices and the products are highly substitutable, it is likely that $P_{UV}^{\;\;t} < P_F^{\,t}$. However, this conclusion does not necessarily hold if there are missing products in period 1. Section 10 derives similar results for the quality-adjusted unit value index, $P_{UV\alpha}^{\;\;t}$.
- Section 11 looks at the relationship of the Lowe index, $P_{Lo}^{\;\;t}$, with other indices. The Lowe index uses the quantities in a base year as weights in a fixed basket type index for months that follow the base year. In using annual weights of a previous year, this index is similar in spirit to the geometric index $P_{J\alpha}^{\;\;t}$ that was analyzed in Section 6. The covariance type identities (128) and (131) are used to suggest that it is likely that the Lowe index lies between the fixed-base Paasche and Laspeyres indices; that is, it is likely that $P_P^{\,t} < P_{Lo}^{\;\;t} < P_L^{\,t}$. The identity (134) is used to suggest that the Lowe index is likely to have an upward bias relative to the fixed-base Fisher index; that is, it is likely that $P_F^{\,t} < P_{Lo}^{\;\;t}$. However, if there are missing products in the base year, then these inequalities do not necessarily hold.
- Section 12 looks at an additional multilateral index, the GK index, $P_{GK}^{\;\;t}$ and shows that $P_{GK}^{\;\;t}$ can be interpreted as a quality-adjusted unit value index and hence using the analysis in Section 10, it is likely that the GK price index has a downward bias relative to the Fisher index; that is, it is likely that $P_{GK}^{\;\;t} < P_F^{\,t}$. However, if there are missing products in the first month of the sample, the above inequality will not necessarily hold.
- Sections 13–16 look at special cases of weighted time product dummy indices, $P_{WTPD}^{\;\;t}$. These sections show how different forms of weighting can generate very different indices. Section 17 finally deals with the general case where there are T periods and missing products. The exact identity (214) is used to show that it is likely that $P_{WTPD}^{\;\;t}$ is less than the corresponding fixed-base Törnqvist–Theil index, $P_T^{\,t}$, provided that the products are highly substitutable and there are no missing products in period 1. However, if there are missing products in period 1, the inequality can be reversed.
- It turns out that the following price indices are not affected by reservation prices: the unit value price indices $P_{UV}^{\;\;t}$ and $P_{UV\alpha}^{\;\;t}$, the GK indices $P_{GK}^{\;\;t}$, and the weighted TPD indices $P_{WTPD}^{\;\;t}$. Thus, these indices are not consistent with the economic approach due to Hicks (1941) to dealing with the problems associated with new and disappearing products and services.
- The final multilateral indices were introduced in Sections 18–20. These indices use bilateral Fisher price indices to link the price and quantity data of the current period to a prior period. The prior period that is chosen minimizes a measure of relative price (or quantity) dissimilarity. Two main measures of relative price dissimilarity were studied: the AL or asymptotic linear measure $\Delta_{AL}(p^r,p^t,q^r,q^t)$ defined by (215) and the SP or predicted share measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ defined by (218). The role of prices and quantities can be interchanged in order to define the predicted share measure $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ of relative quantity dissimilarity which can also be used to generate a set of bilateral Fisher price index links. Finally, the minimum of the $\Delta_{SP}(p^r,p^t,q^r,q^t)$ and $\Delta_{SQ}(p^r,p^t,q^r,q^t)$ measures can be taken to define the $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ measure of relative price and quantity dissimilarity; see definition (221). When observations are linked using this dissimilarity

measure, the resulting price indices satisfy both the identity test for prices and the corresponding identity price for quantities. Thus, the SPQ method explained in Section 20 has attractive axiomatic properties as is explained in Section 21. For our empirical example, relative quantity dissimilarity was always greater than relative price dissimilarity so the SP and SPQ price indices were always identical.

- For our empirical example, the similarity-linked price indices $P_{AL}{}^t$ and $P_{SP}{}^t = P_{SPQ}{}^t$ ended up about 2 percentage points below $P_{GEKS}{}^t$ and $P_{CCDI}{}^t$ which in turn finished about 1 percentage point below $P_F{}^t$ and $P_T{}^t$ and finally $P_{GK}{}^t$ and $P_{WTPD}{}^t$ finished about 1 percentage point above $P_F{}^t$ and $P_T{}^t$; see Table 1.6 and Figure A7.8 in the annex. All of these indices captured the trend in product prices quite well. More research is required in order to determine whether these differences are significant and occur in other examples.
- It is difficult to calculate reservation prices using econometric techniques. Thus, Section 19 looked at methods for replacing reservation prices by inflation-adjusted carry-forward and carry-backward prices which are much easier to calculate.
- For our empirical example, the replacement of the reservation prices by inflation-adjusted carry-forward or carry-backward prices did not make much difference to the multilateral indices.[222] If the products in scope are highly substitutable for each other, then we expect that this invariance result will hold (approximately). However, if products with new characteristics are introduced, then we expect that the replacement of econometrically estimated reservation prices by carry-forward and carry-backward prices would probably lead to an index that has an upward bias.
- Finally, in Section 20, we introduced some similarity-linked Fisher price indices that did not require imputations for missing prices. These indices used the predicted share measure of relative price dissimilarity which is well defined even if the prices of missing products are set equal to 0. The Fisher indices that link pairs of observations that have the lowest measures of dissimilarity are maximum overlap Fisher indices. For our empirical example, it turned out that these indices were very close to their counterparts that used reservation prices for the missing prices. These no imputation indices (denoted by $P_{SP}{}^{t*}$ and $P_{SPM}{}^{t*}$) were calculated for our data set and listed in Table 1.7 and are plotted in Figure A7.9 in the annex.

Conceptually, the price and quantity similarity-linked indices $P_{SPQ}{}^t$ seem to be the most attractive solution for solving the chain drift problem since the strong identity tests for both prices and quantities will always be satisfied using this multilateral method.

The data used for the empirically constructed indices are listed in the annex so that the listed indices can be replicated and alternative solutions to the chain drift problem can be tested out by other statisticians and economists.

[222] See Table 1.8 in the annex.

23. Conclusion

It is evident that there is no easy solution to the chain drift problem. The previous *Consumer Price Index* Manual tended to use the economic approach to index number theory as a guide to choosing between alternative index number formulae; that is, the *Manual* tended to recommend the use of a superlative index number formula as a target index. However, the existence of deeply discounted prices and the appearance and disappearance of products often lead to a substantial chain drift problem. Some of the difficulties stem from the fact that the *economic approach to index number theory* that dates back to Konüs (1924), Konüs and Byushgens (1926), and Diewert (1976) suffers from the following problems:

- The theory assumes that all purchased goods and services are consumed in the period under consideration. But in reality, when a good goes on sale at a deeply discounted price, the quantity purchased will not necessarily be consumed in the current period. If the good can be stored, it will decrease demand for the product in the subsequent period. The traditional economic approach to index number theory does not take the storage problem into account.
- Preferences over goods and services are assumed to be complete. In reality, consumers may not be aware of many new (and old) products; that is, knowledge about products may be subject to a diffusion process.
- Our approach to the treatment of new and disappearing products uses the reservation price methodology due to Hicks (1940), which simply assumes that latent preferences for new products exist in the period before their introduction to the marketplace. Thus, the consumer is assumed to have unchanging preferences over all periods. Before a new product appears, the quantity of the product is set equal to 0 in the consumer's utility function. In reality, a new product may change the consumer's utility function. This makes the estimation of reservation prices very difficult if not impossible.
- Preferences are assumed to be the same across consumers so that they can be represented by a common linearly homogeneous utility function. Moreover, the preferences do not change over time. All of these assumptions are suspect.

In view of the fact that the assumptions of the economic approach to index number theory will not be satisfied precisely in the real world, we cannot rely entirely on this approach to guide advice to statistical agencies on how to deal with the chain drift problem. Thus, it would be useful to develop the test approach to multilateral index number theory in more detail.

So what exactly should statistical agencies do to deal with the chain drift problem when price and quantity are available for a stratum of the CPI? At our current state of knowledge, it seems that the following methods are acceptable:

- Rolling window GEKS and CCDI. Probably the "safest" method of linking the results of one window to the previous window is to use the mean method suggested by Ivancic, Diewert, and Fox (2009) and Diewert and Fox (2017). This is the method used by the Australian Bureau of Statistics (2016). However, in the case of seasonal products that are not present in all periods of the year, rolling window GEKS and CCDI can be problematic and similarity linking is preferred.

- Bilateral linking based on price (and quantity) similarity. This method seems to be very promising. It can be adapted to work in situations where there are imputed prices for missing products or in situations where imputed prices are not allowed. The resulting indices are guaranteed to be free of chain drift.

If only price information is available and there are no missing prices, then the Jevons index is the best alternative to use (at least from the perspective of the test approach to index number theory).

If only price information is available and there are missing prices for some products for some periods, then the TPD method is probably the best index to use. This method reduces to the Jevons index if there are no missing prices.[223]

We conclude this section by noting some priorities for future research:

- We need more studies on price similarity linking, particularly in the context of strongly seasonal commodities.
- What is the "optimal" length of the time period for a CPI? Should statistical agencies produce weekly or daily CPIs in addition to monthly CPIs?[224]
- There is a conceptual problem in using retail outlet prices to construct a CPI, since tourists and governments also purchase consumer goods. It would be preferable to use the purchase data of domestic households in order to construct a CPI for residents of the country so that the welfare of residents in the country could be calculated. However, if we focus on individual households, the matching problems are substantial due to the infrequency of purchases of storable commodities. Thus, it will be necessary to aggregate over demographically and locationally similar households in order to calculate indices that minimize the number of imputations. In the perhaps distant future, it will become possible in a cashless society to utilize the data of banks and credit card companies to track the universe of purchases of individual households and thus to construct more accurate CPIs. However, this development will depend on whether credit and debit card consumer transactions are also coded for the type of purchase.
- A final problem that may require some research is how to combine elementary indices that are constructed using scanner data with elementary indices that use web scraped data on prices or data on prices collected by employees of the statistical agency. This does not seem to be a big conceptual problem: for strata that use scanner data, we end up with an aggregate price and quantity level for each stratum. For strata that use web-scraped data or collector data, we end up with a stratum elementary price level for each period and consumer expenditure survey information will generate an estimated value of consumer expenditures for the stratum in question so the corresponding stratum quantity can be defined as expenditure divided by the elementary price level. Thus, the resulting CPI will be of uneven quality (because the expenditure estimates will not be current for the web scraped categories) but it will probably be of better quality than a traditional price collector generated CPI. However, as mentioned above, another problem is that the scanner data will apply not only to expenditures of domestic households but also to tourists and governments. Thus, there is a need for more research on this topic of combining methods of price collection.

[223] However, in situations where there are many missing prices, it may be preferable to adapt the predicted share similarity linking methodology to the case where only price information is available. We will explore this possibility in another chapter that deals with strongly seasonal products.

[224] The problem with making the time period shorter is that the number of price matches will decline, leading to the need for more imputations. Also, the shorter the period, the more variance there will be in the unit value prices and the associated quantities, leading to indices that will also have high variances. Thus, the shorter the period, the less accurate the resulting indices will be.

Annex: Data Listing and Index Number Tables and Charts

A.7.1 Listing of Data

Here is a listing of the "monthly" quantities sold of 19 varieties of frozen juice (mostly orange juice) from Dominick's Store 5 in the Greater Chicago area, where a "month" consists of sales for four consecutive weeks. These data are available from the Booth School of Business at the University of Chicago (2013).[225] The weekly unit value price and quantity sold data were converted into "monthly" unit value prices and quantities.[226] Finally, the original data came in units where the package size was not standardized. We rescaled the price and quantity data into prices per ounce. Thus, the quantity data are equal to the "monthly" ounces sold for each product.

The actual prices p_2^t and p_4^t are not available for $t = 1,2, . . .,8$ since products 2 and 4 were not sold during these months. However, in Table 1.1, we filled in these missing prices with the imputed reservation prices that were estimated by Diewert and Feenstra (2017). Similarly, p_{12}^t was missing for months $t = 12, 20, 21$, and 22, and again, we replaced these missing prices with the corresponding estimated imputed reservation prices in Table 1.1. The imputed prices appear in italics in Table 1.1.

Table 1.1 "Monthly" Unit Value Prices for 19 Frozen Juice Products

t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t	p_7^t	p_8^t	p_9^t
1	0.122500	*0.145108*	0.147652	*0.148593*	0.146818	0.146875	0.147623	0.080199	0.062944
2	0.118682	*0.127820*	0.116391	*0.128153*	0.117901	0.146875	0.128833	0.090833	0.069167
3	0.120521	*0.128608*	0.129345	*0.148180*	0.131117	0.143750	0.136775	0.090833	0.048803
4	0.126667	*0.128968*	0.114604	*0.115604*	0.116703	0.143750	0.114942	0.088523	0.055842
5	0.126667	*0.130737*	0.140833	*0.141108*	0.140833	0.143304	0.140833	0.090833	0.051730
6	0.120473	*0.113822*	0.157119	*0.151296*	0.156845	0.161844	0.156342	0.090833	0.049167
7	0.164607	*0.144385*	0.154551	*0.158485*	0.156607	0.171875	0.152769	0.084503	0.069167
8	0.142004	*0.160519*	0.174167	*0.179951*	0.174167	0.171341	0.163333	0.089813	0.069167
9	0.135828	0.165833	0.154795	0.159043	0.151628	0.171483	0.160960	0.089970	0.067406
10	0.129208	0.130126	0.153415	0.158167	0.152108	0.171875	0.158225	0.078906	0.067897
11	0.165833	0.165833	0.139690	0.136830	0.134743	0.171875	0.136685	0.079573	0.058841
12	0.165833	0.165833	0.174167	0.174167	0.174167	0.171875	0.174167	0.081902	0.079241
13	0.113739	0.116474	0.155685	0.149942	0.145633	0.171875	0.146875	0.074167	0.048880
14	0.120882	0.125608	0.141602	0.147428	0.142664	0.163750	0.144911	0.090833	0.080000
15	0.165833	0.165833	0.147067	0.143214	0.144306	0.155625	0.147546	0.088410	0.080000
16	0.122603	0.118536	0.135878	0.137359	0.137480	0.155625	0.138146	0.084489	0.080000
17	0.104991	0.104659	0.112497	0.113487	0.110532	0.141250	0.113552	0.082500	0.067104
18	0.088056	0.091133	0.118440	0.120331	0.117468	0.141250	0.124687	0.085000	0.065664
19	0.096637	0.097358	0.141667	0.141667	0.141667	0.141250	0.141667	0.082500	0.080000
20	0.085845	0.090193	0.120354	0.122168	0.113110	0.136250	0.124418	0.085874	0.051003
21	0.094009	0.100208	0.121135	0.122500	0.121497	0.125652	0.121955	0.090833	0.085282
22	0.084371	0.087263	0.120310	0.123833	0.118067	0.125492	0.124167	0.085898	0.063411
23	0.123333	0.123333	0.116412	0.118860	0.113085	0.126250	0.118237	0.085891	0.049167
24	0.078747	0.081153	0.125833	0.125833	0.125833	0.126250	0.125833	0.090833	0.049167
25	0.088284	0.092363	0.098703	0.098279	0.088839	0.126250	0.100640	0.090833	0.049167
26	0.123333	0.123333	0.092725	0.096323	0.095115	0.126250	0.095030	0.090833	0.049167
27	0.101331	0.102442	0.125833	0.125833	0.125833	0.126250	0.125833	0.090833	0.049167
28	0.101450	0.108416	0.092500	0.097740	0.091025	0.126250	0.096140	0.054115	0.049167
29	0.123333	0.123333	0.118986	0.119509	0.115603	0.126250	0.118343	0.096922	0.049167
30	0.094038	0.095444	0.109096	0.113827	0.106760	0.126250	0.113163	0.089697	0.049167

[225] The Office for National Statistics (2020) also used the Dominick's data in order to compare many of the same indices that are compared in this annex.

[226] In practice, statistical agencies will not be able to produce indices for 13 months in a year. There are at least two possible solutions to the problem of aggregating weekly data into monthly data: (i) aggregate the data for the first three weeks in a month or (ii) split the weekly data that spans two consecutive months into imputed data for each month.

t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t	p_7^t	p_8^t	p_9^t
31	0.130179	0.130000	0.110257	0.115028	0.112113	0.134106	0.110579	0.093702	0.049167
32	0.103027	0.103299	0.149167	0.149167	0.149167	0.149375	0.149167	0.098333	0.049167
33	0.148333	0.148333	0.089746	0.097110	0.091357	0.149375	0.094347	0.098333	0.049167
34	0.115247	0.114789	0.123151	0.123892	0.127177	0.149375	0.125362	0.094394	0.049167
35	0.118090	0.120981	0.121191	0.129477	0.128180	0.149375	0.132934	0.096927	0.049167
36	0.132585	0.131547	0.129430	0.128314	0.121833	0.134375	0.128874	0.070481	0.049167
37	0.114056	0.115491	0.138214	0.140090	0.139116	0.146822	0.142770	0.077785	0.053864
38	0.142500	0.142500	0.134677	0.133351	0.133216	0.148125	0.132873	0.108333	0.054167
39	0.121692	0.123274	0.095236	0.102652	0.093365	0.148125	0.101343	0.090180	0.054167

t	p_{10}^t	p_{11}^t	p_{12}^t	p_{13}^t	p_{14}^t	p_{15}^t	p_{16}^t	p_{17}^t	p_{18}^t	p_{19}^t
1	0.062944	0.075795	0.080625	0.087684	0.109375	0.113333	0.149167	0.122097	0.149167	0.124492
2	0.069167	0.082500	0.080625	0.112500	0.109375	0.113333	0.119996	0.109861	0.130311	0.117645
3	0.043997	0.082500	0.078546	0.106468	0.100703	0.110264	0.134380	0.109551	0.131890	0.114933
4	0.055705	0.082500	0.080625	0.099167	0.099375	0.111667	0.109005	0.106843	0.108611	0.118333
5	0.051687	0.071670	0.080625	0.094517	0.099375	0.111667	0.105168	0.106839	0.105055	0.076942
6	0.049167	0.078215	0.080625	0.115352	0.114909	0.130149	0.099128	0.134309	0.118647	0.088949
7	0.069167	0.069945	0.080625	0.124167	0.118125	0.131667	0.102524	0.128471	0.102073	0.160833
8	0.069167	0.082500	0.080625	0.107381	0.121513	0.138184	0.164245	0.141978	0.164162	0.136105
9	0.067401	0.082500	0.074375	0.112463	0.128125	0.141667	0.163333	0.153258	0.163333	0.118979
10	0.067688	0.082500	*0.100545*	0.132500	0.128125	0.141667	0.133711	0.152461	0.133806	0.118439
11	0.060008	0.082500	0.080625	0.120362	0.134151	0.144890	0.163333	0.151033	0.163333	0.120424
12	0.079325	0.071867	0.080625	0.093144	0.136875	0.148333	0.144032	0.148107	0.146491	0.160833
13	0.064028	0.069934	0.067280	0.118009	0.136875	0.148333	0.163333	0.143125	0.163333	0.131144
14	0.080000	0.078491	0.075211	0.131851	0.130342	0.143013	0.123414	0.152937	0.130223	0.122899
15	0.080000	0.082500	0.080625	0.093389	0.128125	0.141667	0.117955	0.147024	0.119786	0.128929
16	0.080000	0.086689	0.080625	0.100592	0.128125	0.141667	0.114940	0.143125	0.126599	0.124620
17	0.065670	0.088333	0.072941	0.115559	0.110426	0.139379	0.107709	0.143125	0.109987	0.145556
18	0.064111	0.091286	0.069866	0.088224	0.105625	0.105529	0.089141	0.130110	0.095463	0.140000
19	0.080000	0.094167	0.088125	0.080392	0.105625	0.131667	0.086086	0.118125	0.091020	0.109424
20	0.048613	0.094167	*0.096177*	0.080643	0.105625	0.131667	0.125000	0.114706	0.125000	0.110921
21	0.085114	0.080262	*0.064774*	0.080245	0.099375	0.125000	0.104513	0.114795	0.104228	0.134014
22	0.062852	0.086115	*0.083132*	0.087551	0.101493	0.127366	0.086484	0.118125	0.088325	0.126667
23	0.049167	0.095833	0.090625	0.089110	0.099375	0.125000	0.086263	0.118125	0.095750	0.100780
24	0.049167	0.095833	0.090625	0.090167	0.099375	0.125000	0.111859	0.114330	0.112296	0.118333
25	0.049167	0.095833	0.090625	0.072861	0.099375	0.125000	0.125000	0.113823	0.125000	0.084817
26	0.049167	0.095833	0.090625	0.086226	0.099375	0.125000	0.086088	0.114190	0.091864	0.118333
27	0.049167	0.077500	0.076875	0.081764	0.099375	0.125000	0.113412	0.114231	0.113241	0.110346
28	0.049167	0.077500	0.076875	0.104167	0.099375	0.125000	0.085803	0.118125	0.086154	0.084604
29	0.049167	0.077500	0.076875	0.086713	0.099375	0.125000	0.087410	0.118125	0.086196	0.085034
30	0.049167	0.077500	0.076875	0.104167	0.099375	0.125000	0.084953	0.114826	0.085156	0.083921
31	0.049167	0.077500	0.076875	0.095613	0.099375	0.125000	0.087372	0.125809	0.087775	0.088304
32	0.049167	0.077500	0.076875	0.112500	0.099375	0.067046	0.091827	0.143125	0.088937	0.103519
33	0.049167	0.077500	0.076875	0.104721	0.099375	0.125000	0.131399	0.143125	0.130253	0.127588
34	0.049167	0.077500	0.076875	0.088935	0.099375	0.125000	0.123037	0.143125	0.123573	0.132500
35	0.049167	0.077500	0.076875	0.112500	0.099375	0.125000	0.125832	0.137837	0.125681	0.112286
36	0.049167	0.077500	0.076875	0.089456	0.099375	0.125000	0.139240	0.141242	0.144390	0.127323

(*Continued*)

Table 1.1 (Continued)

t	p_{10}^t	p_{11}^t	p_{12}^t	p_{13}^t	p_{14}^t	p_{15}^t	p_{16}^t	p_{17}^t	p_{18}^t	p_{19}^t
37	**0.053865**	**0.084549**	**0.083343**	**0.107198**	**0.119368**	**0.151719**	**0.146126**	**0.154886**	**0.146332**	**0.120616**
38	**0.054167**	**0.085000**	**0.084375**	**0.127500**	**0.123125**	**0.156667**	**0.129577**	**0.138823**	**0.130850**	**0.114177**
39	**0.054167**	**0.085000**	**0.084375**	**0.102403**	**0.123125**	**0.156667**	**0.115965**	**0.149219**	**0.114947**	**0.136667**

Table 1.2 "Monthly" Quantities Sold for 19 Frozen Juice Products

t	q_1^t	q_2^t	q_3^t	q_4^t	q_5^t	q_6^t	q_7^t	q_8^t	q_9^t
1	**1704**	**0.000**	**792**	**0.000**	**4428**	**1360**	**1296**	**1956**	**1080**
2	**3960**	**0.000**	**3588**	**0.000**	**19344**	**3568**	**3600**	**2532**	**2052**
3	**5436**	**0.000**	**1680**	**0.000**	**8100**	**3296**	**2760**	**3000**	**1896**
4	**1584**	**0.000**	**5532**	**0.000**	**21744**	**3360**	**5160**	**3420**	**2328**
5	**1044**	**0.000**	**1284**	**0.000**	**5880**	**3360**	**1896**	**3072**	**1908**
6	**8148**	**0.000**	**1260**	**0.000**	**7860**	**2608**	**2184**	**3000**	**2040**
7	**636**	**0.000**	**3120**	**0.000**	**9516**	**2848**	**2784**	**3444**	**1620**
8	**1692**	**0.000**	**1200**	**0.000**	**4116**	**1872**	**1380**	**2088**	**1848**
9	**5304**	**1476.000**	**2292**	**1295.999**	**7596**	**2448**	**1740**	**2016**	**3180**
10	**6288**	**2867.993**	**2448**	**1500.000**	**6528**	**2064**	**2208**	**3840**	**4680**
11	**408**	**228.000**	**2448**	**2147.994**	**9852**	**2096**	**2700**	**5124**	**12168**
12	**624**	**384.000**	**948**	**1020.000**	**2916**	**1872**	**1068**	**2508**	**4032**
13	**6732**	**2964.005**	**1488**	**2064.003**	**8376**	**2224**	**2400**	**4080**	**8928**
14	**6180**	**3192.007**	**2472**	**2244.006**	**7920**	**1920**	**2256**	**1728**	**1836**
15	**1044**	**672.000**	**1572**	**1932.002**	**2880**	**1744**	**1728**	**1692**	**1116**
16	**3900**	**1332.002**	**1560**	**2339.997**	**4464**	**2416**	**2028**	**2112**	**1260**
17	**5328**	**1847.999**	**3528**	**3972.008**	**13524**	**2336**	**3252**	**2628**	**1524**
18	**7056**	**2100.000**	**2436**	**2748.007**	**6828**	**2544**	**1980**	**3000**	**1596**
19	**5712**	**3167.988**	**1464**	**1872.000**	**2100**	**2080**	**1572**	**3384**	**1020**
20	**9960**	**3311.996**	**2376**	**2172.003**	**8028**	**2112**	**1788**	**2460**	**3708**
21	**7368**	**2496.000**	**1992**	**1872.000**	**3708**	**1840**	**1980**	**1692**	**2232**
22	**9168**	**4835.983**	**2064**	**1980.000**	**10476**	**1504**	**2880**	**2472**	**7020**
23	**7068**	**660.000**	**1728**	**1955.999**	**6972**	**1888**	**2172**	**2448**	**12120**
24	**11856**	**5604.017**	**972**	**1464.000**	**2136**	**1296**	**1536**	**3780**	**7584**
25	**7116**	**2831.994**	**2760**	**2207.996**	**12468**	**1776**	**2580**	**2880**	**11220**
26	**660**	**504.000**	**3552**	**3755.995**	**17808**	**1296**	**5580**	**4956**	**7428**
27	**4824**	**3276.011**	**1356**	**1452.000**	**2388**	**1824**	**1524**	**1548**	**10188**
28	**3684**	**971.998**	**4680**	**2832.003**	**11712**	**1712**	**4308**	**4284**	**1140**
29	**684**	**1152.000**	**1884**	**2015.996**	**9252**	**1680**	**3144**	**1020**	**1392**
30	**5112**	**3467.996**	**2256**	**2291.994**	**9060**	**1936**	**2172**	**1452**	**2532**
31	**672**	**840.000**	**4788**	**2951.990**	**9396**	**1856**	**4644**	**1764**	**1260**
32	**7344**	**5843.997**	**1320**	**1128.000**	**2664**	**1744**	**1560**	**1548**	**1416**
33	**480**	**504.000**	**6624**	**5639.996**	**13368**	**1824**	**6888**	**1800**	**1440**
34	**4104**	**3036.001**	**2124**	**3180.009**	**5088**	**1568**	**2820**	**1668**	**1884**
35	**2688**	**1583.997**	**2220**	**2760.008**	**5244**	**1344**	**2532**	**1920**	**4956**
36	**936**	**612.001**	**1824**	**2567.994**	**6684**	**1552**	**2772**	**4740**	**7644**
37	**4140**	**2268.001**	**1932**	**1559.997**	**4740**	**1520**	**2076**	**1752**	**6336**
38	**912**	**264.000**	**1860**	**2844.002**	**4260**	**1808**	**2064**	**1452**	**2952**
39	**1068**	**960.001**	**4356**	**2903.996**	**11052**	**1776**	**4356**	**2220**	**2772**

t	q_{10}^t	q_{11}^t	q_{12}^t	q_{13}^t	q_{14}^t	q_{15}^t	q_{16}^t	q_{17}^t	q_{18}^t	q_{19}^t
1	540	2088	1744.000	30972	3728	792	1512	1712	600	2460
2	1308	4212	3824.000	11796	6480	2712	12720	3312	2376	1788
3	1416	3900	4848.010	18708	10064	2652	4116	3184	1476	3756
4	1716	3156	5152.000	19656	10352	2472	15420	3120	3888	900
5	1452	6168	3360.000	42624	7360	1590	9228	2800	5652	13560
6	1068	5088	3296.000	10380	7712	1884	12012	1808	3348	7824
7	1116	6372	3712.000	11772	7920	1680	29592	3296	11712	708
8	1296	3684	3216.000	21024	5856	1206	11184	1744	4344	6036
9	2220	4512	3024.000	24420	5856	1398	2040	1648	1176	7896
10	4152	4572	0.000	8328	6384	1740	9168	1296	2832	9120
11	9732	3432	3360.000	18372	5808	1638	2412	1568	972	7176
12	3024	6132	1792.000	48648	4672	1770	7512	2208	2052	3564
13	2160	6828	6271.998	15960	4736	1662	1740	2896	1176	3216
14	1356	5088	2991.997	9432	5872	1902	4968	1488	2064	6420
15	1188	4656	2976.000	33936	3872	1452	9060	1744	2712	3876
16	816	3108	4784.000	23772	6272	1578	8496	2832	1488	4128
17	696	3252	4879.997	10656	7648	1836	9000	2704	2292	648
18	720	2940	4848.021	26604	6448	4086	14592	1552	3108	732
19	624	4320	2480.000	27192	4944	1140	19056	1808	5088	5676
20	3288	2784	0.000	23796	5120	1284	2196	2896	1260	3876
21	1848	12324	0.000	25824	5248	1140	8640	1952	2940	588
22	4824	6468	0.000	18168	3872	930	15360	1520	4728	276
23	10092	3708	1744.000	14592	4336	870	14232	1504	2040	1128
24	6372	3264	2016.000	16548	4608	858	6696	1792	2496	792
25	7284	3480	2032.000	38880	4064	750	1836	1232	636	7608
26	6588	3768	2208.000	14724	3760	768	9096	1296	4248	480
27	2832	4692	2592.000	31512	5344	930	5796	2080	5244	1416
28	900	3180	2624.000	8172	5776	810	13896	1328	7536	6744
29	1128	3948	2608.000	19440	5792	954	12360	1552	5796	7296
30	1284	5232	2960.000	6552	6320	924	13932	2304	8064	14520
31	864	5928	3280.000	16896	5888	852	14340	2064	8412	3768
32	948	5784	2496.000	5880	5088	15132	14496	1600	10440	4044
33	708	5232	2704.000	15180	4800	618	4812	976	3204	1812
34	1152	4692	2736.000	25344	5648	600	6552	1360	3876	1344
35	4248	4668	2800.000	8580	5488	498	28104	1872	11292	4152
36	6492	4872	2256.000	30276	5504	510	4080	1328	3768	1860
37	5976	3396	1743.995	8208	2832	384	1092	528	1284	2028
38	1812	3660	2416.000	4392	4144	534	4752	1504	2436	4980
39	2844	3852	1888.000	16704	3488	708	6180	1600	4236	804

It can be seen that there were no sales of products 2 and 4 for months 1–8, and there were no sales of product 12 in month 10 and in months 20–22.

Charts that plot the data in the above tables are presented next.

It can be seen that there is a considerable amount of variability in these per ounce monthly unit value prices for frozen juice products. There are also differences in the average level of the prices of these 19 products. These differences can be interpreted as quality differences.

It can be seen that the volatility in quantity of the products is much higher than the volatility in their prices.

Figure A7.1 Monthly Prices for Products 1–9

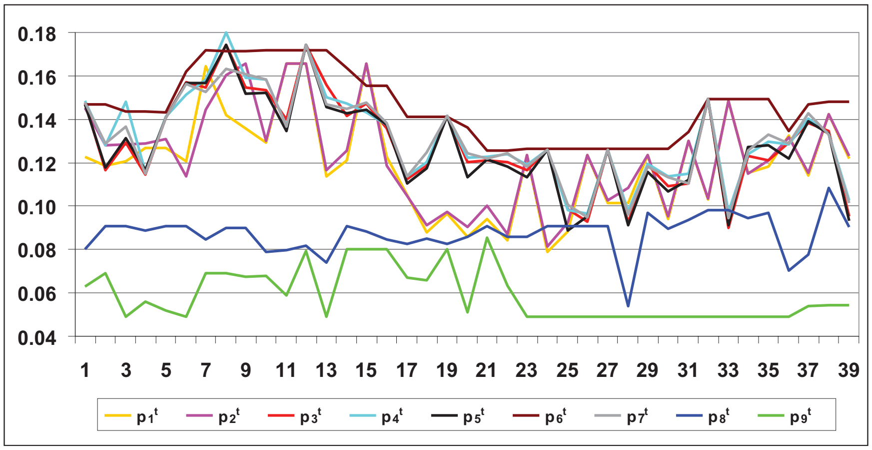


Figure A7.2 Monthly Prices for Products 10–19

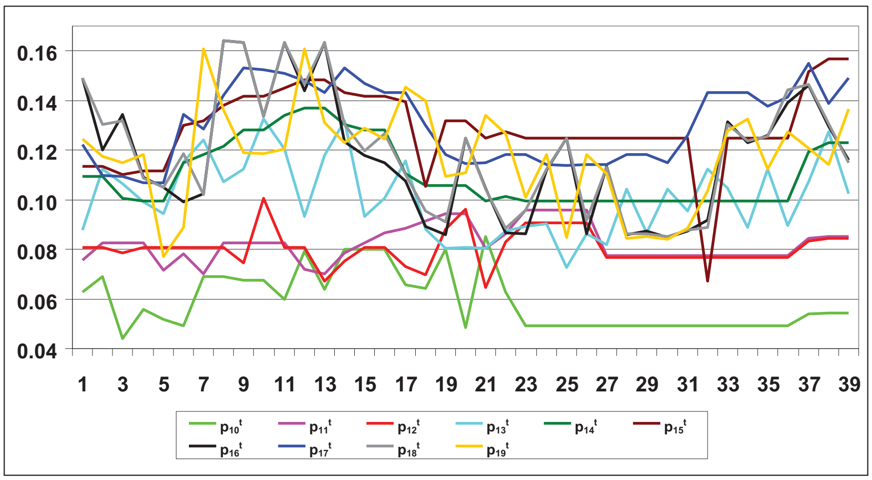


Figure A7.3 Quantities Sold for Products 1–9

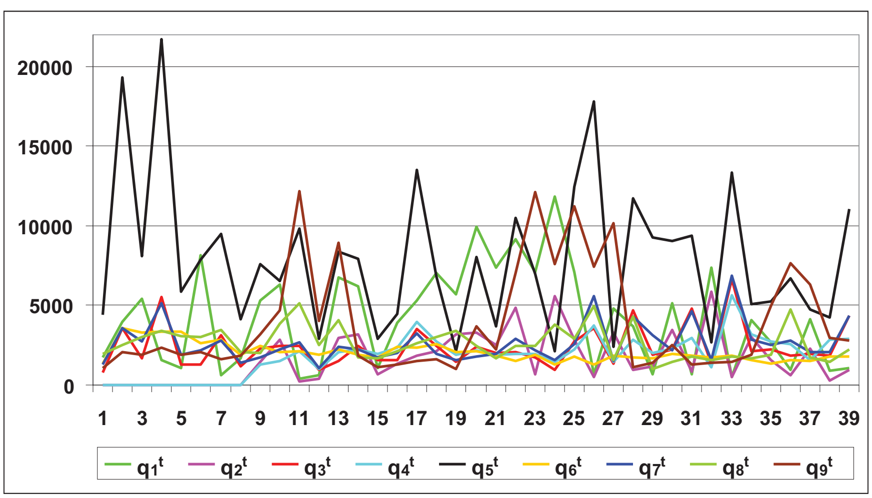
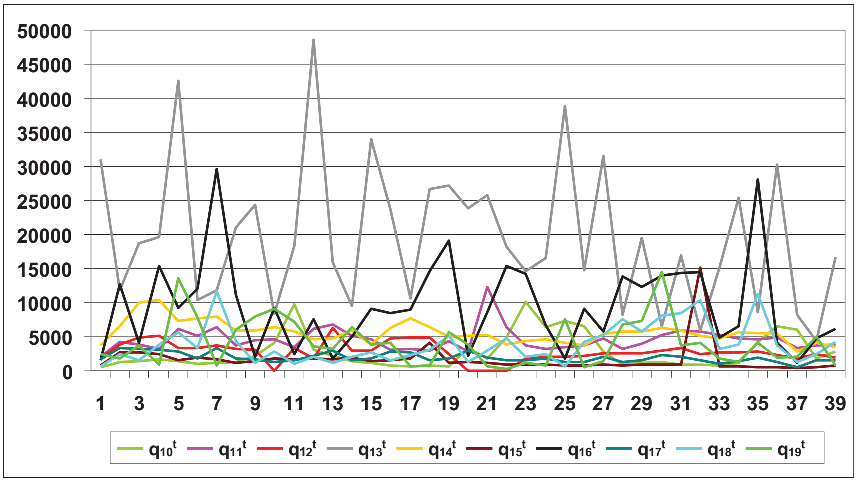


Figure A7.4 Quantities Sold for Products 10–19



A.7.2 Unweighted Price Indices

In this section, we list the unweighted indices[227] that were defined in Sections 2 and 3 in the main text. We used the data that is listed in Annex 1 in order to construct the indices. We list the Jevons, Dutot, Carli, chained Carli, and CES with $r = -1$ and $r = -9$ which we denote by $P_J{}^t$, $P_D{}^t$, $P_C{}^t$, $P_{CCh}{}^t$, $P_{CES,-1}{}^t$, and $P_{CES,-9}{}^t$, respectively, for month t.[228]

The chained Carli index, $P_{CCh}{}^t$, is well above the other indices as is expected. The fixed-base Carli index $P_C{}^t$ is

Table 1.3 Jevons, Dutot, Fixed-Base and Chained Carli and CES Price Indices

t	$P_J{}^t$	$P_D{}^t$	$P_C{}^t$	$P_{CCh}{}^t$	$P_{CES,-1}$	$P_{CES,-9}$
1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**0.96040**	**0.94016**	**0.96846**	**0.96846**	**0.98439**	**1.09067**
3	**0.93661**	**0.94070**	**0.94340**	**0.95336**	**0.92229**	**0.74456**
4	**0.90240**	**0.88954**	**0.91068**	**0.92488**	**0.91540**	**0.90219**
5	**0.90207**	**0.90438**	**0.91172**	**0.93347**	**0.89823**	**0.83603**
6	**0.96315**	**0.97490**	**0.97869**	**1.00142**	**0.94497**	**0.79833**
7	**1.05097**	**1.05468**	**1.06692**	**1.11301**	**1.04802**	**1.06093**
8	**1.13202**	**1.13825**	**1.13337**	**1.21622**	**1.12388**	**1.09382**
9	**1.10373**	**1.10769**	**1.10739**	**1.18820**	**1.09706**	**1.06198**
10	**1.08176**	**1.07574**	**1.09119**	**1.17299**	**1.08685**	**1.07704**
11	**1.07545**	**1.08438**	**1.08516**	**1.17758**	**1.06038**	**0.95660**
12	**1.14864**	**1.15654**	**1.15517**	**1.27479**	**1.13881**	**1.13589**
13	**1.02772**	**1.04754**	**1.03943**	**1.15786**	**0.99848**	**0.84113**
14	**1.06109**	**1.04636**	**1.07433**	**1.21248**	**1.07853**	**1.16587**
15	**1.07130**	**1.06066**	**1.08164**	**1.23459**	**1.08565**	**1.19072**
16	**1.02572**	**1.00635**	**1.03655**	**1.18788**	**1.04979**	**1.19448**
17	**0.93668**	**0.92185**	**0.95548**	**1.09129**	**0.95529**	**1.04194**
18	**0.88243**	**0.86882**	**0.89940**	**1.03405**	**0.90087**	**1.01058**
19	**0.93855**	**0.92175**	**0.96016**	**1.10908**	**0.96244**	**1.15748**
20	**0.88855**	**0.88248**	**0.90225**	**1.07164**	**0.89126**	**0.80633**

(Continued)

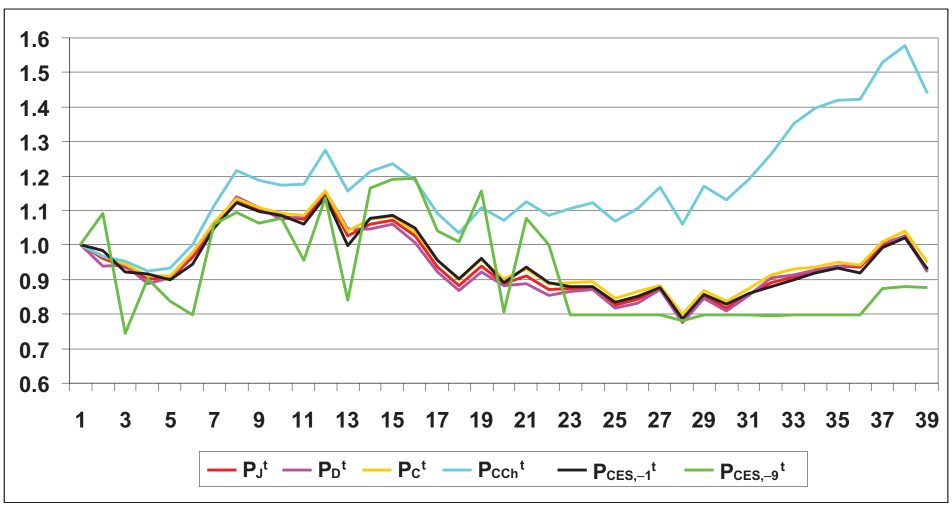
[227] It would be more accurate to call these indices equally weighted indices.

[228] All of these indices were defined in Section 2 except the Carli and chained Carli indices that were defined in Section 3.

Table 1.3 (Continued)

t	$P_J^{\,t}$	$P_D^{\,t}$	$P_C^{\,t}$	$P_{CCh}^{\,t}$	$P_{CES,-1}$	$P_{CES,-9}$
21	**0.91044**	**0.88862**	**0.92930**	**1.12593**	**0.93740**	**1.07775**
22	**0.87080**	**0.85512**	**0.88891**	**1.08595**	**0.89107**	**1.00076**
23	**0.87476**	**0.86577**	**0.89065**	**1.10508**	**0.88042**	**0.79864**
24	**0.87714**	**0.87111**	**0.89384**	**1.12219**	**0.87980**	**0.79810**
25	**0.82562**	**0.81640**	**0.84467**	**1.06793**	**0.83434**	**0.79708**
26	**0.84210**	**0.83168**	**0.86532**	**1.10572**	**0.85123**	**0.79827**
27	**0.87538**	**0.87012**	**0.88197**	**1.16687**	**0.87714**	**0.79760**
28	**0.78149**	**0.77534**	**0.80014**	**1.05919**	**0.78770**	**0.78068**
29	**0.85227**	**0.84699**	**0.86721**	**1.17131**	**0.85568**	**0.79718**
30	**0.81870**	**0.80899**	**0.83656**	**1.13006**	**0.82799**	**0.79688**
31	**0.85842**	**0.85377**	**0.87514**	**1.19113**	**0.86118**	**0.79741**
32	**0.89203**	**0.90407**	**0.91440**	**1.26420**	**0.87884**	**0.79524**
33	**0.90818**	**0.91368**	**0.93127**	**1.35047**	**0.89955**	**0.79775**
34	**0.92659**	**0.92742**	**0.93489**	**1.39685**	**0.91949**	**0.79804**
35	**0.93981**	**0.93944**	**0.94941**	**1.42023**	**0.93256**	**0.79825**
36	**0.93542**	**0.94295**	**0.94087**	**1.42210**	**0.92057**	**0.79654**
37	**1.00182**	**1.00595**	**1.01060**	**1.52914**	**0.99139**	**0.87270**
38	**1.02591**	**1.02295**	**1.04068**	**1.57788**	**1.02072**	**0.87939**
39	**0.92689**	**0.92334**	**0.95090**	**1.44006**	**0.93017**	**0.87789**

Figure A7.5 Unweighted Price Indices



slightly above the corresponding Jevons index $P_J^{\,t}$, which in turn is slightly above the corresponding Dutot index $P_D^{\,t}$. The CES index with $r = -1$ (this corresponds to $s = 2$) is on average between the Jevons and fixed-base Carli indices, while the CES index with $r = -9$ (this corresponds to $\Sigma = 10$) is well below all of the other indices on average (and is extremely volatile).[229]

The Jevons, Dutot, and fixed-base Carli indices, $P_J^{\,t}$, $P_D^{\,t}$, and $P_C^{\,t}$, are quite close to each other. They turn out to end up about 3 percentage points below the fixed-base Fisher indices, $P_F^{\,t}$, at the end of the sample period. However, in the Office for National Statistics (2020) study that also compares unweighted with weighted indices, they find larger differences between these unweighted indices and their superlative index counterparts.[230] The problem with the

[229]The sample means of the $P_J^{\,t}$, $P_D^{\,t}$, $P_C^{\,t}$, $P_{CCh}^{\,t}$, $P_{CES,-1}^{\,t}$, and $P_{CES,-9}^{\,t}$ are 0.9496, 0.9458, 0.9628, 1.1732, 0.9520, and 0.9237, respectively.

[230]The ONS makes the following important point about differences between their GEKS-J unweighted index (essentially our $P_J^{\,t}$ index) and an

unweighted indices is that they do not weight price changes by their economic importance so if weights change dramatically along with dramatic price changes, the unweighted indices can differ significantly from their symmetrically weighted counterpart indices like the Fisher index. For another example of this phenomenon, see the annex to Chapter 6, where it is shown that there are large differences between P_J^t, P_D^t, P_C^t, and P_F^t.

We turn now to a listing of standard bilateral indices using the three years of data and the econometrically estimated reservation prices.

A.7.3 Commonly Used Weighted Price Indices

We list the fixed-base and chained Laspeyres, Paasche, Fisher, and Törnqvist indices in Table 1.4. The geometric Laspeyres and geometric Paasche and unit value indices are also listed in this table.

It can be seen that all nine of the weighted indices that appear in Figure A7.6 capture an underlying general trend in prices. However, there is a considerable dispersion between the indices. Our preferred indices for this group of indices are the fixed-base Fisher and Törnqvist indices, P_F^t and P_T^t. These two indices approximate each other very closely and can barely be distinguished in the chart. The Paasche and geometric Paasche indices, P_P^t and P_{GP}^t, lie below our preferred indices, while the remaining indices generally lie above our preferred indices. The chained Fisher and Törnqvist indices, P_{FCh}^t and P_{TCh}^t, approximate each other very closely, but both indices lie well above their fixed-base counterparts; that is, they exhibit a considerable amount of chain drift. Thus, chained superlative indices are not recommended for use with scanner data, where the products are subject to large fluctuations in prices and quantities. The fixed-base Laspeyres and geometric Laspeyres indices, P_L^t and P_{GL}^t, are fairly close to each other and are well above P_F^t and P_T^t. The unit value price index, P_{UV}^t, is subject to large fluctuations and generally lies above our preferred indices.

We turn now to weighted indices that use annual weights from a base year.

Table 1.4 Fixed-Base and Chained Laspeyres, Paasche, Fisher, and Törnqvist Indices and geometric Laspeyres, geometric Paasche, and Unit Value Indices

t	P_L^t	P_P^t	P_F^t	P_T^t	P_{LCh}^t	P_{PCh}^t	P_{FCh}^t	P_{TCh}^t	P_{GL}^t	P_{GP}^t	P_{UV}^t
1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**1.08991**	**0.92151**	**1.00218**	**1.00036**	**1.08991**	**0.92151**	**1.00218**	**1.00036**	**0.98194**	**1.07214**	**1.10724**
3	**1.06187**	**0.98637**	**1.02342**	**1.02220**	**1.12136**	**0.91193**	**1.01124**	**1.00905**	**0.97979**	**1.05116**	**1.07205**
4	**1.00174**	**0.87061**	**0.93388**	**0.93445**	**1.06798**	**0.83203**	**0.94265**	**0.94077**	**0.91520**	**0.99062**	**1.03463**
5	**0.98198**	**0.89913**	**0.93964**	**0.94387**	**1.11998**	**0.78417**	**0.93715**	**0.93753**	**0.91048**	**0.97176**	**0.95620**
6	**1.13639**	**0.95159**	**1.03989**	**1.04311**	**1.27664**	**0.84845**	**1.04075**	**1.04165**	**0.99679**	**1.11657**	**1.10159**
7	**1.22555**	**0.91097**	**1.05662**	**1.06555**	**1.42086**	**0.85482**	**1.10208**	**1.09531**	**1.07355**	**1.20485**	**1.12167**
8	**1.17447**	**1.14057**	**1.15740**	**1.15743**	**1.75897**	**0.91677**	**1.26987**	**1.26340**	**1.14865**	**1.17300**	**1.25911**
9	**1.17750**	**1.12636**	**1.15164**	**1.15169**	**1.73986**	**0.89414**	**1.24727**	**1.24135**	**1.12700**	**1.17162**	**1.19939**
10	**1.27247**	**1.05895**	**1.16081**	**1.15735**	**1.80210**	**0.86050**	**1.24528**	**1.23902**	**1.12514**	**1.25074**	**1.20900**
11	**1.20770**	**1.07376**	**1.13876**	**1.13875**	**1.86610**	**0.81117**	**1.23034**	**1.22114**	**1.12189**	**1.19276**	**1.06812**
12	**1.12229**	**1.09688**	**1.10951**	**1.10976**	**2.01810**	**0.73863**	**1.22091**	**1.20993**	**1.12209**	**1.11767**	**1.07795**
13	**1.18583**	**1.04861**	**1.11511**	**1.11677**	**2.17862**	**0.66995**	**1.20813**	**1.19943**	**1.09272**	**1.17231**	**1.08595**
14	**1.25239**	**1.05236**	**1.14803**	**1.14485**	**2.30844**	**0.66552**	**1.23948**	**1.22942**	**1.09463**	**1.22682**	**1.21698**
15	**1.06527**	**1.01701**	**1.04086**	**1.04292**	**2.32124**	**0.58025**	**1.16056**	**1.15215**	**1.03397**	**1.06020**	**1.07438**
16	**1.07893**	**1.01866**	**1.04836**	**1.05073**	**2.34342**	**0.56876**	**1.15449**	**1.14720**	**1.01310**	**1.07256**	**1.11895**
17	**1.10767**	**0.89217**	**0.99410**	**0.99352**	**2.28924**	**0.51559**	**1.08642**	**1.07832**	**0.95895**	**1.08127**	**1.06696**
18	**0.95021**	**0.83559**	**0.89105**	**0.89584**	**2.14196**	**0.45252**	**0.98452**	**0.97741**	**0.86911**	**0.94010**	**0.94589**
19	**0.93250**	**0.81744**	**0.87308**	**0.88137**	**2.21416**	**0.44435**	**0.99189**	**0.98454**	**0.88768**	**0.92447**	**0.93364**
20	**0.91010**	**0.85188**	**0.88051**	**0.88230**	**2.37598**	**0.41411**	**0.99193**	**0.98133**	**0.88109**	**0.90423**	**0.92812**
21	**0.90831**	**0.87050**	**0.88920**	**0.89209**	**2.48204**	**0.40411**	**1.00150**	**0.99069**	**0.87548**	**0.90221**	**0.92800**
22	**0.93448**	**0.79545**	**0.86217**	**0.86876**	**2.44050**	**0.37816**	**0.96068**	**0.95081**	**0.85191**	**0.92460**	**0.90448**
23	**0.93852**	**0.82477**	**0.87981**	**0.88494**	**2.54428**	**0.37672**	**0.97902**	**0.96923**	**0.85916**	**0.92722**	**0.86752**
24	**0.95955**	**0.83212**	**0.89357**	**0.90008**	**2.61768**	**0.35461**	**0.96347**	**0.95725**	**0.88900**	**0.95127**	**0.87176**

(*Continued*)

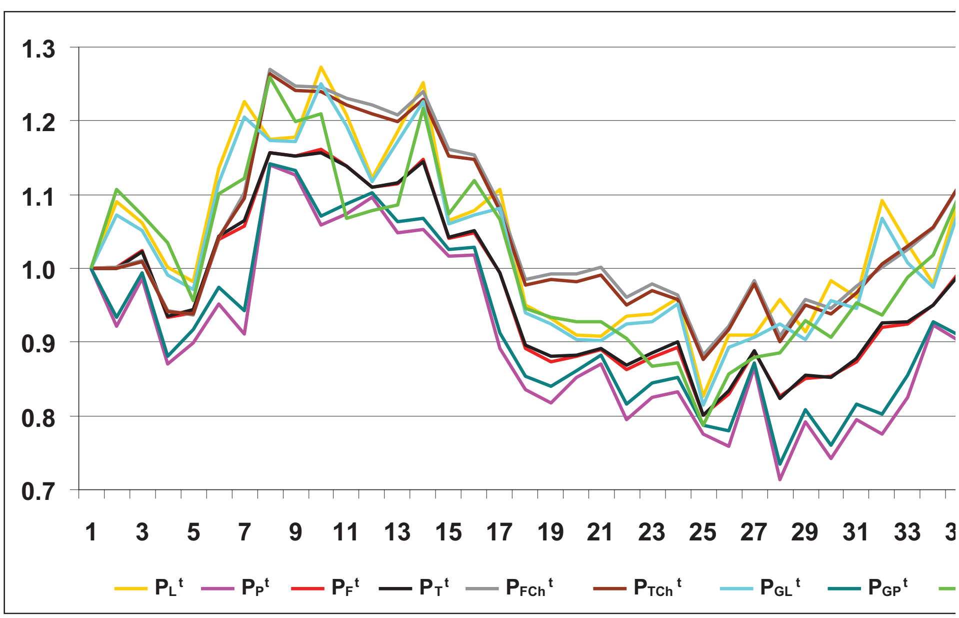
appropriately weighted index: "Two main observations can be made from the observations of these case studies. . . . Secondly, there is an apparent upward bias from the GEKS-J methods in comparison to the weighted methods; this is likely because consumers substitute toward products that are on sale and this is not accounted for when using unweighted methods. This again highlights that having information on sales values, or approximates thereof, is arguably more important than the choice between weighted index number methods themselves" (ONS, 2020, 43).

Table 1.4 (Continued)

t	P_L^t	P_P^t	P_F^t	P_T^t	P_{LCh}^t	P_{PCh}^t	P_{FCh}^t	P_{TCh}^t	P_{GL}^t	P_{GP}^t	P_{UV}^t
25	**0.82659**	**0.77523**	**0.80050**	**0.80120**	**2.54432**	**0.30555**	**0.88172**	**0.87662**	**0.80638**	**0.81529**	**0.78713**
26	**0.90933**	**0.75806**	**0.83026**	**0.83456**	**2.84192**	**0.29847**	**0.92100**	**0.91714**	**0.82419**	**0.89313**	**0.85607**
27	**0.90913**	**0.86638**	**0.88749**	**0.88866**	**3.22816**	**0.29960**	**0.98344**	**0.97818**	**0.87350**	**0.90653**	**0.87957**
28	**0.95748**	**0.71369**	**0.82665**	**0.82378**	**3.27769**	**0.25120**	**0.90739**	**0.90090**	**0.80609**	**0.92446**	**0.88558**
29	**0.91434**	**0.79178**	**0.85086**	**0.85489**	**3.58621**	**0.25612**	**0.95839**	**0.95091**	**0.83824**	**0.90372**	**0.92881**
30	**0.98306**	**0.74159**	**0.85383**	**0.85285**	**3.63285**	**0.24640**	**0.94612**	**0.93848**	**0.83636**	**0.95691**	**0.90674**
31	**0.96148**	**0.79467**	**0.87411**	**0.87827**	**3.82999**	**0.24849**	**0.97557**	**0.96637**	**0.85604**	**0.94519**	**0.95259**
32	**1.09219**	**0.77559**	**0.92038**	**0.92577**	**4.36079**	**0.23020**	**1.00192**	**1.00563**	**0.93404**	**1.06859**	**0.93739**
33	**1.03387**	**0.82587**	**0.92403**	**0.92835**	**5.45066**	**0.19325**	**1.02632**	**1.03039**	**0.92860**	**1.00783**	**0.98847**
34	**0.97819**	**0.92286**	**0.95012**	**0.95072**	**5.95659**	**0.18655**	**1.05412**	**1.05647**	**0.93004**	**0.97390**	**1.01750**
35	**1.09532**	**0.90246**	**0.99422**	**0.99086**	**6.44252**	**0.19130**	**1.11015**	**1.11105**	**0.97904**	**1.07872**	**1.09820**
36	**0.97574**	**0.93603**	**0.95568**	**0.95607**	**6.69005**	**0.17668**	**1.08720**	**1.08989**	**0.94745**	**0.97198**	**0.93645**
37	**1.10952**	**0.99004**	**1.04808**	**1.04846**	**7.50373**	**0.18937**	**1.19204**	**1.19665**	**1.03628**	**1.10176**	**1.02142**
38	**1.21684**	**0.99944**	**1.10280**	**1.09863**	**7.90093**	**0.18768**	**1.21774**	**1.22145**	**1.06914**	**1.19166**	**1.14490**
39	**1.04027**	**0.86886**	**0.95071**	**0.95482**	**7.16398**	**0.15715**	**1.06105**	**1.06219**	**0.93030**	**1.01682**	**0.99999**

Note that the chained Laspeyres index ends up at 7.164, while the chained Paasche index ends up at 0.157. The corresponding fixed-base indices end up at 1.040 and 0.869, so it is clear that these chained indices are subject to tremendous chain drift. The chain drift carries over to the Fisher and Törnqvist indices; that is, the fixed-base Fisher index ends up at 0.9548, while its chained counterpart ends up at 1.061. Chart 1.6 plots these indices with the exceptions of the chained Laspeyres and Paasche indices (these indices exhibit too much chain drift to be considered further).

Figure A7.6 Weighted Price Indices



A.7.4 Indices That Use Annual Weights

The weighted Jevons or geometric Young index, $P_{J\alpha}^{\ t}$ or $P_{GY}^{\ t}$, was defined by (54) in Section 6. This index uses the arithmetic average of the monthly shares in year 1 as weights in a weighted geometric index for subsequent months in the sample. The Lowe index, $P_{Lo}^{\ t}$, was defined by (124) in Section 11. This index is a fixed basket index that uses the average quantities in the base year as the vector of quantity weights. We calculated both of these indices for the months in years 2 and 3 for our sample using the weights from year 1 of our

Table 1.5 geometric Young, Lowe, Laspeyres, Paasche, Fisher, Törnqvist, Jevons, Dutot, and Unit Value Indices for Years 2 and 3

t	$P_{GY}{}^t$	$P_{Lo}{}^t$	$P_L{}^t$	$P_P{}^t$	$P_F{}^t$	$P_T{}^t$	$P_J{}^t$	$P_D{}^t$	$P_{UV}{}^t$
14	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
15	**0.93263**	**0.93494**	**1.00555**	**0.87189**	**0.93634**	**0.93715**	**1.00962**	**1.01366**	**0.88282**
16	**0.92082**	**0.92031**	**0.95041**	**0.90755**	**0.92873**	**0.92986**	**0.96667**	**0.96175**	**0.91945**
17	**0.87997**	**0.88507**	**0.89085**	**0.85384**	**0.87215**	**0.87032**	**0.88275**	**0.88100**	**0.87673**
18	**0.79503**	**0.80022**	**0.82733**	**0.76386**	**0.79496**	**0.79283**	**0.83163**	**0.83032**	**0.77724**
19	**0.80573**	**0.81468**	**0.85664**	**0.76193**	**0.80790**	**0.80865**	**0.88451**	**0.88091**	**0.76718**
20	**0.79981**	**0.80700**	**0.82757**	**0.74904**	**0.78733**	**0.78635**	**0.83739**	**0.84337**	**0.76264**
21	**0.79437**	**0.80164**	**0.83126**	**0.77659**	**0.80346**	**0.80489**	**0.85802**	**0.84924**	**0.76254**
22	**0.77921**	**0.78355**	**0.80911**	**0.75762**	**0.78294**	**0.78149**	**0.82067**	**0.81723**	**0.74322**
23	**0.77876**	**0.78087**	**0.82688**	**0.76468**	**0.79517**	**0.79606**	**0.82440**	**0.82741**	**0.71285**
24	**0.81228**	**0.81680**	**0.83070**	**0.75908**	**0.79408**	**0.79472**	**0.82664**	**0.83251**	**0.71633**
25	**0.72801**	**0.74112**	**0.75452**	**0.66120**	**0.70632**	**0.70498**	**0.77809**	**0.78023**	**0.64679**
26	**0.75011**	**0.75684**	**0.80141**	**0.71350**	**0.75618**	**0.75377**	**0.79362**	**0.79483**	**0.70344**
27	**0.79254**	**0.79375**	**0.82527**	**0.74559**	**0.78442**	**0.78661**	**0.82498**	**0.83156**	**0.72275**
28	**0.73664**	**0.74226**	**0.74893**	**0.71223**	**0.73035**	**0.72970**	**0.73650**	**0.74098**	**0.72769**
29	**0.75964**	**0.76031**	**0.80135**	**0.74576**	**0.77306**	**0.77165**	**0.80321**	**0.80946**	**0.76321**
30	**0.76531**	**0.76828**	**0.77149**	**0.74410**	**0.75767**	**0.75781**	**0.77157**	**0.77315**	**0.74507**
31	**0.77786**	**0.77867**	**0.81448**	**0.76635**	**0.79005**	**0.78811**	**0.80900**	**0.81594**	**0.78275**
32	**0.85506**	**0.86201**	**0.87512**	**0.76018**	**0.81563**	**0.82138**	**0.84067**	**0.86401**	**0.77026**
33	**0.84365**	**0.85499**	**0.88099**	**0.77811**	**0.82795**	**0.82554**	**0.85589**	**0.87320**	**0.81223**
34	**0.84601**	**0.84804**	**0.88159**	**0.82588**	**0.85328**	**0.85422**	**0.87325**	**0.88632**	**0.83608**
35	**0.89199**	**0.89177**	**0.90170**	**0.92254**	**0.91206**	**0.91320**	**0.88570**	**0.89782**	**0.90240**
36	**0.85506**	**0.85983**	**0.90132**	**0.79811**	**0.84815**	**0.84966**	**0.88156**	**0.90117**	**0.76948**
37	**0.94264**	**0.94402**	**0.95898**	**0.90084**	**0.92946**	**0.93135**	**0.94414**	**0.96137**	**0.83931**
38	**0.97419**	**0.97462**	**0.99009**	**0.95811**	**0.97397**	**0.97413**	**0.96684**	**0.97762**	**0.94077**
39	**0.85043**	**0.85908**	**0.88213**	**0.80516**	**0.84277**	**0.84144**	**0.87353**	**0.88242**	**0.82170**
Mean	**0.83338**	**0.83772**	**0.86329**	**0.80014**	**0.83094**	**0.83100**	**0.86080**	**0.86644**	**0.79634**

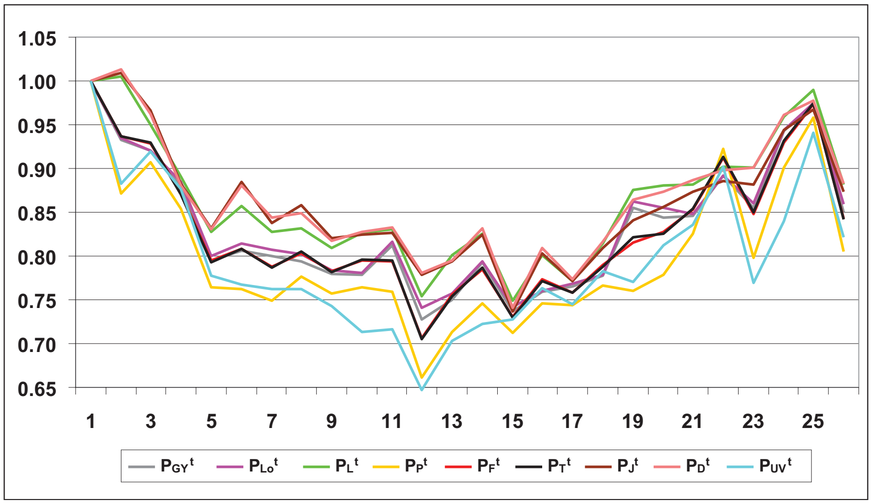
sample. For comparison purposes, we also list the fixed-base Laspeyres, Paasche, Fisher, and Törnqvist indices, $P_L{}^t$, $P_P{}^t$, $P_F{}^t$, and $P_T{}^t$ for the "months" in years 2 and 3 of our sample. It is also of interest to list the Jevons, Dutot, and unit value indices, $P_J{}^t$, $P_D{}^t$, and $P_{UV}{}^t$ for years 2 and 3 in order to see how unweighted indices compare to the weighted indices. The sample averages for these indices are listed in the last row of Table 1.5.

As usual, $P_F{}^t$ and $P_T{}^t$ approximate each other very closely. Indices with substantial upward biases relative to these two indices are the Laspeyres, Jevons, and Dutot indices, $P_L{}^t$, $P_J{}^t$, and $P_D{}^t$. The geometric Young index and the Lowe index, $P_{GY}{}^t$ and $P_{Lo}{}^t$, were about 0.25 and 0.67 percentage points above the superlative indices on average. The Paasche and unit value indices, $P_P{}^t$ and $P_{UV}{}^t$, had substantial downward biases relative to the superlative indices. These inequalities agree with our a priori expectations about biases. The nine indices are plotted in Figure A7.7.

It can be seen that all nine indices capture the trend in the product prices with $P_F{}^t$ and $P_T{}^t$ in the middle of the indices (and barely distinguishable from each other in the chart). The unit value index $P_{UV}{}^t$ is the lowest index followed by the Paasche index $P_P{}^t$. The geometric Young and Lowe indices, $P_{GY}{}^t$ and $P_{Lo}{}^t$, are quite close to each other and close to the superlative indices in the first part of the sample, but then they drift above the superlative indices in the latter half of the sample. We expect the Lowe index to have some upward substitution bias, and with highly substitutable products, we expect the geometric Young index to also have an upward substitution bias. Finally, the Laspeyres, Jevons, and Dutot indices are all substantially above the superlative indices, with $P_J{}^t$ and $P_D{}^t$ approximating each other quite closely.

We turn our attention to multilateral indices.

Figure A7.7 geometric Young, Lowe, and Other Indices for Years 2 and 3



A.7.5 Multilateral Indices

We considered seven main multilateral indices in the main text:[231]

- P_{GEKS}^t (see definition (70) in Section 8);
- P_{CCDI}^t (see definition (77) in Section 8);
- P_{GK}^t (see definition (137) in Section 12);
- P_{WTPD}^t (see definition (149) in Section 13);
- P_{AL}^t, the price similarity-linked indices defined following definition (215), which defined the *asymptotic linear measures of relative price dissimilarity* $\Delta_{AL}(p^r,p^t,q^r,q^t)$;
- P_{SP}^t, the price similarity-linked indices defined following definition (218), which defined the *predicted share measures of relative price dissimilarity* $\Delta_{SP}(p^r,p^t,q^r,q^t)$; and
- P_{SPQ}^t, the price and quantity similarity-linked indices defined following definition (221), which defined the *predicted share measures of relative price and quantity dissimilarity* $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$.

It turned out that the similarity-linked price indices P_{SP}^t were equal to their counterparts P_{SPQ}^t for each time period t, so we list only the P_{SP}^t indices in Table 1.6.[232] The above six multilateral indices are listed in Table 1.6 along with the fixed-base Fisher and Törnqvist indices P_F^t and P_T^t. All of these indices were evaluated using estimated reservation prices for the missing products. The sample mean for each index is listed in the last row of Table 1.6.

If the eight indices are evaluated according to their sample means, the two similarity-linked indices P_{AL}^t and P_{SP}^t = P_{SPQ}^t generated the lowest indices on average. The P_{GEKS}^t, P_{CCDI}^t, P_F^t, and P_T^t indices are tightly clustered in the middle and the P_{GK}^t and P_{WTPD}^t indices are about 2 percentage points above the middle indices on average. Looking at the index levels at the final sample observation, the two indices that use similarity linking end up at 0.9275, which is about 2 percentage points below where the GEKS, CCDI, fixed-base Fisher, and Törnqvist–Theil indices ended up. The GK and weighted TPD indices ended up approximately 4 percentage points above the two similarity-linked indices. These differences are substantial. Figure A7.8 plots the eight indices.

All eight indices capture the trend in product prices reasonably well. It is clear that the GK and weighted TPD indices have some upward bias relative to the remaining six indices. The two similarity-linked indices, P_{AL}^t and P_{SP}^t, both end up at the same index level, and in general, they are very close.

The following table lists the real-time P_{AL}^t and P_{SP}^t again and compares them with their modified counterparts, P_{ALM}^t and P_{SPM}^t. These latter indices use the first 13 "months" as a "training" year where a spanning tree of observations is linked simultaneously. Here is the spanning tree or path of bilateral links that minimizes the sum of the dissimilarity measures associated with the links for P_{ALM}^t:

```
12      13
|       |
1 – 8 – 9 – 11
        |
        3 – 4 – 2 – 10 – 6 – 5
        |
        7
```

Here is the corresponding set of optimal links for P_{SPM}^t for "months" 1–13:

---

[231] We also defined the quantity similarity-linked price indices P_{SQ}^t following definition (219), which were constructed using the predicted share measures of relative quantity dissimilarity $\Delta_{SQ}(p^r,p^t,q^r,q^t)$. However, the indices P_{SQ}^t are absorbed into the definition of the superior indices P_{SPQ}^t, and so we did not list P_{SQ}^t here. We also considered some variants of P_{AL}^t and P_{SP}^t, which will be considered later in this section and in Section A6.

[232] For every pair of observations, the measure of predicted share relative price dissimilarity was always smaller than the corresponding measure of predicted share relative quantity dissimilarity.

Table 1.6 Six Multilateral Indices and the Fixed-Base Fisher and Törnqvist Indices

t	$P_{GEKS}{}^t$	$P_{CCDI}{}^t$	$P_{GK}{}^t$	$P_{WTPD}{}^t$	$P_{AL}{}^t$	$P_{SP}{}^t$	$P_F{}^t$	$P_T{}^t$
1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**1.00233**	**1.00395**	**1.03138**	**1.02468**	**1.00218**	**1.00218**	**1.00218**	**1.00036**
3	**1.00575**	**1.00681**	**1.03801**	**1.03322**	**1.01124**	**1.01124**	**1.02342**	**1.02220**
4	**0.93922**	**0.94020**	**0.97021**	**0.96241**	**0.94262**	**0.94262**	**0.93388**	**0.93445**
5	**0.92448**	**0.92712**	**0.94754**	**0.94505**	**0.92812**	**0.92812**	**0.93964**	**0.94387**
6	**1.02249**	**1.02595**	**1.06097**	**1.05893**	**1.03073**	**1.03073**	**1.03989**	**1.04311**
7	**1.06833**	**1.06995**	**1.06459**	**1.06390**	**1.07314**	**1.09146**	**1.05662**	**1.06555**
8	**1.19023**	**1.19269**	**1.24385**	**1.24192**	**1.15740**	**1.15740**	**1.15740**	**1.15743**
9	**1.15115**	**1.15206**	**1.18818**	**1.18231**	**1.13680**	**1.13680**	**1.15164**	**1.15169**
10	**1.14730**	**1.15007**	**1.19184**	**1.18333**	**1.15156**	**1.15156**	**1.16081**	**1.15735**
11	**1.13270**	**1.13301**	**1.14662**	**1.14308**	**1.12574**	**1.12574**	**1.13876**	**1.13875**
12	**1.11903**	**1.12079**	**1.11332**	**1.12082**	**1.10951**	**1.10951**	**1.10951**	**1.10976**
13	**1.10247**	**1.10487**	**1.11561**	**1.11838**	**1.09229**	**1.09229**	**1.11511**	**1.11677**
14	**1.12136**	**1.12345**	**1.16579**	**1.15912**	**1.12489**	**1.12489**	**1.14803**	**1.14485**
15	**1.04827**	**1.04883**	**1.06958**	**1.06608**	**1.04237**	**1.04086**	**1.04086**	**1.04292**
16	**1.04385**	**1.04539**	**1.08842**	**1.08044**	**1.03692**	**1.04704**	**1.04836**	**1.05073**
17	**0.97470**	**0.97550**	**0.99512**	**0.99145**	**0.97013**	**0.97013**	**0.99410**	**0.99352**
18	**0.88586**	**0.88695**	**0.91319**	**0.90765**	**0.88455**	**0.89319**	**0.89105**	**0.89584**
19	**0.89497**	**0.89597**	**0.90990**	**0.90923**	**0.89118**	**0.89702**	**0.87308**	**0.88137**
20	**0.88973**	**0.89126**	**0.90822**	**0.90578**	**0.88051**	**0.88051**	**0.88051**	**0.88230**
21	**0.89904**	**0.89990**	**0.92641**	**0.92503**	**0.88482**	**0.89346**	**0.88920**	**0.89209**
22	**0.87061**	**0.87363**	**0.90145**	**0.89880**	**0.87151**	**0.88001**	**0.86217**	**0.86876**
23	**0.88592**	**0.88868**	**0.92421**	**0.92158**	**0.88280**	**0.88280**	**0.87981**	**0.88494**
24	**0.89282**	**0.89799**	**0.91127**	**0.91198**	**0.88502**	**0.88502**	**0.89357**	**0.90008**
25	**0.81132**	**0.81115**	**0.81875**	**0.81913**	**0.79966**	**0.79966**	**0.80050**	**0.80120**
26	**0.83799**	**0.83914**	**0.85168**	**0.85089**	**0.83378**	**0.83378**	**0.83026**	**0.83456**
27	**0.89063**	**0.89246**	**0.91906**	**0.91398**	**0.88481**	**0.88481**	**0.88749**	**0.88866**
28	**0.81304**	**0.81411**	**0.82600**	**0.82419**	**0.81336**	**0.81336**	**0.82665**	**0.82378**
29	**0.85763**	**0.85934**	**0.88821**	**0.88248**	**0.86271**	**0.86271**	**0.85086**	**0.85489**
30	**0.84103**	**0.84305**	**0.86121**	**0.85556**	**0.85166**	**0.85230**	**0.85383**	**0.85285**
31	**0.87495**	**0.87639**	**0.90123**	**0.89600**	**0.87568**	**0.87568**	**0.87411**	**0.87827**
32	**0.89936**	**0.90831**	**0.88553**	**0.89332**	**0.91368**	**0.91398**	**0.92038**	**0.92577**
33	**0.92670**	**0.92878**	**0.91672**	**0.92625**	**0.91517**	**0.91517**	**0.92403**	**0.92835**
34	**0.95721**	**0.95846**	**0.99507**	**0.98974**	**0.94435**	**0.94435**	**0.95012**	**0.95072**
35	**1.01848**	**1.02026**	**1.07728**	**1.06779**	**1.00422**	**1.00422**	**0.99422**	**0.99086**
36	**0.96507**	**0.96601**	**0.98339**	**0.98282**	**0.96122**	**0.96122**	**0.95568**	**0.95607**
37	**1.05250**	**1.05448**	**1.08019**	**1.07514**	**1.07953**	**1.03556**	**1.04808**	**1.04846**
38	**1.08819**	**1.08961**	**1.11648**	**1.10963**	**1.07546**	**1.07546**	**1.10280**	**1.09863**
39	**0.94591**	**0.94834**	**0.96156**	**0.96453**	**0.92575**	**0.92575**	**0.95071**	**0.95482**
Mean	**0.97417**	**0.97602**	**0.99764**	**0.99504**	**0.97069**	**0.97109**	**0.97434**	**0.97607**

```
      13
      |
12 – 1 – 8 – 9 – 11
              |
10 – 2 – 4 – 3 – 5 – 6 – 7.
```

These spanning trees are similar but not identical. Nevertheless, the index levels generated by the two alternative measures of price dissimilarity end up being the same.

At the end of Section 20, the fixed-base maximum overlap Fisher indices P_F^{t*} were defined along with the GEKS index that uses the geometric mean of the maximum overlap Fisher indices for each choice of a base, $P_{GEKS}{}^{t*}$. We also defined the counterparts to the predicted share multilateral indices $P_{SP}{}^{t}$ and $P_{SPM}{}^{t}$ using maximum overlap Fisher indices to do the linking of observations in place of regular Fisher indices. These maximum overlap indices (which do not use imputations) were denoted by $P_{SP}{}^{t*}$ and $P_{SPM}{}^{t*}$. All of these indices are listed in Table 1.8. The set of optimal links for $P_{SPM}{}^{t*}$ for "months" 1–13 are as follows:

13
|
12 – 1 – 8 – 9 – 11 – 10
|
2 – 4 – 3 – 5 – 6 – 7

All 10 of the above indices are listed in Table1.7 and are plotted in Figure A7.9.

The four similarity-linked indices that used reservation prices, $P_{AL}{}^{t}$, $P_{ALM}{}^{t}$, $P_{SP}{}^{t}$, and $P_{SMP}{}^{t}$, ended up at the same level for the last observation, 0.92575. The predicted share similarity-linked indices that did not use imputations for the prices of missing products, $P_{SP}{}^{t*}$ and $P_{SMP}{}^{t*}$, ended up at the slightly higher level, 0.92612. Thus, all of the similarity-linked indices behaved in a similar manner for our particular data set.

Figure A7.8 Six Multilateral and Two Superlative Indices

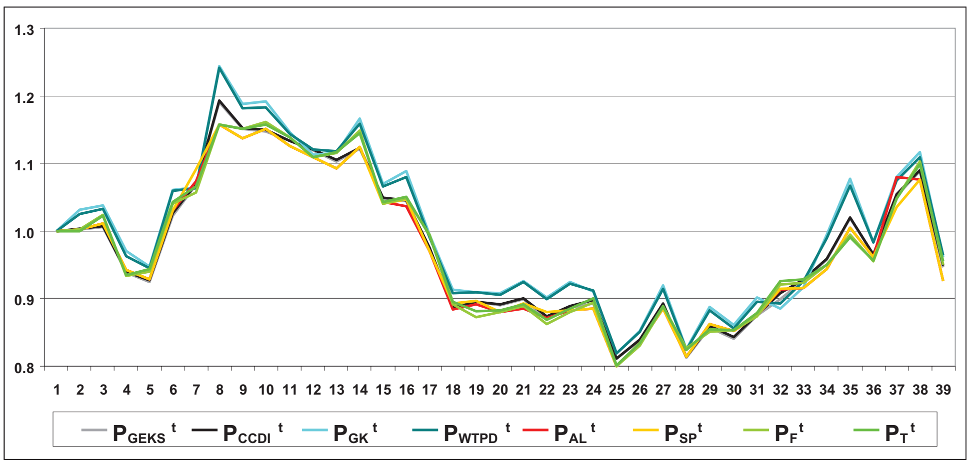


Table 1.7 Six Similarity-Linked Multilateral, Two GEKS, and Two Fisher Indices

t	$P_{AL}{}^{t}$	$P_{ALM}{}^{t}$	$P_{SP}{}^{t}$	$P_{SPM}{}^{t}$	$P_{SP}{}^{t*}$	$P_{SPM}{}^{t*}$	$P_{GEKS}{}^{t}$	$P_{GEKS}{}^{t*}$	$P_{F}{}^{t}$	$P_{F}{}^{t*}$
1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**1.00218**	**0.99067**	**1.00218**	**0.99067**	**1.00218**	**0.99213**	**1.00233**	**1.00546**	**1.00218**	**1.00218**
3	**1.01124**	**0.99960**	**1.01124**	**0.99960**	**1.01124**	**1.00108**	**1.00575**	**1.00673**	**1.02342**	**1.02342**
4	**0.94262**	**0.93180**	**0.94262**	**0.93180**	**0.94262**	**0.93317**	**0.93922**	**0.94156**	**0.93388**	**0.93388**
5	**0.92812**	**0.90620**	**0.92812**	**0.91744**	**0.92812**	**0.91879**	**0.92448**	**0.92384**	**0.93964**	**0.93964**
6	**1.03073**	**1.00638**	**1.03073**	**1.01886**	**1.03073**	**1.02037**	**1.02249**	**1.02505**	**1.03989**	**1.03989**
7	**1.07314**	**1.06081**	**1.09146**	**1.07890**	**1.09146**	**1.08049**	**1.06833**	**1.06965**	**1.05662**	**1.05662**
8	**1.15740**	**1.15740**	**1.15740**	**1.15740**	**1.15740**	**1.15740**	**1.19023**	**1.19015**	**1.15740**	**1.15740**
9	**1.13680**	**1.13680**	**1.13680**	**1.13680**	**1.13726**	**1.13726**	**1.15115**	**1.15502**	**1.15164**	**1.15209**
10	**1.15156**	**1.13833**	**1.15156**	**1.13833**	**1.13142**	**1.13707**	**1.14730**	**1.15094**	**1.16081**	**1.16529**
11	**1.12574**	**1.12574**	**1.12574**	**1.12574**	**1.12620**	**1.12620**	**1.13270**	**1.13707**	**1.13876**	**1.14153**
12	**1.10951**	**1.10951**	**1.10951**	**1.10951**	**1.10876**	**1.10876**	**1.11903**	**1.12242**	**1.10951**	**1.10876**
13	**1.09229**	**1.09229**	**1.09229**	**1.09229**	**1.09273**	**1.09273**	**1.10247**	**1.10798**	**1.11511**	**1.12264**
14	**1.12489**	**1.11196**	**1.12489**	**1.11196**	**1.10948**	**1.11502**	**1.12136**	**1.12651**	**1.14803**	**1.15567**
15	**1.04237**	**1.04237**	**1.04086**	**1.04086**	**1.04167**	**1.04167**	**1.04827**	**1.05159**	**1.04086**	**1.04105**
16	**1.03692**	**1.03692**	**1.04704**	**1.03502**	**1.03622**	**1.03622**	**1.04385**	**1.04814**	**1.04836**	**1.05283**

t	P_{AL}^{t}	P_{ALM}^{t}	P_{SP}^{t}	P_{SPM}^{t}	P_{SP}^{t*}	P_{SPM}^{t*}	P_{GEKS}^{t}	P_{GEKS}^{t*}	P_{F}^{t}	P_{F}^{t*}
17	**0.97013**	**0.95899**	**0.97013**	**0.95899**	**0.96764**	**0.97246**	**0.97470**	**0.97951**	**0.99410**	**1.00156**
18	**0.88455**	**0.88455**	**0.89319**	**0.88293**	**0.88396**	**0.88396**	**0.88586**	**0.88943**	**0.89105**	**0.89486**
19	**0.89118**	**0.89118**	**0.89702**	**0.88672**	**0.88775**	**0.88775**	**0.89497**	**0.89780**	**0.87308**	**0.87462**
20	**0.88051**	**0.88051**	**0.88051**	**0.88051**	**0.86666**	**0.86666**	**0.88973**	**0.89037**	**0.88051**	**0.88462**
21	**0.88482**	**0.88482**	**0.89346**	**0.88319**	**0.87503**	**0.87503**	**0.89904**	**0.90403**	**0.88920**	**0.89505**
22	**0.87151**	**0.87151**	**0.88001**	**0.86991**	**0.86764**	**0.86764**	**0.87061**	**0.87296**	**0.86217**	**0.86759**
23	**0.88280**	**0.87265**	**0.88280**	**0.87265**	**0.87100**	**0.87100**	**0.88592**	**0.88869**	**0.87981**	**0.88008**
24	**0.88502**	**0.88502**	**0.88502**	**0.88502**	**0.87164**	**0.87164**	**0.89282**	**0.89785**	**0.89357**	**0.90877**
25	**0.79966**	**0.79966**	**0.79966**	**0.79966**	**0.78672**	**0.78672**	**0.81132**	**0.81419**	**0.80050**	**0.80492**
26	**0.83378**	**0.82421**	**0.83378**	**0.82421**	**0.82264**	**0.82264**	**0.83799**	**0.84106**	**0.83026**	**0.83325**
27	**0.88481**	**0.88481**	**0.88481**	**0.88481**	**0.87500**	**0.87500**	**0.89063**	**0.89395**	**0.88749**	**0.89223**
28	**0.81336**	**0.80401**	**0.81336**	**0.80401**	**0.81126**	**0.81531**	**0.81304**	**0.81584**	**0.82665**	**0.82771**
29	**0.86271**	**0.85280**	**0.86271**	**0.85280**	**0.85118**	**0.85118**	**0.85763**	**0.86015**	**0.85086**	**0.85009**
30	**0.85166**	**0.84188**	**0.85230**	**0.84250**	**0.84063**	**0.84482**	**0.84103**	**0.84407**	**0.85383**	**0.85566**
31	**0.87568**	**0.86562**	**0.87568**	**0.86562**	**0.86398**	**0.86398**	**0.87495**	**0.87775**	**0.87411**	**0.87393**
32	**0.91368**	**0.89210**	**0.91398**	**0.90346**	**0.88825**	**0.89268**	**0.89936**	**0.90222**	**0.92038**	**0.92131**
33	**0.91517**	**0.91517**	**0.91517**	**0.91517**	**0.91554**	**0.91554**	**0.92670**	**0.93126**	**0.92403**	**0.93241**
34	**0.94435**	**0.94435**	**0.94435**	**0.94435**	**0.93388**	**0.93388**	**0.95721**	**0.96113**	**0.95012**	**0.95662**
35	**1.00422**	**0.99266**	**1.00422**	**0.99266**	**1.00461**	**1.00963**	**1.01848**	**1.02253**	**0.99422**	**0.99561**
36	**0.96122**	**0.96122**	**0.96122**	**0.96122**	**0.95057**	**0.95057**	**0.96507**	**0.96835**	**0.95568**	**0.95746**
37	**1.07953**	**1.06710**	**1.03556**	**1.03556**	**1.03597**	**1.03597**	**1.05250**	**1.05702**	**1.04808**	**1.05585**
38	**1.07546**	**1.06308**	**1.07546**	**1.06308**	**1.07588**	**1.08124**	**1.08819**	**1.09293**	**1.10280**	**1.10739**
39	**0.92575**	**0.92575**	**0.92575**	**0.92575**	**0.92612**	**0.92612**	**0.94591**	**0.94987**	**0.95071**	**0.95610**
Mean	**0.97069**	**0.96437**	**0.97109**	**0.96461**	**0.96464**	**0.96410**	**0.97417**	**0.97731**	**0.97434**	**0.97745**

Figure A7.9 Similarity-Linked, GEKS, and Fisher Price Indices

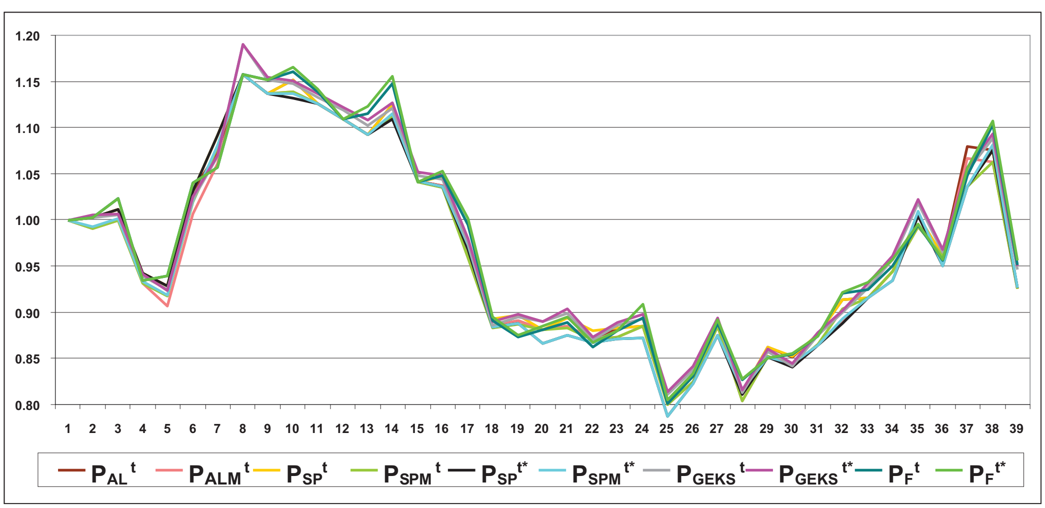
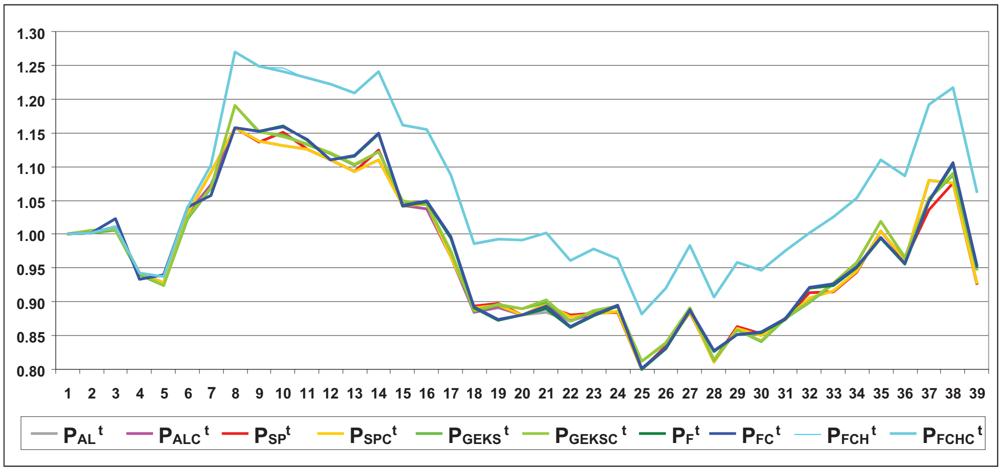


Table 1.8 Six Multilateral Indices and Four Fisher Indices Using Reservation Prices and Inflation-Adjusted Carry-Forward or Carry-Backward Prices

t	P_{AL}^{t}	P_{ALC}^{t}	P_{SP}^{t}	P_{SPC}^{t}	P_{GEKS}^{t}	P_{GEKSC}^{t}	P_{F}^{t}	P_{FC}^{t}	P_{FCH}^{t}	P_{FCHC}^{t}
1	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**	**1.00000**
2	**1.00218**	**1.00218**	**1.00218**	**1.00218**	**1.00233**	**1.00600**	**1.00218**	**1.00218**	**1.00218**	**1.00218**
3	**1.01124**	**1.01124**	**1.01124**	**1.01124**	**1.00575**	**1.00765**	**1.02342**	**1.02342**	**1.01124**	**1.01124**
4	**0.94262**	**0.94262**	**0.94262**	**0.94262**	**0.93922**	**0.94238**	**0.93388**	**0.93388**	**0.94265**	**0.94265**
5	**0.92812**	**0.92812**	**0.92812**	**0.92812**	**0.92448**	**0.92458**	**0.93964**	**0.93964**	**0.93715**	**0.93715**
6	**1.03073**	**1.03073**	**1.03073**	**1.03073**	**1.02249**	**1.02595**	**1.03989**	**1.03989**	**1.04075**	**1.04075**
7	**1.07314**	**1.07314**	**1.09146**	**1.09146**	**1.06833**	**1.06926**	**1.05662**	**1.05662**	**1.10208**	**1.10208**
8	**1.15740**	**1.15740**	**1.15740**	**1.15740**	**1.19023**	**1.19049**	**1.15740**	**1.15740**	**1.26987**	**1.26987**
9	**1.13680**	**1.13743**	**1.13680**	**1.13743**	**1.15115**	**1.15235**	**1.15164**	**1.15265**	**1.24727**	**1.24796**
10	**1.15156**	**1.13117**	**1.15156**	**1.13117**	**1.14730**	**1.14432**	**1.16081**	**1.15847**	**1.24528**	**1.24110**
11	**1.12574**	**1.12637**	**1.12574**	**1.12637**	**1.13270**	**1.13413**	**1.13876**	**1.14017**	**1.23034**	**1.23142**
12	**1.10951**	**1.11015**	**1.10951**	**1.11015**	**1.11903**	**1.12033**	**1.10951**	**1.11015**	**1.22091**	**1.22199**
13	**1.09229**	**1.09290**	**1.09229**	**1.09290**	**1.10247**	**1.10348**	**1.11511**	**1.11667**	**1.20813**	**1.20919**
14	**1.12489**	**1.10982**	**1.12489**	**1.10982**	**1.12136**	**1.12230**	**1.14803**	**1.14991**	**1.23948**	**1.24057**
15	**1.04237**	**1.04298**	**1.04086**	**1.04215**	**1.04827**	**1.04951**	**1.04086**	**1.04215**	**1.16056**	**1.16159**
16	**1.03692**	**1.03752**	**1.04704**	**1.04435**	**1.04385**	**1.04502**	**1.04836**	**1.04993**	**1.15449**	**1.15551**
17	**0.97013**	**0.96643**	**0.97013**	**0.96643**	**0.97470**	**0.97582**	**0.99410**	**0.99631**	**1.08642**	**1.08738**
18	**0.88455**	**0.88507**	**0.89319**	**0.89089**	**0.88586**	**0.88680**	**0.89105**	**0.89233**	**0.98452**	**0.98539**
19	**0.89118**	**0.89169**	**0.89702**	**0.89471**	**0.89497**	**0.89577**	**0.87308**	**0.87401**	**0.99189**	**0.99277**
20	**0.88051**	**0.88066**	**0.88051**	**0.88066**	**0.88973**	**0.88931**	**0.88051**	**0.88066**	**0.99193**	**0.99178**
21	**0.88482**	**0.89189**	**0.89346**	**0.89776**	**0.89904**	**0.90338**	**0.88920**	**0.89369**	**1.00150**	**1.00135**
22	**0.87151**	**0.87235**	**0.88001**	**0.87809**	**0.87061**	**0.87144**	**0.86217**	**0.86337**	**0.96068**	**0.96053**
23	**0.88280**	**0.88115**	**0.88280**	**0.88115**	**0.88592**	**0.88697**	**0.87981**	**0.88078**	**0.97902**	**0.97871**
24	**0.88502**	**0.88616**	**0.88502**	**0.88616**	**0.89282**	**0.89324**	**0.89357**	**0.89470**	**0.96347**	**0.96316**
25	**0.79966**	**0.80045**	**0.79966**	**0.80045**	**0.81132**	**0.81211**	**0.80050**	**0.80141**	**0.88172**	**0.88144**
26	**0.83378**	**0.83223**	**0.83378**	**0.83223**	**0.83799**	**0.83906**	**0.83026**	**0.83184**	**0.92100**	**0.92071**
27	**0.88481**	**0.88608**	**0.88481**	**0.88608**	**0.89063**	**0.89137**	**0.88749**	**0.88840**	**0.98344**	**0.98313**
28	**0.81336**	**0.81025**	**0.81336**	**0.81025**	**0.81304**	**0.81400**	**0.82665**	**0.82783**	**0.90739**	**0.90710**
29	**0.86271**	**0.86110**	**0.86271**	**0.86110**	**0.85763**	**0.85859**	**0.85086**	**0.85183**	**0.95839**	**0.95809**
30	**0.85166**	**0.85007**	**0.85230**	**0.85007**	**0.84103**	**0.84177**	**0.85383**	**0.85488**	**0.94612**	**0.94582**
31	**0.87568**	**0.87405**	**0.87568**	**0.87405**	**0.87495**	**0.87600**	**0.87411**	**0.87539**	**0.97557**	**0.97526**
32	**0.91368**	**0.90516**	**0.91398**	**0.90516**	**0.89936**	**0.89984**	**0.92038**	**0.92116**	**1.00192**	**1.00161**
33	**0.91517**	**0.91568**	**0.91517**	**0.91568**	**0.92670**	**0.92799**	**0.92403**	**0.92695**	**1.02632**	**1.02600**
34	**0.94435**	**0.94571**	**0.94435**	**0.94571**	**0.95721**	**0.95811**	**0.95012**	**0.95213**	**1.05412**	**1.05379**
35	**1.00422**	**1.00400**	**1.00422**	**1.00400**	**1.01848**	**1.01961**	**0.99422**	**0.99551**	**1.11015**	**1.10980**
36	**0.96122**	**0.96261**	**0.96122**	**0.96261**	**0.96507**	**0.96626**	**0.95568**	**0.95713**	**1.08720**	**1.08686**
37	**1.07953**	**1.07929**	**1.03556**	**1.07929**	**1.05250**	**1.05337**	**1.04808**	**1.04986**	**1.19204**	**1.19167**
38	**1.07546**	**1.07522**	**1.07546**	**1.07522**	**1.08819**	**1.08970**	**1.10280**	**1.10574**	**1.21774**	**1.21735**
39	**0.92575**	**0.92626**	**0.92575**	**0.92626**	**0.94591**	**0.94704**	**0.95071**	**0.95246**	**1.06105**	**1.06071**
Mean	**0.97069**	**0.96968**	**0.97109**	**0.97082**	**0.97417**	**0.97526**	**0.97434**	**0.97542**	**1.0589**	**1.0589**

Figure A7.10 Indices Using Reservation Prices and Carry-Forward Prices



A.7.6 Multilateral and Fisher Indices Using Reservation Prices versus Carry-Forward Prices

Finally, we compare $P_{AL}^{\ t}$ (asymptotic linear), $P_{SP}^{\ t}$ (predicted share), $P_{GEKS}^{\ t}$ (GEKS), P_{F}^{t} (fixed-base Fisher), and $P_{FCH}^{\ t}$ (chained Fisher) using reservation prices, with their counterparts using inflation-adjusted carry-forward or carry-backward prices, $P_{ALC}^{\ t}$, $P_{SPC}^{\ t}$, $P_{GEKSC}^{\ t}$, $P_{FC}^{\ t}$, and $P_{FCHC}^{\ t}$, in Table 1.8. The ten indices are plotted on Chart 1.10.

Basically, each index that uses reservation prices is close to its counterpart index that uses inflation-adjusted carry-forward or carry-backward prices. This is to be expected since there are only 20 missing product prices out of a sample of $19 \cdot 39 = 741$ price and quantity observations.

The two chained Fisher indices, $P_{FCH}^{\ t}$ (uses reservation prices) and $P_{FCHC}^{\ t}$ (uses inflation-adjusted carry-forward or carry-backward prices), cannot be distinguished from each other in Figure A7.10. These indices are subject to substantial upward chain drift. The remaining indices (which are not subject to chain drift) are quite close to each other.

A7. Conclusion

Conceptually, the price and quantity similarity-linked indices $P_{SPQ}^{\ t}$ based on the combined price and quantity dissimilarity measure $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ seem to be the most attractive solution for solving the chain drift problem.[233] In practice, $\Delta_{SPQ}(p^r,p^t,q^r,q^t)$ will typically equal the predicted share price dissimilarity measure $\Delta_{SP}(p^r,p^t,q^r,q^t)$ so that $P_{SPQ}^{\ t}$ will typically equal $P_{SP}^{\ t}$. The indices $P_{SPQ}^{\ t}$ and $P_{SP}^{\ t}$ can be implemented using either reservation prices or some form of carry-forward prices, or if the statistical agency does not want to use explicit imputations for missing product prices, these indices can be calculated without using imputations.

[233] They can deal with seasonal products more adequately than the other indices that are considered in this paper. They also satisfy the strong identity test (and thus are not subject to chain drift) as well as the fixed basket test.

References

Aizcorbe, Ana, Carol Corrado, and Mark Doms. 2000. *Constructing Price and Quantity Indexes for High Technology Goods*. Washington, DC: Industrial Output Section, Division of Research and Statistics, Board of Governors of the Federal Reserve System.

Allen, Robert C., and W. Erwin Diewert. 1981. "Direct versus Implicit Superlative Index Number Formulae." *Review of Economics and Statistics* 63: 430–35.

Alterman, William F., W. Erwin Diewert, and Robert C. Feenstra. 1999. *International Trade Price Indexes and Seasonal Commodities*. Washington, DC: Bureau of Labor Statistics.

Armknecht, Paul, and Mick Silver. 2014. "Post-Laspeyres: The Case for a New Formula for Compiling Consumer Price Indexes." *Review of Income and Wealth* 60 (2): 225–44.

Arrow, Kenneth J., Hollis B. Chenery, Bagicha S. Minhas, and Robert M. Solow. 1961. "Capital-Labor Substitution and Economic Efficiency." *Review of Economics and Statistics* 63: 225–50.

Aten, Bettina, and Alan Heston. 2009. "Chaining Methods for International Real Product and Purchasing Power Comparisons: Issues and Alternatives." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao, 245–73. Cheltenham, UK: Edward Elgar.

Australian Bureau of Statistics. 2016. "Making Greater Use of Transactions Data to Compile the Consumer Price Index." Information Paper 6401.0.60.003, ABS, Canberra, November 29.

Balk, Bert M. 1980. "A Method for Constructing Price Indices for Seasonal Commodities." *Journal of the Royal Statistical Society, Series A* 143: 68–75.

———. 1981. "A Simple Method for Constructing Price Indices for Seasonal Commodities." *Statistische Hefte* 22 (1): 1–8.

———. 1996. "A Comparison of Ten Methods for Multilateral International Price and Volume Comparisons." *Journal of Official Statistics* 12: 199–222.

———. 2008. *Price and Quantity Index Numbers*. New York: Cambridge University Press.

Bortkiewicz, L. von. 1923. "Zweck und Struktur einer Preisindexzahl." *Nordisk Statistisk Tidsskrift* 2: 369–408.

Boskin, Michael J., Ellen R. Dulberger, Robert J. Gordon, Zvi Griliches, and Dale Jorgenson. 1996. *Toward a More Accurate Measure of the Cost of Living*. Final Report to the U.S. Senate Finance Committee. Washington, DC: US Government Printing Office.

Carli, Gian-Rinaldo. 1804. "Del valore e della proporzione de' metalli monetati." In *Scrittori classici italiani di economia* *politica*, vol. 13, 297-366. Milano: G.G. Destefanis (originally published in 1764).

Carruthers, A.G., D.J. Sellwood, and P.W. Ward. 1980. "Recent Developments in the Retail Prices Index." *The Statistician* 29: 1–32.

Caves, Douglas W., Laurits R. Christensen, and W. Erwin Diewert. 1982. "Multilateral Comparisons of Output, Input, and Productivity using Superlative Index Numbers." *Economic Journal* 92: 73–86.

Chessa, Antonio G. 2016. "A New Methodology for Processing Scanner Data in the Dutch CPI." *Eurona* 2016 (1): 49–69.

Cobb, Charles W., and Paul H. Douglas. 1928. "A Theory of Production." *American Economic Review* 18 (1): 139–65.

Court, Andrew T. 1939. "Hedonic Price Indexes with Automotive Examples." In The *Dynamics of Automobile Demand*, 99–117. New York: General Motors Corporation.

Dalén, Jörgen. 1992. "Computing Elementary Aggregates in the Swedish Consumer Price Index." *Journal of Official Statistics* 8: 129–47.

———. 2001. "Statistical Targets for Price Indexes in Dynamic Universes." Paper presented at the Sixth Meeting of the Ottawa Group, Canberra, April 2–6.

———. 2017. "Unit Values and Aggregation in Scanner Data—Towards a Best Practice." Paper presented at the 15th Meeting of the Ottawa Group, Eltville am Rhein, Germany, May 10–12.

Davies, George R. 1924. "The Problem of a Standard Index Number Formula." *Journal of the American Statistical Association* 19: 180–88.

———. 1932. "Index Numbers in Mathematical Economics." *Journal of the American Statistical Association* 27: 58–64.

de Haan, Jan. 2004a. "The Time Dummy Index as a Special Case of the Imputation Törnqvist Index." Paper presented at The Eighth Meeting of the International Working Group on Price Indices (the Ottawa Group), Helsinki, Finland.

———. 2004b. "Estimating Quality-Adjusted Unit Value Indices: Evidence from Scanner Data." Paper presented at the Seventh EMG Workshop, Sydney, Australia, December 12–14.

———. 2004c. "Estimating Quality-Adjusted Unit Value Indexes: Evidence from Scanner Data." Paper presented at the SSHRC International Conference on Index Number Theory and the Measurement of Prices and Productivity, Vancouver, Canada, June 30–July 3.

———. 2008. "Reducing Drift in Chained Superlative Price Indexes for Highly Disaggregated Data." Paper presented at the Economic Measurement Workshop, Centre for Applied Economic Research, University of New South Wales, December 10.

———. 2010. "Hedonic Price Indexes: A Comparison of Imputation, Time Dummy and Re-pricing Methods." *Jahrbücher für Nationökonomie und Statistik* 230: 772–91.

———. 2015. "Rolling Year Time Dummy Indexes and the Choice of Splicing Method." Room Document at the 14th meeting of the Ottawa Group, Tokyo, May 22, http://www.stat.go.jp/english/info/meetings/og2015/pdf/t1s3room.

de Haan, Jan, and Frances Krsinich. 2014. "Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes." *Journal of Business and Economic Statistics* 32: 341–58.

———. 2018. "Time Dummy Hedonic and Quality-Adjusted Unit Value Indexes: Do They Really Differ?." *Review of Income and Wealth* 64 (4): 757–76.

de Haan, Jan, and Heymerik A. van der Grient. 2011. "Eliminating Chain Drift in Price Indexes Based on Scanner Data." *Journal of Econometrics* 161: 36–46.

Diewert, W. Erwin. 1976. "Exact and Superlative Index Numbers." *Journal of Econometrics* 4: 114–45.

———. 1978. "Superlative Index Numbers and Consistency in Aggregation." *Econometrica* 46: 883–900.

———. 1988. "Test Approaches to International Comparisons." In *Measurement in Economics: Theory and Applications of Economic Indices*, edited by Wolfgang Eichhorn, 67–86. Heidelberg: Physica-Verlag.

———. 1992. "Fisher Ideal Output, Input and Productivity Indexes Revisited." *Journal of Productivity Analysis* 3: 211–48.

———. 1995. "Axiomatic and Economic Approaches to Elementary Price Indexes." Discussion Paper No. 95–01, Department of Economics, University of British Columbia, Vancouver, Canada.

———. 1998. "Index Number Issues in the Consumer Price Index." *Journal of Economic Perspectives* 12 (1): 47–58.

———. 1999a. "Index Number Approaches to Seasonal Adjustment." *Macroeconomic Dynamics* 3: 1–21.

———. 1999b. "Axiomatic and Economic Approaches to International Comparisons." In *International and Interarea Comparisons of Income, Output and Prices*, edited by Alan Heston and Robert E. Lipsey, Studies in Income and Wealth, vol. 61, 13–87. Chicago: The University of Chicago Press.

———. 2002. "Weighted Country Product Dummy variable Regressions and Index Number Formulae." Department of Economics, Discussion Paper 02–15, University of British Columbia, Vancouver, Canada.

———. 2003a. "Hedonic Regressions: A Consumer Theory Approach." In *Scanner Data and Price Indexes*, edited by Robert C. Feenstra and Matthew D. Shapiro, Studies in Income and Wealth, vol. 61, 317–48. Chicago: University of Chicago Press.

———. 2003b. "Hedonic Regressions: A Review of Some Unresolved Issues." Paper presented at the Seventh Meeting of the Ottawa Group, Paris, May 27–29.

———. 2004. "On the Stochastic Approach to Linking the Regions in the ICP." Discussion Paper no. 04–16, Department of Economics, The University of British Columbia, Vancouver, Canada.

———. 2005a. "Weighted Country Product Dummy variable Regressions and Index Number Formulae." *Review of Income and Wealth* 51: 561–70.

———. 2005b. "Adjacent Period Dummy variable Hedonic Regressions and Bilateral Index Number Theory." *Annales D'Économie et de Statistique* 79–80: 759–86.

———. 2009. "Similarity Indexes and Criteria for Spatial Linking." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao, 183–216. Cheltenham, UK: Edward Elgar.

———. 2012. *Consumer Price Statistics in the UK*. Government Buildings, Cardiff Road, Newport, UK, NP10 8XG: Office for National Statistics, http://www.ons.gov.uk/ons/guide-method/userguidance/prices/cpi-and-rpi/index.html.

———. 2013. "Methods of Aggregation above the Basic Heading Level within Regions." In *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, 121–67. Washington, DC: The World Bank.

———. 2014. "An Empirical Illustration of Index Construction using Israeli Data on Vegetables." Discussion Paper 14–04, School of Economics, The University of British Columbia, Vancouver, Canada.

———. 2018. "Scanner Data, Elementary Price Indexes and the Chain Drift Problem." Discussion Paper 18–06, Vancouver School of Economics, University of British Columbia, Vancouver, Canada.

———. 2022a. "The Economic Approach to Index Number Theory." In *Consumer Price Index Theory*, Chapter 5. Washington, DC: International Monetary Fund, https://www.imf.org/en/Data/Statistics/cpi-manual.

———. 2022b. "Elementary Indexes." In *Consumer Price Index Theory*, Chapter 6. Washington, DC: International Monetary Fund, https://www.imf.org/en/Data/Statistics/cpi-manual.

———. 2022c. "Quality Adjustment Methods." In *Consumer Price Index Theory*, Chapter 8. Washington, DC: International

Monetary Fund, https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W. Erwin, and Robert C. Feenstra. 2017. "Estimating the Benefits and Costs of New and Disappearing Products." Discussion Paper 17–10, Vancouver School of Economics, University of British Columbia, Vancouver, Canada.

———. 2022. "Estimating the Benefits of New Products." In *Big Data for Twenty-First-Century Economic Statistics*, edited by Katharine G. Abraham, Ron S. Jarmin, Brian C. Moyer, and Matthew D. Shapiro, 437–73. Chicago: University of Chicago Press.

Diewert, W. Erwin, Yoel Finkel, and Yevgeny Artsev. 2009. "Empirical Evidence on the Treatment of Seasonal Products: The Israeli Experience." In *Price and Productivity Measurement: Volume 2: Seasonality*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 53–78. Victoria: Trafford Press.

Diewert, W. Erwin, and Kevin J. Fox. 2021. "Substitution Bias in Multilateral Methods for CPI Construction Using Scanner Data." *Journal of Business and Economic Statistics* 40 (1): 355–69.

Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2017. "The Digital Economy, New Products and Consumer Welfare." Discussion Paper 17–09, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada.

Diewert, W. Erwin, Marco Huwiler, and Ulrich Kohli. 2009. "Retrospective Price Indices and Substitution Bias." *Swiss Journal of Economics and Statistics* 145 (20): 127–35.

Diewert, W. Erwin, and Peter von der Lippe. 2010. "Notes on Unit Value Index Bias." *Journal of Economics and Statistics* 230: 690–708.

Diewert, W. Erwin, and Chihiro Shimizu. 2022. "The Treatment of Durable Goods and Housing." In *Consumer Price Index Theory*, Chapter 10. Washington, DC: International Monetary Fund, https://www.imf.org/en/Data/Statistics/cpi-manual.

Drobisch, Moritz W. 1871. "Über die Berechnung der Veränderung der Waarenpreis und des Geldwertes." *Jahrbücher für Nationalökonomie und Statistik* 16: 416–27.

Dutot, Charles. 1738. *Réflexions politiques sur les finances et le commerce*, vol. 1. La Haye: Les frères Vaillant et N. Prevost.

Eltetö, Ödon., and Pal Köves. 1964. "On a Problem of Index Number Computation Relating to International Comparisons." (in Hungarian), *Statisztikai Szemle* 42: 507–18.

Feenstra, Robert C. 1994. "New Product varieties and the Measurement of International Prices." *American Economic Review* 84: 157–77.

Feenstra, Robert C., and Matthew D. Shapiro. 2003. "High-Frequency Substitution and the Measurement of Price Indexes." In *Scanner Data and Price Indexes*, edited by Robert C. Feenstra and Matthew D. Shapiro, Studies in Income and Wealth, vol. 64, 123–46. Chicago: The University of Chicago Press.

Fisher, Irving. 1911. *The Purchasing Power of Money*. London: Macmillan.

———. 1922. *The Making of Index Numbers*. Boston: Houghton-Mifflin.

Frisch, Ragnar. 1936. "Annual Survey of General Economic Theory: The Problem of Index Numbers." *Econometrica* 4: 1–39.

Geary, Roy C. 1958. "A Note on Comparisons of Exchange Rates and Purchasing Power between Countries." *Journal of the Royal Statistical Society Series A* 121: 97–99.

Gini, Corrado. 1931. "On the Circular Test of Index Numbers." *Metron* 9 (9): 3–24.

Gorajek, Adam. 2018. "Econometric Perspectives on Economic Measurement." Research Discussion Paper 2018–08, Reserve Bank of Australia, 65 Martin Pl, Sydney NSW 2000.

Griliches, Zvi. 1971. "Introduction: Hedonic Price Indexes Revisited." In *Price Indexes and Quality Change*, edited by Zvi Griliches, 3–15. Cambridge, MA: Harvard University Press.

Handbury, Jessie, Tsutomu TWatanabe, and David E. Weinstein. 2013. "How Much Do Official Price Indexes Tell Us about Inflation." NBER Working Paper 19504, National Bureau of Economic Research, Cambridge, MA.

Hardy, Godfrey Harold, John Edensor Littlewood, and György Pólya. 1934. *Inequalities*. Cambridge: Cambridge University Press.

Hicks, John R. 1940. "The Valuation of the Social Income." *Economica* 7: 105–40.

Hill, Robert J. 1997. "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities." *Review of Income and Wealth* 43 (1): 49–69.

———. 1999a. "Comparing Price Levels across Countries Using Minimum Spanning Trees." *The Review of Economics and Statistics* 81: 135–42.

———. 1999b. "International Comparisons using Spanning Trees." In *International and Interarea Comparisons of Income, Output and Prices*, edited by Alan Heston and Robert E. Lipsey, Studies in Income and Wealth, vol. 61, 109–20. NBER. Chicago: The University of Chicago Press.

———. 2001. "Measuring Inflation and Growth Using Spanning Trees." *International Economic Review* 42: 167–85.

———. 2004. "Constructing Price Indexes Across Space and Time: The Case of the European Union." *American Economic Review* 94: 1379–410.

———. 2009. "Comparing Per Capita Income Levels Across Countries Using Spanning Trees: Robustness, Prior Restrictions, Hybrids and Hierarchies." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao, 17–244. Cheltenham, UK: Edward Elgar.

Hill, Robert J., D.S. Prasada Rao, Sriram Shankar, and Reza Hajargasht. 2017. "Spatial Chaining as a Way of Improving International Comparisons of Prices and Real Incomes." Paper presented at the Meeting on the International Comparisons of Income, Prices and Production, Princeton University, May 25–26.

Hill, Robert J., and Marcel P. Timmer. 2006. "Standard Errors as Weights in Multilateral Price Indexes." *Journal of Business and Economic Statistics* 24 (3): 366–77.

Hill, Peter. 1988. "Recent Developments in Index Number Theory and Practice." *OECD Economic Studies* 10: 123–48.

———T. 1993. "Price and Volume Measures." In *System of National Accounts 1993*, 379–406. Luxembourg, Washington, DC, Paris, New York and Washington, DC: Eurostat, IMF, OECD, UN and World Bank.

Huang, Ning, Waruna Wimalaratne, and Brent Pollard. 2015. "Choice of Index Number Formula and the Upper Level Substitution Bias in the Canadian CPI." Paper presented at the 14th Ottawa Group Meeting, Tokyo, Japan, May 20–22.

ILO/IMF/OECD/UNECE/Eurostat/The World Bank. 2004. *Consumer Price Index Manual: Theory and Practice*, edited by Peter Hill. Geneva: International Labour Office.

Inklaar, Robert, and W. Erwin Diewert. 2016. "Measuring Industry Productivity and Cross-Country Convergence." *Journal of Econometrics* 191: 426–33.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2009. "Scanner Data, Time Aggregation and the Construction of Price Indexes." Discussion Paper 09–09, Department of Economics, University of British Columbia, Vancouver, Canada.

———. 2010. "Using a Constant Elasticity of Substitution Index to estimate a Cost of Living Index: from Theory to Practice." Australian School of Business Research Paper No. 2010 ECON 15, University of New South Wales, Sydney 2052 Australia.

———. 2011. "Scanner Data, Time Aggregation and the Construction of Price Indexes." *Journal of Econometrics* 161: 24–35.

Jevons, William Stanley. 1865. "The variation of Prices and the Value of the Currency since 1782." *Journal of the Statistical Society of London* 28: 294–320.

Keynes, John M. 1909. "The Method of Index Numbers with Special Reference to the Measurement of General Exchange Value." Reprinted as in *The Collected Writings of John Maynard Keynes* (1983), edited by Don Moggridge, vol. 11, 49–156. Cambridge: Cambridge University Press.

———. 1930. *Treatise on Money*, vol. 1. London: Macmillan.

Khamis, Salem H. 1970. "Properties and Conditions for the Existence of a New Type of Index Number." *Sankhya B* 32: 81–98.

———. 1972. "A New System of Index Numbers for National and International Purposes." *Journal of the Royal Statistical Society Series A* 135: 96–121.

Konüs, Alexander A. 1924. "The Problem of the True Index of the Cost of Living." translated in *Econometrica* 7 (1939): 10–29.

Konüs, Alexander A., and Sergei S. Byushgens. 1926. "K probleme pokupatelnoi cili deneg." *Voprosi Konyunkturi* 2: 151–72.

Kravis, Irving B., Alan Heston, and Robert Summers. 1982. *World Product and Income: International Comparisons of Real Gross Product*. Statistical Office of the United Nations and the World Bank. Baltimore: The Johns Hopkins University Press.

Krsinich, Frances. 2016. "The FEWS Index: Fixed Effects with a Window Splice." *Journal of Official Statistics* 32: 375–404.

Laspeyres, Etienne. 1871. "Die Berechnung einer mittleren Waarenpreissteigerung." *Jahrbücher für Nationalökonomie und Statistik* 16: 296–314.

Leontief, Wassily. 1936. "Composite Commodities and the Problem of Index Numbers." *Econometrica* 4: 39–59.

Lowe, Esq Joseph. 1823. *The Present State of England in Regard to Agriculture, Trade and Finance*, 2nd ed. London: Longman, Hurst, Rees, Orme and Brown.

Marris, Robin. 1984. "Comparing the Incomes of Nations: A Critique of the International Comparison Project." *Journal of Economic Literature* 22 (1): 40–57.

Muellbauer, John. 1974. "Household Production Theory, Quality and the Hedonic Technique." *American Economic Review* 64 (6): 977–94.

Nordhaus, William D. 1997. "Do Real Output and Real Wage Measures Capture Reality? The History of Lighting Suggests Not." In *The Economics of New Goods*, edited by Timothy F. Bresnahan and Robert J. Gordon, 29–66. Chicago: University of Chicago Press.

Office for National Statistics (ONS). 2020. *New Index Number Methods in Consumer Price Statistics*. Newport: Office for National Statistics.

Paasche, Hermann 1874. "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen." *Jahrbücher für Nationalökonomie und Statistik* 12: 168–78.

Pakes, Ariel. 2001. "A Reconsideration of Hedonic Price Indices with and Application to PCs." NBER Working Paper 8715, National Bureau of Economic Research, Cambridge, MA.

Persons, Warren M. 1921. "Fisher's Formula for Index Numbers." *Review of Economics and Statistics* 3 (5): 103–13.

———. 1928. "The Effect of Correlation between Weights and Relatives in the Construction of Index Numbers." *The Review of Economics and Statistics* 10 (2): 80–107.

Rao, D.S. Prasada. 1995. "On the Equivalence of the Generalized Country-Product-Dummy (CPD) Method and the Rao-System for Multilateral Comparisons." Working Paper No. 5, Centre for International Comparisons, University of Pennsylvania, Philadelphia.

———. 2004. "The Country-Product-Dummy Method: A Stochastic Approach to the Computation of Purchasing Power parities in the ICP." Paper presented at the SSHRC Conference on Index Numbers and Productivity Measurement, Vancouver, Canada, June 30–July 3.

———. 2005. "On the Equivalence of the Weighted Country Product Dummy (CPD) Method and the Rao System for Multilateral Price Comparisons." *Review of Income and Wealth* 51 (4): 571–80.

Rao, D.S. Prasada, and Gholamreza Hajargasht. 2016. "Stochastic Approach to Computation of Purchasing Power Parities in the International Comparison Program." *Journal of Econometrics* 191 (2): 414–25.

Rao, D.S. Prasada, and Marcel P. Timmer. 2003. "Purchasing Power Parities for Industry Comparisons Using Weighted Eltetö-Köves-Szulc (EKS) Methods." *Review of Income and Wealth* 49: 491–511.

Reinsdorf, Marshall B. 2007. "Axiomatic Price Index Theory." In *Measurement in Economics: A Handbook*, edited by Marcel Boumans, 153–88. Amsterdam: Elsevier.

Rosen, Sherwin. 1974. "Hedonic Prices and Implicit Markets: Product Differentiation in Pure Competition." *Journal of Political Economy* 82: 34–55.

Schlömilch, Oskar. 1858. "Über Mïttelgrössen verschiedener Ordnungen." *Zeitschrift für Mathematik und Physik* 3: 308–10.

Sergeev, Sergey. 2001. "Measures of the Similarity of the Country's Price Structures and their Practical Application." Conference on the European Comparison Program, U.N. Statistical Commission. Economic Commission for Europe, Geneva, November 12–14.

———. 2009. "Aggregation Methods Based on Structural International Prices." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao, 274–97. Cheltenham, UK: Edward Elgar.

Shapiro, Matthew D., and David W. Wilcox. 1997. "Alternative Strategies for Aggregating Prices in the CPI." *Federal Reserve Bank of St. Louis Review* 79 (3): 113–25.

Silver, Mick. 2010. "The Wrongs and Rights of Unit Value Indices." *Review of Income and Wealth* 56: S206–23.

———. 2011. "An Index Number Formula Problem: the Aggregation of Broadly Comparable Items." *Journal of Official Statistics* 27 (4): 1–17.

Silver, Mick, and Saeed Heravi. 2005. "A Failure in the Measurement of Inflation: Results from a Hedonic and Matched Experiment using Scanner Data." *Journal of Business and Economic Statistics* 23: 269–81.

———. 2007. "Why Elementary Price Index Number Formulas Differ: Evidence on Price Dispersion." *Journal of Econometrics* 140: 874–83.

Summers, Robert. 1973. "International Comparisons with Incomplete Data." *Review of Income and Wealth* 29 (1): 1–16.

Szulc, Bohdan J. 1964. "Indices for Multiregional Comparisons." (in Polish), *Przeglad Statystyczny* 3: 239–54.

———. 1983. "Linking Price Index Numbers." In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette, 537–66. Ottawa: Statistics Canada.

———. 1987. "Price Indices Below the Basic Aggregation Level." *Bulletin of Labour Statistics* 2: 9–16.

Theil, Henri. 1967. *Economics and Information Theory*. Amsterdam: North-Holland Publishing.

Törnqvist, Leo. 1936. "The Bank of Finland's Consumption Price Index." *Bank of Finland Monthly Bulletin* 10: 1–8.

Törnqvist, Leo, and Erik Törnqvist. 1937. "Vilket är förhällandet mellan finska markens och svenska kronans köpkraft?." *Ekonomiska Samfundets Tidskrift* 39: 1–39 reprinted as in *Collected Scientific Papers of Leo Törnqvist*, 121–60. Helsinki: The Research Institute of the Finnish Economy, 1981.

Triplett, Jack E. 1987. "Hedonic Functions and Hedonic Indexes." In *The New Palgrave: A Dictionary of Economics*, edited by John Eatwell, Murray Milgate, and Peter Newman, vol. 2, 630–34. New York, NY: Stockton Press.

———. 2004. *Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes*. Directorate for Science, Technology and Industry, DSTI/DOC(2004)9. Paris: OECD.

Triplett, Jack E., and Richard J. McDonald. 1977. "Assessing the Quality Error in Output Measures: The Case of Refrigerators." *The Review of Income and Wealth* 23 (2): 137–56.

University of Chicago. 2013. *Dominick*'s *Data Manual*. Chicago: Kilts Center for Marketing, Booth School of Business.

vartia, Yrjö O. 1978. "Fisher's Five-Tined Fork and Other Quantum Theories of Index Numbers." In *Theory and Applications of Economic Indices*, edited by Wolfgang Eichhorn, Ralph Henn, Otto Opitz, and Ronald W. Shephard, 271–95. Wurzburg: Physica-Verlag.

vartia, Yrjö O., and Antti Suoperä. 2018. "Contingently Biased, Permanently Biased and Excellent Index Numbers for Complete Micro Data." Unpublished paper, http://www.stat.fi/static/media/uploads/meta_en/menetelmakehitystyo/contingently_biased_vartia_suopera_updated.pdf.

von Auer, Ludwig. 2014. "The Generalized Unit Value Index Family." *Review of Income and Wealth* 60: 843–61.

———. 2019. "The Nature of Chain Drift: Implications for Scanner Data Price Indices." Paper presented at the 16th Meeting of the Ottawa Group, Rio de Janeiro, Brazil, May 8.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

———. 1921a. *The Problem of Estimation*. London: P.S. King & Son.

———. 1921b. "Discussion." *Journal of the American Statistical Association* 17: 537–44.

Whittaker, Edmund TaylorT, and George Robinson. 1940. *The Calculus of Observations*, 3rd ed. London: Blackie & Sons.

Young, Arthur. 1812. *An Inquiry into the Progressive Value of Money as Marked by the Price of Agricultural Products*. London: Macmillan.

Zhang, Li-Chun, Ingvild Johansen, and Ragnhild Nygaard. 2019. "Tests for Price Indices in a Dynamic Item Universe." *Journal of Official Statistics* 35 (3): 683–97.

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QUALITY ADJUSTMENT METHODS* 8

1. Introduction

This chapter will attempt to place most methods used by statistical agencies to quality adjust prices into a common economic framework. The economic framework is based on purchasers maximizing a linearly homogeneous utility function subject to a budget constraint on their purchases of a group of related products. This framework is far from a perfect description of reality, but it captures an important empirical phenomenon: When the price of a product drops a lot, purchasers of the product buy more of it! Moreover, the theory allows us to provide a welfare interpretation for the quantity indices that are generated by this approach. The very concept of comparing the relative quality of two related products means that we are comparing the relative usefulness or *utility* of the products to the purchaser. Thus, it seems to be necessary to take an economic approach to the problem of quality adjustment.

The theory of quality adjustment to be presented in this chapter is meant to be applied at the level where subindices are constructed at the first stage of aggregation; that is, at what is called the *elementary level of aggregation* by price statisticians. Furthermore, the methods for quality adjustment to be discussed in this chapter are largely aimed at the *scanner data context*; that is, we will assume that the statistical agency has access to detailed price and quantity (or value) information at the product code level, either from retail outlets or from the detailed purchases of a group of similar households.[1] Thus, our focus will be on both the construction of CPIs at the elementary level and the companion consumer quantity indices.

The assumption of linearly homogeneous utility or valuation functions is an important restriction, so one may ask: Why impose it? The reason is that economic models constructed by private and public sector economists generally do not make use of disaggregated information; instead, they use the elementary indices that are produced by national statistical agencies in their models. However, the price levels that correspond to these elementary indices are treated as "normal" prices by applied economists; that is, the elementary prices are not regarded as prices that are conditional on particular levels of the corresponding quantity levels. In order to construct unconditional price levels, we need to assume that the underlying aggregator or utility functions are linearly homogeneous.[2]

Marshall (1887) was one of the first to introduce *the new goods problem*: How exactly should price indices be adjusted to account for the introduction of new and hopefully improved products? [3] Marshall suggested that chaining period-to-period indices would provide a partial solution to the problem. Keynes (1909) endorsed Marshall's suggestion as a step in the right direction but noted that chaining alone will not solve the fundamental problem: Increased product choice will generally increase the utility of purchasers of products, but it is very difficult to measure this increase.[4] This is the essence of the *quality adjustment problem*; how can statistical agencies construct price and quantity indices over two or more periods when there are new and disappearing products?

Hicks (1940, 114) suggested a general approach to this measurement problem in the context of the economic approach to index number theory. His approach was to apply normal index number theory but estimate (somehow) hypothetical prices that would induce utility-maximizing purchasers of a related group of products to demand 0 units of unavailable products.[5] With these *reservation* or *imputed prices* in

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[1] As cash transactions diminish in importance, credit and debit card companies will have detailed price and quantity information on household purchases. Once this information on consumer transactions also includes product bar codes, statistical agencies will eventually be able to access this information and use it to produce high-quality CPIs .

[2] The underlying index number theory using linearly homogeneous aggregator functions was developed by Shephard (1953), Samuelson and Swamy (1974), and Diewert (1976). This theory was explained in Chapter 5 and will be summarized in Section 2.

[3] "This brings us to consider the great problem of how to modify our unit so as to allow for the invention of new commodities. The difficulty is insuperable, if we compare two distant periods without access to the detailed statistics of intermediate times, but it can be got over fairly well by systematic statistics" (Alfred Marshall, 1887, 373). Lehr (1885, 45–46) also introduced the chain system as a way of mitigating the new goods problem. For more on the early history of the new goods problem, see Diewert (1993, 59–63).

[4] "The [chaining] method has another advantage. It enables us to introduce new commodities and to drop others which have fallen out of use. . . . For most practical purposes, therefore, this is the method to be recommended. . . . Yet we must not exaggerate its merits" (John M. Keynes, 1909, 80). "We cannot hope to find a ratio of equivalent substitution for gladiators against cinemas, or for the conveniences of being able to buy motor cars against the conveniences of being able to buy slaves" (John M. Keynes, 1930, 96).

[5] "The same kind of device can be used in another difficult case, that in which new sorts of goods are introduced in the interval between the two situations we are comparing. If certain goods are available in the II situation which were not available in the I situation, the p_1's corresponding to these goods become indeterminate. The p_2's and q_2's are given by the data and the q_1's are zero. Nevertheless, although the p_1's cannot be determined from the data, since the goods are not sold in the I situation, it is apparent from the preceding argument what p_1's ought to be introduced in order to make the index-number tests hold. They are those prices which, in the I situation, would *just* make the demands for these commodities (from the whole community) equal to zero" (John *R*. Hicks, 1940, 114). Von Hofsten (1952, 95–97) extended Hicks' methodology to cover the case of disappearing goods as well.

hand, one can just apply normal index number theory using the augmented price data and the observed quantity data (which impute zero quantities to unavailable products). This is the economic framework we will use in this chapter.[6] The practical problem facing statistical agencies is: *How exactly are these reservation prices to be estimated*?

The approach to the estimation of reservation prices that will be taken here is to use consumer demand theory to estimate preferences. Suppose that purchasers maximize a utility function $f(q)$ subject to the budget constraint $p{\cdot}q \equiv \Sigma_{n=1}^{N} p_nq_n = v > 0$, where the price and quantity of commodity n are p_n and q_n for $n = 1, \ldots, N$. Define the price and quantity vectors $p \equiv [p_1, \ldots, p_N]$ and $q \equiv [q_1, \ldots, q_N]$. Suppose that p, q, and v are observed, and q is a solution to the utility maximization problem $\max_q \{f(q) : p{\cdot}q = v\}$. Then, given a functional form for f, the solution q to the utility maximization problem will satisfy the usual consumer demand functions, $q_n = d_n(p,v)$ for $n = 1, \ldots, N$, where $d_n(p,v)$ is the nth consumer demand function. Given price and quantity for many periods, the unknown parameters for the utility function that are imbedded in these consumer demand functions can be estimated using econometric methods. Duality theory can be used to simplify the derivation of the consumer demand functions.[7] This is the approach used by Hausman (1981, 1996, 1999, 2003) to estimate reservation prices. However, the econometrics of this method are complex. To illustrate these problems, suppose that in the first sample period, product 1 was not available. The observed demand for product 1 in period 1 is 0. Thus, the first estimating equation in the sample would take the form $0 = d_1(p_1^{1*}, p_2^1, \ldots, p_N^1, v^1) + e_1^1$, where $d_1(p,v)$ is the demand function for commodity 1, $p_2^1, \ldots, p_N^1$ are the observed prices for products 2,3, . . .,N in period 1, v^1 is the observed period 1 expenditure on the N products, e_1^1 is an error term, and p_1^{1*} is the unknown period 1 reservation price for product 1. It can be seen that p_1^{1*} is now an extra parameter that must be estimated. Hence, the usual approach that conditions on prices (on the right-hand sides of the estimating equations) and treats quantities as random variables on the left-hand sides of the estimating equations does not apply due to the endogeneity of the reservation price. Moreover, the variable on the left-hand side of the preceding equation is 0, and this is not a random variable. Thus, simple econometric techniques cannot be used in this situation.

To deal with the preceding econometric problem, one can abandon the estimation of traditional consumer demand functions and switch to the estimation of the system of *inverse consumer demand functions*. The nth inverse demand function gives the observed price for product n, p_n, as a function of the vector of quantities chosen by the purchasers, q, and total expenditure on the products v; that is, we have $p_n = g_n(q,v)$ for $n = 1, \ldots, N$, where g_n is the nth inverse demand function.[8] Again suppose product 1 was not available in period 1. Then, the first inverse demand function in period 1 becomes $p_1^{1*} = g_1(0, q_2^1, \ldots, q_N^1, v^1) + e_1^1$ using the notation in the previous paragraph. Thus, we simply drop this equation from the system of inverse demand estimating equations and use the remaining equations to estimate the unknown parameters in the direct utility function $Q(q)$. Once these unknown parameters have been estimated, the period 1 reservation price for product 1 can be defined as $p_1^{1*} = g_1(0, q_2^1, \ldots, q_N^1, v^1)$. This methodology will be described in Sections 9 and 10 in more detail.[9]

It turns out that a special case of this inverse demand function methodology is the case of a *linear utility function*; that is, set $f(q) = \Sigma_{n=1}^{N} \alpha_n q_n \equiv \alpha{\cdot}q$, where α_n are *quality adjustment factors*. Thus, α_n give the increase in utility of purchasers due to the acquisition of an extra unit of product n. The case of a linear utility function will be used as an underlying economic model in Sections 3 and 5–8. Furthermore, it turns out that the assumption of an underlying linear utility function provides a rationale for *hedonic regression models*, which will be studied in Sections 5–8.

In Sections 3 and 4, we apply the linear utility function assumption to some special situations where it is possible to generate missing prices without using any econometrics. These sections introduce *inflation-adjusted carry-forward and carry-backward prices*, which have been used for many years by statistical agencies to replace missing prices.[10]

In Section 5, we also assume an underlying linear utility function, but we no longer assume that the underlying economic model holds exactly. Thus, error terms make their appearance in this section (and in subsequent sections). The resulting model is called the *time product dummy hedonic regression model*. This model is an application of Summer's (1973) *country product dummy model* to the time series context. The underlying TPD hedonic regression model is $p_{tn} = \pi_t\alpha_n$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$, where α_n are the *quality adjustment factors* that appear in the purchasers' linear utility function and π_t turn out to be period *t aggregate price levels*.[11] However, in real-life applications, these equations will not hold exactly, and thus it is necessary to introduce error terms. The preceding exact equations can be replaced by $ln p_{tn} = ln\pi_t + ln\alpha_n + e_{tn}$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$, where e_{tn} are error terms. This is a stochastic model, which was discussed in Chapter 7. It is also a special case of a *hedonic regression model* where prices are regressed on the characteristics of products. In this simple special case framework, each product has its own separate characteristic. Estimators for the logarithms of π_t and a_n are found by minimizing the sum of squared errors, $\Sigma_{t=1}^{T} \Sigma_{n=1}^{N} [e_{tn}]^2 = \Sigma_{t=1}^{T} \Sigma_{n=1}^{N} [\ln p_{tn} - \ln\pi_t - \ln\alpha_n]^2$. However, the log prices are not weighted according to their economic importance, and the model does not allow for

[6] Two major problems with this framework are (i) it does not take into account the fact that purchasers may stockpile goods on sale and this will affect demand in subsequent periods and (ii) the introduction of a new revolutionary product may change purchaser preferences over existing goods. However, until a better welfare-oriented model of purchaser behavior comes along, we are stuck with using the Hicksian approach.

[7] See, for example, Diewert (1974, 120–133).

[8] Suppose that the utility function f(q) is differentiable and linearly homogeneous, and we have an interior solution to the purchaser's utility maximization problem. Then using Wold's (1944, 69–71) identity, $p_n = [\partial f(q)/\partial q_n]v/f(q) \equiv g_n(q,v)$ for $n = 1, \ldots, N$. We will derive these equations in more detail in Section 2. See also Section 4 in Chapter 5.

[9] This methodology was first suggested by Diewert (1980, 498–503) and implemented by Diewert and Feenstra (2017, 2022).

[10] See von Hofsten (1952), Triplett (2004), de Haan and Krsinich (2012, 31–32), and Diewert, Fox, and Schreyer (2017). Inflation-adjusted carry-forward and carry-backward prices were discussed in Section 19 of Chapter 7; see Diewert (2021).

[11] A *bilateral price index* between period t relative to period r is defined as the ratio of the relevant price levels, π_t/π_r.

missing products as was seen in Chapter 7. In Section 17 of Chapter 7, we found a satisfactory stochastic model that allowed for missing observations and weighted prices by their economic importance. This model is reviewed in Section 5 of the present chapter.

It should be noted that the hedonic regression models that will be studied in Sections 5–8 are fairly complicated since they deal with expenditure weights and missing observations. For readers who are not familiar with hedonic regression models and want a simpler introduction to the topic, see the excellent book by Aizcorbe (2014).

The model described in Section 5 generates price levels that have some good axiomatic properties, but the model has an important drawback: A product that is available only in one period out of the T periods has no influence on the estimated aggregate price levels π_t^* for all periods. Thus, the introduction of a new product in period T will have no effect on the estimated price level for period T, π_T^*. This goes against the spirit of the Hicksian approach to the treatment of new goods. The hedonic regression models considered in Sections 6 and 7 do not suffer from this drawback.

Sections 6 and 7 deal with hedonic regression models that make use of information on the *characteristics* of the N products under consideration. The models in these two sections are more satisfactory than the weighted TPD model discussed in Section 5 because now-isolated prices play a role in the determination of the estimated price levels π_t^* for $t = 1, \ldots, T$. However, the hedonic regression models considered in Sections 6 and 7 do require information on product characteristics, information that may be difficult to collect. The important results obtained by de Haan and Krsinich (2018) using this class of hedonic regression models applied to electronic products are discussed in Section 7. They compare weighted and unweighted versions of the same hedonic regression models and show that weighting leads to improved results.

The problems raised by taste change in the two period cases are addressed in Section 8. The treatment of the problem in this section was suggested by Diewert, Heravi, and Silver (2009), and it uses the tastes of each period to construct separate bilateral price indices between the two periods. The two indices, each of which hold tastes constant, are then averaged to form a final index.

Finally, in Sections 9 and 10, two alternative methods for constructing reservation prices are discussed. In these methods, the underlying utility function is *not* assumed to be a linear function. In Section 9, the reservation price model due to Feenstra (1994) is presented. This model assumes that the underlying preferences are Constant Elasticity of Substitution (CES).[12] The model presented in Section 10 assumes that the underlying preferences are a certain flexible functional form (that is exact for the Fisher (1922) ideal quantity index). This model was developed by Diewert and Feenstra (2017, 2022).

Section 11 considers three additional methods for quality adjustment, and Section 12 offers some conclusions.

[12] See Arrow et al. (1961) for the first use of this functional form in the economics literature. Chapter 5 considered alternative estimation methods for this functional form.

2. A Framework for Evaluating Quality Change in the Scanner Data Context

In this section, we provide a framework for the construction of consumer price and quantity indices in the scanner data context using the economic approach to index number theory. We assume that transaction data for the sales or purchases of N products over T time periods are available.[13] The N products will typically be a group of related products so that the goal is the construction of price and quantity indices at the first stage of aggregation. The transactions data are aggregated over time within each period so that the prices for each period are unit value prices. Let $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ denote the price and quantity vectors for time periods $t = 1, \ldots, T$. The period t quantity for product n, q_{tn}, is equal to total purchases of product n by purchasers or it is equal to the sales of product n by the outlet (or group of outlets) for period t, while the corresponding period t price for product n, p_{tn}, is equal to the value of sales (or purchases) of product n in period t, v_{tn}, divided by the corresponding total quantity sold (or purchased), q_{tn}. Thus, $p_{tn} \equiv v_{tn}/q_{tn}$ is the *unit value price* for product n in period t for $t = 1, \ldots, T$ and $n = 1, \ldots, N$. In this section, we assume that all prices, quantities, and values are positive; in subsequent sections, this assumption will be relaxed.

Let $q \equiv [q_1, \ldots, q_N]$ be a generic quantity vector. In order to compare various methods for comparing the value of alternative combinations of the N products, it is necessary that a *valuation function* or *aggregator function* or *utility function* $f(q)$ exist. This function allows us to value alternative combinations of products; if $f(q^2) > f(q^1)$, then purchasers of the products place a higher utility value on the vector of purchases q^2 than they place on the vector of purchases q^1. The function $f(q)$ can also act as an *aggregate quantity level* for the vector of purchases, q. Thus, $f(q^t)$ can be interpreted as an aggregate quantity level for the period t vector of purchases, q^t, and the ratios, $f(q^t)/f(q^1)$, $t = 1, \ldots, T$, can be interpreted as *fixed-base quantity indices* covering periods 1 to T.

In the following analysis, we assume that $f(q)$ has the following properties: (i) $f(q) > 0$ if $q >> 0_N$;[14] (ii) $f(q)$ is nondecreasing in its components; (iii) $f(\lambda q) = \lambda f(q)$ for $q \geq 0_N$ and $\lambda \geq 0$; (iv) $f(q)$ is a continuous concave function over the nonnegative orthant. Assumption (iii), linear homogeneity of $f(q)$, is a somewhat restrictive assumption. However, this assumption is required to ensure that the aggregate price level, $P(p,q)$, that corresponds to $f(q)$ does not depend on the scale of q.[15] Property (iv) will ensure that the first-order necessary conditions for the budget-constrained maximization of $f(q)$ are also sufficient.

Let $p \equiv [p_1, \ldots, p_N] > 0_N$ and $q \equiv [q_1, \ldots, q_N] > 0_N$ with $p \cdot q \equiv \Sigma_{n=1}^N p_n q_n > 0$. Then the *aggregate price level*, $P(p,q)$, that

[13] The data could be price and quantity (or value and quantity) on sales of the N products from a retail outlet (or group of outlets in the same region) or it could be price and quantity data for the purchases of the N products by a group of similar households.

[14] Notation: $q >> 0_N$ means each component of q is positive, $q \geq 0_N$ means each component of q is nonnegative, and $q > 0_N$ means $q \geq 0_N$ but $q \neq 0_N$.

[15] $P(p,q) \equiv p \cdot q/f(q)$, where $p \cdot q \equiv \Sigma_{n=1}^N p_n q_n$. Thus, using property (iii) of $f(q)$, we have $P(p,\lambda q) = p \cdot \lambda q/f(\lambda q) = \lambda p \cdot q/\lambda f(q) = P(p,q)$.

corresponds to the aggregate quantity level $f(q)$ is defined as follows:

$$P(p,q) \equiv p{\cdot}q/f(q). \quad (1)$$

Thus, the implicit price level $P(p,q)$, which is generated by the generic price and quantity vectors, p and q, is equal to the value of purchases, $p{\cdot}q$, deflated by the aggregate quantity level, $f(q)$. Note that using these definitions, the product of the aggregate price and quantity levels equals the value of purchases during the period; that is, we have $P(p,q)f(q) = p{\cdot}q$.

Once the functional form for the aggregator function $f(q)$ is known, then the *aggregate quantity level for period t*, Q^t, can be calculated as follows:

$$Q^t \equiv f(q^t);\ t = 1, \ldots, T. \quad (2)$$

Using definition (1), the corresponding period t aggregate price level, P^t, can be calculated as follows:

$$P^t \equiv p^t{\cdot}q^t/f(q^t);\ t = 1, \ldots, T. \quad (3)$$

Note that if $f(q)$ turns out to be a *linear aggregator function*, so that $f(q^t) \equiv \alpha{\cdot}q^t = \Sigma_{n=1}^{N} \alpha_n q_{tn}$, then the corresponding period t price level P^t is equal to $p^t{\cdot}q^t/\alpha{\cdot}q^t$, which is a *quality-adjusted unit value price level*.[16]

In order to make further progress, it is necessary to make some additional assumptions. The two additional assumptions are (v) $f(q)$ is once differentiable with respect to the components of q and (vi) the observed strictly positive quantity vector for period t, $q^t >> 0_N$,[17] is a solution to the following period t constrained maximization problem: [18]

$$\max_q \{f(q) : p^t{\cdot}q = v^t;\ q \geq 0_N\};\ t = 1, \ldots, T. \quad (4)$$

The first-order conditions for solving (4) for period t are as follows:[19]

$$\nabla_q f(q^t) = \lambda_t p^t;\ t = 1, \ldots, T; \quad (5)$$
$$p^t{\cdot}q^t = v^t;\ t = 1, \ldots, T. \quad (6)$$

Since $f(q)$ is assumed to be linearly homogeneous with respect to q, Euler's Theorem on homogeneous functions implies that the following equations hold:

$$q^t{\cdot}\nabla_q f(q^t) = f(q^t);\ t = 1, \ldots, T. \quad (7)$$

Take the inner product of both sides of equations (5) with q^t and use the resulting equations along with equations (7) to solve for the Lagrange multipliers, λ_t:

$$\lambda_t = f(q^t)/p^t{\cdot}q^t\ \ t = 1, \ldots, T \quad (8)$$
$$= 1/P^t \text{ using definitions (3).}$$[20]

Thus, if we assume utility-maximizing behavior on the part of purchasers of the N products using the collective utility function $f(q)$ that satisfies these regularity conditions, then the period t quantity aggregate is $Q^t \equiv f(q^t)$ and the companion period t price level defined as $P^t \equiv p^t{\cdot}q^t/Q^t$ is equal to $1/\lambda_t$, where λ_t is the Lagrange multiplier for problem t in the constrained utility maximization problems (4) and where q^t and λ_t solve equations (5) and (6) for period t. Equations (8) also imply that the product of P^t and Q^t is exactly equal to observed period t expenditure v^t; that is, we have

$$P^tQ^t = p^t{\cdot}q^t = v^t;\ t = 1, \ldots, T. \quad (9)$$

Substitute equations (8) into equations (5), and after suitable rearrangement, the following *fundamental equations* are obtained:[21]

$$p^t = P^t\nabla_q f(q^t);\ t = 1, \ldots, T. \quad (10)$$

In the following section, we will assume that the aggregator function, $f(q)$, is a linear function, and we will show how this assumption along with equations (9) for the case where $T = 2$ and $N = 3$ can lead to a simple well-known method for quality adjustment that does not involve any econometric estimation of the parameters of the linear function. In subsequent sections, equation (10) will be utilized in the hedonic regression context. In the final sections of the chapter, the assumption that $f(q)$ is a linear function will be relaxed.

3. A Nonstochastic Method for Quality Adjustment: A Simple Model

A major problem that arises when statistical agencies use scanner data to construct an elementary index is that some products are sold or purchased in one period but not in a subsequent period. Conversely, new products appear in the present period; these were not present in previous periods.

[16] See Section 10 of Chapter 7 for the properties of quality-adjusted unit value indices.

[17] The assumption that $q^t >> 0_N$ can be replaced by the assumptions $q^t > 0_N$ and $p^t{\cdot}q^t > 0$.

[18] The theory that follows dates back to Konüs and Byushgens (1926). This approach blends standard consumer demand theory based on the maximization of a linearly homogeneous utility function with index number theory. It was further developed by Shephard (1953) (in the context of a producer cost minimization framework) and by Samuelson and Swamy (1974) and Diewert (1976) in the consumer context. The price indices that result from this theory are special cases of the Konüs (1924) true cost of living index. What is new in the present chapter is the application of this theory to hedonic regression models.

[19] Using the assumption of concavity of $f(q)$ and the assumption that $q^t >> 0_N$, these conditions are also sufficient to solve (4). Notation: $\nabla_q f(q) \equiv [\partial f(q)/\partial q_1, \ldots, \partial f(q)/\partial q_N]$.

[20] Note that equations (8) imply that $p^t{\cdot}q^t = P^tf(q^t)$. Since f is linearly homogeneous, we also know that $p^t{\cdot}q^t = c(p^t)f(q^t)$ where $c(p)$ is the unit cost function that is dual to $f(q)$. Hence, $P^t = c(p^t)$; that is, the period t price level P^t is equal to the unit cost function c that corresponds to the utility function f evaluated at the period t price vector p^t.

[21] Multiply the right-hand side of equation t in (10) by $1 = Q^t/f(q^t)$ and use $P^tQ^t = v^t$ to obtain the following system of equations: $p^t = v^t\nabla_q f(q^t)/f(q^t)$ for $t = 1, \ldots, T$. For each t, this system of equations is the consumer's system of *inverse demand functions* that give the period t prices that are consistent with the observed period t demands q^t as functions of p^t and period t expenditure v^t. Konüs and Byushgens (1926) obtained a system of equations that is equivalent to this system of inverse demand functions. Linear homogeneity of the utility function is required in order to obtain these equations and the equivalent equations defined by (9) and (10).

How should price and quantity indices be constructed under these circumstances? Equation (10) in the previous section can be used to provide an answer to this question.

Consider the special case where the number of periods T is equal to 2 and the number of products in scope for the elementary index is N equal to 3. Product 1 is present in both periods, product 2 is present in period 1 but not in period 2 (a disappearing product), and product 3 is not present in period 1 but is present in period 2 (a new product).[22] We assume that purchasers of the three products behave as if they collectively maximized the following linear aggregator function:

$$f(q_1,q_2,q_3) \equiv \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3, \tag{11}$$

where α_n are positive constants. Under these assumptions, equations (10) written out in scalar form become[23]

$$p_{tn} = P^t \alpha_n;\ n = 1,2,3;\ t = 1,2. \tag{12}$$

Equations (12) are six equations in the five parameters P^1 and P^2 which can be interpreted as *aggregate price levels* for periods 1 and 2, and α_1, α_2, and α_3, which can be interpreted as *quality adjustment factors* for the three products; that is, each α_n measures the relative usefulness of an additional unit of product n to purchasers of the three products. However, product 3 is not observed in the marketplace during period 1, and product 2 is not observed in the marketplace in period 2, and so there are only four equations in (12) to determine five parameters. However, P^t and α_n cannot all be identified using observable data; that is, if P^1, P^2, α_1, α_2, and α_3 satisfy equations (12) and λ is any positive number, then λP^1, λP^2, $\lambda^{-1}\alpha_1$, $\lambda^{-1}\alpha_2$, and $l^{-1}\alpha_3$ will also satisfy equations (12). Thus, it is necessary to place a normalization (like $P^1 = 1$ or $\alpha_1 = 1$) on the five parameters that appear in equations (12) in order to obtain a unique solution. In the index number context, it is natural to set the price level for period 1 equal to unity, and so we impose the following normalization on the five unknown parameters that appear in equations (12):

$$P^1 = 1. \tag{13}$$

The four equations in (12) that involve observed prices and the single equation (13) are five equations in five unknowns. The unique solution to these equations is

$$P^1 = 1;\ P^2 = p_{21}/p_{11};\ \alpha_1 = p_{11};\ \alpha_2 = p_{12};$$
$$\alpha_3 = p_{23}/(p_{21}/p_{11}) = p_{23}/P^2. \tag{14}$$

Note that the resulting *price index*, P^2/P^1, is equal to p_{21}/p_{11}, the price ratio for the commodity that is present in both periods. Thus, the price index for this very simple model turns out to be a *maximum overlap price index*.[24]

Once the P^t and α_n have been determined, equations (12) for the missing products can be used to define the following *imputed prices* p_{tn}^* for commodity 3 in period 1 and product 2 in period 2:

$$p_{13}^* \equiv P^1\alpha_3 = p_{23}/(P^2/P^1);\ p_{22}^* \equiv P^2\alpha_2$$
$$= (p_{21}/p_{11})p_{12} = (P^2/P^1)p_{12}. \tag{15}$$

These imputed prices can be interpreted as Hicksian (1940, 12) *reservation prices*;[25] that is, they are the lowest possible prices that would still deter purchasers from purchasing the products during periods if the unavailable products hypothetically became available.[26]

Note that $p_{13}^* = p_{23}/(P^2/P^1)$ is an *inflation-adjusted carry-backward price*; that is, the observed price for product 3 in period 2, p_{23}, is divided by the maximum overlap price index P^2/P^1 in order to obtain a "reasonable" valuation for a unit of product 3 in period 1. Similarly, $p_{22}^* = (P^2/P^1)p_{12}$ is an *inflation-adjusted carry-forward price* for product 2 in period 2; that is, the observed price for product 2 in period 1, p_{12}, is multiplied by the maximum overlap price index P^2/P^1 in order to obtain a "reasonable" valuation for a unit of product 2 in period 2.[27]

Note that the preceding algebra can be implemented without a knowledge of quantities sold or purchased. Assuming that quantity information is available, we now consider how companion quantity levels, Q^1 and Q^2, for the price levels, P^1 and P^2, can be determined. Note that $q_{13} = 0$ and $q_{22} = 0$ since consumers cannot purchase products that are not available. Use the imputed prices defined by (15) to obtain complete price vectors for each period; that is, define the period 1 complete price vector by $p^1 \equiv [p_{11}, p_{12}, p_{13}^*]$ and the complete period 2 price vector by $p^2 \equiv [p_{21}, p_{22}^*, p_{23}]$. The corresponding complete quantity vectors are $q^1 \equiv [q_{11}, q_{12}, 0]$ and $q^2 \equiv [q_{21}, 0, q_{23}]$. The period t aggregate quantity level Q^t can be calculated directly using only information on q^t and the vector of quality adjustment factors, $\alpha \equiv [\alpha_1, \alpha_2, \alpha_3]$, or indirectly by deflating period t expenditure $v^t \equiv p^t \cdot q^t$ by the estimated period t price level, P^t. Thus, we have the following two possible methods for constructing the Q^t:

$$Q^t \equiv \alpha \cdot q^t;\ \text{or}\ Q^t \equiv p^t \cdot q^t/P^t;\ t = 1,2. \tag{16}$$

[22] The "new" product may not be a truly new product; it may be the case that product 3 was temporarily not available in period 1. Similarly, product 2 may not permanently disappear in period 2; it may reappear in a subsequent period.

[23] This is a special case of the TPD regression model that was studied in Chapter 7 and will be summarized in Section 5. Thus, equations (12), which are the inverse consumer demand functions that result from the maximization of a linear utility function, lead directly to a particular hedonic regression model. It is this result that allows us to claim that our present approach is a way of reconciling hedonic regression models with classical consumer demand theory.

[24] Keynes (1930, 94) was an early author who advocated this method for dealing with new goods by restricting attention to the goods that were present in both periods being compared. He called his suggested method the *highest common factor method*. Marshall (1887, 373) implicitly endorsed this method. Triplett (2004, 18) called it the *overlapping link method*.

[25] Hicks (1940) dealt only with the case of new goods; von Hofsten (1952, 95–97) extended his approach to cover the case of disappearing goods as well.

[26] Strictly speaking, it would be necessary to add a tiny amount to these prices to deter consumers from purchasing these products if they were made available.

[27] The use of carry-forward and carry-backward prices to estimate missing prices is widespread in statistical agencies. For additional materials on this method for estimating missing prices, see Triplett (2004), de Haan and Krsinich (2012), Diewert, Fox, and Schreyer (2017), and Section 19 of Chapter 7.

However, using the complete price vectors p^t with imputed prices filling in for the missing prices, equations (12) hold exactly, and thus it is straightforward to show that $Q^t = \alpha \cdot q^t = p^t \cdot q^t / P^t$ for $t = 1,2$. Thus, it does not matter whether we use the direct or indirect method for calculating the quantity levels; both methods give the same answer in this simple model.[28]

A problem with this simple model is that there is only one product that is present in both periods. In the following section, we generalize the present model to allow for multiple overlapping products.

4. A Nonstochastic Method for Quality Adjustment: A More Complex Model

In order to generalize the very simple model for dealing with new and disappearing products that was presented in the previous section, it is first necessary to develop another application of the fundamental equations (10) in Section 2.

Define the aggregator function $f(q)$ as follows:

$$f_{KBF}(q^*) \equiv [q^* \cdot Aq^*]^{1/2} \equiv [\Sigma_{i=1}^{N} \Sigma_{j=1}^{N} a_{ij} q_i^* q_j^*]^{1/2}, \quad (17)$$

where q^* is defined as the N-dimensional quantity vector $[q_1^*, \ldots, q_N^*]$ and $A \equiv [a_{ij}]$ is an N by N symmetric matrix of parameters that satisfies certain regularity conditions.[29] Suppose further that the observed price and quantity vectors for periods 1 and 2 are the positive price and quantity vectors, $p^{t*} \equiv [p_{t1}^*, \ldots, p_{tN}^*]$ and $q^{t*} \equiv [q_{t1}^*, \ldots, q_{tN}^*]$ for $t = 1,2$. We assume that q^{t*} solves $\max_q \{f_{KBF}(q) : p^{t*} \cdot q = v^{t*}; q \geq 0_N\}$ for $t = 1,2$, where $v^{t*} \equiv p^{t*} \cdot q^{t*}$ is the observed expenditure on the N products for periods $t = 1,2$. The inverse demand functions (10) that correspond to this particular aggregator function are the following ones:

$$p^{t*} = P^{t*} \nabla_q f_{KBF}(q^{t*}) = P^t [q^{t*} \cdot Aq^{t*}]^{-1/2} Aq^{t*}; t = 1,2. \quad (18)$$

Using the framework described in Section 2, the period t aggregate quantity level for the present model is $Q^{t*} \equiv [q^{t*} \cdot Aq^{t*}]^{1/2}$, and the corresponding period t price level is $P^{t*} \equiv p^{t*} \cdot q^{t*} / Q^{t*}$ for $t = 1,2$. The *Fisher* (1922) *ideal quantity index* is a function of the observable price and quantity data and is defined as follows:

$$Q_F(p^{1*}, p^{2*}, q^{1*}, q^{2*}) \equiv [p^{1*} \cdot q^{2*}\ p^{2*} \cdot q^{2*} / p^{1*} \cdot q^{1*}\ p^{2*} \cdot q^{1*}]^{1/2}. \quad (19)$$

Use equations (18) to eliminate p^{1*} and p^{2*} from the right-hand side of (19). We find that

$$(p^{1*} \cdot q^{2*} p^{2*} \cdot q^{2*}) / (p^{1*} \cdot q^{1*} p^{2*} \cdot q^{1*}) = q^{2*} \cdot Aq^{2*} / q^{1*} \cdot Aq^{1*}. \quad (20)$$

Take positive square roots on both sides of (20). Using definitions (17) and (19), the resulting equation is

$$f_{KBF}(q^{2*}) / f_{KBF}(q^{1*}) = Q_F(p^{1*}, p^{2*}, q^{1*}, q^{2*}). \quad (21)$$

Thus, $Q^{2*}/Q^{1*} = f_{KBF}(q^{2*}) / f_{KBF}(q^{1*})$ is equal to the Fisher ideal quantity index $Q_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$, which can be calculated using observable price and quantity data for the two periods. We know from Section 2 that

$$P^{t*} Q^{t*} = p^{t*} \cdot q^{t*}; t = 1,2. \quad (22)$$

Now make the normalization $P^{1*} = 1$. Using this normalization and equations (21) and (22), the aggregate price and quantity levels for the two periods can be defined in terms of observable data as follows:

$$P^{1*} \equiv 1;\ Q^{1*} \equiv p^{1*} \cdot q^{1*};\ Q^{2*} \equiv Q^{1*} Q_F(p^{1*}, p^{2*}, q^{1*}, q^{2*});$$
$$P^{2*} \equiv p^{2*} \cdot q^{2*} / Q^{2*}. \quad (23)$$

These results can be combined with the three-product model that was described in the previous section: relabel this aggregate data as a composite product 1 so that the new product 1 that corresponds to the first product in Section 3 has prices and quantities defined as $p_{t1} \equiv P^{t*}$ and $q_{t1} \equiv Q^{t*}$ for $t = 1,2$. Products 2 and 3 are a disappearing product and a new product, respectively, as in Section 3. The aggregate price levels for the two periods (which use all $N + 2$ products) are P^1 and P^2, and the new α_n parameters are defined by the following counterparts to equations (14):

$$P^1 = 1;\ P^2 = P^{2*}/P^{1*} = P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*});$$
$$\alpha_1 = 1;\ \alpha_2 = p_{12};\ \alpha_3 = p_{23}/(P^{2*}/P^{1*}), \quad (24)$$

where $P^{2*}/P^{1*} \equiv [v^{2*}/v^{1*}]/[Q^{2*}/Q^{1*}] \equiv P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$ is the Fisher (1922) ideal price index that compares the prices of N products that are present in both periods, p^{1*}, p^{2*}, for the two periods under consideration. The imputed prices for the missing products, p_{13}^* and p_{22}^*, are obtained by using equations (15) for our present model:

$$p_{13}^* \equiv p_{23} / P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*});$$
$$p_{22}^* \equiv P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*}) p_{12}. \quad (25)$$

Comparing (24) and (25) with the corresponding equations (14) and (15) for the three-product model, it can be seen that the price ratio for product 1 that was present in both periods, p_{21}/p_{11}, is replaced by the Fisher index $P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$, which is now defined over the set of products that are present in both periods. The type of inflation-adjusted carry-backward price p_{13}^* and the inflation-adjusted carry-forward price p_{22}^* defined by (25) are widely used by statistical agencies to estimate missing prices, but agencies usually use the Lowe, Laspeyres, or Paasche index in place of the Fisher price index.[30]

[28] In subsequent sections when we no longer assume that equations (12) hold exactly, then the direct and indirect methods for calculating Q^t will in general differ.

[29] Thus, $A = A^T$ and A is assumed to have one positive eigenvalue with a corresponding strictly positive eigenvector and N–1 negative or zero eigenvalues. This functional form was introduced into the economics literature by Konüs and Byushgens (1926), who showed its connection with the Fisher (1922) ideal index. This explains why $f(q^*)$ is labeled as $f_{KBF}(q^*)$. For further discussion of the regularity conditions on $f_{KBF}(q^*)$, see Diewert (1976) and Diewert and Hill (2010) or Section 5 of Chapter 5.

[30] Note that the aggregate price index that is generated by this model is $P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$ which does not use the unmatched prices for the two periods.

The aggregator function that is consistent with the new model with N continuing products, one disappearing product, and one new product is defined as follows:

$$Q(q_1^*, \ldots, q_N^*, q_2, q_3) \equiv \alpha_1 f_{KBF}(q^*) + \alpha_2 q_2 + \alpha_3 q_3, \qquad (26)$$

where $f_{KBF}(q^*)$ is the KBF aggregator function defined by (17) and a_1 is set equal to 1.[31] Note that the model defined by (26) is restrictive from the economic perspective because the additive nature of definition (26) implies that the composite first commodity is perfectly substitutable with the new and disappearing commodities (which are also perfect substitutes for each other after quality adjustment). However, if the products under consideration *are* highly substitutable for each other, the implicit assumption of perfect substitutes for missing products will be acceptable. Moreover, the advantage of this form of quality adjustment is that it is relatively easy to explain to the public, and it is fairly straightforward to implement.

The restriction that there is only one new product and one disappearing product is readily relaxed. The overall price index will continue to be $P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$, and counterparts to equations (25) can be used to generate imputed prices for the missing products. To summarize how the many new products and many disappearing products model works, let V^0 and V^1 be the aggregate value of all transactions in periods 0 and 1, respectively. Then the aggregate price levels generated by the above model of quality adjustment are given by $P^0 \equiv 1$ and $P^1 \equiv P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$, which is equal to the Fisher index defined over all continuing products. The corresponding aggregate quantity levels for periods 0 and 1 are set equal to $Q^0 \equiv V^0$ and $Q^1 \equiv V^1/P^1 = V^1/P_F(p^{1*}, p^{2*}, q^{1*}, q^{2*})$. This is a very simple model to implement.

We turn now to applications of the basic framework explained in Section 2, where conditions (10) only hold approximately rather than exactly.

5. Weighted Time Product Dummy Regressions

In this section, we consider a special case of the model of economic behavior explained in Section 2, where there are N products in the aggregate and T periods. Let $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ denote the price and quantity vectors for time periods $t = 1, \ldots, T$. Initially, it is assumed that there are no missing prices or quantities so that all N times T prices and quantities are positive. We assume that the quantity aggregator function $f(q)$ is the following *linear function*:

$$f(q) = f(q_1, q_2, \ldots, q_N) \equiv \Sigma_{n=1}^{N} \alpha_n q_n = \alpha \cdot q, \qquad (27)$$

where α_n are positive parameters, which can be interpreted as quality adjustment factors. Under the assumption of maximizing behavior on the part of purchasers of N commodities, assumption (27) applied to equations (10) implies that the following NT equations should hold exactly:

$$p_{tn} = \pi_t \alpha_n;\ n = 1, \ldots, N;\ t = 1, \ldots, T, \qquad (28)$$

where we have redefined the period t price levels P^t in equations (10) as the parameters π_t for $t = 1, \ldots, T$.

Note that equations (28) form the basis for the *time dummy hedonic regression model*, which was developed by Court (1939).[32] It can be seen that these equations are a special case of the general model of consumer behavior that was explained in Section 2.

At this point, it is necessary to point out that our consumer theory derivation of equations (28) is not accepted by all economists. Rosen (1974), Triplett (1987, 2004), and Pakes (2001)[33] have argued for a more general approach to the derivation of hedonic regression models that is based on supply conditions as well as on demand conditions. The present approach is obviously based on only consumer (or purchaser) preferences. This consumer-oriented approach was endorsed by Griliches (1971, 14–15), Muellbauer (1974, 988), and Diewert (2003a, 2003b).[34] Of course, the functional form assumptions that justify the present consumer approach are quite restrictive, but, nevertheless, it is useful to imbed hedonic regression models in a traditional consumer demand setting.

Empirically, equations (28) are unlikely to hold exactly. Thus, following Court (1939), we assume that the exact model defined by (28) holds only to some degree of approximation, and so we could add error terms e_{tn} to the right-hand sides of equations (28). The unknown parameters, $\pi \equiv [\pi_1, \ldots, \pi_T]$ and $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$, could be estimated as solutions to the following (nonlinear) least squares minimization problem:

$$\min_{\alpha,\, \pi} \Sigma_{n=1}^{N} \Sigma_{t=1}^{T} [p_{tn} - \pi_t \alpha_n]^2. \qquad (29)$$

[31] It is not necessary to use the KBF aggregator function in the above model; any aggregator function that has an exact index number associated with it will work. See Diewert (1976) for examples of exact index number formulae.

[32] This was Court's (1939, 109–111) hedonic suggestion number two. He transformed the underlying equations (28) by taking logarithms of both sides of these equations (which will be done later). He chose to transform the prices by the log transformation because the resulting regression model fit his data on automobiles better. Diewert (2003b) also recommended the log transformation on the grounds that multiplicative errors were more plausible than additive errors.

[33] "The derivatives of a hedonic price function should not be interpreted as either willingness to pay derivatives or cost derivatives; rather they are formed from a complex equilibrium process" (Ariel Pakes, 2001, 14).

[34] Diewert (2003b, 97) justified the consumer demand approach as follows: "After all, the purpose of the hedonic exercise is to find how demanders (and not suppliers) of the product value alternative models in a given period. Thus for the present purpose, it is the preferences of consumers that should be decisive, and not the technology and market power of producers. The situation is similar to ordinary general equilibrium theory where an equilibrium price and quantity for each commodity is determined by the interaction of consumer preferences and producer's technology sets and market power. However, there is a big branch of applied econometrics that ignores this complex interaction and simply uses information on the prices that consumers face, the quantities that they demand and perhaps demographic information in order to estimate systems of consumer demand functions. Then these estimated demand functions are used to form estimates of consumer utility functions and these functions are often used in applied welfare economics. What producers are doing is entirely irrelevant to these exercises in applied econometrics with the exception of the prices that they are offering to sell at. In other words, we do not need information on producer marginal costs and markups in order to estimate consumer preferences: all we need are selling prices." Footnote 25 on page 82 of Diewert (2003b) explains how the present hedonic model can be derived from Diewert's (2003a) consumer-based model by strengthening the assumptions in the 2003a paper.

However, in Section 13 of Chapter 7, we showed that the estimated price levels π_t^* that solve the minimization problem (29) had unsatisfactory axiomatic properties. Thus, we took logarithms of both sides of the exact equations (28) and added error terms to the resulting equations. This led to the following least squares minimization problem:[35]

$$\min_{\rho,\,\beta} \Sigma_{n=1}^{N} \Sigma_{t=1}^{T} [lnp_{tn} - \rho_t - \beta_n]^2, \qquad (30)$$

where the new parameters ρ_t and β_n were defined as the logarithms of π_t and α_n; that is, define

$$\rho_t \equiv ln\pi_t;\ t = 1, \ldots, T; \qquad (31)$$
$$\beta_n \equiv ln\alpha_n;\ n = 1, \ldots, N. \qquad (32)$$

However, the least squares minimization problem defined by (30) does not weight the log price terms $[lnp_{tn} - \rho_t - \beta_n]^2$ by their *economic importance*, and so in Section 15 of Chapter 7, we considered the following *weighted least squares minimization problem*:[36]

$$\min_{\rho,\,\beta} \Sigma_{n=1}^{N} \Sigma_{t=1}^{T} s_{tn}[lnp_{tn} - \rho_t - \beta_n]^2, \qquad (33)$$

where s_{tn} is the expenditure share of product n in period t. The first-order necessary conditions for $\rho^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\beta^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ to solve (33) simplify to the following T equations (34) and N equations (35):[37]

$$\rho_t^* = \Sigma_{n=1}^{N} s_{tn}[lnp_{tn} - \beta_n^*];\ t = 1, \ldots, T; \qquad (34)$$
$$\beta_n^* = \Sigma_{t=1}^{T} s_{tn}[lnp_{tn} - \rho_t^*]/(\Sigma_{t=1}^{T} s_{tn});\ n = 1, \ldots, N. \qquad (35)$$

The solution to (34) and (35) is not unique: If $\rho^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\beta^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ solve (34) and (35), then so do $[\rho_1^* + \lambda, \ldots, \rho_T^* + \lambda]$ and $[\beta_1^* - \lambda, \ldots, \beta_N^* - \lambda]$ for all λ. Thus, we can set $\rho_1^* = 0$ in equations (35) and drop the first equation in (34) and use linear algebra to find a unique solution for the resulting equations.[38] Once the solution is found, define the estimated *price levels* π_t^* and *quality adjustment factors* α_n^* as follows:

$$\pi_t^* \equiv \exp[\rho_t^*];\ t = 1, \ldots, T;\ \alpha_n^* \equiv \exp[\beta_n^*];$$
$$n = 1, \ldots, N. \qquad (36)$$

The price levels π_t^* defined by (36) are called the *weighted time product dummy price levels*. That the resulting *price index* between periods t and τ is given by

$$\pi_t^*/\pi_\tau^* = \prod_{n=1}^{N} \exp[s_{tn}ln(p_{tn}/\alpha_n^*)]/\prod_{n=1}^{N}$$
$$\exp[s_{\tau n}ln(p_{\tau n}/\alpha_n^*)];\ 1 \le t, \tau \le T. \qquad (37)$$

If $s_{tn} = s_{\tau n}$ for $n = 1, \ldots, N$, then $\pi_t^*/\pi_{!\tau}^*$ will equal a weighted geometric mean of the price ratios $p_{tn}/p_{\tau n}$, where the weight for $p_{tn}/p_{\tau n}$ is the common expenditure share $s_{tn} = s_{\tau n}$. Thus, π_t^*/π_τ^* will not depend on α_n^* in this case.

Once the estimates for π_t and α_n have been computed, we have two methods for constructing period-by-period price and quantity levels, P^t and Q^t for $t = 1, \ldots, T$. The π_t^* estimates can be used to form the aggregates using equations (38) or the α_n^* estimates can be used to form the aggregates using equations (39):[39]

$$P^{t*} \equiv \pi_t^*;\ Q^{t*} \equiv p^t \cdot q^t/\pi_t^*;\ t = 1, \ldots, T; \qquad (38)$$
$$Q^{t**} \equiv \alpha^* \cdot q^t;\ P^{t**} \equiv p^t \cdot q^t/\alpha^* \cdot q^t;\ t = 1, \ldots, T. \qquad (39)$$

Define the error terms $e_{tn} \equiv lnp_{tn} - ln\pi_t^* - lna_n^*$ for $t = 1, \ldots, T$ and $n = 1, \ldots, N$. If all $e_{tn} = 0$, then P^{t*} will equal P^{t**} and Q^{t*} will equal Q^{t**} for $t = 1, \ldots, T$.[40] However, if the error terms are not all equal to zero, then the statistical agency will have to decide on pragmatic grounds which option to use to form period t price and quantity levels, (38) or (39).[41]

It is straightforward to generalize the weighted least squares minimization problem (33) to the case where there are missing prices and quantities. As in Section 17 of Chapter 7, we assume that there are N products and T time periods but not all products are purchased (or sold) in all time periods. For each period t, define the set of products n that are present in period t as $S(t) \equiv \{n: p_{tn} > 0\}$ for $t = 1, 2, \ldots, T$. It is assumed that these sets are not empty; that is, at least one product is purchased in each period. For each product n, define the set of periods t where product n is present as $S^*(n) \equiv \{t: p_{tn} > 0\}$. Again, assume that these sets are not empty; that is, each product is sold in at least one time period. The

[35] This model is an adaptation of Summer's (1973) country product dummy model to the time series context. See Aizcorbe, Corrado, and Doms (2000) for an early application of this model in the time series context.

[36] Rao (1995; 2004; 2005, 574) was the first to consider this model using expenditure share weights; see also Diewert (2004). However, Balk (1980, 70) suggested this class of models much earlier using somewhat different weights. For the case of two periods, see Diewert (2004, 2005a) and de Haan (2004a).

[37] If information on expenditures or quantities is not available, then the weighted least squares problem is replaced by the unweighted least squares problem (30). The first-order conditions for the simplified problem (30) are given by (34) and (35), where the shares s_{tn} are replaced by the numbers $1/N$ for all t and n. In this unweighted case, the price index defined by (37) collapses down to the Jevons index.

[38] Alternatively, one can set up the linear regression model defined by $(s_{tn})^{1/2}\ln p_{tn} = (s_{tn})^{1/2}\rho_t + (s_{tn})^{1/2}\beta_n + e_{tn}$ for $t = 1, \ldots, T$ and $n = 1, \ldots, N$, where we set $\rho_1 = 0$ to avoid exact multicollinearity. Iterating between equations (34) and (35) will also generate a solution to these equations, and the solution can be normalized so that $r_1 = 0$.

[39] Note that the price level P^{t**} defined in (39) is a quality-adjusted unit value index of the type studied by de Haan (2004b).

[40] If all $e_{tn} = 0$, then the unweighted (or more accurately, the equally weighted) least squares minimization problem defined by (30) will generate the same solution as that generated by the weighted least squares minimization problem defined by (33). This fact gives rise to the following rule of thumb: If the unweighted problem (30) fits the data very well, then it is not necessary to work with the more complicated weighted problem (33).

[41] In Section 21 of Chapter 7, the following multilateral test was considered: *Test 2*: *The fixed basket test for prices* or *the strong identity test for quantities*: If $q^r = q^t \equiv q$, then the price index for period t relative to period r is $p_M{}^t(P,Q)/p_M{}^r(P,Q) = p^t \cdot q/p^r \cdot q$. If the price and quantity aggregates are formed using equations (39) rather than (38), then this test will be satisfied. However, the more usual approach is to define the period t price and quantity aggregates using equations (38). If this is done, then in general, the weighted TPD price level functions, $p_{WTPD}{}^t(P,Q)$, will not satisfy the basket test, Test 2.

generalization of (33) to the case of missing products is the following *weighted least squares minimization problem*:[42]

$$\min_{\rho,\beta} \Sigma_{t=1}^{T} \Sigma_{n\in S(t)} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2$$
$$= \min_{\rho,\beta} \Sigma_{n=1}^{N} \Sigma_{t\in S^*(n)} s_{tn}[\ln p_{tn} - \rho_t - \beta_n]^2. \quad (40)$$

Note that there are two equivalent ways of writing the least squares minimization problem; the first way uses the definition for the set of products n present in period t, $S(t)$, while the second way uses the definition for the set of periods t where product n is present, $S^*(n)$. The first-order necessary conditions for $\rho_1, \ldots, \rho_T$ and $b_1, \ldots, b_N$ to solve (40) are the following counterparts to (34) and (35):[43]

$$\Sigma_{n\in S(t)} s_{tn}[\rho_t^* + \beta_n^*] = \Sigma_{n\in S(t)} s_{tn} lnp_{tn};\ t = 1, \ldots, T; \quad (41)$$
$$\Sigma_{t\in S^*(n)} s_{tn}[\rho_t^* + \beta_n^*] = \Sigma_{t\in S^*(n)} s_{tn} lnp_{tn};\ n = 1, \ldots, N. \quad (42)$$

As usual, the solution to (41) and (42) is not unique: If $\rho^* \equiv [\rho_1^*, \ldots, \rho_T^*]$ and $\beta^* \equiv [\beta_1^*, \ldots, \beta_N^*]$ solve (41) and (42), then so do $[\rho_1^* + \lambda, \ldots, \rho_T^* + \lambda]$ and $[\beta_1^* - \lambda, \ldots, \beta_N^* - \lambda]$ for all λ. Thus, we can set $\rho_1^* = 0$ in equations (42), drop the first equation in (41), and use linear algebra to find a unique solution for the resulting equations.[44]

Define the estimated *price levels* π_t^* and *quality adjustment factors* α_n^* by definitions (31) and (32). Substitute these definitions into equations (41) and (42). After some rearrangement, equations (41) and (42) become the following ones:

$$\pi_t^* = \exp[\Sigma_{n\in S(t)} s_{tn} ln(p_{tn}/\alpha_n^*)];\ t = 1, \ldots, T; \quad (43)$$
$$\alpha_n^* = \exp[\Sigma_{t\in S^*(n)} s_{tn} ln(p_{tn}/\pi_t^*)/\Sigma_{t\in S^*(n)} s_{tn}];$$
$$n = 1, \ldots, N. \quad (44)$$

Once the estimates for π_t and α_n have been computed, we have the usual two methods for constructing period-by-period price and quantity levels, P^t and Q^t for $t = 1, \ldots, T$. The counterparts to definitions (38) are as follows:

$$P^{t*} \equiv \pi_t^* = \exp[\Sigma_{n\in S(t)} s_{tn} ln(p_{tn}/\alpha_n^*)];$$
$$t = 1, \ldots, T; \quad (45)$$
$$Q^{t*} \equiv \Sigma_{n\in S(t)} p_{tn}q_{tn}/P^{t*};\ t = 1, \ldots, T. \quad (46)$$

Thus, P^{t*} is a weighted geometric mean of the quality-adjusted prices p_{tn}/α_n^* that are present in period t, where the weight for p_{tn}/α_n^* is the corresponding period t expenditure (or sales) share for product n in period t, s_{tn}. The counterparts to definitions (39) are as follows:

$$Q^{t**} \equiv \Sigma_{n\in S(t)} \alpha_n^* q_{tn};\ t = 1, \ldots, T; \quad (47)$$
$$P^{t**} \equiv \Sigma_{n\in S(t)} p_{tn}q_{tn}/Q^{t**}\ t = 1, \ldots, T; \quad (48)$$
$$= \Sigma_{n\in S(t)} p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^* q_{tn} \text{ using (47)}$$
$$= \Sigma_{n\in S(t)} p_{tn}q_{tn}/\Sigma_{n\in S(t)} \alpha_n^*(p_{tn})^{-1}p_{tn}q_{tn}$$
$$= [\Sigma_{n\in S(t)} s_{tn}(p_{tn}/a_n^*)^{-1}]^{-1}$$
$$\leq \exp[\Sigma_{n\in S(t)} s_{tn} ln(p_{tn}/\alpha_n^*)]$$
$$= P^{t*},$$

where the inequality follows from Schlömilch's inequality;[45] that is, a weighted harmonic mean of the quality-adjusted prices p_{tn}/α_n^* that are present in period t, P^{t**}, will always be less than or equal to the corresponding weighted geometric mean of the prices where both averages use the same share weights s_{tn} when forming the two weighted averages. The inequalities $P^{t**} \leq P^{t*}$ imply the inequalities $Q^{t**} \geq Q^{t*}$ for $t = 1, \ldots, T$. This algebra was developed by de Haan (2004b, 2010) and de Haan and Krsinich (2018, 763). The model used by de Haan and Krsinich is a more general hedonic regression model that includes the time dummy model used in the present section as a special case. Thus, their algebra can be applied to all of the subsequent hedonic regression models in the following two sections that use time dummies, share weights, and log prices.

If the estimated errors $e_{tn}^* \equiv lnp_{tn} - \rho_t^* - \beta_n^*$ that implicitly appear in the weighted least squares minimization problem turn out to equal 0, then the underlying model, $p_{tn} = \pi_t a_n$ for $t = 1, \ldots, T$, $n\in S(t)$, holds without error and thus provides a good approximation to reality. Moreover, under these conditions, P^{t*} will equal P^{t**} for all t. If the fit of the model is not good, then it may be necessary to look at other models such as those to be considered in subsequent sections.

The solution to the weighted least squares regression problem defined by (40) can be used to generate imputed prices for the missing products. Thus, if product n in period t is missing, define $p_{tn} \equiv \pi_t^*\alpha_n^*$. The corresponding missing quantity is defined as $q_{tn} \equiv 0$. Some statistical agencies use hedonic regression models to generate imputed prices for missing prices and then use these imputed prices in their chosen index number formula. This imputation procedure is an alternative to the inflation-adjusted carry-forward price procedure explained in Sections 3 and 4. From the viewpoint of the economic approach to index number theory, the Section 4 procedure seems to be preferable since the Fisher index used in Section 4 is a fully flexible functional form, whereas the preferences that are exact for the weighted TPD model must be either linear in quantities or be Cobb–Douglas (in which case the expenditure shares are constant over time and there will be no missing products). However, as indicated earlier, if the error terms in (40) are small, the missing product prices generated by the solution to (40) can be used with some confidence.

The axiomatic properties of the price level functions π_t^* generated by the solution to (40) were studied in Section 21 of Chapter 7 and will be noted in the following section. One unsatisfactory property of the WTPD price levels π_t^* is the

[42] If only price information is available, then replace s_{tn} in (40) by $1/N(t)$, where $N(t)$ is the number of products that are observed in period t.

[43] The unweighted (that is, equally weighted) counterpart least squares minimization problem to (40) sets all $s_{tn} = 1$ for $n\in S(t)$. The resulting first-order conditions are equations (41) and (42) with the positive s_{tn} replaced with a 1.

[44] The resulting system of $T - 1 + N$ equations needs to be of full rank in order to obtain a unique solution.

[45] See Hardy, Littlewood, and Pólya (1934, 26).

following one: A product that is available in only one period out of the T periods has no influence on the aggregate price levels p_t^*. This means that the price of a new product that appears in period T has no influence on the price levels. The hedonic regression models in the next section that make use of information on the characteristics of the products do not have this unsatisfactory property of the weighted time dummy hedonic regression models studied in this section.

6. The Time Dummy Hedonic Regression Model with Characteristics Information

In this section, it is again assumed that there are N products that are available over a window of T periods. As in the previous sections, we again assume that the quantity aggregator function for the N products is the linear function, $f(q) \equiv \alpha \cdot q = \Sigma_{n=1}^{N} \alpha_n q_n$, where q_n is the quantity of product n purchased or sold in the period under consideration and α_n is the quality adjustment factor for product n. What is new is the assumption that the quality adjustment factors are functions of a vector of K *characteristics* of the products. Thus, it is assumed that product n has the vector of characteristics $z^n \equiv [z_{n1}, z_{n2}, \ldots, z_{nK}]$ for $n = 1, \ldots, N$. We assume that this information on the characteristics of each product has been collected.[46] The new assumption in this section is that the quality adjustment factors α_n are functions of the vector of characteristics z^n for each product, and the same function, $g(z)$ can be used for each quality adjustment factor; that is, we have the following assumptions:

$$\alpha_n \equiv g(z^n) = g(z_{n1}, z_{n2}, \ldots, z_{nK});\ n = 1, \ldots, N. \qquad (49)$$

Thus, each product n has its own unique mix of characteristics z^n, but the *same function g* can be used to determine the relative utility to purchasers of the products.[47] Define the period t quantity vector as $q^t = [q_{t1}, \ldots, q_{tN}]$ for $t = 1, \ldots, T$. If product n is missing in period t, then define $q_{tn} \equiv 0$. Using these assumptions, the aggregate quantity level Q^t for period t is defined as

$$Q^t \equiv f(q^t) \equiv \Sigma_{n=1}^{N} \alpha_n q_{tn} = \Sigma_{n=1}^{N} g(z^n) q_{tn};\ t = 1, \ldots, T. \qquad (50)$$

Using our assumption of (exact) utility-maximizing behavior with the linear utility function defined by (50), equations (10) become:

$$p_{tn} = \pi_t g(z^n);\ t = 1, \ldots, T;\ n \in S(t). \qquad (51)$$

The assumption of approximate utility-maximizing behavior is more realistic, so error terms need to be appended to equations (51). We also need to choose a functional form for the *quality adjustment function* or *hedonic valuation function* $g(z)$. Consider the following functional form for the hedonic valuation function:

$$g(z) = g(z_1, \ldots, z_K) \equiv e^{\gamma_0} \prod_{k=1}^{K} Z_k^{\gamma_k}. \qquad (52)$$

Define the logarithms of the *quality adjustment factors* α_n as follows:

$$\beta_n \equiv ln\alpha_n = lng(z^n) = \gamma_0 + \Sigma_{k=1}^{K} \gamma_k lnz_{nk};$$
$$n = 1, \ldots, N, \qquad (53)$$

where we have used assumptions (50) and (53). Now take logarithms of both sides of equations (51) and add error terms e_{tn} to the resulting equations. Using equations (53), we obtain the following system of estimating equations:[48]

$$lnp_{tn} = \rho_t + g_0 + \Sigma_{k=1}^{K} \gamma_k lnz_{nk} + e_{tn};$$
$$t = 1, \ldots, T;\ n \in S(t), \qquad (54)$$

where, as usual, we have defined ρ_t as $\ln\pi_t$ for $t = 1, \ldots, T$. Equations (54) characterize the classic *log linear time dummy hedonic regression model*.[49] Note that our derivation of this model rests on the assumption of approximate utility-maximizing behavior on the part of purchasers of the N products. Note also that our underlying economic model, which sets the error terms equal to zero, assumes that the N products are perfect substitutes once they have been quality adjusted, where the logarithms of the quality adjustment factors are defined by (53).[50]

Estimates for $\rho \equiv [\rho_1, \ldots, \rho_T]$ and $\gamma \equiv [\gamma_0, \gamma_1, \ldots, \gamma_K]$ can be obtained by minimizing the sum of the squared errors e_{tn} which appear in equations (54). This leads to the following least squares minimization problem:

$$\min_{\rho,\gamma} \Sigma_{t=1}^{T} \Sigma_{n \in S(t)} [lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k lnz_{nk}]^2. \qquad (55)$$

A solution ρ, γ to the minimization problem (55) will satisfy the following first-order conditions:

$$\Sigma_{n \in S(t)} [lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k lnz_{nk}] = 0;$$
$$t = 1, \ldots, T; \qquad (56)$$
$$\Sigma_{t=1}^{T} \Sigma_{n \in S(t)} [lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k lnz_{nk}] = 0; \qquad (57)$$
$$\Sigma_{t=1}^{T} \Sigma_{n \in S(t)} [lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k lnz_{nk}] lnz_{nk} = 0;$$
$$k = 1, \ldots, K. \qquad (58)$$

[46] Basically, we want to collect information on the most important price-determining characteristics of each product; see Triplett (2004) and Aizcorbe (2014) for many examples of this type of hedonic regression and references to the applied literature on this topic.

[47] In this section, we require that each of the N products possess a positive amount of each characteristic; that is, we require that $z^n >> 0_K$ for $n = 1, \ldots, N$. This assumption will be relaxed in the following section.

[48] If both sides of equation tn in equations (54) are differentiated with respect to $\ln z_{nk}$, we find that $\partial \ln p_{tn} / \partial \ln z_{nk} = \gamma_k$ for $n \in S(t)$. Thus, γ_k is the percentage change in the price of a product with respect to a 1 percent increase in the amount of characteristic k in a product. In general, this (constant) elasticity will be positive; that is, a small increase in the amount of characteristic k that is present in a generic product will increase the price of the product.

[49] This model was first introduced by Court (1939) as his hedonic suggestion number 2. It was popularized by Griliches (1971, 7) and others. See Triplett (2004) and Aizcorbe (2014) for hundreds of references to the literature on the use of this model.

[50] Thus, smaller in magnitude errors e_{tn} in the hedonic regression imply that the underlying economic model provides a closer approximation to actual behavior; that is, a higher R^2 for the linear regression model defined by (54) means that the underlying economic model provides a closer approximation to actual behavior.

Equations (56)–(58) are $T + 1 + K$ equations in the $T + 1 + K$ unknown parameters in the vectors r and γ. However, solutions to these equations are not unique; if ρ_t for $t = 1, \ldots, T$ and g_k for $k = 0, 1, \ldots, K$ is a solution to (56)–(58), then $\rho_t + \lambda$ for $t = 1, \ldots, T$, $\gamma_0 - \lambda$ and γ_k for $k = 1, \ldots, K$ is also a solution for any number λ. Thus, a normalization on these parameters is required for a unique solution to (56)–(58).[51] Choose the normalization $\rho_1^* = 0$ which is equivalent to $\pi_1^* = 1$. Thus, set $\rho_1^* = 0$ in equations (56)–(58), drop the first equation in equations (56) and solve the remaining $T + K$ equations for $\rho_2^*, \ldots, \rho_T^*$ and $\gamma_0^*, \gamma_1^*, \ldots, \gamma_K^*$.[52] Once these parameters have been determined, the estimated $\beta_n^* \equiv ln\alpha_n^*$ can be defined using definitions (53) as follows:

$$\beta_n^* \equiv ln\alpha_n^* = lng(z^n) = \gamma_0^* + \Sigma_{k=1}^K \gamma_k^* lnz_{nk}; \quad n = 1, \ldots, N. \tag{59}$$

Using equations (56) evaluated at ρ^* and γ^* and definitions (59), we see that $ln\pi_t^* \equiv \rho_t^*$ is equal to the following expression:

$$ln\pi_t^* = [1/N(t)]\, \Sigma_{n\in S(t)}\, ln(p_{tn}/\alpha_n^*); \quad t = 1, \ldots, T, \tag{60}$$

where $\alpha_n^* \equiv \exp[\beta_n^*]$ for $n = 1, \ldots, N$ and where $N(t)$ is equal to the number of products that are available in period t. Thus, the estimated period t price level, π_t^*, is an *equally weighted geometric average of the quality-adjusted prices* p_{tn}/α_n^* for the products that are present in period t.[53] Once the π_t^* have been calculated, the *price index* between periods t and τ is defined as p_t^*/π_τ^* for $1 \le t, \tau \le T$. If quantity data are available, then we have the usual two methods for constructing period-by-period price and quantity levels, P^t and Q^t for $t = 1, \ldots, T$; see (45)–(48).

It is useful to compare the present time dummy hedonic regression that uses characteristics information with the time dummy product regression in the previous section where the only characteristic of each product was the product itself; that is, recall the least squares minimization problem defined by (30). It seems that this earlier model is more general than the present model. To see this, define β_n^* by definitions (59) for $n = 1, \ldots, N$. Substitute these β_n^* into the objective function for the minimization problem defined by (30) in Section 5. Thus, these β_n^* are feasible β_n that could be inserted into (30) but they may not be optimal; that is, in general, we can expect the time dummy product least squares minimization problem defined by (30) to deliver a *lower* sum of squared residuals than the solution to (55) delivers. Thus, we might ask at this point why consider the least squares problem (55) when, in general, the least squares problem (30) will deliver a better outcome in terms of fitting the data? The problem with (30) is that there may be *no unique solution* to the least squares minimization problem (even after setting $\rho_1 = 0$) if product turnover is rapid; that is, if there are very few matched models in the window of observations, then the regression associated with (30) may not have enough degrees of freedom to provide a solution to the first-order condition equations that are associated with this model. An extreme case where there is no unique solution to (30) is the case where every product is a new one which appears in only one period.[54] In this case, there are $T + N - 1$ unknown ρ_t and β_n parameters (after making one normalization) and only T observed prices. Thus, the use of hedonic regressions with characteristics information is particularly useful in situations where there is rapid product turnover and there are relatively few matched models.

The price levels π_t^* defined by (60) are not satisfactory for the following reason: Suppose periods τ and t have exactly the same set of products that are available for those two periods. Then the price index between those two periods is given by:

$$\pi_t^*/\pi_\tau^* = \prod_{n\in S(t)} (p_{tn}/p_{\tau n})^{1/N(t)}. \tag{61}$$

Thus, the price index between the two periods is equal to a simple (equally weighted) geometric average of the price ratios $p_{tn}/p_{\tau n}$ for the products that are present in both periods; that is, the economic importance of the products is not taken into account.[55]

In the previous section, we noted that weighting prices by their economic importance was generally recommended (but not necessary if the fit of the corresponding unweighted hedonic regression was good). The same conclusion applies in the present context. Thus, if quantity information is available (in addition to price and product characteristic information), then it is preferable to generate ρ and γ estimates by solving the following *weighted least squares minimization problem*:[56]

$$\min_{\rho,\,\gamma} \Sigma_{t=1}^T \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^K \gamma_k lnz_{nk}]^2 \tag{62}$$

where the expenditure or sales shares s_{tn} are defined as $s_{tn} \equiv p_{tn}q_{tn}/S_{i\in S(t)}\, p_{ti}q_{ti}$ for $t = 1, \ldots, T$ and $n \in S(t)$. A solution ρ, γ to the minimization problem (62) will satisfy the following first-order conditions:

[51] We also need the modified equations (56)–(58) to satisfy a full rank condition so that the matrix of coefficients associated with these equations can be inverted. Thus, in particular, K, the number of characteristics, cannot be too big relative to N, the number of products.

[52] Alternatively, set $\rho_1 = 0$ in equations (54) and run a simple linear regression to obtain estimates for the remaining parameters.

[53] An equivalent result was derived in Triplett and McDonald (1977, 150).

[54] Housing is an example of such a unique product. Every dwelling unit is uniquely determined by its location and over time, the structure associated with the housing unit depreciates in value with age (or it may appreciate in value due to renovations and improvements). Thus, hedonic regressions with housing characteristics information *must* be used in order to obtain useful price indices for housing. For applications of hedonic regressions to property prices, see Eurostat (2013), Diewert, de Haan, and Hendriks (2015), Hill (2013), Diewert and Shimizu (2015, 2016, 2022), Diewert, Huang, and Burnett-Issacs (2017), and Silver (2018).

[55] As in Section 5, we note that if the estimated squared residuals for this model are small, then the estimated π_t^* defined by (60) will be satisfactory since in this case, $p^t \approx \pi_t^*\alpha^*$ so that prices vary (approximately) proportionally over time, and thus $\prod_{n=1}^N (p_{tn}/\alpha_n^*)^{1/N} \approx \pi_t^*$ for $t = 1, \ldots, T$. Any missing price for period t and product n is defined as $p_{tn} \equiv \pi_t^*\alpha_n^*$ in the products $\prod_{n=1}^N (p_{tn}/\alpha_n^*)^{1/N}$. The idea of using R^2 or the fit of a hedonic regression model to judge its adequacy can be traced back to Silver (2010, S220; 2011, 561). He implicitly suggested that hedonic regressions should only be used when the products under consideration are highly substitutable and hence when the R^2 value for the relevant hedonic regression is high.

[56] Diewert (2003b, 2005b) considered this model for the bilateral case where $T = 2$. Silver and Heravi (2005) and de Haan and Krsinich (2014, 2018) considered the general model.

$$\Sigma_{n\in S(t)}\, s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K}\gamma_k lnz_{nk}] = 0; \quad t = 1,\ldots,T; \qquad (63)$$

$$\Sigma_{t=1}^{T}\Sigma_{n\in S(t)}\, s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K}\gamma_k lnz_{nk}] = 0; \qquad (64)$$

$$\Sigma_{t=1}^{T}\Sigma_{n\in S(t)}\, s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K}\gamma_k lnz_{nk}]lnz_{nk} = 0;\ k = 1,\ldots,K. \qquad (65)$$

Equations (63)–(65) are $T + 1 + K$ equations in the $T + 1 + K$ unknown parameters in the vectors ρ and γ. However, solutions to these equations are not unique; if ρ_t for $t = 1,\ldots,T$ and γ_k for $k = 0,1,\ldots,K$ is a solution to (63)–(65), then $\rho_t + 1$ for $t = 1,\ldots,T$, $\gamma_0 - \lambda$ and γ_k for $k = 1,\ldots,K$ is also a solution for any number λ. Thus, a normalization on these parameters is required for a unique solution to (63)–(65).[57] Choose the normalization $\rho_1^* = 0$ which is equivalent to $\pi_1^* = 1$. Thus, set $\rho_1^* = 0$ in equations (63)–(65), drop the first equation in equations (63), and solve the remaining $T + K$ equations for $\rho_2^*,\ldots,\rho_T^*$ and $\gamma_0^*, \gamma_1^*,\ldots,\gamma_K^*$. Once these parameters have been determined, the estimated b_n^* can be defined as $\beta_n^* \equiv \gamma_0^* + \Sigma_{k=1}^{K}\gamma_k^* lnz_{nk}$ for $n = 1,\ldots,N$. Once β_n^* have been defined, the corresponding quality adjustment factors are defined as $\alpha_n^* \equiv \exp[\beta_n^*] > 0$ for $n = 1,\ldots,N$.

Using equations (63) evaluated at r^* and γ^*, we see that $\pi_t^* \equiv \exp[\rho_t^*]$ is equal to[58]

$$\pi_t^* = \exp[\Sigma_{n\in S(t)}\, s_{tn} ln(p_{tn}/\alpha_n^*)];\ t = 1,\ldots,T \qquad (66)$$

with $\pi_1^* \equiv 1$. Thus, the period t estimated price level, π_t^*, is an expenditure share-weighted geometric mean of the quality-adjusted period t prices, p_{tn}/α_n^*, for the products n that are present in period t. Once π_t^* have been calculated, the *price index* between periods t and τ is defined as p_t^*/π_τ^* for $1 \le t, \tau \le T$. Note that (62) depends on the availability of expenditure share information. If, in addition, quantity data are available, then we have the usual two methods for constructing period-by-period price and quantity levels, P^t and Q^t for $t = 1,\ldots,T$; see (45)–(48).

The new price indices are a clear improvement over their unweighted counterparts defined earlier by equations (60). In the present situation, using equations (66), we see that π_t^*/π_τ^* is a share-weighted geometric mean of the quality-adjusted period t prices, p_{tn}/α_n^*, for the products n that are present in period t with weights s_{tn} in the numerator divided by the share-weighted geometric mean of the quality-adjusted period τ prices, $p_{\tau n}/\alpha_n^*$, for the products n that are present in period τ with weights $s_{\tau n}$ in the denominator. Thus, economic importance of each product counts in the present model, whereas it did not in the corresponding unweighted model.

Note that equations (66) are the same as equations (43) in the previous section. The new quality adjustment parameters α_n^* are defined by the following counterparts to equations (44):

$$\alpha_n^* \equiv \exp[\gamma_0^* + \Sigma_{k=1}^{K}\gamma_k^* lnz_{nk}];\ n = 1,\ldots,N. \qquad (67)$$

Now use definitions (45)–(48) to define P^{t*}, Q^{t*}, P^{t**}, and Q^{t**} where the new p_t^* and α_n^* are defined by (66) and (67). We can again deduce the inequality in (48) using these new definitions; that is, the following inequalities were developed by de Haan (2004b, 2010) and de Haan and Krsinich (2018, 763):

$$P^{t**} \equiv \Sigma_{n\in S(t)}\, p_{tn}q_{tn}/\Sigma_{n\in S(t)}\,\alpha_n^* q_{tn} \le \pi_t^* \equiv P^{t*};\quad t = 1,\ldots,T. \qquad (68)$$

As in the previous section, P^{t*} is a weighted geometric mean of the quality-adjusted prices p_{tn}/α_n^* that are present in period t where the weight for p_{tn}/α_n^* is the period t expenditure (or sales) share for product n in period t, s_{tn}, and P^{t**} is the corresponding weighted harmonic mean of the quality-adjusted prices p_{tn}/α_n^* using the same weights.

The solution to the weighted least squares minimization problem defined by (62) along with the normalization $\rho_1 = 0$ can also be obtained by running the following linear regression with ρ_1 set to zero :

$$(s_{tn})^{1/2}lnp_{tn} = (s_{tn})^{1/2}\rho_t + (s_{tn})^{1/2}\gamma_0 + (s_{tn})^{1/2}\Sigma_{k=1}^{K}\gamma_k lnz_{nk} + e_{tn};\ t = 1,\ldots,T;\ n\in S(t). \qquad (69)$$

The solution to the weighted least squares regression problem defined by (62) can be used to generate imputed prices for the missing products. Thus, if product n in period t is missing, define $p_{tn} \equiv \pi_t^*\alpha_n^*$. The corresponding missing quantity is defined as $q_{tn} \equiv 0$. As was mentioned in the previous section, some statistical agencies use hedonic regression models to generate imputed prices for missing prices and then use these imputed prices in their chosen index number formula. If the weighted sum of squared errors, $\Sigma_{t=1}^{T}\Sigma_{n\in S(t)}\, s_{tn}[\ln p_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K}\gamma_k lnz_{nk}]^2$, is small or equivalently if the R^2 value for the linear regression defined by (69) is large, then using the imputed prices generated by this model to fill in for missing prices is justified.

Using the solution functions for the price levels π_t^* given by (66) plus the definition of the weighted least squares minimization problem (62), it can be shown that π_t^* regarded as a function of $P \equiv [p^1,\ldots,p^T]$, $Q \equiv [q^1,\ldots,q^T]$, and $Z \equiv [z^1,\ldots,z^K]$ satisfies the following eight tests:[59]

Test 1: *The weak identity test for prices*. If $p^\tau = p^t$ and $q^\tau = q^t$, then $\pi_t^*(P,Q,Z) = \pi_\tau^*(P,Q,Z)$.

Test 2: *The weak fixed basket test for prices* or *the weak identity test for quantities*. If $q^\tau = q^t \equiv q$ and $p^\tau = p^t$ then the price index for period t relative to period t is $\pi_t^*(P,Q,Z)/\pi_\tau^*(P,Q,Z) = p^t\cdot q/p^\tau\cdot q$.

Test 3: *Linear homogeneity test for prices*. Let $\lambda > 0$. Then $\pi_t^*(p^1,\ldots,p^{t-1},\lambda p^t,p^{t+1},\ldots,p^T,Q,Z) = \lambda\pi_t^*(P,Q,Z)$ for $t = 1,\ldots,T$. Thus, if all prices in period t are multiplied by a common scalar factor λ, then the price level of period t relative to the price level of any other period r will increase by the multiplicative factor λ.[60]

[57] As usual, we need a full-rank condition to be satisfied so that the matrix of coefficients in the system of linear equations involving ρ and γ can be inverted.

[58] These equations are equivalent to equations (8) in de Haan and Krsinich (2018, 760).

[59] See Diewert (2004, 2005b) for materials on the test approach applied to time product hedonic regressions with and without characteristics information.

[60] Furthermore, the price levels $\pi_\tau^*(P,Q,Z)$ for $\tau \ne t$ are homogeneous of degree 0 in the components of p^t; that is, we have $\pi_\tau^*(p^1,\ldots,p^{t-1},\lambda p^t,p^{t+1},\ldots,p^T,Q,Z) = \pi_\tau^*(P,Q,Z)$ for all $\tau \ne t$.

Test 4: *Homogeneity test for quantities*. Let $\lambda > 0$. Then $\pi_t^*(P,q^1, \ldots,q^{t-1},\lambda q^t,q^{t+1}, \ldots,q^T,Z) = \pi_t^*(P,Q,Z)$ for $t = 1, \ldots,T$. Thus, if all quantities in period t are multiplied by a common scalar factor λ, then the price level of any period r remains unchanged.

Test 5: *Invariance to changes in the units of measurement of the characteristics*. The price level functions $\pi_t^*(P,Q,Z)$ for $t = 1, \ldots,T$ remain unchanged if the K characteristics are measured in different units.

Test 6: *Invariance to changes in the ordering of the commodities*. The price level functions $\pi_t^*(P,Q,Z)$ for $t = 1, \ldots,T$ remain unchanged if the ordering of the N commodities is changed.

Test 7: *Invariance to changes in the ordering of the time periods*. If the T time periods are reordered by some permutation of the first T integers, then the new price level functions are equal to the same permutation of the initial price level functions.

Test 8: *Responsiveness to isolated products test*: If a product is available in only one period in the window of T periods, this test asks that the price level functions $\pi_t^*(P,Q,Z)$ *respond to changes* in the prices of these isolated products; that is, the test asks that the price level functions *are not constant* as the prices for isolated products change. This test is a variation of Test 5 suggested by Zhang, Johansen, and Nygaard (2019), who suggested a bilateral version of this test.[61]

The weighted hedonic regression price levels using characteristics information, $\pi_t^*(P,Q.Z)$, that solve (62), do *not* satisfy the following Tests 9–12.

Test 9: *The strong identity test for prices*. If $p^\tau = p^t$, then $\pi_\tau^*(P,Q,Z) = \pi_t^*(P,Q,Z)$.

Thus, Test 9 is similar to Test 1 but Test 9 asks that the price levels for two periods be equal if the price vectors for the two periods are identical even if the quantity vectors for the two periods are different, whereas Test 1 asks that the price levels for two periods be equal if the price *and* quantity vectors for the two periods are identical.

Test 10: *The strong fixed basket test for prices* or *the strong identity test for quantities*. If $q^\tau = q^t \equiv q$, then the price index for period t relative to period τ is $\pi_t^*(P,Q,Z)/\pi_\tau^*(P,Q,Z) = p^t{\cdot}q/p^\tau{\cdot}q$.[62]

Test 11: *Invariance to changes in the units of measurement for the quantities*. The price level functions $\pi_t^*(P,Q,Z)$ for $t = 1, \ldots,T$ remain unchanged if the N commodities are measured in different units of measurement.

Test 12: *Responsiveness to changes in imputed prices for missing products test*: If there are missing products in one or more periods, then one can define imputed prices for these missing products. This test asks that the price level functions $p_t^*(P,Q,Z)$ *respond to changes* in these imputed prices; that is, the test asks that the price level functions *are not constant* as the imputed prices change. This test allows a price level to decline if new products enter the marketplace during the period and for consumer utility to increase as the number of available products increases. If this test is not satisfied, then the price levels will be subject to *new products bias*. This is an important source of bias in a dynamic product universe.

Many multilateral index number methods do not satisfy the strong identity Tests 9 and 10 and the responsiveness Test 12, so the failure of the hedonic regression price levels to pass these tests is not catastrophic. At first sight, the failure of $\pi_\tau^*(P,Q,Z)$ to pass the invariance to changes in the units of measurement for the N quantities q_n is more worrisome. The failure of this test suggests that the use of hedonic regressions to adjust for quality changes *should be restricted to classes of products that are similar* and have a dominant characteristic that all of the products possess. The quantity q_n of each product should be measured in units of this dominant characteristic. Thus, if the product class is candy bars, the quantity of each product should be measured by its weight. If the product class is a beverage, each product's quantity should be measured by its volume. If this advice is followed, then the unit of measurement for all quantities in the aggregate will be the same. Thus, if the units of measurement change, the change of units should affect all quantities in the same way. It can be shown that the hedonic regression price levels using characteristics information, $\pi_\tau^*(P,Q,Z)$, satisfy the following test:

Test 13: *Restricted change of units test*. If the units of measurement for all products are changed by the same factor, the price levels $\pi_t^*(P,Q,Z)$ remain invariant; that is, the price levels satisfy $p_t^*(\delta^{-1}P,\delta Q,Z) = p_t^*(P,Q,Z)$ for all scalars $\delta > 0$ for $t = 1, \ldots,T$.[63]

Thus, the failure of the hedonic regression price levels to pass the unrestricted change of units test, Test 6, is not catastrophic because for closely related products, these price levels will pass the restricted change of units test, Test 13.

Recall that the weighted TPD price levels defined in the previous section had the undesirable property that a product that is available in only one period out of the T periods had no influence on the aggregate price levels π_t^*. This meant that the price of a new product that appears in period T had no influence on the resulting price levels. The weighted time dummy hedonic price levels $\pi_t^*(P,Q,Z)$ defined in this section no longer have this undesirable property since they satisfy Test 8.

It is possible to apply the tests listed here to the weighted time dummy price levels defined in the previous section. However, in order to do this, the $g(z)$ function defined by (52) needs to be replaced by the linear function $g(z) \equiv \alpha{\cdot}z$ where z is now an N-dimensional vector of characteristics (instead of a K-dimensional vector). Assume that there are N models and the characteristics vector for product n is $z^n \equiv e^n$ for $n = 1, \ldots,N$, where e^n is the nth unit vector; that is, e^n is an N-dimensional vector which has a 1 in component

[61] This test was explicitly suggested by Claude Lamboray.

[62] The price levels $\pi_t^*(P,Q,Z)$ that are directly defined from the solution to (62) using equations (66) will not in general satisfy Test 10. However, if we use the solution to (62) to define the a_n^* and then use definitions (47) and (48) to define the period t price and quantity levels, P^{t**} and Q^{t**}, then the P^{t**} will satisfy Test 2. However, the present set of tests applies to the price levels $\pi_t^*(P,Q,Z)$ that are directly defined by the solution to (62).

[63] Notation: $\delta Q = [\delta q^1,\delta q^2, \ldots, \delta q^T]$; that is, if the N by T matrix Q is multiplied by the scalar δ, then all NT elements in the matrix Q are multiplied by this scalar.

n and zeros elsewhere. Thus, in this case, the Z matrix is the N by N matrix $Z \equiv [z^1,z^2,\ldots,z^N] = I_N$ where I_N is the N by N identity matrix. With this new definition for $g(z)$ and for the matrix Z, we have $g(z^n) = g(e^n) = \alpha \cdot e^n = \alpha_n$ for $n = 1,\ldots,N$, which are equations (49). Equations (51) become $p_{tn} = \pi_t g(z^n) = \pi_t \alpha_n$ for $t = 1,\ldots,T$ and $n \in S(t)$. From these equations, we can follow the steps in the previous section and the counterpart to the weighted least squares minimization problem (62) is (40), the final model in the previous section. Thus, we can apply the preceding tests to the price levels that result from solving (40). We find that the weighted time dummy hedonic price levels without characteristics satisfies Tests 1–7, 11, and 13; they fail Tests 8–10 and 12. Thus, the test performance of both methods is identical except that the price levels from the weighted hedonic TPD model that result from solving (40) pass Test 11 (invariance to changes in the units of measurement for quantities) and fail Test 8 (responsiveness to isolated products test) and the weighted hedonic TPD model that uses characteristics information that result from solving (62) pass Test 8 and fail Test 11.[64]

It is possible to derive some approximate equalities for the α_n^* that are counterparts to the exact equalities (44) for a_n^* that were satisfied for the weighted TPD quality adjustment parameters for the model defined by (40) in the previous section. Recall that the estimated quality adjustment factors for the N products in the present model are α_n^* defined by (67) for $n = 1,\ldots,N$. The logarithms of these estimated quality adjustment factors are $\beta_n^* \equiv ln a_n^* = \gamma_0^* + \Sigma_{k=1}^{K} \gamma_k^* \ln z_{nk}$ for $n = 1,\ldots,N$. Once the $\rho^* \equiv [\rho_1^*,\rho_2^*,\ldots,\rho_T^*]$ and $\gamma^* \equiv [\gamma_0^*,\gamma_1^*,\ldots,\gamma_K^*]$ solution to (62) has been determined (with $\rho_1^* = 1$), the sample residuals e_{tn}^* can be defined by equations (70):

$$\begin{aligned} e_{tn}^* &\equiv lnp_{tn} - \rho_t^* - \gamma_0^* - \Sigma_{k=1}^{K} \gamma_k^* lnz_{nk};\ t = 1,\ldots,T;\ n \in S(t) \qquad (70) \\ &= lnp_{tn} - \rho_t^* - \beta_n^* \\ &= ln(p_{tn}/\pi_t^*) - \beta_n^* \text{ since } \rho_t^* \equiv ln\pi_t^*. \end{aligned}$$

Rearranging equations (70), it can be seen that β_n^* satisfy the following equations:

$$\beta_n^* = ln(p_{tn}/\pi_t^*) - e_{tn}^*;\ n = 1,\ldots,N;\ t \in S^*(n). \qquad (71)$$

For each n, multiply both sides of (71) by the share s_{tn} for each $t \in S^*(n)$ and sum the resulting equations over all t that belong to the set $S^*(n)$. The following system of N equations is obtained:

$$\begin{aligned} \Sigma_{t\in S^*(n)} s_{tn}\beta_n^* &= \Sigma_{t\in S^*(n)} s_{tn}[ln(p_{tn}/\pi_t^*) - e_{tn}^*];\ n = 1,\ldots,N \qquad (72) \\ &\approx \Sigma_{t\in S^*(n)} s_{tn} ln(p_{tn}/\pi_t^*), \end{aligned}$$

where the approximate equalities in (72) will follow since the minimization problem defined by (62) will make the squared errors $(e_{tn}^*)^2$ small within the constraints of the hedonic model. Thus, we have the following approximation for β_n^*:[65]

$$\beta_n^* \approx [\Sigma_{t\in S^*(n)} s_{tn} ln(p_{tn}/\pi_t^*)]/\Sigma_{t\in S^*(n)} s_{tn};\ n = 1,\ldots,N. \qquad (73)$$

Thus, the logarithm of the product n quality adjustment factor, β_n^*, is approximately equal to a share-weighted average of the logarithms of the inflation-adjusted prices p_{tn}/p_t^* for product n over the periods t when this product was sold (or purchased) on the marketplace. Note that the averages on the right-hand sides of the approximate equalities (73) are taken over the entire sample period.

The next few paragraphs will be devoted to addressing a problem that was first posed by de Haan and Krsinich (2018, 760): Are hedonic regression models consistent with the use of unit values to aggregate over narrowly defined products at the first stage of aggregation?

Equations (70) and the definitions $b_n^* \equiv ln\alpha_n^*$ for $n = 1,\ldots,N$ can be used to establish the following equalities:

$$p_{tn} = \alpha_n^* \pi_t^* \, exp[e_{tn}^*];\ t = 1,\ldots,T. \qquad (74)$$

Suppose that the underlying hedonic model holds exactly so that each error term e_{tn}^* is equal to 0. Finally, suppose that all of the products are *perfect substitutes* so that all of the quality adjustment factors α_n^* are equal. Thus, the following equations hold:

$$\alpha_1^* = \alpha_2^* = \ldots = \alpha_N^*. \qquad (75)$$

Thus, all of the estimated a_n^* will equal a_1^* for $n = 2,\ldots,N$. Since $e_{tn}^* = 0$ by assumption, $\exp[e_{tn}^*] = 1$ for $t = 1,\ldots,T$; $n \in S(t)$. Substitute these relationships into equations (74). Now multiply both sides of equation p_{tn} in equations (74) by q_{tn} for $t = 1,\ldots,T$; $n \in S(t)$. We obtain the following system of equations after a certain amount of summation within each period:

$$\Sigma_{n\in S(t)} p_{tn} q_{tn} = a_1^* \pi_t^* \, \Sigma_{n\in S(t)} q_{tn};\ t = 1,\ldots,T. \qquad (76)$$

Now take ratios of equations (76) for $t = 1$ and a general t. After suitable rearrangement, we obtain the following equation for the price index between periods 1 and t:

$$\pi_t^*/\pi_1^* = \{\Sigma_{n\in S(t)} p_{tn}q_{tn}/\Sigma_{n\in S(t)} q_{tn}\}/\{\Sigma_{n\in S(1)} p_{1n}q_{1n}/\Sigma_{n\in S(1)} q_{1n}\};\ t = 1,\ldots,T. \qquad (77)$$

The right-hand side of (77) for period t can be recognized as the *unit value price index* between periods 1 and t.

The preceding algebra resolves the index number discontinuity problem recognized by de Haan and Krsinich (2018, 760). These authors noted that the weighted geometric mean representation for $\pi_t^* = \exp[\Sigma_{n\in S(t)} s_{tn} ln(p_{tn}/\alpha_n^*)]$ (recall equations (66)) did not seem to collapse down to a unit value

[64] However, as indicated earlier, often statistical agencies have to choose the hedonic regression model with characteristics over the TPD model explained in the previous section due to frequent model changes or to the fact that some products are unique (like housing). In the case of unique products, the time dummy approach fails and the characteristics approach is the only viable approach.

[65] These equations provide approximate counterparts to equations (44), which were exact for the weighted TPD model discussed in Section 5.

index if all of the estimated α_n^* were equal, which is disconcerting because if the products are perfect substitutes (without quality adjustment), then the appropriate index should collapse down to a unit value index (because each additional unit of any product gives the purchaser the same utility). However, if the products are perfect substitutes and markets are functioning properly, the price of every product in the group under consideration should be the same in each period. Under these conditions, the estimated α_n^* will all be equal and equations (74) will become $p_{tn} = \alpha_1^*\pi_t^*$ and equations (77) will hold. Thus, under these conditions, there is no discontinuity problem.

As was noted earlier, once the estimated coefficients $\pi^* \equiv [\pi_1^*, \ldots, \pi_T^*]$ and $\alpha^* \equiv [\alpha_1^*, \ldots, \alpha_N^*]$ have been determined, these estimates can be used to determine *imputed prices* for the missing observations; that is, if product n in period t is missing, define $p_{tn} \equiv \pi_t^*\alpha_n^*$. The corresponding missing quantities and shares are defined as $q_{tn} \equiv 0$ and $s_{tn} \equiv 0$. Using these imputed prices and quantities, we can form complete price, quantity, and share vectors for all N products for each period t. Denote these vectors as p^t, q^t, and s^t for $t = 1, \ldots, T$. Using the fact that the share for a missing product is equal to zero, we can rewrite equations (66) as follows:

$$\pi_t^* = \prod_{n=1}^N (p_{tn}/\alpha_n^*)^{s_{tn}};\ t = 1, \ldots, T. \tag{78}$$

Define the sequence of *hedonic price indices*, P_H^t, as $P_H^t \equiv \pi_t^*/\pi_1^*$ for $t = 1, \ldots, T$.[66] Using equations (66) and $\beta_n^* \equiv ln\alpha_n^*$ for $n = 1, \ldots, N$, we have the following expressions for the logarithms of the hedonic price indices:

$$lnP_H^t = \Sigma_{n=1}^N s_{tn}(lnp_{tn} - \beta_n^*) - \Sigma_{n=1}^N s_{1n}(lnp_{1n} - \beta_n^*);\quad t = 1, \ldots, T. \tag{79}$$

It is now possible to compare the sequence of price indices to the corresponding Törnqvist–Theil fixed-base indices that make use of the imputed prices generated by the present model for the missing products. The logarithm of the *fixed-base Törnqvist–Theil price index* between periods 1 and t, P_T^t, is defined as follows:[67]

$$\begin{aligned} lnP_T^t &\equiv \Sigma_{n=1}^N \tfrac{1}{2}(s_{tn} + s_{1n})(lnp_{tn} - lnp_{1n})\ t = 1, \ldots, T \\ &= \Sigma_{n=1}^N \tfrac{1}{2}(s_{tn} + s_{1n})[(lnp_{tn} - \beta_n^*) - (lnp_{1n} - \beta_n^*)]. \end{aligned} \tag{80}$$

Taking the difference between (79) and (80), we can derive the following expressions for $t = 1, 2, \ldots, T$:

$$lnP_H^t - lnP_T^t = \Sigma_{n=1}^N \tfrac{1}{2}(s_{tn} - s_{1n})(lnp_{tn} - \beta_n^*) + \Sigma_{n=1}^N \tfrac{1}{2}(s_{tn} - s_{1n})(lnp_{1n} - \beta_n^*). \tag{81}$$

Since $\Sigma_{n=1}^N (s_{tn} - s_{1n}) = 0$ for each t, the two sets of terms on the right-hand side of equation t in (81) can be interpreted as normalizations of the covariances between $s^t - s^1$ and $lnp^t - \beta^*$ for the first set of terms and between $s^t - s^1$ and $lnp^1 - \beta^*$ for the second set of terms. If the products are highly substitutable with each other, then a low p_{tn} will usually imply that lnp_{tn} is less than the average log price β_n^* and it is also likely that s_{tn} is greater than s_{1n} so that $(s_{tn} - s_{1n})(lnp_{tn} - \beta_n^*)$ is likely to be negative. Hence the covariance between $s^t - s^1$ and $lnp^t - \beta^*$ will tend to be negative. On the other hand, if p_{1n} is unusually low, then lnp_{1n} will be less than the average log price β_n^*, and it is likely that s_{1n} is greater than s_{tn} so that $(s_{tn} - s_{1n})(lnp_{1n} - \beta_n^*)$ is likely to be positive. Hence, the covariance between $s^t - s^1$ and $lnp^1 - \beta^*$ will tend to be positive. Thus, the first set of terms on the right-hand side of (81) will tend to be negative, while the second set will tend to be positive. If there are no divergent trends in log prices and sales shares, then it is likely that these two terms will largely offset each other and under these conditions, P_H^t is likely to approximate P_T^t reasonably well. However, with divergent trends and highly substitutable products, it is likely that the first set of negative terms will be larger in magnitude than the second set of terms and thus P_H^t is likely to be below P_T^t under these conditions. On the other hand, if there are missing products in period 1, then the second set of covariance terms can become very large and positive and outweigh the first set of generally negative terms.[68] The bottom line is that P_H^t and P_T^t can diverge substantially. In such a case, it may be preferable to use the hedonic regression to simply fill in the missing prices and use a superlative index to generate price indices rather than use the price levels π_t^* generated by the hedonic time dummy regression as the price indices.[69]

The hedonic valuation function $g(z)$ defined by (49) has a useful property: One can impose constant returns to scale in the characteristics (the property $g(\lambda z) = \lambda g(z)$ for all $\lambda > 0$) if the g_k satisfy the restriction $\Sigma_{k=1}^K \gamma_k = 1$. However, if we want to apply equations (63)–(65) or equations (69) as estimating equations for the unknown parameters in $g(z)$, we need *positive amounts of all characteristics in all models* so that lnz_{nk} is well defined; that is, we need $z_{nk} > 0$ for all $n = 1, \ldots, N$ and $k = 1, \ldots, K$. The alternative hedonic regression model to be considered at the beginning of the following section relaxes this positivity restriction.

7. Alternative Hedonic Regression Models with Characteristics Information

As noted in the previous section, the hedonic valuation function $g(z)$ defined by (52) requires that positive amounts of all characteristics be present in all N models. It would be useful to have a hedonic regression model that could in principle deal with the introduction of new characteristics over the sample period. This can be done if we replace $g(z)$ defined by (52) by the following functional form for $g(z)$:

[66] Recall that we set $\rho_1^* = 0$ when solving equations (63)–(65), and hence $\pi_1^* = 1$. This fact and the first equation in (66) implies that $\pi_1^* = 1 = \exp[\Sigma_{n\in S(1)} s_{1n}\ln(p_{1n}/\alpha_n^*)] = \exp[\Sigma_{n=1}^N s_{1n}\ln(p_{1n}/\alpha_n^*)]$, and thus $P_H^t \equiv \pi_t^*/\pi_1^* = \pi_t^*$ for $t = 1, \ldots, T$. However, when we compare P_H^t to the corresponding fixed-base Törnqvist index P_T^t, it proves to be more convenient to define P_H^t as π_t^*/π_1^* for $t = 1, \ldots, T$, where π_1^* is defined by the first equation in (66).

[67] The imputed prices and shares defined in equations (78) are used to fill in any missing prices and shares in the Törnqvist formula.

[68] See Diewert (2021, 39) for just such an example.

[69] However, if the fit in the hedonic regression is good, then prices are close to being proportional over time, and the price levels generated by the hedonic regression will generate satisfactory results.

$$g(z_1,z_2,\ldots,z_K) \equiv \exp[\gamma_0 + \Sigma_{k=1}{}^K \gamma_k z_k]. \quad (82)$$

Using this new hedonic valuation function and making the same assumptions (49)–(51) as were made in the previous section along with the new assumption (82), we obtain a new weighted least squares minimization problem that is a counterpart to (62). The new system of estimating equations which are counterparts to equations (69) are the following ones:

$$(s_{tn})^{1/2} lnp_{tn} = (s_{tn})^{1/2}[\rho_t + \gamma_0 + \Sigma_{k=1}{}^K \gamma_k z_{nk}] + e_{tn}; \quad t = 1,\ldots,T;\ n \in S(t), \quad (83)$$

where $\rho_t \equiv ln\mathrm{p}_t$ for $t = 1,\ldots,T$. We can find estimators for the unknown parameters in equations (83) by running the linear regression defined by (83) (with ρ_1 set equal to 0) or by minimizing the following sum of weighted squared residuals e_{tn} with respect to the components of the parameter vectors ρ and γ:[70]

$$\min_{\rho,\gamma} \Sigma_{t=1}{}^T \Sigma_{n \in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}{}^K \gamma_k z_{nk}]^2. \quad (84)$$

A solution ρ, γ to the minimization problem (84) will satisfy the first-order conditions (63)–(65) in the previous section, except that z_{nk} replaces lnz_{nk} for all n and k. The rest of the analysis of the hedonic regression model defined by (84) follows along the same lines as the share-weighted model (62) defined in the previous section. In particular, in order to obtain a unique solution to the modified equations (63)–(65), we impose the normalization $\rho_1 = 0$ and drop the first equation in the modified equations (63).[71] The new product n quality adjustment parameters $\mathrm{b}_n{}^*$ and $\alpha_n{}^*$ are defined by equations (85) and the new sample residuals are defined by equations (86):

$$\beta_n{}^* \equiv ln\alpha_n{}^* = lng(z^n) = \gamma_0{}^* + \Sigma_{k=1}{}^K \gamma_k{}^* z_{nk}; \quad n = 1,\ldots,N; \quad (85)$$

$$e_{tn}{}^* \equiv lnp_{tn} - \rho_t{}^* - \gamma_0{}^* - \Sigma_{k=1}{}^K \gamma_k{}^* z_{nk}; \quad t = 1,\ldots,T;\ n \in S(t); \quad (86)$$
$$= lnp_{tn} - \rho_t{}^* - \beta_n{}^*.$$

The new period t price levels, $\pi_t{}^*$, are still defined by equations (66). The remaining equations (72)–(81) in Section 6 apply to the hedonic regression model defined by (84). Once $\pi_t{}^*$ have been calculated, the *price index* between periods t and τ is defined as $p_t{}^*/\pi_\tau{}^*$ for $1 \le t, \tau \le T$.

As usual, we can use definitions (45)–(48) to define P^{t*}, Q^{t*}, P^{t**}, and Q^{t**}, where the new terms $\pi_t{}^*$ and $\alpha_n{}^*$ are used in these definitions. We can again deduce the de Haan inequalities $P^{t**} \le P^{t*}$ for $t = 1,\ldots,T$. defined by (66) and (67). The axiomatic properties of the new price levels $\pi_t{}^*(P,Q,Z)$ are the same as the properties for the weighted TPD model that was defined by (62) in the previous section.

The hedonic regression models defined by (84) and its equally weighted counterpart which set all $s_{tn} = 1$ were implemented by de Haan and Krsinich (2018) using monthly New Zealand data over three years (so that $T = 36$) for the following seven classes of electronic products: desktop computers, laptop computers, portable media players, DVD players, digital cameras, camcorders, and televisions. For each product class, they had data on approximately 40 characteristics. The data were aggregated across outlets and basically covered the New Zealand market. New products entered each of the seven markets at monthly rates that ranged from 24 to 29 percent and old products disappeared at rates that ranged from 23 to 29 percent. Thus, there was a tremendous amount of product churn in each of the seven categories. Once the weighted and unweighted regressions defined by (84) were run for each category, the alternative price levels, P^{t*} and P^{t**}, were computed for each of the seven categories and compared.[72] They found that P^{t*} was very close to P^{t**} for each category when the weighted regressions were used. This suggests that it may not matter that much which method for computing the P^t is used, since the direct hedonic regression price level estimates $\pi_t{}^*$ were always very close to the indirect estimates based on deflating period t values by $\Sigma_{n \in S(t)} \alpha_n{}^* q_{tn}$. This is a very encouraging result. However, it was a different story for the unweighted hedonic regressions: they were much more volatile than their weighted counterparts, and the direct and indirect price levels that they generated were frequently noticeably different. Moreover the unweighted regressions generated a sequence of price levels that had substantially different trends than the corresponding trends for the weighted regressions. Our conclusion is that the results obtained by de Haan and Krsinich support the use of weighted hedonic regressions over their unweighted counterparts.

The preceding results were for regressions that covered the entire sample period. Statistical agencies that produce CPIs need to produce monthly indices that do not revise the data for the previous months. In order to deal with these constraints, Ivancic, Diewert, and Fox (2009) suggested the use of a rolling window time dummy regression approach with a window length of 13 months (so that strongly seasonal commodities could play a role in the resulting indices). De Haan and Krsinich (2018, 773) implemented this rolling window approach for their seven product categories with a window length of 13 consecutive months for each weighted hedonic regression. The month-to-month change in the estimated price levels (using the P^{t**} option) for the last two months in the new window was used to update the results of the previous regression. Thus, in the end, they could compare this rolling window approach to

[70] This is precisely the model studied by de Haan and Krsinich (2018). The results we derive below are identical to their results.

[71] As usual, we need a full-rank condition to be satisfied so that the matrix of coefficients in the system of linear equations involving ρ and γ can be inverted.

[72] The average unadjusted R^2 for the seven weighted models was 0.981. The corresponding R^2 for the equally weighted models was 0.885. This suggests that the popular products were close substitutes with each other while the unpopular models were not as close substitutes. The fact that the R^2_{values} for the seven classes of products were so high means that the underlying assumption of a linear aggregator function (after quality adjustment) is adequate to describe the data, and thus it is not necessary to explore the alternative models for estimating reservation prices that will be explained in subsequent sections. Of course, the drawback to the hedonic regression models with characteristics is that it is necessary to collect information on characteristics, whereas the reservation price models that will be explained in subsequent sections do not require information on characteristics.

the generation of a price level series for each of the seven categories with the corresponding one big weighted regression approach. For three of the seven categories, they found that the rolling window series ended up well below the corresponding single regression series and for one category, the rolling window series ended up well above the corresponding single regression series. This shows evidence of chain drift in these four rolling window series. For these four series, it may be best to lengthen the window length for the rolling window hedonic regressions. This will usually cure the chain drift problem.

For our next hedonic model, we introduce a *discrete characteristic category*; that is, each product n has a characteristic where there are M separate states for this characteristic. For example, the product may come in 3 distinct package sizes: small, medium, and large. In this case, $M = 3$. In addition, there are K continuous price-determining characteristics, and each product n has varying amounts of these characteristics. As usual, denote the vector of continuous characteristics for product n by $z^n \equiv [z_{n1}, \ldots, z_{nK}]$ for $n = 1, \ldots, N$. If product n belongs to discrete category m, define the M-dimensional vector x^n for this product as $x^n \equiv [x_{n1}, \ldots, x_{nM}] = e^m$, where e^m is a unit vector with a 1 in component m and zeros elsewhere. We assume that there is at least one product that belongs to each of the M discrete categories. We assume the existence of a hedonic product valuation function, $g(z^n, x^n)$, that gives us the relative values for the N products where the logarithm of $g(z^n, x^n)$ is defined as follows:

$$lng(z^n, x^n) \equiv \gamma_0 + \Sigma_{k=1}^{K} \gamma_k z_{nk} + \Sigma_{m=1}^{M} \delta_m x_{nm}; \quad n = 1, \ldots, N. \tag{87}$$

As usual, the exact hedonic model for the prices is $p_{tn} = \pi_t g(z^n, x^n)$ for $t = 1, \ldots, T$ and $n \in S(t)$. By taking logarithms of both sides of these price equations, using $\rho_t \equiv \ln\pi_t$ for $t = 1, \ldots, T$ and using definitions (87) for the N products in the sample, we obtain the following *weighted hedonic regression model*:

$$(s_{tn})^{1/2} lnp_{tn} = (s_{tn})^{1/2} [\rho_t + \gamma_0 + \Sigma_{k=1}^{K} \gamma_k z_{nk} + \Sigma_{m=1}^{M} \delta_m x_{nm}] + e_{tn}; \; t = 1, \ldots, T; \; n \in S(t). \tag{88}$$

Rather than running this linear regression (after imposing the normalizations $\rho_1 = 0$ and $\delta_1 = 0$), we could instead minimize the following *weighted* sum of squared residuals:

$$\min_{\rho,\gamma,\delta} \Sigma_{t=1}^{T} \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k z_{nk} - \Sigma_{m=1}^{M} \delta_m x_{nm}]^2, \tag{89}$$

where $\rho \equiv [\rho_1, \ldots, \rho_T]$, $g \equiv [\gamma_0, \gamma_1, \ldots, \gamma_K]$, and $\delta \equiv [\delta_1, \ldots, \delta_M]$. A solution ρ, γ, δ to the minimization problem (89) will satisfy the following first-order conditions:

$$\Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k z_{nk} - \Sigma_{m=1}^{M} \delta_m x_{nm}] = 0; \quad t = 1, \ldots, T; \tag{90}$$

$$\Sigma_{t=1}^{T} \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k z_{nk} - \Sigma_{m=1}^{M} \delta_m x_{nm}] = 0; \tag{91}$$

$$\Sigma_{t=1}^{T} \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k z_{nk} - \Sigma_{m=1}^{M} \delta_m x_{nm}] z_{nk} = 0; \; k = 1, \ldots, K; \tag{92}$$

$$\Sigma_{t=1}^{T} \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k z_{nk} - \Sigma_{m=1}^{M} \delta_m x_{nm}] x_{nm} = 0; \; m = 1, \ldots, M. \tag{93}$$

Equations (90)–(93) are $T + 1 + K + M$ equations in the $T + 1 + K + M$ unknown parameters in the vectors ρ, γ, and δ. However, solutions to these equations are not unique: The variables associated with the ρ_t, g_0, and the δ_m parameters are collinear. In order to obtain a unique solution to equations (90)–(93), it is necessary to impose *two normalizations* on these parameters. Choose the normalizations $\rho_1^* = 0$ (which is equivalent to $\pi_1^* = 1$) and $\delta_1^* = 0$. Thus, set $\rho_1^* = 0$ and $\delta_1^* = 0$ in equations (90)–(93), drop the first equation in equations (90), drop the first equation in (93), and solve the remaining $T + K + M - 1$ equations for $r_2^*, \ldots, \rho_T^*, \gamma_0^*, \gamma_1^*, \ldots, \gamma_K^*, \delta_2^*, \ldots, \delta_M^*$.[73] Once these parameters have been determined, define the estimated *logarithm of the quality adjustment factor for product n* as

$$\beta_n^* \equiv \gamma_0^* + \Sigma_{k=1}^{K} \gamma_k^* z_{nk} + \Sigma_{m=1}^{M} \delta_m^* x_{nm} = ln\alpha_n^*; \quad n = 1, \ldots, N. \tag{94}$$

Once β_n^* have been defined, the corresponding *quality adjustment factors* are defined as $\alpha_n^* \equiv \exp[\beta_n^*] > 0$ for $n = 1, \ldots, N$. Evaluate equations (90)–(93) at the solution $\rho^*, \gamma^*, \delta^*$, where $\rho_1^* = 0$ and $\delta_1^* = 0$.[74] Using definitions (94), equations (90) evaluated at the preceding solution become the following equations:

$$\rho_t^* = \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \beta_n^*] = ln\pi_t^*; \; t = 1, \ldots, T. \tag{95}$$

Thus, the period t estimated price level $\pi_t^* \equiv \exp[\rho_t^*]$ is a period t share-weighted geometric average of the period t quality-adjusted prices, p_{tn}/a_n^*, for $n \in S(t)$.

With some new definitions, it is possible to provide fairly transparent interpretations for the discrete variable parameters, δ_m^*. Define the set of observations t,n that are in the discrete product group m as $S^{**}(m)$ for $m = 1, \ldots, M$. For each model n, define the *partial log adjustment factor* μ_n^* for the continuous characteristics as follows:

$$\mu_n^* \equiv \gamma_0^* + \Sigma_{k=1}^{K} \gamma_k^* lnz_{nk}; \; n = 1, \ldots, N. \tag{96}$$

Using these new definitions, it can be seen that equations (93), evaluated at the normalized solution to the weighted least squares minimization problem (89), can be rewritten as follows:

$$\delta_m^* = \Sigma_{t,n\in S^{**}(m)} s_{tn}[\ln p_{tn} - \rho_t^* - \mu_n^*]/\Sigma_{t,n\in S^{**}(m)} s_{tn}; \quad m = 1, \ldots, M. \tag{97}$$

[73] The number of observations in the window of observations must be equal to or greater than $T + K + M - 1$. More generally, the rank of the coefficient matrix that is associated with the $T + K + M - 1$ remaining equations in the system of equations defined by (90)–(93) is assumed to be full so that the coefficient matrix has an inverse.

[74] All $T + K + M + 1$ of equations (90)–(93) will be satisfied at this solution.

Define $\theta_n^* = \exp[\mu_n^*]$ for $n = 1, \ldots, N$. Then $\exp[\delta_m^*]$ is equal to a share-weighted geometric average of the partially quality-adjusted prices $p_{tn}/\pi_t^*\theta_n^*$ for all t,n that belong to the set $S^{**}(m)$; that is, for all observations over all periods on products that are in group m for the discrete characteristic. Thus, the characteristics of d_m^* given by equations (97) are intuitively plausible.

The analysis in the previous section can be adapted to the model defined by (89). Once π_t^* have been calculated using definitions (95), the *price index* between periods t and τ is defined as π_t^*/π_τ^* for $1 \le t, \tau \le T$. Once α_n^* and π_t^* have been calculated using (94) and (95), we have the usual two alternative methods for constructing period-by-period price and quantity levels, P^t and Q^t, for $t = 1, \ldots, T$. The first uses the π_t^* estimates as follows:

$$P^{t*} \equiv \pi_t^*;\ t = 1, \ldots, T; \tag{98}$$

$$Q^{t*} \equiv \Sigma_{n \in S(t)}\, p_{tn} q_{tn}/P^{t*};\ t = 1, \ldots, T. \tag{99}$$

The second method uses the a_n^* estimates as follows:

$$Q^{t**} \equiv \Sigma_{n \in S(t)}\, \alpha_n^* q_{tn};\ t = 1, \ldots, T; \tag{100}$$

$$P^{t**} \equiv \Sigma_{n \in S(t)}\, p_{tn} q_{tn}/Q^{t**};\ t = 1, \ldots, T. \tag{101}$$

As usual, we have the inequalities $P^{t**} \le P^{t*}$ for $t = 1, \ldots, T$.

As was the case for the previous hedonic regression models, the present model can be used to generate estimates for missing prices using the equations $p_{tn} \equiv \pi_t^*\alpha_n^*$ if product n is missing in period t. Using these estimates for missing prices, the analysis following equation (81) can be used to analyze the difference between $P^{t*} = \pi_t^*/\pi_1^*$ and the Törnqvist–Theil index $P_T^{\,t}$ for period t.

We conclude this section by providing one more extension of the basic hedonic regression model using characteristics defined by (84).

In many cases, the continuous characteristics that describe a product or model range from very low values to very high values. In such cases, it is unlikely that a single parameter γ_k could provide an adequate approximation to the value of additional amounts of the characteristic over the entire range of feasible characteristic values. To deal with this difficulty, *piecewise linear spline functions* can be introduced into the hedonic model. Thus, let y be the amount of a continuous characteristic that takes on a wide range of values. We again assume that there are N models or products and T time periods and we can observe the amounts $z_1, \ldots, z_K$ of K continuous characteristics (where a single parameter γ_k can capture the value of an additional unit of z_k for $k = 1, \ldots, K$) and the highly variable characteristic y that each product n has.

In order to obtain more flexibility with respect to the y characteristic, the observed products could be grouped into say three groups with respect to the amounts of y that they possess: low, medium, and high amounts of y. In order to parameterize this grouping, pick y^* and y^{**} such that approximately one-third of the sample observations have $y \le y^*$, one-third have $y^* < y$ £ y^{**}, and one-third have $y^{**} < y$. Define the following *dummy variable functions*, $D_i(y)$ for i = 1,2,3, which depend on y:

$$D_1(y) \equiv 1 \textit{ if } y \le y^* \text{ and is equal to 0 elsewhere;} \tag{102}$$

$$D_2(y) \equiv 1 \textit{ if } y^* < y \le y^{**} \text{ and is equal to 0 elsewhere;} \tag{103}$$

$$D_3(y) \equiv 1 \textit{ if } y^{**} < y \text{ and is equal to 0 elsewhere.} \tag{104}$$

The preceding functions can be used to define the logarithm of the following *partial hedonic valuation function* h(y):

$$lnh(y) \equiv D_1(y)\varphi_1 y + D_2(y)[\varphi_1 y^* + \varphi_2(y - y^*)] + D_3(y)[\varphi_1 y^* + \varphi_2(y^{**} - y^*) + \varphi_3(y - y^{**})]. \tag{105}$$

Note that the logarithm of $h(y)$ is a piecewise linear function of y.[75] If $\varphi_1 = \varphi_2 = f_3$, then $lnh(y) = \varphi_1 y$; that is, under these conditions, $lnh(y)$ becomes a linear function of y.

We assume the existence of an *overall hedonic valuation function*, $g(z^n, y^n)$, that defines the relative utility for the N products where product n has characteristics defined by vector $z^n \equiv [z_{n1}, \ldots, z_{nK}]$ and the scalar y^n. The logarithm of $g(z^n, y^n)$ is defined as follows:

$$lng(z^n, y^n) \equiv \gamma_0 + \Sigma_{k=1}^{K} \gamma_k z_{nk} + lnh(y^n);\ n = 1, \ldots, N. \tag{106}$$

As usual, the exact hedonic model for the sample prices is $p_{tn} = \pi_t g(z^n, y^n)$ for $t = 1, \ldots, T$ and $n \in S(t)$. By taking logarithms of both sides of these price equations, using $\rho_t \equiv ln\mathrm{p}_t$ for $t = 1, \ldots, T$ and using definitions (105) and (106), we obtain the following *hedonic regression model*:

$$lnp_{tn} = \rho_t + g_0 + \Sigma_{k=1}^{K} \gamma_k z_{nk} + lnh(y^n) + e_{tn};\quad t = 1, \ldots, T;\ n \in S(t), \tag{107}$$

where $\ln h(y^n)$ is defined by evaluating (105) at $y = y^n$. It can be seen that the unknown parameters, $\rho \equiv [\rho_1, \ldots, \rho_T]$, $g \equiv [\gamma_0, \gamma_1, \ldots, \gamma_K]$, and $\varphi \equiv [\varphi_1, \varphi_2, \varphi_3]$, appear on the right-hand sides of equations (107) in a linear fashion so the unknown parameters can be estimated using linear regression techniques.

In order to take into account the economic importance of each model, estimates for the unknown parameters in equations (107) can be obtained by minimizing the following *weighted* sum of squared residuals:

$$\min_{\rho,\gamma,\varphi} \Sigma_{t=1}^{T} \Sigma_{n \in S(t)}\, s_{tn}[lnp_{tn} - \rho_t - \gamma_0 - \Sigma_{k=1}^{K} \gamma_k z_{nk} - lnh(y^n)]^2. \tag{108}$$

We leave the further analysis of this model to the reader after noting that in order to obtain a unique solution to (108), we require a normalization on ρ_t and γ_0 such as $\rho_1 = 0$.

It is not necessary to restrict ourselves to hedonic regression models where the hedonic valuation function $g(z,y)$ is such that $lng(z,y)$ is linear in the unknown parameters. One can choose functions $g(z,y)$ such that $lng(z,y)$ is a nonlinear function of the unknown parameters and use nonlinear

[75] This function is known as a *linear spline function* in the literature on nonparametric approximations. The points y^* and y^{**} are called break points or knots. With a sufficient number of break points, any continuous function can be arbitrarily well approximated by a linear spline function. See Poirier (1976) for applications of regression models using splines.

estimation techniques to estimate the parameters. However, when estimating nonlinear regression models that are fairly complex, it is not wise to attempt to estimate the final model right away. It is best if there are very simple models that can be nested in the final model so that one starts by estimating the simplest model and gradually, more bells and whistles are added until one arrives at the final model. The final parameter values for a simpler model should be used as starting parameter values in the next stage model if possible.[76]

All of the models for quality adjustment that we have considered thus far have assumed constant tastes; that is, the functional form for the aggregator function $f(q)$ and for the hedonic valuation functions $g(z^n,y^n,x^n)$ have remained constant over the sample period. In the following section, this assumption will be relaxed.

8. Hedonics and the Problem of Taste Change: Hedonic Imputation Indices

A problem with hedonic regression models that are applied over many periods is that consumer tastes may change over time. In this section, we will outline three possible methods for dealing with the problem of taste change.

The first method that could be used to deal with taste change is to restrict the time dummy hedonic regression models to the case of two adjacent periods. Each pair of periods allows for a different set of tastes.[77] As each adjacent period time dummy regression model is run for say periods t–1 and t, the estimated price level ratio, say π_t^*/π_{t-1}^*, is used as an update factor for the price level of period t – 1. Each bilateral regression will generate a set of quality adjustment factors which can be used to fill in missing prices. Over time, these quality adjustment factors will change. It can be seen that this model of taste change is somewhat inconsistent over time but it does allow for taste change.

The second method for dealing with taste change is similar to the first method, except instead of holding tastes constant for two consecutive periods, we hold tastes constant for T consecutive periods. When the data for a subsequent period becomes available, the data for the first period is dropped, the data for the new period is added to form a new window of T observations and a new time dummy hedonic regression is run. This method assumes that tastes change more slowly than the first method. This *rolling window time dummy hedonic regression model* [78] has a new problem that did not arise with the adjacent period model: How should the results of the new regression be linked to the results of the previous regression? Thus, suppose the first window of observations generates the sequence of price levels, π_1^1, π_2^1, . . ., π_T^1 and these levels are labeled as official indices for the first T periods. Suppose the time dummy hedonic regression for the second window generates the sequence of price levels π_2^2, π_3^2, . . ., π_{T+1}^2. How exactly should the official index for period T + 1 be constructed? Ivancic, Diewert, and Fox (2009, 2011) suggested using period T as the linking observation. Krsinich (2016, 383) called this the *movement splice* method for linking the two windows. Krsinich (2016, 383) also suggested that a better choice of the linking observation in the context of her multilateral model was t = 2, and she called this the *window splice* method. De Haan (2015, 26) suggested that the link period t should be chosen to be in the middle of the first window time span; that is, choose t = T/2 if T is an even integer or t = (T + 1)/2 if T is an odd integer. The Australian Bureau of Statistics (2016, 12) called this the *half splice* method for linking the results of the two windows. Ivancic, Diewert, and Fox (2011, 33) and Diewert and Fox (2021) argued that each choice of a linking period t running from t = 2 to t = T is an equally valid choice of a period to link the two sets of price levels. Thus, they suggested the *mean splice*, defined as the geometric mean of all of the possible estimates for π_{T+1} using each of the T – 1 possible link periods. The first three methods of linking one window to the next window are easy to explain to the public, but the mean splice seems to be the least "risky" and follows standard statistical practice; that is, if one has many estimators for the same thing that are equally plausible, then taking an average of these estimators is recommended. It can be seen that this model of taste change is again slightly inconsistent; the models are internally consistent within each window of observations but when we move from one window to another, this internal consistency is lost.

The third method for dealing with taste change is to simply estimate a separate hedonic regression for each time period. This method is called the *hedonic imputation method*. In order to explain this method and its connection to the adjacent period time dummy model, it is necessary to develop the algebra for both methods for the case of two time periods.

We first develop the algebra for the adjacent period time dummy hedonic regression model. Recall the model defined in the previous section by solving the weighted least squares minimization problem defined by (84). Consider the special case of this model with only two periods so that T = 2. We reparameterize this problem defined by (84) for the case T = 2 and consider the following equivalent problem:

$$\min_{\theta,\gamma} \Sigma_{t=1}^{2} \Sigma_{n\in S(t)} s_{tn}[lnp_{tn} - \theta_t - \Sigma_{k=1}^{K} \gamma_k z_{nk}]^2, \qquad (109)$$

where $\theta \equiv [\theta_1,\theta_2]$ and $g \equiv [\gamma_1, . . .,\gamma_K]$. Comparing (109) with (84) for T = 2, it can be seen that $\theta_1 = \rho_1 + \gamma_0 = g_0$ (since we set ρ_1 = 0 when using the model defined by (84)) and $\theta_2 = \rho_2 + \gamma_0$. Thus, the two problems are completely equivalent once we impose the normalization r_1 = 0 on (84) for the case where T = 2. The first-order conditions that determine a unique solution to (109)[79] are the following 2 + K equations:

[76] For examples of nonlinear hedonic models that make use of this nesting technique, see Chapter 10 or Diewert, de Haan, and Hendriks (2015), Diewert and Shimizu (2015, 2016, 2022), or Diewert, Huang, and Burnett-Issacs (2017).

[77] This method was developed by Court (1939) and popularized by Griliches (1971). It is called the adjacent period time dummy hedonic regression model.

[78] This rolling window time dummy hedonic model was implemented by Ivancic, Diewert, and Fox (2009) and Shimizu, Nishimura, and Watanabe (2010).

[79] As usual, the coefficient matrix for the unknown parameters in equations (110) and (111) must be of full rank (which is K + 2) in order to obtain a unique solution. This means that the number of observations must be equal to or greater than K + 2.

$$\Sigma_{n\in S(t)}\, s_{tn}[lnp_{tn} - \theta_t^* - \Sigma_{k=1}^{K}\gamma_k^* z_{nk}] = 0;\ t = 1,2; \quad (110)$$
$$\Sigma_{t=1}^{2}\Sigma_{n\in S(t)}\, s_{tn}[lnp_{tn} - \theta_t^* - \Sigma_{k=1}^{K}\gamma_k^* z_{nk}]z_{nk} = 0;$$
$$k = 1,\ldots,K. \quad (111)$$

Denote the solution to (110) and (111) by $\theta^* \equiv [\theta_1^*,\theta_2^*]$ and $\gamma^* \equiv [\gamma_1^*,\ldots,\gamma_K^*]$. Estimates for the parameters γ_0 and ρ_2 that were used in our initial parameterization of the model defined by (84) for the case where $T = 2$ can be recovered from the solution to (110) and (111) as follows:[80]

$$\gamma_0^* \equiv \theta_1^*;\ \rho_1^* \equiv 0;\ \rho_2^* \equiv \theta_2^* - \theta_1^*. \quad (112)$$

The estimated quality adjustment parameters, β_n^* and α_n^*, for the model defined by (84) can be recovered from the estimated q_t^* and γ_k^* by using the equations $\beta_n^* \equiv \theta_1^* + \Sigma_{k=1}^{K}\gamma_k^* z_{nk}$; $\alpha_n^* \equiv \exp[\beta_n^*]$ for $n = 1,\ldots,N$.

However, for the remainder of this section, it will prove to be more convenient to define new quality adjustment parameters, β_n^{**} and α_n^{**}, as follows:

$$\beta_n^{**} \equiv \Sigma_{k=1}^{K}\gamma_k^* z_{nk};\ \alpha_n^{**} \equiv \exp[\beta_n^{**}];\ n = 1,\ldots,N. \quad (113)$$

Equations (110), definitions (113), and the equations $\Sigma_{n\in S(t)}\, s_{tn} = 1$ for each t imply that the estimated θ_1^* and θ_2^* satisfy the following equations:

$$\theta_t^* = \Sigma_{n\in S(t)}\, s_{tn}[lnp_{tn} - \Sigma_{k=1}^{K}\gamma_k^* z_{nk}]\ t = 1,2; \quad (114)$$
$$= \Sigma_{n\in S(t)}\, s_{tn} ln(p_{tn}/\alpha_n^{**}).$$

Using equations (112) and (113), we obtain the following expressions for ρ_2^* which is the logarithm of the price index π_2^*/π_1^* generated by the time dummy adjacent period hedonic regression model:[81]

$$\rho_2^* \equiv \theta_2^* - \theta_1^* \quad (115)$$
$$= \Sigma_{n\in S(2)}\, s_{2n} ln(p_{2n}/\alpha_n^{**}) - \Sigma_{n\in S(1)}\, s_{1n} ln(p_{1n}/\alpha_n^{**})$$
$$= \Sigma_{n\in S(2)}\, s_{2n}[lnp_{2n} - \Sigma_{k=1}^{K}\gamma_k^* z_{nk}] - \Sigma_{n\in S(1)}\, s_{1n}[lnp_{1n} - \Sigma_{k=1}^{K}\gamma_k^* z_{nk}].$$

This completes the algebra for the reparameterization of the time dummy adjacent period hedonic regression model. In what follows, we will develop the algebra for entirely separate hedonic regression models for each period. In this model, the hedonic surfaces for the two periods, $\theta_1^* + \Sigma_{k=1}^{K}\gamma_k^* z_{nk}$ and $\theta_2^* + \Sigma_{k=1}^{K}\gamma_k^* z_{nk}$, differed only in their constant terms. In the following model, the hedonic surfaces can shift in a non-parallel fashion.

Consider the following two weighted least squares minimization problems:

$$\min_{\theta,\gamma} \Sigma_{n\in S(1)}\, s_{1n}[lnp_{1n} - \theta^1 - \Sigma_{k=1}^{K}\gamma_k^1 z_{nk}]^2; \quad (116)$$
$$min_{\theta,\gamma} \Sigma_{n\in S(2)}\, s_{2n}[lnp_{2n} - \theta^2 - \Sigma_{k=1}^{K}\gamma_k^2 z_{nk}]^2, \quad (117)$$

where the unknown parameters in (116) are θ^1 and $\gamma^1 \equiv [\gamma_1^1,\ldots,\gamma_K^1]$ and the unknown parameters in (117) are θ^2 and $\gamma^2 \equiv [\gamma_1^2,\ldots,\gamma_K^2]$. In the previous model defined by (109), there was only one vector of γ parameters to model prices in both periods, while the new models defined by (116) and (117) have separate quality adjustment parameter vectors, γ^1 and γ^2.

The first-order conditions for (116) are equations (118) and (119), while the first-order conditions for (117) are equations (120) and (121) below:

$$\Sigma_{n\in S(1)}\, s_{1n}[lnp_{1n} - \theta^{1*} - \Sigma_{k=1}^{K}\gamma_k^{1*} z_{nk}] = 0; \quad (118)$$
$$\Sigma_{n\in S(1)}\, s_{1n}[lnp_{1n} - \theta^{1*} - \Sigma_{k=1}^{K}\gamma_k^{1*} z_{nk}]z_{nk}$$
$$= 0;\ k = 1,\ldots,K; \quad (119)$$
$$\Sigma_{n\in S(2)}\, s_{2n}[lnp_{2n} - \theta^{2*} - \Sigma_{k=1}^{K}\gamma_k^{2*} z_{nk}] = 0; \quad (120)$$
$$\Sigma_{n\in S(2)}\, s_{2n}[lnp_{2n} - \theta^{2*} - \Sigma_{k=1}^{K}\gamma_k^{2*} z_{nk}]z_{nk} = 0;$$
$$k = 1,\ldots,K. \quad (121)$$

Let θ^{1*} and $\gamma_1^{1*},\ldots,\gamma_K^{1*}$ solve (118) and (119) and let θ^{2*} and $\gamma_1^{2*},\ldots,\gamma_K^{2*}$ solve (120) and (121). There are now *two sets of quality adjustment factors*: $\alpha_1^{1*},\ldots,\alpha_N^{1*}$ for period 1 and $\alpha_1^{2*},\ldots,\alpha_N^{2*}$ for period 2. The logarithms of these parameters are defined as follows:

$$ln\alpha_n^{1*} \equiv \Sigma_{k=1}^{K}\gamma_k^{1*} z_{nk};\ ln\alpha_n^{2*} \equiv \Sigma_{k=1}^{K}\gamma_k^{2*} z_{nk};$$
$$n = 1,\ldots,N. \quad (122)$$

Using (118), (120), and definitions (122), we obtain the following expressions for θ^{1*} and θ^{2*} as *quality-adjusted log prices* for periods 1 and 2:

$$\theta^{1*} = \Sigma_{n\in S(1)}\, s_{1n} ln(p_{1n}/\alpha_n^{1*}) = \Sigma_{n\in S(1)}\, s_{1n}[lnp_{1n} - \Sigma_{k=1}^{K}\gamma_k^{1*} z_{nk}]; \quad (123)$$
$$\theta^{2*} = \Sigma_{n\in S(2)}\, s_{2n} ln(p_{2n}/\alpha_n^{2*}) = \Sigma_{n\in S(2)}\, s_{2n}[lnp_{2n} - \Sigma_{k=1}^{K}\gamma_k^{2*} z_{nk}]. \quad (124)$$

The average measure of log price change going from period 1 to 2 using the adjacent period time dummy hedonic model was $\rho_2^* = \theta_2^* - \theta_1^*$; see (115). Note that the same quality adjustment factors, α_n^*, were used to quality adjust prices in both periods. At first glance, we might think that an analogous measure of average constant quality log price in our new model could be defined as $\theta^{2*} - \theta^{1*}$. However, looking at (123) and (124), we see that the quality adjustment factors are not held constant in constructing this measure. The underlying exact models are now $p_{1n} = \exp[q^{1*}]\alpha_n^{1*}$ for $n\in S(1)$ and $p_{2n} = \exp[\theta^{2*}]\alpha_n^{2*}$ for $n\in S(2)$. Thus, the period 1 quality-adjusted prices, p_{1n}/α_n^{1*}, are not comparable to their period 2 counterparts, p_{2n}/α_n^{2*}, unless $\alpha_n^{1*} = \alpha_n^{2*}$. Hence, π_2^*/π_1^* is not a useful price index that compares like with like.

[80] The new γ_k^* are equal to the old g_k^* for $k = 1,\ldots,K$.

[81] If the model defined by (109) held exactly so that all error terms were equal to 0, then $\ln p_{1n} = \theta_1^* + \ln\alpha_n^{**}$ for $n\in S(1)$ and $\ln p_{2n} = \theta_2^* + \ln\alpha_n^{**}$ for $n\in S(2)$. Thus, $p_{1n}/\alpha_n^{**} = \exp[\theta_1^*]$ for each $n\in S(1)$ and $p_{2n}/\alpha_n^{**} = \exp[\theta_2^*]$ for each $n\in S(2)$. Thus, each quality-adjusted period t price, p_{tn}/α_n^{**} for $n\in S(t)$, is an estimator for $\exp[\theta_t^*]$, and thus a weighted geometric mean of these quality-adjusted prices (where the weights sum to 1) is also an estimator for $\exp[\theta_t^*]$.

At this point, the analysis could go in at least three different directions:

- Use the two hedonic regressions to fill in the missing prices; that is, if $n \in S(1)$ but $n \in S(2)$, define $p_{2n} \equiv \exp[\theta^{2*}]\alpha_n^{2*}$ and $q_{2n} = 0$. If $n \in S(2)$ but $n \in S(1)$, define $p_{1n} \equiv \exp[\theta^{1*}]\alpha_n^{1*}$ and $q_{1n} = 0$. Using these estimated prices, we would have complete overlapping price and quantity data for the two periods. Now use the actual data along with the imputed data to calculate a favorite price index and define the companion quantity index residually by deflating the value ratio by the price index. The problem with this strategy is that the quantity index that emerges using this strategy cannot be given a welfare interpretation because preferences are allowed to change over the two periods.
- A product or model with characteristics vector $z^* \equiv [z_1^*, \ldots, z_K^*]$ should have a log price which is approximately equal to $q^{1*} + \Sigma_{k=1}^K \gamma_k^{1*} z_k^* \equiv lnp^{1*}$ in period 1 and a log price which is approximately equal to $\theta^{2*} + \Sigma_{k=1}^K \gamma_k^{2*} z_k^* \equiv lnp^{2*}$ in period 2. Choose z^* to be a characteristics vector that is *representative* for the set of products that exist in periods 1 and 2. Then, the exponential of $\ln(p^{2*}/p^{1*}) = q^{2*} - \theta^{1*} + S_{k=1}^K (\gamma_k^{2*} - \gamma_k^{1*}) z_k^*$ can serve as a measure of average logarithmic inflation over the period. The problem with this method is that there are many possible choices for the reference vector z^*.[82]
- Use each set of quality adjustment factors to generate two consistent measures of inflation over the two periods and then take the average of the two measures.

In what follows, we will work out the algebra for the third alternative.[83] Let δ^{1*} be the share-weighted average of the quality-adjusted log prices for period 1, p_{1n}/α_n^{2*}, using the period 2 quality adjustment factors α_n^{2*} defined in (122) and let δ^{2*} be the share-weighted average of the quality-adjusted log prices for period 2, p_{2n}/α_n^{1*}, using the period 1 quality adjustment factors α_n^{1*} defined in (122):

$$\delta^{1*} \equiv \Sigma_{n \in S(1)} s_{1n} ln(p_{1n}/\alpha_n^{2*});\ \delta^{2*} \equiv \Sigma_{n \in S(2)} s_{2n} ln(p_{2n}/\alpha_n^{1*}). \quad (125)$$

It can be seen that $\theta^{2*} - \delta^{1*}$ is a *constant quality measure* of overall log price change which uses the quality adjustment factors α_n^{2*} for period 2 to deflate prices in both periods. Similarly, $\delta^{2*} - \theta^{1*}$ is a *constant quality measure* of overall log price change which uses the quality adjustment factors α_n^{1*} for period 1 to deflate prices in both periods. It is natural to take the arithmetic mean of these two measures of constant quality log price change in order to obtain the *following counterpart*, r_2^{**}, to the adjacent period time dummy measure of constant quality log price change, ρ_2^* defined by (115).

$$\rho_2^{**} \equiv \tfrac{1}{2}[\theta^{2*} - \delta^{1*}] + \tfrac{1}{2}[\delta^{2*} - \theta^{1*}] \quad (126)$$

$$= \tfrac{1}{2}[\Sigma_{n \in S(2)} s_{2n} ln(p_{2n}/\alpha_n^{2*}) - \Sigma_{n \in S(1)} s_{1n} ln(p_{1n}/\alpha_n^{2*})]$$

$$+ \tfrac{1}{2}[\Sigma_{n \in S(2)} s_{2n} ln(p_{2n}/\alpha_n^{1*}) - \Sigma_{n \in S(1)} s_{1n} ln(p_{1n}/\alpha_n^{1*})] \text{ using (123)–(125)}$$

$$= \Sigma_{n \in S(2)} s_{2n}[lnp_{2n} - \tfrac{1}{2}(lna_n^{1*} + ln\alpha_n^{2*})]$$

$$- \Sigma_{n \in S(1)} s_{1n}[lnp_{1n} - \tfrac{1}{2}(lna_n^{1*} + ln\alpha_n^{2*})]$$

$$= \Sigma_{n \in S(2)} s_{2n}[lnp_{2n} - \Sigma_{k=1}^K (\tfrac{1}{2}\gamma_k^{1*} + \tfrac{1}{2}\gamma_k^{2*}) z_{nk}]$$

$$- \Sigma_{n \in S(1)} s_{1n}[lnp_{1n} - \Sigma_{k=1}^K (\tfrac{1}{2}\gamma_k^{1*} + \tfrac{1}{2}\gamma_k^{2*}) z_{nk}] \text{ using definitions (122).}$$

Using (115), ρ_2^* can be expressed as follows:

$$\rho_2^* = \Sigma_{n \in S(2)} s_{2n}[lnp_{2n} - \Sigma_{k=1}^K \gamma_k^* z_{nk}] - \Sigma_{n \in S(1)} s_{1n}[lnp_{1n} - \Sigma_{k=1}^K \gamma_k^* z_{nk}]. \quad (127)$$

The time dummy hedonic regression model defined by the minimization problem (109) uses the hedonic coefficients, γ_k^* for $k = 1, \ldots, K$ to form the quality adjustment factors α_n^* for $n = 1, \ldots, N$. The single-period hedonic regressions are defined by the minimization problems defined by (116) and (117), which in turn generate the two sets of hedonic coefficients, the g_k^{1*} and the g_k^{2*} for $k = 1, \ldots, K$. But in the end, these two sets of hedonic coefficients are averaged when the overall measure of log price change defined by ρ_2^{**} is calculated. Thus, the only difference between ρ_2^* defined by (115) or (127) and ρ_2^{**} defined by (126) is that the average hedonic coefficients $\tfrac{1}{2}\gamma_k^{1*} + \tfrac{1}{2}\gamma_k^{2*}$ are used in (126) while ρ_2^* uses the single set of coefficients γ_k^*. Thus, (127) lets the single regression do the job of constructing a set of hedonic coefficients that covers both periods while (126) averages the results of the two single-period regressions.

Which approach is "better"? The hedonic imputation approach requires the estimation of $2 + 2K$ parameters, while the adjacent period time dummy hedonic approach requires only $2 + K$ parameters. Thus, if the number of price observations in the two periods is plentiful, then the hedonic imputation approach will fit the data better and thus, in general, will be the preferred approach. However, if the number of observations is small and K is relatively large, then the adjacent period time dummy approach may be less vulnerable to multicollinearity and outlier problems and hence may be the preferred approach.[84] In particular, if the number of observations for the two periods is less than $2 + 2K$, then the hedonic imputation approach cannot be used. On the other hand, if the fit is very good in the two weighted least squares minimization problems defined by (115) and (116) (and there are ample degrees of freedom) and not good in the single weighted least squares minimization problem defined by (109), then it is preferable to estimate price change between the two periods using the hedonic imputation estimates for logarithmic price change defined by (126), since this difference in fit for the two models is evidence of taste change, and

[82] Note that if γ^{1*} happens to equal γ^{2*}, then $\ln(p^{2*}/p^{1*}) = \theta^{2*} - \theta^{1*}$, and $\theta^{2*} - \theta^{1*}$ turns out to equal ρ_2^* defined by (115).

[83] The analysis which follows was performed by Silver and Heravi (2007), Diewert, Heravi, and Silver (2009), and de Haan (2009). For additional materials on hedonic imputation methods, see Aizcorbe (2014).

[84] "In practice, while one may want to use the most recent cross section to derive the relevant price weights, such estimates may fluctuate too much for comfort as the result of multicollinearity and sampling fluctuations. They should be smoothed in some way, either by choosing $w_i = (1/2)[w_i(t) + w_i(t+1)]$, or by using adjacent year regressions in estimating these weights" (Zvi Griliches, 1971, 7). Thus, Griliches suggested the time dummy approach if the separate hedonic regressions led to substantial fluctuation in the parameter estimates.

thus it will be safer to use (126) over (127) to measure price change.

A problem with all of the hedonic regression models that we have considered thus far is that the underlying economic model is quite restrictive; that is, the underlying exact model is $p_{tn} = \pi_t\alpha_n$, which implies that purchasers of the products have *linear preferences* over the N products under consideration.[85] Linear preferences mean that the quality-adjusted products are perfect substitutes for each other. In the following two sections, we will consider economic models that relax this assumption of perfect substitutes.

9. Estimating Reservation Prices: The Case of CES Preferences

In this section, we will explain Feenstra's (1994) CES methodology that he proposed to measure the benefits and costs to consumers due to the appearance of new products and the disappearance of existing products.[86]

The Feenstra methodology starts out by making the same assumptions as were made in Section 2; that is, it is assumed that purchasers of a group of N products collectively maximize the linearly homogeneous, concave, and nondecreasing aggregator or utility function $f(q)$ subject to a budget constraint. Given that purchasers face the positive vector of prices $p \equiv (p_1, \ldots, p_N)$, the *unit cost function* $c(p)$ that is dual to the utility function f is defined as the minimum cost of attaining the utility level that is equal to one:

$$c(p) \equiv \min_q\{f(q) \geq 1;\ q \geq 0_N\}. \qquad (128)$$

If the unit cost function $c(p)$ is known, then using duality theory, it is possible to recover the underlying utility function $f(q)$.[87] Feenstra assumed that the unit cost function has the following *CES functional form*:

$$\begin{aligned} c(p) &\equiv \alpha_0\,[\Sigma_{n=1}^{N}\,\alpha_n p_n^{\,1-\sigma}]^{1/(1-\sigma)}\ \textit{if}\ \Sigma \neq 1; \\ &\equiv \alpha_0\,\textstyle\prod_{n=1}^{N} p_n^{\alpha_n}\ \textit{if}\ \Sigma = 1, \end{aligned} \qquad (129)$$

where α_i and Σ are nonnegative parameters with $\Sigma_{i=1}^{N}\,\alpha_i = 1$. The unit cost function defined by (129) is a *CES utility function* that was introduced into the economics literature by Arrow et al. (1961)[88].

The parameter Σ is the *elasticity of substitution*;[89] when $\Sigma = 0$, the unit cost function defined by (129) becomes linear in prices and hence corresponds to a fixed coefficients aggregator function that exhibits 0 substitutability between all commodities. When $\Sigma = 1$, the corresponding aggregator or utility function is a Cobb–Douglas function. When Σ approaches $+\infty$, the corresponding aggregator function f approaches a linear aggregator function that exhibits infinite substitutability between each pair of inputs. The CES unit cost function defined by (129) is of course *not* a fully flexible functional form (unless the number of commodities being aggregated is $N = 2$), but it is considerably more flexible than the zero substitutability aggregator function (this is the special case of (129) where Σ is set equal to zero) or the linear aggregator function (which corresponds to $\Sigma = +\infty$).

In order to simplify the notation, we set $r \equiv 1 - \Sigma$. Under the assumption of cost-minimizing behavior on the part of purchasers of the N products for periods $t = 1, \ldots, T$, Shephard's (1953, 11) Lemma tells us that the observed period t consumption of commodity i, $q_i^{\,t}$, will be equal to $u^t\partial c(p^t)/\partial p_i$, where $\partial c(p^t)/\partial p_i$ is the first-order partial derivative of the unit cost function with respect to the ith commodity price evaluated at the period t prices, and $u^t = f(q^t)$ is the aggregate (unobservable) level of period t utility. As usual, denote the share of product i in total sales of the N products during period t as $s_{ti} \equiv p_{ti}q_{ti}/p^t{\cdot}q^t$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T$, where $p^t{\cdot}q^t \equiv \Sigma_{n=1}^{N}\,p_{tn}q_{tn}$. We initially assume that there are no missing products. Note that the assumption of cost-minimizing behavior during each period implies that the following equations will hold:

$$p^t{\cdot}q^t = u^t c(p^t);\ t = 1, \ldots, T, \qquad (130)$$

where c is the CES unit cost function defined by (129).

Using the CES functional form defined by (129) and assuming that $\Sigma \neq 1$ (or $r \neq 0$),[90] the following equations are obtained using Shephard's Lemma:

$$\begin{aligned} q_{ti} &= u^t\alpha_0\,[\Sigma_{n=1}^{N}\,\alpha_n\,(p_{tn})^{r}]^{(1/r)-1}\alpha_i\,(p_{ti})^{r-1}; \\ &\qquad i = 1, \ldots, N;\ t = 1, \ldots, T \\ &= u^t c(p^t)\,\alpha_i\,(p_{ti})^{r-1}/\Sigma_{n=1}^{N}\,\alpha_n\,(p_{tn})^{r}. \end{aligned} \qquad (131)$$

Premultiply equation i for period t in (131) by $p_{ti}/p^t{\cdot}q^t$. Using (129) and (131), the resulting equations can be rewritten as follows:

[85] This criticism of hedonic regression models is similar to that of Hausman (2003, 32): "In the presence of the introduction of new goods and quality improvement of existing goods, both prices and quantities (or alternatively, prices and expenditures) must be used to calculate a correct cost of living index. Using only prices and ignoring information in quantity data will never allow for a correct estimate of a cost of living index in the presence of new goods and improvements in existing goods." However, if the fit of a hedonic regression model is good, then the hedonic regression model is justified, and there is no need to move to a more complicated consumer demand framework.

[86] The exposition in this section follows that of Diewert and Feenstra (2017).

[87] It can be shown that for $q >> 0_N$, $f(q) = 1/\max_p\,\{c(p){:}\ \Sigma_{n=1}^{N}\,p_nq_n \leq 1;\ p \geq 0_N\}$; see Chapter 5 or Diewert (1974, 110–112) on the duality between linearly homogeneous aggregator functions $f(q)$ and unit cost functions $c(p)$.

[88] In the mathematics literature, this aggregator function or utility function is known as a mean of order $r \equiv 1 - \Sigma$; see Hardy, Littlewood, and Pólya (1934, 12–13). For more on estimating CES utility functions, see Chapter 5.

[89] Let $c(p)$ be an arbitrary unit cost function that is twice continuously differentiable. The Allen (1938, 504)-Uzawa (1962) *elasticity of substitution* $\Sigma_{nk}(p)$ between products n and k is defined as $c(p)c_{nk}(p)/c_n(p)c_k(p)$ for $n \neq k$, where the first- and second-order partial derivatives of $c(p)$ are defined as $c_n(p) \equiv \partial c(p)/\partial p_n$ and $c_{nk}(p) \equiv \partial^2 c(p)/\partial p_n\partial p_k$. For the CES unit cost function defined by (129), $\Sigma_{nk}(p) = \Sigma$ for all pairs of products; that is, the elasticity of substitution between all pairs of products is a constant for the CES unit cost function.

[90] When $\Sigma = 1$, we have the case of Cobb–Douglas preferences. In the remainder of this section, we will assume that $\Sigma > 1$ (or equivalently, that $r < 0$). This assumption means that the products under consideration are either highly substitutable (Σ is considerably greater than one) or moderately substitutable (Σ is greater than one but fairly close to one).

$$s_{ti} = \alpha_i (p_{ti})^r / \Sigma_{n=1}^{N} \alpha_n (p_{tn})^r;\ i = 1, \ldots, N;\ t = 1, \ldots, T. \quad (132)$$

The NT share equations defined by (132) can be used as estimating equations using a nonlinear regression approach. Note that the positive scale parameter α_0 cannot be identified using equations (132), which of course is normal: Utility can only be estimated up to an arbitrary scaling factor. Henceforth, we will assume $\alpha_0 = 1$. The share equations (132) are homogeneous of degree one in the parameters $\alpha_1, \ldots, \alpha_N$, and thus the identifying restriction on these parameters, $\Sigma_{i=1}^{N} \alpha_i = 1$ can be replaced with an equivalent restriction such as $\alpha_N = 1$.

The sequence of *period t CES price indices* (relative to the level of prices for period 1), $P_{CES}{}^t$, can be defined as the following ratios of unit costs in period t relative to period 1:

$$P_{CES}{}^t \equiv [\Sigma_{n=1}^{N} \alpha_n (p_{tn})^r]^{(1/r)} / [\Sigma_{n=1}^{N} \alpha_n (p_{1n})^r]^{(1/r)};\ t = 1, \ldots, T. \quad (133)$$

Suppose further that the observed price and quantity data vectors, p^t and q^t for $t = 1, \ldots, T$, satisfy equations (130), where $c(p)$ is defined by (129) and the quantity data vectors q^t satisfy the Shephard's Lemma equations (131). This means that the observed price and quantity data are consistent with cost-minimizing behavior on the part of purchasers, where all purchasers have CES preferences that are dual to the CES unit cost function defined by (129). Then, Sato (1976) and vartia (1976) showed that the sequence of CES price indices defined by (133) *could be numerically calculated just using the observed price and quantity data*; that is, it is not necessary to estimate the unknown α_n and Σ (or r) parameters in equations (132).[91] The logarithm of the period t fixed-base *Sato-vartia Index* $P_{SV}{}^t$ is defined by the following equation:

$$lnP_{SV}{}^t \equiv \Sigma_{n=1}^{N} w_n{}^t\, ln(p_{tn}/p_{1n});\ t = 1, \ldots, T. \quad (134)$$

The weights $w_n{}^t$ that appear in equations (134) are calculated in two stages. The first stage set of weights is defined as $w_n{}^{t*} \equiv (s_{tn} - s_{1n})/(lns_{tn} - lns_{1n})$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$ provided that $s_{tn} \neq s_{1n}$. If $s_{tn} = s_{1n}$, then define $w_n{}^{t*} \equiv s_{tn} = s_{1n}$. The second stage set of weights are defined as $w_n{}^t \equiv w_n{}^{t*}/\Sigma_{i=1}^{N} w_i{}^{t*}$ for $n = 1, \ldots, N$ and $t = 1, \ldots, T$. Note that in order for $lnP_{SV}{}^t$ to be well defined, we require that $s_{tn} > 0$, $s_{1n} > 0$, $p_{tn} > 0$ and $p_{1n} > 0$ for all $n = 1, \ldots, N$ and $t = 1, \ldots, T$; that is, all prices and quantities must be positive for all products and for all periods.

With this background information in hand, we can explain Feenstra's (1994) model where "new" commodities can appear and "old" commodities can disappear from period to period.

Feenstra (1994) assumed CES preferences with $\Sigma > 1$ (or equivalently, $r < 0$). He applied the reservation price methodology first introduced by Hicks (1940); that is, as mentioned earlier, Hicks assumed that the consumer had preferences over all goods but for the goods which had not yet appeared, there was a reservation price that would be just high enough that consumers would not want to purchase the good in the period under consideration.[92] This assumption works rather well with CES preferences *because we do not have to estimate these reservation prices*; they will all be equal to $+\infty$ when $\Sigma > 1$.

Feenstra allowed for new products to appear and for existing products to disappear from period to period.[93] Feenstra assumed that the set of commodities that are available in period t is $S(t)$ for $t = 1, \ldots, T$. The (imputed) prices for the unavailable commodities in each period are set equal to $+\infty$, and thus if $r < 0$, an infinite price p_{tn} raised to a negative power generates a 0—that is, if product n is unavailable in period t—then $(p_{tn})^r = (\infty)^r = (1/\infty)^{-r} = 0$ if r is negative.

The CES period t true price level under these conditions when $r < 0$ turns out to be the following CES unit cost function that is defined over only products that are available during period t:

$$c(p^t) \equiv [\Sigma_{n=1}^{N} \alpha_n (p_{tn})^r]^{(1/r)} = [\Sigma_{n\in S(t)} \alpha_n (p_{tn})^r]^{1/r}. \quad (135)$$

Using equations (131) for this new model with some missing products and multiplying the period t demand q_{ti} if product i is present in period t by the corresponding price p_{ti} leads to the following equations, which describe the purchasers' nonzero expenditures on product i in period t:

$$p_{ti}q_{ti} = u^t [\Sigma_{n\in S(t)} \alpha_n (p_{tn})^r]^{(1/r)-1} \alpha_i (p_{ti})^r;\ t = 1, \ldots, T;\ i\in S(t)$$
$$= u^t c(p^t)\, \alpha_i (p_{ti})^r / \Sigma_{n\in S(t)} \alpha_n (p_{tn})^r. \quad (136)$$

In each period t, the sum of observed expenditures, $\Sigma_{n\in S(t)} p_{tn}q_{tn}$, equals the period t utility level, u^t, times the CES unit cost $c(p^t)$ defined by (135):

$$\Sigma_{n\in S(t)} p_{tn}\, q_{tn} = u^t c(p^t) = u^t [\Sigma_{i\in S(t)} \alpha_i (p_{ti})^r]^{1/r};\ t = 1, \ldots, T. \quad (137)$$

Recall that the ith sales share of product i in period t was defined as $s_{ti} \equiv p_{ti}q_{ti}/\Sigma_{n\in S(t)} p_{tn}q_{tn}$ for $t = 1, \ldots, T$ and $i\in S(t)$. Using these share definitions and equations (137), we can rewrite equations (136) in the following form:

$$s_{ti} = \alpha_i (p_{ti})^r / \Sigma_{n\in S(t)} \alpha_n (p_{tn})^r;\ t = 1, \ldots, T;\ i\in S(t)$$
$$= \alpha_i (p_{ti})^r / c(p^t)^r, \quad (138)$$

where the second set of equations follows from definitions (135).

Now we can work out Feenstra's (1994) model for measuring the benefits and costs of new and disappearing commodities. Start out with the period t CES exact price level defined by (135), and define the CES fixed-base price index

[91] See Chapter 5 for a proof of this result.

[92] The same logic is applied to disappearing products.

[93] In many cases, a "new" product is not a genuinely new product; it is just a product that was not in stock in the previous period. Similarly, in many cases, a disappearing product is not necessarily a truly disappearing product; it is simple a product that was not in stock for the period under consideration. Many retail chains rotate products, temporarily discontinuing some products in favor of competing products in order to take advantage of manufacturer-discounted prices for selected products.

for period t, $P_{CES}{}^{t}$, as the ratio of the period t CES price level to the corresponding period 1 price level:[94]

$$\begin{aligned} P_{CES}{}^{t} &\equiv c(p^t)/c(p^1);\ t = 2,3,\ldots,T \\ &= [\Sigma_{i\in S(t)}\ \alpha_i\ (p_{ti})^r]^{1/r} / [\Sigma_{i\in S(1)}\ \alpha_i\ (p_{1i})^{\ r}]^{1/r} \\ &= [\text{ Index 1}]\times[\text{Index 2}]\times[\text{Index 3}], \end{aligned} \quad (139)$$

where the three indices in equations (139) are defined as follows:[95]

$$\text{Index 1} \equiv [\Sigma_{i\in S(t)\cap S(1)}\ \alpha_i\ (p_{ti})^r]^{1/r} / [\Sigma_{i\in S(1)\cap S(t)}\ \alpha_i\ (p_{1i})^{\ r}]^{1/r};\ t = 2,3,\ldots,T; \quad (140)$$

$$\text{Index 2} \equiv [\Sigma_{i\in S(t)}\ \alpha_i\ (p_{ti})^r]^{1/r} / [\Sigma_{i\in S(1)\cap S(t)}\ \alpha_i\ (p_{ti})^{\ r}]^{1/r};\ t = 2,3,\ldots,T; \quad (141)$$

$$\text{Index 3} \equiv [\Sigma_{i\in S(1)\cap S(t)}\ \alpha_i\ (p_{1i})^r]^{1/r} / [\Sigma_{i\in S(1)}\ \alpha_i(p_{1i})^r]^{1/r};\ t = 2,3,\ldots,T. \quad (142)$$

Note that Index 1 defines a CES price index over the set of commodities that are available in both periods t and 1. Denote the CES cost function c^{t*} that has the same α_n parameters as before but is now defined over only products that are available in periods 1 and t:

$$c^{t*}(p) \equiv [\Sigma_{i\in S(t)\cap S(1)}\ \alpha_i\ (p_i)^r]^{1/r};\ t = 1,2,\ldots,T. \quad (143)$$

The period t expenditure share equations defined by equations (138) using the unit cost functions defined by (143) are the following ones:

$$\begin{aligned} s_i^{\ t*} &\equiv p_{ti}q_{ti}/\Sigma_{n\in S(t)\ \cap S(1)}\ p_{tn}q_{tn}\ t = 1,\ldots,T;\ i\in S(1)\cap S(t); \\ &= \alpha_i\ (p_{ti})^r/\text{å}_{n\in S(t)\ \cap S(1)}\ \alpha_n\ (p_{tn})^{\ r}; \\ &= \alpha_i\ (p_{ti})^r/c^{t*}(p^t)^r, \end{aligned} \quad (144)$$

where the third equality follows from definitions (143).

Note that Index 1 is equal to $c^{t*}(p^t)/c^{t*}(p^1)$ and the Sato-vartia formula (134) (restricted to commodities n that are present in periods 1 and t) can be used to calculate this index using the observed price and quantity data for the products that are available in both periods 1 and t.

We turn now to the evaluation of Indices 2 and 3. It turns out that we will need an estimate for the elasticity of substitution Σ (or equivalently of $r \equiv 1–\Sigma$) in order to find empirical expressions for these indices.[96] It is convenient to define the following *observable expenditure or sales ratios*:

$$\lambda^t \equiv \Sigma_{n\in S(t)}\ p_{tn}q_{tn}/\text{å}_{n\in S(1)\cap S(t)}\ p_{tn}q_{tn};\ t = 2,3,\ldots,T; \quad (145)$$

$$\mu^t \equiv \Sigma_{n\in S(1)\cap S(t)}\ p_{1n}q_{1n}/\Sigma_{n\in S(1)}\ p_{1n}\ q_{1n};\ t =2,3,\ldots\ T. \quad (146)$$

We assume that there is at least one product that is present in periods 1 and t for each $t \geq 2$. Let product i be any one of these common products for a given $t \geq 2$. Then, the share equations (138) and (144) hold for this product. These share equations can be rearranged to give us the following two sets of equations:

$$\alpha_i(p_{ti})^r = [\Sigma_{n\in S(t)}\ \alpha_n\ (p_{tn})^r]p_{ti}q_{ti}/[\Sigma_{n\in S(t)}\ p_{tn}q_{tn}];\ t = 2,3,\ldots,T; \quad (147)$$

$$\alpha_i(p_{ti})^r = [\Sigma_{n\in S(1)\cap S(t)}\ \alpha_n\ (p_{tn})^r]p_{ti}q_{ti}/[\Sigma_{n\in S(1)\cap S(t)}\ p_{tn}q_{tn}];\ t = 2,3,\ldots,T. \quad (148)$$

For each $t \geq 2$, equating (147) to (148) for the common product i leads to the following equations:

$$\Sigma_{n\in S(t)}\ \alpha_n\ (p_{tn})^r/\Sigma_{n\in S(1)\cap S(t)}\ \alpha_n\ (p_{tn})^r = \Sigma_{n\in S(t)}\ p_{tn}q_{tn}/\Sigma_{n\in S(1)\cap S(t)}\ p_{tn}q_{tn};\ t = 2,3,\ldots,T; = \lambda^t, \quad (149)$$

where the second set of equalities follows from definitions (145). Now take the $1/r$ root of both sides of (149) and use definitions (141) in order to obtain the following equalities:

$$\text{Index 2} = [\lambda^t]^{1/r} = [\Sigma_{i\in S(t)}\ p_{ti}\ q_{ti}/\Sigma_{i\in S(1)\cap S(t)}\ p_{ti}q_{ti}]^{1/r};^{97}\ t = 2,3,\ldots,T. \quad (150)$$

Again assume that product i is available in periods 1 and $t \geq 2$. Rearrange the share equations (138) and (144) for $t = 1$ and product i, and we obtain the following two equations:

$$\alpha_i(p_{1i})^r = [\Sigma_{n\in S(1)}\ \alpha_n\ (p_{1n})^r]\ p_{1i}q_{1i}/[\Sigma_{n\in S(1)}\ p_{1n}q_{1n}];\ t = 2,3,\ldots,T; \quad (151)$$

$$\alpha_i(p_{1i})^r = [\Sigma_{n\in S(1)\cap S(t)}\ \alpha_n\ (p_{1n})^r]\ p_{1i}q_{1i}/[\Sigma_{n\in S(1)\cap S(t)}\ p_{1n}q_{1n}];\ t = 2,3,\ldots,T. \quad (152)$$

Equating (151) to (152) leads to the following equations:

$$\Sigma_{n\in S(1)\cap S(t)}\ \alpha_n\ (p_{1n})^r/\Sigma_{n\in S(1)}\ a_n\ (p_{1n})^r = \Sigma_{n\in S(1)\cap S(t)}\ p_{1n}q_{1n}/\Sigma_{n\in S(1)}\ p_{1n}q_{1n};\ t = 2,3,\ldots,T; = \mu^t, \quad (153)$$

where the last set of equalities follows from definitions (146). Now take the $1/r$ root of both sides of (153) and use definitions (143) in order to obtain the following equalities:[98]

[94] In the algebra which follows, the prices and quantities of period 1 can be replaced with the prices and quantities of any period. Feenstra (1994) developed his algebra for $c(p^t)/c(p^{t-1})$.

[95] The Indices 1–3 depend on period t, but we suppressed the Index t from the left-hand side of definitions (140)–(142).

[96] See Chapter 5 or Diewert and Feenstra (2017) for a variety of methods for estimating the elasticity of substitution.

[97] If new products become available in period t that were not available in period 1, then $\lambda^t > 1$. Recall that $r = 1 – \Sigma$ and $r < 0$. Index 2 evaluated at period t prices equals $(\lambda^t)^{1/r} = (\lambda^t)^{1/(1-\sigma)}$ and thus is an increasing function of s for $1 < \Sigma < +\infty$. With $\lambda^t > 1$, the limit of $(\lambda^t)^{1/(1-\sigma)}$ as Σ approaches 1 from above is 0, and the limit of $(\lambda^t)^{1/(1-\sigma)}$ as Σ approaches $+\infty$ is 1. Thus, the gains in utility from increased product variety are huge if Σ is slightly greater than 1 and diminish to tiny gains as Σ becomes very large. Suppose that $\lambda^t = 1.05$ and Σ = 1.01, 1.1, 1.5, 2, 3, 5, 10, and 100. Then Index 2 will equal 0.0076, 0.614, 0.907, 0.952, 0.976, 0.988, 0.995, and 0.9995, respectively. Thus, the gains from increased product variety are very sensitive to the estimate for the elasticity of substitution. The gains are gigantic if Σ is close to 1.

[98] If some products that were available in period 1 become unavailable in period t, then $\mu^t < 1$. Index 3 evaluated at period 1 prices equals $(\mu^t)^{1/r} = (\mu^t)^{1/(1-\sigma)}$ and is a decreasing function of s for $1 < \Sigma < +\infty$. With $\mu^t < 1$, the limit of $(m^t)^{1/(1-\sigma)}$ as Σ approaches 1 is $+\infty$, and the limit of $(\mu^t)^{1/(1-\sigma)}$ as Σ approaches $+\infty$ is 1. Thus, the losses in utility from decreased product variety are huge if Σ is slightly greater than 1 and diminish to tiny gains as Σ becomes very large. Suppose that $\mu^t = 0.95$ and Σ takes

$$\text{Index } 3 = [\mu^t]^{1/r} = [\Sigma_{n\in S(1)\cap S(t)}\, p_{1n}q_{1n}/\Sigma_{n\in S(1)}\, p_{1n}q_{1n}]^{1/r};\ t = 2,3,\ldots,T. \tag{154}$$

Thus, if r is known or has been estimated, then Index 2 and Index 3 can readily be calculated as simple ratios of sums of observable expenditures raised to the power $1/r$. Note that $[\Sigma_{i\in S(t)}\, p_{ti}q_{ti}/\Sigma_{i\in S(1)\cap S(t)}\, p_{ti}q_{ti}] \leq 1$. If period t has products that were not available in period 1, then the strict inequality will hold, and since $1/r < 0$, it can be seen that Index 2 will be less than unity. Thus, Index 2 is a measure of how much the true cost of living index is *reduced* in period t due to the introduction of products that were not available in period 1. Similarly, $[\Sigma_{i\in S(1)\cap S(t)}\, p_{1i}q_{1i}/\Sigma_{i\in S(1)}\, p_{1i}q_{1i}] \leq 1$. If period 1 has products that are not available in period t, then the strict inequality will hold, and since $1/r < 0$, it can be seen that Index 3 will be greater than unity, Thus, Index 3 is a measure of how much the true cost of living index has *increased* in period t due to the disappearance of products that were available in period 1 but are not available in period t.

Turning briefly to the problems associated with estimating r (and α_n) when not all products are available in all periods, it can be seen that the initial estimating share equations (132) need to be replaced by the estimating equations (138). However, there are many methods that have been suggested in the literature to estimate r (or the elasticity of substitution Σ) when there are missing products; see, for example, Diewert and Feenstra (2017) or the extensive discussion of estimation issues in Chapter 5.

The Feenstra methodology is easy to implement once an estimate for Σ is available, and so it has been widely used in the macroeconomic literature. However, if the elasticity of substitution is low and new products outnumber disappearing products, then this methodology will lead to quality-adjusted price indices that will decrease by amounts that are not plausible, and this point should be kept in mind.[99] The Feenstra methodology will tend to be biased for elasticities of substitution that are close to one and should not be used in this case.[100] Thus, in the next section, we will study a model that is similar to Feenstra's model, but the reservation prices generated by the model are finite and a flexible functional form for $f(q)$ is used in place of the CES functional form.

10. Estimating Reservation Prices: The Case of KBF Preferences

The functional form for the aggregator function $f(q)$ that we will use in this section is the *KBF function form*, $f_{KBF}(q) \equiv [q\cdot Aq]^{1/2}$ defined by (17) in Section 4.[101] The system of *inverse demand functions* for this functional form for our data set with no missing observations is given by the following system of equations:

$$p^t = P^t\, \nabla_q f_{KBF}(q^t) = P^t\, [q^t\cdot Aq^t]^{-1/2}\, Aq^t;\quad t = 1,\ldots,T, \tag{155}$$

where the N by N matrix $A \equiv [a_{nk}]$ is symmetric (so that $A^T = A$) and thus has $N(N + 1)/2$ unknown a_{nk} elements. As in Section 4, we also assume that A has one positive eigenvalue with a corresponding strictly positive eigenvector, and the remaining N–1 eigenvalues are negative or zero. These conditions will ensure that the aggregator function has indifference surfaces with the correct curvature.

The period t aggregate price level is P^t, and the corresponding aggregate quantity level is $Q^t \equiv [q^t\cdot Aq^t]^{1/2}$ for $t = 1,\ldots,T$. Multiply the right-hand side of equation t in (155) by $1 = Q^t/[q^t\cdot Aq^t]^{1/2}$ for $t = 1,\ldots,T$, and we obtain the following system of estimating equations:

$$p^t = P^tQ^tAq^t/q^t\cdot Aq^t = v^tAq^t/q^t\cdot Aq^t;\quad t = 1,\ldots,T, \tag{156}$$

where we have used equations (9), $P^tQ^t = p^t\cdot q^t = v^t$ for $t = 1,\ldots,T$, to derive the second set of equations in (156). Now convert equations (156) into a set of share equations by taking component n in the vector p^t, p_{tn}, and multiplying both sides of this equation by q_{tn} and dividing by $v^t = p^t\cdot q^t$. We obtain the following system of estimating equations:

$$s_{tn} = \Sigma_{m=1}^{N}\, q_{tn}a_{nm}q_{tm}/\Sigma_{n=1}^{N}\,\Sigma_{m=1}^{N}\, q_{tm}a_{nm}q_{tm};\quad t = 1,\ldots,T;\ n = 1,\ldots,N. \tag{157}$$

When estimating systems of consumer demand equations, it is common to use share equations such as equations (157) as the estimating equations. However, in our particular situation, it may be preferable to use the system of inverse demand functions defined by equations (156) as estimating equations as we shall see later.[102]

Now introduce missing products into the model. Let $S(t)$ be the set of products n that are present in period t for $t = 1,\ldots,T$. If product n is missing in period t, define $q_{tn} \equiv 0$ and $s_{tn} = 0$. Define q^t and s^t as the period t vectors of quantities and shares, where $q_{tn} \equiv 0$ and $s_{tn} \equiv 0$ if product n is missing in period t. It can be seen that equations (156) and (157) are still valid when there are missing products, except that instead of $t = 1,\ldots,T$ and $n = 1,\ldots,N$, we have $t = 1,\ldots,T$ and $n\in S(t)$.

on the same values as in the previous footnote. Then Index 3 will equal 168.9, 1.670, 1.108, 1.053, 1.026, 1.013, 1.0057, and 1.00052, respectively. Thus, the losses are gigantic if Σ is close to 1 and negligible if Σ is very large.

[99] Also keep in mind that the Feenstra methodology does not work at all if the elasticity of substitution is equal to or less than one.

[100] Another feature of the Feenstra methodology is that the reservation prices are infinite. Typically, it does not take an infinitely high price to deter consumers from buying the product under consideration.

[101] The analysis in this section follows that of Diewert and Feenstra (2017). The same theoretical framework was suggested by Diewert (1980, 498–503), but a different flexible functional form was used to illustrate the methodology. The Diewert and Feenstra functional form is a better choice since the correct curvature conditions can be imposed on the KBF functional form without destroying its flexibility.

[102] When there are missing prices, estimating systems of inverse demand functions with prices as the dependent variables is econometrically convenient. The advantages and disadvantages of alternative methods for estimating consumer preferences are discussed at some length in Section 10 of Chapter 5.

Thus, we use equation t,n in (157) as an estimating equation only if the corresponding product n is present in period t.

The $N(N + 1)/2$ unknown parameters a_{nm} in the symmetric $A \equiv [a_{nm}]$ matrix can be determined by solving the following nonlinear least squares minimization problem:[103]

$$\min_A S_{t=1}^{T} \Sigma_{n\in S(t)} [s_{tn} - \{\Sigma_{m=1}^{N} q_{tn}a_{nm}q_{tm}/\Sigma_{i=1}^{N} \Sigma_{j=1}^{N} q_{ti}a_{ij}q_{tj}\}]^2. \quad (158)$$

Note that the minimization problem defined by (158) is run as a single nonlinear regression rather than as a system of N share equations, which is the more traditional approach when estimating systems of consumer demand functions. The unusual specification is due to the fact that there are missing products in the T time periods, and so the traditional systems approach cannot be applied. A second point to note is that not all of the parameters a_{nm} can be identified: If a_{nm}^* solves (158), then so does λa_{nm}^* for $1 \le n \le m \le N$ for all $\lambda \ne 0$. Thus, a normalization on the matrix of parameters is required for a unique solution to (158). A final point to note is that the error terms in (158) are not weighted by their economic importance. There is no need to do this because the dependent variables in (158), the shares, are already weighted by their economic importance, and so there is no need for further weighting. Put another way, each share is equally important (and is measured in comparable units), and hence it makes sense to fit the observed shares by model-predicted shares using a least squares approach.

Once the parameters a_{nm}^* have been determined, we can use the price equations defined by (156) to determine the *Hicksian reservation prices* p_{tn}^* for the missing products for $t = 1, \ldots, T$ and n does not belong to $S(t)$:

$$p_{tn}^* \equiv v^t\Sigma_{m=1}^{N} a_{nm}^*q_{tm}/\{\Sigma_{i=1}^{N} \Sigma_{j=1}^{N} q_{ti}a_{ij}^*q_{tj}\}; \quad t = 1, \ldots, T; n\in S(t). \quad (159)$$

Note that the reservation prices defined by (159) will be finite. Using the observed prices and quantities for each period t along with the imputed prices p_{tn}^*, complete price and quantity vectors for each period can be formed. These complete price and quantity vectors can be used to form price and quantity levels for each period using a preferred index number formula. Alternatively, the estimated parameters a_{nm}^* can be used to form the matrix of parameters, $A^* \equiv [a_{nm}^*]$. Use the estimated A^* matrix to form the period t quantity levels, $Q^{t*} \equiv [q^t \cdot A^*q^t]^{1/2}$ for $t = 1, \ldots, T$ and the corresponding period t price levels, $P^{t*} \equiv v^t/Q^{t*}$ for $t = 1, \ldots, T$.

There are two problems with the preceding methodology that need to be addressed: (i) how can we be sure that the estimated A matrix satisfies the eigenvalue restrictions listed earlier and (ii) how can we estimate the parameters of the A matrix when N is large?

The number of unknown parameters in the A matrix is $N(N + 1)/2$ if there are N products in the window of observations. If $N = 10$, $N(N + 1)/2 = 55$; if $N = 100$, $N(N + 1)/2 = 5050$. Thus, it will be impossible to estimate all of the parameters in the A matrix if N is large.

These two difficulties with this methodology can be addressed if we make use of the following reparameterization of the A matrix. Thus, we set A equal to the following expression:[104]

$$A = bb^T + B;\ b >> 0_N;\ B = B^T;\ B \text{ is negative semidefinite; } Bq^* = 0_N. \quad (160)$$

The vector $b^T \equiv [b_1, \ldots, b_N]$ is a row vector of positive constants, and so bb^T is a rank one positive semidefinite N by N matrix. The symmetric matrix B has $N(N + 1)/2$ independent elements b_{nk}, but the N constraints $Bq^* = 0_N$ reduce this number by N. Thus, there are N independent parameters in the b vector and $N(N–1)/2$ independent parameters in the B matrix so that bb^T + B has the same number of independent parameters as the A matrix.

The reparameterization of A by bb^T + B is useful in the present context because this reparameterization can be used to estimate the unknown parameters in stages. Thus, initially set $B = O_{N\times N}$, a matrix of 0s. The resulting aggregator function becomes $f(q) = (q^Tbb^Tq)^{1/2} = (b^Tqb^Tq)^{1/2} = b^Tq$, a *linear utility function*. Thus, this special case of (160) boils down to the linear utility function model that has been used repeatedly in this chapter.

The matrix B is required to be negative semidefinite. The procedure used by Wiley, Schmidt, and Bramble (1973) and Diewert and Wales (1987) can be used to impose negative semidefiniteness on B by setting B equal to $-CC^T$, where C is a lower triangular matrix.[105] Write C as $[c^1, c^2, \ldots, c^N]$, where c^k is a column vector for $k = 1, \ldots, K$. If C is lower triangular, then the first $k–1$ elements of c^k are equal to 0 for $k = 2,3, \ldots, N$. The following representation for B will hold:

$$B = -CC^T = -\Sigma_{n=1}^{N} c^nc^{nT}, \quad (161)$$

where the following restrictions on the vectors c^n are imposed in order to impose the restrictions $Bq^* = 0_N$ on B:[106]

$$c^n \cdot q^* = 0;\ n = 1, \ldots, N. \quad (162)$$

As mentioned earlier, if N is not small, then usually it will not be possible to estimate all of the parameters in the C matrix. Furthermore, frequently nonlinear estimation is not

[103] Alternative estimating equations are considered in Diewert and Feenstra (2017), which has a worked example. Diewert and Feenstra found that it was preferable to use the system of estimating equations (156) rather than (157) since the goal of the regressions was to find the best-fitting system of inverse demand functions rather than to find the best-fitting system of share equations. More research on the econometrics associated with estimating reservation prices is necessary.

[104] Notation: b is regarded as a column vector and b^T is its transpose, which is a row vector.

[105] $C = [c_{nk}]$ is a lower triangular matrix if $c_{nk} = 0$ for $k > n$; that is, there are 0s in the upper triangle. Wiley, Schmidt, and Bramble showed that setting $B = -CC^T$, where C as a lower triangular matrix was sufficient to impose negative semidefiniteness, while Diewert and Wales showed that any negative semidefinite matrix could be represented in this fashion.

[106] The restriction that C be upper triangular means that c^N will have at most one nonzero element, namely c_N^N. However, the positivity of q^* and the restriction $c^{NT}q^* = 0$ will imply that $c^N = 0_N$. Thus, the maximal rank of B is $N–1$.

successful if one attempts to estimate all of the parameters at once. Thus, it is necessary to estimate the parameters in the utility function $f(q) = (q^TAq)^{1/2}$ in stages. In the first stage, estimate the linear utility function $f(q) = b^Tq$.[107] In the second stage, estimate $f(q) = (q^T[bb^T - c^1c^{1T}]q)^{1/2}$, where $c^{1T} \equiv [c_1^1,c_2^1, \ldots,c_N^1]$ and $c^{1T}q^* = 0$. For starting coefficient values in the second nonlinear regression, use the final estimates for b from the first nonlinear regression, and set the starting $c^1 \equiv 0_N$.[108] In the third stage, estimate $f(q) = (q^T[bb^T - c^1c^{1T} - c^2c^{2T}]q)^{1/2}$, where $c^{1T} \equiv [c_1^1,c_2^1, \ldots,c_N^1]$, $c^{1T}q^* = 0$, $c^{2T} \equiv [0,c_2^2, \ldots,c_N^2]$, and $c^{2T}q^* = 0$. The starting coefficient values are the final values from the second stage with $c^2 \equiv 0_N$. At each stage, the log likelihood will generally increase.[109] Stop adding columns to the C matrix when the increase in the log likelihood becomes small (or the number of degrees of freedom becomes small). At stage k of this procedure, it turns out that a substitution matrix of rank k–1 is estimated to be the most negative semidefinite substitution matrix that the data will support.[110] This is the same type of procedure that Diewert and Wales (1987, 1988) used in order to estimate normalized quadratic preferences, and they termed the final functional form a *semiflexible functional form*. The preceding treatment of the KBF functional form also generates a semiflexible functional form.

This functional form for the aggregator function is more general than the linear utility function that has been used throughout this chapter, and it is conceptually more general than the CES aggregator function that was used in the previous section. Moreover, the reservation prices that the method generates are finite. Finally, the present model can deal with situations where a new product has a low elasticity of substitution with all existing products; that is, it provides a more satisfactory solution to the new goods problem and the problem of adjusting for quality change. However, it has the drawback of being rather complex, and hence it may be resistant to large-scale applications of the method. More research is required in order to develop methods that are simpler to implement.

11. Other Approaches to Quality Adjustment

In this section, we will briefly review three approaches to quality adjustment that have not been discussed explicitly in the previous sections of this chapter. The three approaches are as follows:

- Clustering or grouping approaches
- The dominant characteristic approach
- Experimental economics approaches

11.1 Clustering or Grouping Approaches

Before explaining this approach, it will be useful to ask exactly should product prices (and quantities) that enter into a CPI be defined. Suppose that scanner data on products are available for time periods that correspond to the frequency of the CPI; that is, price and quantity data on household purchases or by retailers are available to the Statistical Office. The period t product prices that are used in a bilateral or multilateral index number formula should be *representative* of household purchases of the class of products under consideration. Walsh (1901, 96) and Fisher (1922, 318) suggested that the *representative quantity* should be equal to the total quantity of each product (in scope) that the households (in scope) purchased during the period, and the corresponding product price should be equal to total expenditures on the product during the period divided by the total quantity purchased; that is, the corresponding price should be a *unit value price*. This formation of a unit value price for a narrowly defined product is the *first stage of aggregation* in the construction of a CPI.

A *second stage of aggregation* might be the formation of a broader unit value price by aggregating transactions in the same product over *time* periods[111] or over *space* (geographical location of households or of retail outlets).

A *third stage of aggregation* is to form broader unit value prices by aggregating over closely related products.

A *fourth stage of aggregation* is to use the unit value prices and total quantities that have been formed in the prior stages of aggregation for a number of product groups as inputs into an index number formula that constructs aggregate prices (and quantities or volumes) for the group of commodities under consideration.[112]

The second and third stages of aggregation listed earlier may seem to be unnecessary. Since the first stage of aggregation leads to product prices and quantities for each period that can be inserted into a bilateral or multilateral index number formula, why should we undertake further unit value aggregation in stages 2 and 3 which could lead to considerable amounts of unit value bias?[113] The reason for unit value aggregation of products over time, location, and product type is to improve *the matching of product prices* across different time periods. To give an extreme example, suppose we choose the length of the accounting period to be 1 minute. In this case, as we go from 1 minute to the next, there will be very few or no product matches, and hence, it becomes impossible to construct a price index based on the existence of matched products! Using a broader unit value pricing concept will lead to more product matches across time and hence improve index accuracy. But the broader unit value may lead to more unit value bias.[114]

National Statistical Offices will not face much criticism if they aggregate the same product over time and space, but they may face criticism over their decisions if they use

[107] In order to identify all of the parameters, set one component of the b vector to 1.

[108] We also use the constraint $c^{1T}q^* = 0$ to eliminate one of the c_n^1 from the nonlinear regression.

[109] If it does not increase, then the data do not support the estimation of a higher rank substitution matrix, and we stop adding columns to the C matrix. The log likelihood cannot decrease since the successive models are nested.

[110] For a worked example of this methodology, see Diewert and Feenstra (2017).

[111] See Diewert, de Haan, and Fox (2016) on possible unit value aggregation that could result from aggregating over time.

[112] For additional material on stages of aggregation, see Lamboray (2022) and Dalmaans (2022).

[113] For materials on unit value bias, see Silver (2010), Diewert and von der Lippe (2010), and Diewert (2022).

[114] Chessa (2021) brought this tradeoff to the attention of price statisticians. He also provides an extensive discussion on grouping methods.

unit value aggregation over products.[115] Aggregation over products is riskier than aggregating the same product over time and space because in the former case, the resulting unit value prices and quantities are not invariant to changes in the units of measurement for the products. Moreover, it will be difficult to choose between alternative aggregations over products, so it will become difficult to explain to the public exactly how the final aggregates were determined.[116]

In some cases, it does make sense to broaden the scope of unit value aggregation. Chessa (2021) noted the importance of the *product relaunch problem*. This problem arises when a product that is present in a prior period is discontinued and replaced with a new product label (and sold usually at a higher price) in the present period, but it is essentially the same as the discontinued product. Using normal matched model index numbers, this increase in price would not be picked up. In this case, it is reasonable to treat the "new" product as being the same as the "old" product. The practical problem that the price statistician faces is "How can these more or less fake product relaunches be detected when there are hundreds or thousands of products in scope?"[117] The answer lies in having information on the characteristics of the products, but collecting information on product characteristics is costly and somewhat subjective. The solution probably lies in using the services of market specialists who can detect these "fake" product relaunches.

In his empirical work, Chessa (2021, 5) found that his broadly aggregated unit value price indices for four classes of product showed a higher inflation rate than his GK and TPD indices using the same data. The GK and TPD indices are both examples of quality-adjusted unit value indices.[118] A quality-adjusted unit value index is consistent with the aggregate period t quantity or volume index being proportional to $f(q^t) = \alpha \cdot q^t \equiv \Sigma_{n=1}^{N} \alpha_n q_{tn}$, where $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ is the period t quantity vector for the N products in scope and $\alpha \equiv [\alpha_1, \ldots, \alpha_N]$ is a vector of positive quality adjustment parameters. Here, we derive a simple relationship between a unit value price index and a quality-adjusted unit value price index. This relationship could explain Chessa's empirical results.

Define $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ as the period t price and quantity vectors for the products in scope for the index for $t = 1, \ldots, T$. If a product n is not present in period t, define $p_{tn} \equiv 0$ and $q_{tn} \equiv 0$. The *period t unit value price level*, $p_{UV}^{\ t}$, and the *period t quality-adjusted unit value price level*, $p_{UV\alpha}^{\ t}$, are defined as follows:

$$p_{UV}^{\ t} \equiv p^t \cdot q^t / 1_N \cdot q^t;\ p_{UV\alpha}^{\ t} \equiv p^t \cdot q^t / \alpha \cdot q^t;\ t = 1, \ldots, T, \qquad (163)$$

where 1_N is a vector of ones of dimension N. The *aggregate quantity levels* that correspond to the period t price levels defined by (163) are defined as $q_{UV}^{\ t} \equiv 1_N \cdot q^t$ and $q_{UV\alpha}^{\ t} \equiv \alpha \cdot q^t$ for $t = 1, \ldots, T$. We normalize the quality adjustment parameters so that the period 1 price levels defined by (163) for $t = 1$ are equal so that $p_{UV\alpha}^{\ 1} = p_{UV}^{\ 1}$. Thus, the vector of quality adjustment parameters a satisfies the following linear restriction:

$$\alpha \cdot q^1 = 1_N \cdot q^1. \qquad (164)$$

The *period t unit value price in*dex $P_{UV}^{\ t}$ and the *period t quality-adjusted unit value price index* $P_{UV\alpha}^{\ t}$ are defined as follows:

$$P_{UV}^{\ t} \equiv p_{UV}^{\ t} / p_{UV}^{\ 1};\ P_{UV\alpha}^{\ t} \equiv p_{UV\alpha}^{\ t} / p_{UV\alpha}^{\ 1};\quad t = 1, \ldots, T. \qquad (165)$$

Using definitions (163) and (165), it can be seen that the following equalities hold:

$$\begin{aligned} P_{UV\alpha}^{\ t} &= (p^t \cdot q^t / p^1 \cdot q^1)(1_N \cdot q^1 / 1_N \cdot q^t)(1_N \cdot q^t / 1_N \cdot q^1)(\alpha \cdot q^1 / \alpha \cdot q^t) \\ &= P_{UV}^{\ t}(1_N \cdot q^t / 1_N \cdot q^1)(\alpha \cdot q^1 / \alpha \cdot q^t) \qquad (166) \\ &= P_{UV}^{\ t}(1_N \cdot q^t / \alpha \cdot q^t) \text{ using (164).} \end{aligned}$$

In general, if the new products entering the aggregate in period t are of higher quality than continuing or disappearing products, then the corresponding α_n for the new products will be higher than the average a_n for the continuing products, and thus $1_N \cdot q^1 / \alpha \cdot q^t$ will tend to decrease as t increases, which in turn implies that the unit value price index $P_{UV}^{\ t}$ will tend to be higher than the corresponding quality-adjusted unit value price index $P_{UV\alpha}^{\ t}$. This upward bias of the unit value index will tend to also apply to the corresponding GK and TPD price indices since these indices are equal to quality-adjusted unit value indices.[119] On the other hand, if poorer quality product relaunches are prevalent, then the entering α_n will tend to be below average, and thus $1_N \cdot q^1 / \alpha \cdot q^t$ will tend to increase as t increases, which in turn implies that the unit value price index $P_{UV}^{\ t}$ will tend to be lower than the corresponding quality-adjusted unit value price index $P_{UV\alpha}^{\ t}$.

115 This comment is consistent with the following advice: "For instance, it can be necessary to cluster daily data into weekly or monthly data (aggregation in time), or to cluster individual stores into a chain store level. If the first level of aggregation is done on narrowly defined products, unit value bias should not be too much of an issue. The risks of product clustering are the highest for the second step in which price indices are compiled from a set of essentially different products" (Jacco Dalmaans, 2022, 21).

116 Chessa (2021) attempted to derive a scientific method for determining how to aggregate unit values over related products. His MARS *measure of product matching* is not invariant to the units of measurement, but more importantly, his *measure of product homogeneity* focuses only on the variance of current period prices rather than focusing on how *proportional* the prices in the current period are relative to the corresponding base period prices. However, Chessa deserves credit for attempting to solve a very difficult problem in a systematic way.

117 Amazing progress has been made in recent years in the ability of statistical agencies to compute various bilateral and multilateral indices when there are hundreds or thousands of products in a product class. For *R* software that can compute many standard indices at scale, see Bialek (2021, 2022) and Graham White's IndexNumR, which is available for download from CRAN and Github as well as the MAP software package of Stansfield and Krsinich (2022), which draws on IndexNumR. For information on IndexNumR, visit https://cran.*r*-project.org/web/packages/IndexNumR/vignettes/indexnumr.html.

118 See Section 5 for this consistency result for TPD price indices.

119 *If* the fit in the TPD or weighted TPD indices is perfect, then these indices will be *exactly* equal to a quality-adjusted unit value index. The GK indices are exactly equal to a quality-adjusted unit value index.

11.2 Dominant Characteristic Quality Adjustment

This method of quality adjustment involves the use of quality-adjusted unit values and is also related to the use of hedonic regressions. The method assumes that the related products in scope have a single dominant utility determining characteristic which can be measured. For example, suppose the index product category is chocolate bars. The dominant utility determining characteristic might be the volume or weight of chocolate in each bar of chocolate. Suppose that there are N chocolate bars in scope, product n is sold at unit value price p_{tn} in period t, and the amount of chocolate in bar n is $\alpha_n > 0$ for $n = 1, \ldots, N$. As in Section 11.1, let $p^t \equiv [p_{t1}, \ldots, p_{tN}]$ and $q^t \equiv [q_{t1}, \ldots, q_{tN}]$ as the period t price and quantity vectors for the chocolate bars in scope for the index for $t = 1, \ldots, T$. If a bar n is not present in period t, define $p_{tn} \equiv 0$ and $q_{tn} \equiv 0$. The next step is to simply assume that consumers only care about the amount of chocolate in each bar so that consumer *utility* or *aggregate quantity* of chocolate bars in period t is set equal to

$$q_{UV\alpha}{}^t \equiv \alpha \cdot q^t;\ t = 1, \ldots, T. \qquad (167)$$

Now use the measurement methodology explained in Section 2, and simply define the *chocolate bar price level* in period t, $p_{UV\alpha}{}^t$, as the period t expenditure on chocolate bars, $p^t \cdot q^t$, divided by the period t aggregate quantity defined by (167):

$$p_{UV\alpha}{}^t \equiv p^t \cdot q^t / q_{UV\alpha}{}^t = p^t \cdot q^t / \alpha \cdot q^t;\ t = 1, \ldots, T. \qquad (168)$$

Thus, the assumption of a single measurable dominant product characteristic leads to the period t price level for chocolate bars being equal to a *quality-adjusted unit value*.

The assumption of a single dominant characteristic of the product group can be modeled as a hedonic regression with a single characteristic. The hedonic regression model explained in Section 7 with only one characteristic boils down to the model $p_{tn} = \pi_t a_n + e_{nt}$ for $t = 1, \ldots, T$ and $n \in S(t)$, where $S(t)$ is the set of products available in period t, π_t is the period t price level, α_n are the quality adjustment parameters, and e_{tn} are error terms. But in the present case, we know α_n, so we can divide each p_{tn} by the corresponding α_n and obtain an estimator for π_t. Thus, the hedonic regression model becomes the following regression model:

$$p_{tn}/\alpha_n = p_t + e_{tn};\ t = 1, \ldots, T;\ n \in S(t). \qquad (169)$$

But the hedonic regression model defined by (169) does not take into account the economic importance of each chocolate bar to households. Thus, if we weight each quality-adjusted price p_{tn}/α_n by the market share amount of product n chocolate consumed by households during period t, we obtain the following *share-weighted estimator for the period t price level*:

$$\begin{aligned}\pi_t^* &\equiv \Sigma_{n \in S(t)}\ (p_{tn}/\alpha_n)[(\alpha_n q_{tn})/\Sigma_{i \in S(t)}\ (\alpha_i q_{ti})];\\ &\qquad t = 1, \ldots, T \qquad (170)\\ &= \Sigma_{n=1}^{N}\ (p_{tn}/\alpha_n)[(\alpha_n q_{tn})/\Sigma_{i=1}^{N}\ (\alpha_i q_{ti})] \text{ since } q_{tn} = 0 \textit{ if } n \in S(t)\\ &= p^t \cdot q^t / \alpha \cdot q^t\\ &= p_{UV\alpha}{}^t.\end{aligned}$$

Thus, the period t quality-adjusted price level, $p_{UV\alpha}{}^t$, can be viewed as an estimator for a very simple hedonic regression model.

The use of dominant characteristic quality adjustment is widespread. Some examples of this method of quality adjustment are listed here.

- **Package size.** Adjusting prices for the package size (or volume or weight of the product) is routinely applied by most Statistical Offices.[120]
- **Generic drugs**. When the patent on a new drug expires, usually a generic version of the drug comes on the marketplace at a lower price than the price for the branded product. The chemical composition of the branded and generic drug is the same (but the packaging can be different), and so rather than treating the generic drug as a new product, it makes sense to do a unit value aggregation in this situation where the dominant characteristic is the quantity of the drug.[121]
- **Lumens**. Nordhaus (1997) argued that what consumers valued was a measure of the amount of light that a device could deliver, and this assumption allowed him to make comparisons of the price of light over very long periods of time.
- **Minutes communicating.** Instead of following the price of a telecom plan, use the number of minutes spent using a telecommunications device (volume of calls) as the dominant characteristic.[122]
- **Bytes downloaded**. Instead of following the cost of a monthly subscription to access the internet, use the number of bytes downloaded (a volume measure) as the dominant characteristic.[123]
- **Floor space area**. In constructing price indices for rental properties, it is common to quality adjust the rental price by dividing it by the floor space of the unit.

Even though the dominant characteristic method of quality adjustment is widely used, Statistical Offices should be cautious in using this method. In many cases where it is used, there will be more than one price-determining characteristic of the product class. Thus, while it may be sensible to regard the amount of lumens emitted by a light bulb as a good indicator of quality, it may not be sensible to compare a modern light bulb with a kerosene oil burning lamp by looking at lumens emitted. Thus, in this case, it may be preferable to treat kerosene lamps as a separate product. Similarly, using only the floor space of a rental property as the only price-determining characteristic ignores other important price-determining characteristics such as the location of the property, the age of the structure, and the area of the land plot area associated with the structure. In this case, since each rental property is a unique product, the use of a hedonic regression with multiple price-determining

[120] Usually, the per volume price of a small package is greater than the per volume price of a large package. This fact can be modeled by using a more complicated hedonic regression model; see Diewert (2003a, 328).

[121] See Griliches and Cockburn (1994) for an example of this type of quality adjustment.

[122] See Abdirahman et al. (2022).

[123] See Byrne and Corrado (2021).

characteristics is recommended.[124] Unfortunately, any method of quality adjustment will have some subjective elements associated with it. National Statistical Offices will have to use their judgment on what is an appropriate method of quality adjustment to use in each specific application.

11.3 Experimental Economics Approaches and the Valuation of Free Products

It is particularly difficult to measure the welfare contribution of new digital services (such as Facebook) that are provided to households at prices that are close to zero. Brynjolfsson et al. (2019, 2020) (BCDEF in what follows) worked out a methodology for measuring the welfare effects of new free products using a blend of the Hicksian reservation price methodology for measuring the welfare benefits of new goods and services and an experimental economics approach involving estimating the willingness to pay for new products.[125]

Schreyer (2022) took the BCDEF approach one step further. He made use of a household production framework that took into account the fact that households face not only a budget constraint but also a time constraint.[126] Thus, spending time on Facebook is not completely free: The household's time constraints add an implicit cost to the household's consumption of Facebook services. Schreyer's analysis extended the BCDEF approach to include network effects. He also discussed the problems associated with including household production as part of GDP.

The BCDEF and Schreyer approach to the valuation of new free digital services requires many assumptions in order to generate concrete estimates of the benefits of these services. Thus, this approach is not ready to be implemented by National Statistical Offices at this stage of our knowledge. But the benefit estimates made by BCDEF are significant, and thus more research on this methodology should be undertaken. Moreover, the analysis of Schreyer highlights the importance of obtaining more accurate information on the household allocation of time.

12. Conclusion

This chapter has taken a consumer demand perspective to addressing the problem of adjusting price and quantity indices to take into account the benefits and costs of the introduction of new goods and services and the disappearance of existing commodities. This perspective allows all of the major methods that address the new and disappearing goods problem to be compared in a common framework.

There are three main methods that have been suggested in the literature to address the new goods problem: (i) the use of inflation-adjusted carry-forward and carry-backward prices; (ii) hedonic regression methods; and (iii) the estimation of consumer preferences and Hicksian reservation prices using both price and quantity data. The first two methods will work well if the new and disappearing products are highly substitutable with continuing products. However, if substitution is low, then the use of the first two methods can lead to substantial biases in price and quantity indices for the class of products under consideration. In the low elasticity of substitution case, the third class of methods should be used; that is, one should use either the cost or expenditure function methods suggested by Hausman[127] or the direct utility function estimation methods suggested by Diewert and Feenstra in Section 10. Unfortunately, these methods are not easy to implement. Thus, more research on these methods is required before statistical agencies can implement these methods on a large scale.

Some of the more important points made in the chapter are summarized here.

- Using the theoretical framework explained in Section 2 and applying it to hedonic regressions in Section 5 (when price and quantity data are available) shows that the hedonic regression approach generates two distinct estimates for the resulting price and quantity levels generated by the regression (unless the regression fits the data perfectly, in which case the two methods generate identical estimates). Thus, statistical agencies will have to choose between these two alternative index number estimates.
- The use of weights that reflect economic importance is recommended when running hedonic regressions; see the summary of the work by de Haan and Krsinich (2018) in Section 7.
- Weighted time dummy hedonic regression models that use characteristics information are recommended for dealing with quality adjustment problems provided that the products are moderately or highly substitutable; see Sections 6 and 7.
- Section 7 developed a test approach for evaluating the properties of hedonic regressions.
- Section 8 dealt with hedonic regressions in the context of taste change. Two useful methods for estimating price levels when there is considerable product churn were suggested: adjacent period time product hedonic regressions and the hedonic imputation method. The latter method runs separate hedonic regressions for each period and

[124] For examples of price-determining characteristics of property price indices, see Diewert, de Haan, and Hendriks (2015) and Diewert and Shimizu (2015, 2022).

[125] See also Diewert, Fox, and Schreyer (2020) for a simplified explanation of the BCDEF approach.

[126] This is a classic paper on integrating the time constraint with the budget constraint is Becker (1965). For extensions of his approach, see Schreyer and Diewert (2014) and Diewert, Fox, and Schreyer (2018).

[127] "Ultimately, data on price and product attributes alone will not allow correct estimation of the compensating variation adjustment to a cost of living index. Quantity data are also needed so that estimates of the demand functions (or equivalently, the expenditure or utility functions) can occur. For this reason, I disagree with the panel's conclusion that hedonic methods are 'probably the best hope' for improving quality adjustments (Schultze and Mackie 2002, 64 and 122) since hedonic methods do not use quantity data to estimate consumer valuation of a product, and consumer demand must be the basis of a cost of living index" (Jerry Hausman, 2003, 37). We agree with Hausman's criticisms of hedonic regression techniques to deal with the quality change problem except that we note that hedonic regressions can work well if the class of products under consideration are close substitutes for each other. Also, in some situations, we have no choice but to work with hedonic regressions rather than estimate consumer demand systems. For example, when constructing property price indices, each property is a *unique good*, both over time and space. A property has a unique location and over time the structure on the property changes due to renovations and depreciation. Thus, as noted earlier, hedonic regressions with characteristics information must be used in this situation.

averages the results of these separate regressions to obtain estimated price levels. If degrees of freedom are ample, the hedonic imputation method is recommended.

- Hedonic regression models viewed from the Hicksian approach to the treatment of new products have a fundamental problem: The underlying economic model assumes that the products are perfect substitutes after the implied quality adjustment. This is not a problem if, in fact, the quality-adjusted products are close to being perfect substitutes, but it can be a problem if this is not the case.
- The CES methodology for accounting for the benefits of new products due to Feenstra explained in Section 9 can work well if the elasticity of substitution between the products under consideration is high. If it is not high, the method will tend to lead to price indices that have a downward bias.
- The econometric method explained in Section 10 for dealing with new and disappearing products in the context of the Hicksian reservation price methodology avoids the problems associated with the Feenstra methodology but at the cost of a great deal of econometric complexity. A robust simplified version of this methodology is required before it can be applied by statistical agencies on a routine basis.
- Section 11 reviewed three additional methods that have been suggested to deal with the problems of quality adjustment and the introduction of new goods and services: (i) product clustering, (ii) dominant characteristic quality adjustment, and (iii) an experimental economics approach. Method (i) should be used with caution, but it can be justified in certain cases. Method (ii) is widely used, but again, this method should be used with caution if there is more than one important price-determining characteristic. Method (iii) is not yet ready for application by National Statistical Offices, but it does suggest that NSOs collect more information on the household allocation of time.

This chapter has taken an economic approach to the problem of quality adjustment that is based on the basic model of household behavior explained in Section 2. This economic model is not without its problems, but it does lead to a unified approach to the treatment of quality change from an economic perspective.

References

Abdirahman, Mo, Diane Coyle, Richard Heys and Will Stewart. 2022. “Telecoms Deflators: A Story of Volume and Revenue Weights.” *Economie et Statistique/Economics and Statistics*, 530–31: 43–59.

Aizcorbe, Ana. 2014. *A Practical Guide to Price Index and Hedonic Techniques*. Oxford: Oxford University Press.

Aizcorbe, Ana, Carol Corrado, and Mark Doms. 2000. “Constructing Price and Quantity Indexes for High Technology Goods.” Industrial Output Section, Division of Research and Statistics, Board of Governors of the Federal Reserve System, Washington, DC.

Allen, Roy George Douglas. 1938. *Mathematical Analysis for Economists*, London: Macmillan.

Arrow, Kenneth J., Hollis B. Chenery, Bagicha S. Minhas, and Robert M. Solow. 1961. “Capital-Labor Substitution and Economic Efficiency.” *Review of Economics and Statistics* 63: 225–250.

Australian Bureau of Statistics. 2016. “Making Greater Use of Transactions Data to Compile the Consumer Price Index.” Information Paper 6401.0.60.003, November 29, Canberra: ABS.

Balk, Bert M. 1980. “A Method for Constructing Price Indices for Seasonal Commodities.” *Journal of the Royal Statistical Society, Series A* 143: 68–75.

Becker, Gary S. 1965. “A Theory of the Allocation of Time.” *Economic Journal* 75: 493–517.

Bialek, Jacek. 2021. “Priceindices – A new R Package for Bilateral and Multilateral Price Index Calculations.” *Statistika – Statistics and Economy Journal* 36(2): 122–141.

———. 2022. “The General Class of Multilateral Indices and its Two Special Cases.” Paper presented at the 17th Meeting of the Ottawa Group on Price Indices, 7–10 June, Rome, Italy.

Brynjolfsson, Erik, Avinash Collis, W. Erwin Diewert, Felix Eggers, and Kevin J. Fox. 2019. “The Digital Economy, GDP and Household Welfare: Theory and Evidence.” NBER Working Paper 25695, National Bureau of Economic Research, Cambridge, MA.

———. 2020. “Measuring the Impact of Free Goods on Real Household Consumption.” *American Economic Association Papers and Proceedings* 110: 25–30.

Byrne, David, and Carol Corrado. 2021. “Accounting for Innovations in Consumer Digital Services: IT Still Matters.” In *Measuring and Accounting for Innovation in the Twenty-First Century*, edited by Carol Corrado, Jonathan Haskel, Javier Miranda, and Daniel Sichel, NBER Studies in Income and Wealth 78, 19–59, University of Chicago Press.

Chessa, Antonio G. 2021. “A Product Match Adjusted R Squared Method for Defining Products with Transaction Data.” *Journal of Official Statistics* 37: 411–432.

Court, Andrew T. 1939. “Hedonic Price Indexes with Automotive Examples.” In The *Dynamics of Automobile Demand*, 99–117. New York: General Motors Corporation.

Dalmaans, Jacco. 2022. “Multilateral Indices and the Relaunch Problem: Product Clustering and Alternative Solutions.” Paper presented at the 17th Meeting of the Ottawa Group on Price Indices, 7–10 June, Rome, Italy.

de Haan, Jan. 2004a. “The Time Dummy Index as a Special Case of the Imputation Törnqvist Index.” Paper presented at The Eighth Meeting of the International Working Group on Price Indices (the Ottawa Group), Helsinki, Finland.

———. 2004b. “Estimating Quality-Adjusted Unit Value Indices: Evidence from Scanner Data.” Paper presented at the Seventh EMG Workshop, Sydney, Australia, December 12–14.

———. 2009. “Comment on “Hedonic Imputation Versus Time Dummy Hedonic Indexes.” In *Price Index Concepts and Measurements*, edited by W. Erwin Diewert, John Greenlees, and Charles R. Hulten, 196–200. Chicago: University of Chicago Press.

———. 2010. “Hedonic Price Indexes: A Comparison of Imputation, Time Dummy and Re-pricing Methods.” *Jahrbücher für Nationökonomie und Statistik* 230: 772–791.

———. 2015. “A Framework for Large Scale Use of Scanner Data in the Dutch CPI.” Paper presented at the 14th meeting of the Ottawa Group, May 22, Tokyo.

de Haan, Jan, and Frances Krsinich. 2012. “The Treatment of Unmatched Items in Rolling Year GEKS Price Indexes: Evidence from New Zealand Scanner Data.” Paper presented at the Economic Measurement Group Workshop 2012, Australian School of Business, University of New South Wales, November 23.

———. 2014. “Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes.” *Journal of Business and Economic Statistics* 32(3): 341–358.

———. 2018. “Time Dummy Hedonic and Quality Adjusted Unit Value Indexes: Do They Really Differ?.” *Review of Income and Wealth* 64(4): 757–776.

Diewert, W. Erwin. 1974. "Applications of Duality Theory." In *Frontiers of Quantitative Economics*, vol. II, edited by Michael D. Intriligator and David A. Kendrick, 106–171. Amsterdam: North-Holland.

———. 1976. "Exact and Superlative Index Numbers." *Journal of Econometrics* 4: 114–145.

———. 1980. "Aggregation Problems in the Measurement of Capital." In *The Measurement of Capital*, edited by Dan Usher, 433–528. Chicago: The University of Chicago Press.

———. 1992. "Fisher Ideal Output, Input and Productivity Indexes Revisited." *Journal of Productivity Analysis* 3: 211–248.

———. 1993. "The Early History of Price Index Research." In *Essays in Index Number Theory*, edited by W. Erwin Diewert and Alice O. Nakamura, vol. 1, 33–65. Amsterdam: North-Holland.

———. 1998. "Index Number Issues in the Consumer Price Index." *Journal of Economic Perspectives* 12(1): 47–58.

———. 2003a. "Hedonic Regressions: A Consumer Theory Approach." In *Scanner Data and Price Indexes*, edited by Robert C. Feenstra and Matthew D. Shapiro, Studies in Income and Wealth, vol. 61, 317–348. Chicago: University of Chicago Press.

———. 2003b. "Hedonic Regressions: A Review of Some Unresolved Issues." Paper presented at the Seventh Meeting of the Ottawa Group, Paris, 27–29 May.

———. 2004. "On the Stochastic Approach to Linking the Regions in the ICP." Discussion Paper No. 04–16, Department of Economics, The University of British Columbia, Vancouver, Canada.

———. 2005a. "Weighted Country Product Dummy variable Regressions and Index Number Formulae." *Review of Income and Wealth* 51: 561–570.

———. 2005b. "Adjacent Period Dummy variable Hedonic Regressions and Bilateral Index Number Theory." *Annales D'Économie et de Statistique* 79/80: 759–786.

———. 2022. "Scanner Data, Elementary Price Indexes and the Chain Drift Problem." In *Consumer Price Index Theory*, Draft Chapter 7. Washington, DC: International Monetary Fund, https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W. Erwin, and Robert C. Feenstra. 2017. "Estimating the Benefits and Costs of New and Disappearing Products." Vancouver School of Economics, Discussion Paper 17–10, University of British Columbia, Vancouver, B.C., Canada, V6T 1L4.

———. 2022. "Estimating the Benefits of New Products." In *Big Data for Twenty-First-Century Economic Statistics*, edited by Katharine G. Abraham, Ron S. Jarmin, Brian C. Moyer, and Matthew D. Shapiro, 437–473. Chicago: University of Chicago Press.

Diewert, W. Erwin, and Kevin J. Fox (2021) "Substitution Bias in Multilateral Methods for CPI Construction Using Scanner Data." *Journal of Business and Economic Statistics* 40(1): 355–369.

Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2017. "The Digital Economy, New Products and Consumer Welfare." Vancouver School of Economics Discussion Paper 17–09, University of British Columbia.

———. 2018. "The Allocation and Valuation of Time." Vancouver School of Economics Discussion Paper 18–10, University of British Columbia, Vancouver Canada.

———. 2022. "Experimental Economics and the New Commodities Problem." *Review of Income and Wealth*, forthcoming.

Diewert, W. Erwin, Jan de Haan, and Kevin J. Fox. 2016. "A Newly Identified Source of Potential CPI Bias: Weekly versus Monthly Unit Value Price Indexes." *Economics Letters* 141: 169–172.

Diewert, W. Erwin, Jan de Haan, and Rens Hendriks. 2015. "Hedonic Regressions and the Decomposition of a House Price index into Land and Structure Components." *Econometric Reviews* 34: 106–126.

Diewert, W. Erwin, Saeed Heravi and Mick Silver. 2009. "Hedonic Imputation versus Time Dummy Hedonic Indexes." In *Price Index Concepts and Measurement*, edited by W. Erwin Diewert, John Greenlees, and Charles R. Hulten, Studies in Income and Wealth, vol. 70, 87–116. Chicago: University of Chicago Press.

Diewert, W. Erwin, and Robert J. Hill. 2010. "Alternative Approaches to Index Number Theory." In *Price and Productivity Measurement*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura (eds.), 263–278. Victoria Canada: Trafford Press.

Diewert, W. Erwin, Ning Huang, and Kate Burnett-Issacs. 2017. "Alternative Approaches for Resale Housing Price Indexes." Discussion Paper 17–05, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.

Diewert, W. Erwin, and Peter von der Lippe. 2010. "Notes on Unit Value Index Bias." *Jahrbücher für Nationalökonomie und Statistik* 230: 690–708.

Diewert, W. Erwin, and Chihiro Shimizu. 2015. "Residential Property Price Indices for Tokyo." *Macroeconomic Dynamics* 19: 1659–1714.

———. 2016. "Hedonic Regression Models for Tokyo Condominium Sales." *Regional Science and Urban Economics* 60: 300–315.

———. 2022. "Residential Property Price Indexes: Spatial Coordinates versus Neighbourhood Dummy variables." *Review of Income and Wealth*, forthcoming.

Diewert, W. Erwin, and Terence J. Wales. 1987. "Flexible Functional Forms and Global Curvature Conditions." *Econometrica* 55: 43–68.

———. 1988. "A Normalized Quadratic Semiflexible Functional Form." *Journal of Econometrics* 37: 327–342.

Eurostat. 2013. *Handbook on Residential Property Prices Indices (RPPIs)*. Luxembourg: Publications Office of the European Union.

Feenstra, Robert C. 1994. "New Product varieties and the Measurement of International Prices." *American Economic Review* 84(1): 157–177.

Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton-Mifflin.

Griliches, Zvi. 1971. "Introduction: Hedonic Price Indexes Revisited." In *Price Indexes and Quality Change*, edited by Zvi Griliches, 3–15. Cambridge, MA: Harvard University Press.

Griliches, Zvi, and I. Cockburn. 1994. "Generics and New Goods in Pharmaceutical Price Indexes." *American Economic Review* 84: 1213–1232.

Hardy, Godfrey Harold, John Edensor Littlewood, and György Pólya. 1934. *Inequalities*. Cambridge: Cambridge University Press.

Hausman, Jerry A. 1981. "Exact Consumer's Surplus and Deadweight Loss." *American Economic Review* 71(4): 662–676.

———. 1996. "Valuation of New Goods under Perfect and Imperfect Competition." In *The Economics of New Goods*, edited by Timothy F. Bresnahan and Robert J. Gordon, 20–236. Chicago: University of Chicago Press.

———. 1999. "Cellular Telephone, New Products and the CPI." *Journal of Business and Economic Statistics* 17(2): 188–194.

———. 2003. "Sources of Bias and Solutions to Bias in the Consumer Price Index." *Journal of Economic Perspectives* 17(1): 23–44.

Hicks, John R. 1940. "The Valuation of the Social Income." *Economica* 7: 105–124.

Hill, Robert J. 2013. "Hedonic Price Indexes for Residential Housing: A Survey, Evaluation and Taxonomy." *Journal of Economic Surveys* 27: 879–914.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2009. "Scanner Data, Time Aggregation and the Construction of Price Indexes." Discussion Paper 09–09, Department of Economics, University of British Columbia, Vancouver, Canada.

———. 2011. "Scanner Data, Time Aggregation and the Construction of Price Indexes." *Journal of Econometrics* 161: 24–35.

Keynes, John M. 1909. "The Method of Index Numbers with Special Reference to the Measurement of General Exchange Value." Reprinted as in *The Collected Writings of John Maynard Keynes* (1983), edited by Don Moggridge, vol. 11, 49–156. Cambridge: Cambridge University Press.

———. 1930. *Treatise on Money*, vol. 1. London: Macmillan.

Konüs, Alexander A. 1924. "The Problem of the True Index of the Cost of Living." translated in 1939 in *Econometrica* 7: 10–29.

Konüs, Alexander A. and Sergei S. Byushgens. 1926. "K Probleme Pokupatelnoi Cili Deneg." *Voprosi Konyunkturi* 2: 151–172.

Krsinich, Frances. 2016. "The FEWS Index: Fixed Effects with a Window Splice." *Journal of Official Statistics* 32: 375–404.

Lamboray, Claude. 2022. "What Impact does Product Specification have on a Fisher Price Index?." Paper presented at the 17th Meeting of the Ottawa Group on Price Indices, 7–10 June, Rome, Italy.

Lehr, Julius. 1885. *Beiträge zur Statistik der Preise*. Frankfurt: J.D. Sauerländer.

Marshall, Alfred. 1887. "Remedies for Fluctuations of General Prices." *Contemporary Review* 51: 355–375.

Muellbauer, John. 1974. "Household Production Theory, Quality and the .Hedonic Technique." *American Economic Review* 64(6): 977–994.

Nordhaus, William D. 1997. "Do Real Output and Real Wage Measures Capture Reality? The History of Lighting Suggests Not." In *The Economics of New Goods*, edited by Timothy F. Bresnahan and Robert J. Gordon, 29–66. Chicago: University of Chicago Press.

Pakes, Ariel. 2001. "A Reconsideration of Hedonic Price Indices with and Application to PCs." NBER Working Paper 8715, National Bureau of Economic Research, Cambridge, MA.

Poirier, Dale J. 1976. *The Econometrics of Structural Change*. Amsterdam: North-Holland Publishing Company.

Rao, D.S. Prasada. 1995. "On the Equivalence of the Generalized Country-Product-Dummy (CPD) Method and the Rao-System for Multilateral Comparisons." Working Paper No. 5, Centre for International Comparisons, University of Pennsylvania, Philadelphia.

———. 2004. "The Country-Product-Dummy Method: A Stochastic Approach to the Computation of Purchasing Power parities in the ICP." Paper presented at the SSHRC Conference on Index Numbers and Productivity Measurement, June 30–July 3, 2004, Vancouver, Canada.

———. 2005. "On the Equivalence of the Weighted Country Product Dummy (CPD) Method and the Rao System for Multilateral Price Comparisons." *Review of Income and Wealth* 51(4): 571–580.

Rosen, Sherwin. 1974. "Hedonic Prices and Implicit Markets: Product Differentiation in Pure Competition." *Journal of Political Economy* 82: 34–55.

Samuelson, Paul A. and Subramanian Swamy. 1974. "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis." *American Economic Review* 64: 566–593.

Sato, Kazuo. 1976. "The Ideal Log-Change Index Number." *Review of Economics and Statistics* 58: 223–228.

Schreyer, Paul. 2022. "Accounting for Free Digital Services and Household Production – An Application to Facebook (Meta)." *Eurostat Review on National Accounts and Macroeconomic Indicators*, forthcoming.

Schreyer, Paul, and W. Erwin Diewert. 2014. "Household Production, Leisure and Living Standards." In *Measuring Economic Sustainability and Progress*, edited by Dale W. Jorgenson, J. Steven Landefeld, and Paul Schreyer, 89–114. Chicago, IL: University of Chicago Press.

Schultze, Charles, and Christopher Mackie, eds. 2002. *At What Price?*. Washington, DC: National Academy Press.

Shephard, Ronald William. 1953. *Cost and Production Functions*. Princeton: Princeton University Press.

Shimizu, Chihiro, Kiyohiko G. Nishimura, and Tsutomu Watanabe. 2010. "Housing Prices in Tokyo: A Comparison of Hedonic and Repeat Sales Measures." *Journal of Economics and Statistics* 230: 792–813.

Silver, Mick. 2010. "The Wrong and Rights of Unit Value Indices." *Review of Income and Wealth* 56:S1: 206–206.

———. 2011. "An Index Number Formula Problem: The Aggregation of Broadly Comparable Items." *Journal of Official Statistics* 27(4): 1–17.

———. 2018. "How to Measure Hedonic Property Price Indexes Better." *EURONA* 1/2018: 35–66.

Silver, Mick, and Saeed Heravi. 2005. "A Failure in the Measurement of Inflation: Results from a Hedonic and Matched Experiment using Scanner Data." *Journal of Business and Economic Statistics* 23: 269–281.

———. 2007. "The Difference Between Hedonic Imputation Indexes and Time Dummy Hedonic Indexes." *Journal of Business and Economic Statistics* 25: 239–246.

Stansfield, Matthew, and Frances Krsinich. 2022. "A MAP for the Future of Price Indexes at Stats NZ." Paper presented at the 17th Ottawa Group on Price Indices, Rome 7–10 June.

Summers, Robert. 1973. "International Price Comparisons Based Upon Incomplete Data." *Review of Income and Wealth* 19: 1–16.

Triplett, Jack E. 1987. "Hedonic Functions and Hedonic Indexes." In *The New Palgrave: A Dictionary of Economics*, edited by John Eatwell, Murray Milgate, and Peter Newman, vol. 2, 630–634. New York, NY: Stockton Press.

———. 2004. *Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes*, Directorate for Science, Technology and Industry, DSTI/DOC(2004)9. Paris: OECD.

Triplett, Jack E., and Richard J. McDonald. 1977. "Assessing the Quality Error in Output Measures: The Case of Refrigerators." *The Review of Income and Wealth* 23(2): 137–156.

Uzawa, Hirofumi. 1962. "Production Functions with Constant Elasticities of Substitution." *Review of Economic Studies* 29: 291–299.

vartia, Yrjö O. 1976. "Ideal Log-Change Index Numbers." *Scandinavian Journal of Statistics* 3: 121–126.

von Hofsten, Erland. 1952. *Price Indexes and Quality Change*. London: George Allen and Unwin.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Wiley, David E., William H. Schmidt, and William J. Bramble. 1973. "Studies of a Class of Covariance Structure Models." *Journal of the American Statistical Association* 68: 317–323.

Wold, Herman. 1944. "A Synthesis of Pure Demand Analysis, Part 3." *Skandinavisk Aktuarietidskrift* 27: 69–120.

Zhang, Li-Chun, Ingvild Johansen, and Ragnhild Nygaard. 2019. "Tests for Price Indices in a Dynamic Item Universe." *Journal of Official Statistics*, 35(3): 683–697.

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SEASONAL PRODUCTS 9

1. The Problem of Seasonal Products

The existence of seasonal products (goods or services) poses some significant challenges for price statisticians. *Seasonal products* are products or services that are either (a) not available in the marketplace during certain seasons of the year or (b) available throughout the year but there are regular fluctuations in prices or quantities that are synchronized with the season or the time of the year.[1] A product that satisfies (a) is termed a *strongly seasonal product*, whereas a product that satisfies (b) is termed a *weakly seasonal product*. It is strongly seasonal products that create the biggest problems for price statisticians in the context of producing a monthly or quarterly CPI because if a product price is available in only one of the two months (or quarters) being compared, then obviously it is not possible to calculate a relative price for the product, and traditional bilateral index number theory breaks down. In other words, if a product is present in one month but not in the next, how can the month-to-month amount of price change for that product be computed?[2] There is no easy solution to this lack of comparability problem. This chapter will present various attempts at finding solutions to this problem.

There are two main sources of seasonal fluctuations in prices and quantities: (a) climate and (b) custom.[3] In the first category, fluctuations in temperature, precipitation, and hours of daylight cause fluctuations in the demand for or supply of many products; for example, think of summer versus winter clothing, the demand for light and heat, and vacations. With respect to custom and convention as a cause of seasonal fluctuations, consider the following quotation:

> Conventional seasons have many origins—ancient religious observances, folk customs, fashions, business practices, statute law. . . . Many of the conventional seasons have considerable effects on economic behaviour. We can count on active retail buying before Christmas, on the Thanksgiving demand for turkeys, on the first of July demand for fireworks, on the preparations for June weddings, on heavy dividend and interest payments at the beginning of each quarter, on an increase in bankruptcies in January, and so on.
>
> Wesley C. Mitchell (1927; 237)

Examples of important seasonal products are many food items, alcoholic beverages, many clothing and footwear items, water, heating oil, electricity, flowers and garden supplies, vehicle purchases, vehicle operation, many entertainment and recreation expenditures, books, insurance expenditures, wedding expenditures, recreational equipment, and air travel and tourism expenditures. For a "typical" country, seasonal expenditures will often amount to one-fifth to one-third of all consumer expenditures.[4]

In the context of producing a monthly or quarterly CPI, it must be recognized that there is no completely satisfactory way for dealing with strongly seasonal products. If a product is present in one month but missing from the marketplace in the next month, then many of the index number theories that were considered in earlier chapters cannot be applied because these theories assumed that the dimensionality of the product space was constant for the two periods being compared. However, if seasonal products are present in the market for certain months of the year on a regular basis, then traditional index number theory can be applied in order to construct *year-over-year indices for the same month*. This approach is discussed in Sections 2 and 3. In the initial sections of this chapter, it will be assumed that price and quantity information for the seasonal products is available. The various indices that are considered in this chapter will be illustrated using actual data on fresh fruit consumption for Israel for the six years 2012–2017. The underlying data are listed in the annex along with tables using these data that list the various indices that are considered in the main text.

The methods that are suggested in Sections 2–6 of this chapter to deal with seasonal products assume that the statistical agency is able to collect *price* and *expenditure* information on these seasonal products by month.[5] In Sections 7 and

[1] This classification of seasonal products corresponds to Balk's narrow and wide sense seasonal products; see Balk (1980a; 7) (1980b; 110) (1980c; 68). Diewert (1998; 457) used the terms type 1 and type 2 seasonality.

[2] Zarnowitz (1961; 238) was perhaps the first to note the importance of this problem: "But the main problem introduced by the seasonal change is precisely that the market basket is different in the consecutive months (seasons), not only in weights but presumably often also in its very composition by products. This is a general and complex problem which will have to be dealt with separately at later stages of our analysis."

[3] This classification dates back to Mitchell (1927; 236) at least: "Two types of seasons produce annually recurring variations in economic activity—those which are due to climates and those which are due to conventions."

[4] Alterman, Diewert, and Feenstra (1999; 151) found that over the 40 months between September 1993 and December 1996, somewhere between 23 and 40 percent of US imports and exports exhibited seasonal variations in *quantities*, whereas only about 5 percent of US export and import *prices* exhibited seasonal fluctuations.

[5] Hardly any statistical agencies have comprehensive monthly expenditure surveys, and so many of the methods suggested in this chapter are not feasible at present. However, an increasing number of agencies are collecting weekly scanner data from retailers, which have detailed

8, the construction of month-to-month indices using *only price information* will be considered.

The indices discussed in the various sections of this chapter are different depending on the following differences that characterize the method used to deal with the seasonality problem and the availability of data:

- Price and quantity (or expenditure) data are available versus only price information is available.
- Carry-forward prices[6] are used as imputations for missing prices versus methods that do not use imputations.
- A year-over-year index for the same month is constructed versus a month-to-month index. Annual indices that measure all prices in one year relative to another year provide another source of difference.
- A traditional fixed-base or chained bilateral Laspeyres, Paasche, Fisher, or Törnqvist index is constructed versus a multilateral index.
- The index uses monthly weights or annual weights.

With these five main sources of differences in index concept in mind, an overview of the various sections of this chapter follows.

Sections 2–7 deal with methods that make use of monthly *price* and *quantity* information. Section 2 constructs traditional fixed-base and chained *year-over-year monthly indices* using year-over-year *carry-forward prices* for any missing prices. Section 3 *constructs year-over-year monthly indices* using fixed-base or chained or multilateral indices *without using imputations* for missing prices and quantities. Sections 4 and 5 consider the production of *annual indices*. These annual indices treat each monthly product as a separate product in a yearly index. Section 4 indices use carry-forward prices for missing prices, while Section 5 indices do not use any imputed prices. It turns out that some of the Laspeyres and Paasche annual indices that use carry-forward prices can be related to the year-over-year Laspeyres and Paasche monthly indices studied in Sections 2 and 3.

Section 6 constructs traditional *month-to-month indices* using carry-forward prices for the missing prices. Section 7 constructs month-to-month fixed-base and chained Laspeyres, Paasche, and Fisher indices as well as some multilateral indices (with no imputations for missing prices).

Sections 8 and 9 construct indices *using only information on prices*. Section 8 uses carry-forward prices for the missing prices, while Section 9 uses multilateral methods with no imputations for any missing prices. A new multilateral method of linking price observations based on *relative price similarity* (when quantity or expenditure information is not available) is suggested in Section 9.

Section 10 assumes that some expenditure or quantity information is available in addition to price information. The additional expenditure information that is assumed available is *annual expenditure information by product for a base year*. With this extra information (and the use of carry-forward prices for any missing prices), a Lowe (1823) or Young (1812) index can be calculated and compared to some of the alternative indices that were calculated in earlier sections.

Section 11 returns to the problems associated with forming annual indices. The annual indices studied in Sections 4 and 5 are annual indices for calendar years. In Section 11, these annual indices are generalized to form *Rolling Year annual indices*; that is, the prices of 12 consecutive months are compared with the prices of a base period run of 12 consecutive months, and the price comparisons are such that the January prices in the current rolling year are compared with the January prices in the base year; the February prices in the current rolling year are compared with the February prices in the base year, and so on. It turns out that these Rolling Year indices are related to measures of trend inflation.

Section 12 concludes by summarizing the more important results in the light of the calculations using the Israeli data set. Before proceeding to the technical definitions of the various indices, it is useful to discuss the notation that will be used and the interpretation of the variables. The following algebra assumes that the statistical agency has information on the monthly prices and quantities for the N products that enter the scope of the index. However, not all products will be present in each month. Denote the set of products n that are present in the marketplace during month m of year y as $S(y,m)$. Data on prices and quantities are available for Y years and say $M = 12$ months.[7] Denote the price of product n in month m of year y as $p_{y,m,n}$, the corresponding quantity as $q_{y,m,n}$, and the corresponding expenditure share as

$$s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/\Sigma_{k\in S(y,m)}\, p_{y,m,k}q_{y,m,k};\; y = 1,\ldots,Y;\quad m = 1,2,\ldots,M;\; n\in S(y,m).^{8} \qquad (1)$$

It is assumed that $q_{y,m,n}$ is the total quantity of product n sold to households in scope for the index in month m of year y, and $p_{y,m,n}$ is the corresponding monthly unit value price. In the following four sections, various index number formulae will be defined using this notation. However, the resulting indices could refer to several situations:

price and quantity information on sales by individual product, including seasonal products. Thus, at the first stage of aggregation for these categories of consumer expenditures, it is possible to utilize current price and quantity information to construct an elementary index. In addition, in the future, it may become possible to collect electronic data on consumer products directly from households. In general, it will be possible to implement the methods suggested in this chapter for at least parts of a country's CPI.

[6] Most statistical agencies do not use simple carry-forward prices for missing prices. Instead, they use *inflation-adjusted carry-forward prices*; that is, the price movement of a closely related product or group of products is used to update the last available price for a missing product. For example, the Eurostat 2020 regulation for computing an inflation-adjusted price for a missing seasonal price specifies that it should be set "equal to the previous month's price adjusted by the average change in observed prices over all individual products in the same ECOICOP group, class, subclass or same aggregate at any level below the subclass"; see Article 2 (25) (*b*) in the European Commission (2020; 15). See Diewert, Fox, and Schreyer (2018) for examples of how this method for price imputation works in the context of particular index number formulae. Due to the large number of ways inflation-adjusted carry-forward prices could be calculated, we will just discuss carry-forward prices in this chapter.

[7] It is possible to construct "monthly" indices that consist of 13 "months" that consist of four consecutive weeks. Thus, when we define various indices, we will generally assume that there are data for M "months" in the year. This also allows M to equal 4 for cases where quarterly price indices are constructed. However, for our empirical example, $M = 12$.

[8] The summation $\Sigma_{k\in S(y,m)}\, p_{y,m,k}q_{y,m,k}$ means that we sum expenditures in month m of year y over products k that are actually present in month m of year y; that is, strongly seasonal products that are not present in month m of year y are excluded in this sum.

- N is the total number of separate items that are to be distinguished in the overall CPI; that is, the underlying assumption here is that we have complete price and quantity information on the universe of expenditures for the reference population.
- N refers to the number of items in one particular *stratum* of the overall CPI. Standard index number theory is also applicable in this situation.
- The various methodologies to deal with seasonal products could be applied at higher levels of aggregation. Data on expenditures by category could be available along with elementary price indices for the categories in scope. Implicit quantities (or volumes) by category could be constructed by deflating the expenditure categories by the respective elementary price indices. These deflated expenditures are treated as the quantities $q_{y,m,n}$, and the corresponding elementary price indices $p_{y,m,n}$ are treated as the corresponding prices.

Obviously, application of the first interpretation of the indices is unrealistic; the statistical agency will typically not have access to true microeconomic data at the finest level of aggregation. However, application of the second interpretation of the indices is quite possible; the existence of scanner data sets has led to the possibility of computing say true Fisher indices for some strata of the CPI.[9]

2. Year-over-Year Monthly Indices Using Carry-Forward Prices

For over a century,[10] it has been recognized that making year-over-year comparisons[11] of prices in the same month provides the simplest method for making comparisons that are (mostly) free from the contaminating effects of seasonal fluctuations. For example, the economist Flux and the statistician Yule endorsed the idea of making year-over-year comparisons to minimize the effects of seasonal fluctuations:

> Each month the average price change compared with the corresponding month of the previous year is to be computed. . . . The determination of the proper seasonal variations of weights, especially in view of the liability of seasons to vary from year to year, is a task from which, I imagine, most of us would be tempted to recoil.
>
> A. *W*. Flux (1921; 184–185)

> My own inclination would be to form the index number for any month by taking ratios to the corresponding month of the year being used for reference, the year before presumably, as this would avoid any difficulties with seasonal products. *I* should then form the annual average by the geometric mean of the monthly figures.
>
> *G*. Udny Yule (1921; 199)

Zarnowitz also endorsed the use of year-over-year monthly indices:

> There is of course no difficulty in measuring the average price change between the same months of successive years, if a month is our unit season, and if a constant seasonal market basket can be used, for traditional methods of price index construction can be applied in such comparisons.
>
> Victor Zarnowitz (1961; 266)

However, using year-over-year monthly indices does not completely solve the seasonality problem. Diewert, Finkel, and Artsev found that strongly seasonal fresh fruits in Israel did not always appear in the same months:[12]

> Seasonal fluctuations are not completely synchronized with the calendar months for products with strong seasonality. Thus a product may appear/disappear a month before/later than in the previous year.
>
> *W*. Erwin Diewert, Yoel Finkel, and Yevgeny Artsev (2011; 63)

In this section, we will deal with the possibility that the strongly seasonal products do not always appear in the same month of each year by using *carry-forward prices* from the previous year (for the same month)[13] for any missing prices. The corresponding missing quantities are set to 0. With these conventions, the set of available products for month m in year y, $S(y,m)$, is defined to include any temporarily missing products so that for any month m, the set of available products for month m in year y will always be the same. Thus, the set of "available" products for month m in year y, $S(y,m)$, will be constant over the years y.[14] Thus, we can denote the common set of "available" products for month m in any year y as $S(m)$. With this new notation that accommodates the carry-forward prices for missing products in a given month, the *fixed-base Laspeyres, Paasche,*

[9] See Ivancic, Diewert, and Fox (2011), de Haan and van der Grient (2011), and the Australian Bureau of Statistics (2016) for early applications of this type.

[10] "In the daily market reports, and other statistical publications, we continually find comparisons between numbers referring to the week, month, or other parts of the year, and those for the corresponding parts of a previous year. The comparison is given in this way in order to avoid any variation due to the time of the year. And it is obvious to everyone that this precaution is necessary. Every branch of industry and commerce must be affected more or less by the revolution of the seasons, and we must allow for what is due to this cause before we can learn what is due to other causes" (*W*. Stanley Jevons (1884; 3)).

[11] In the seasonal price index literature, this type of index corresponds to Bean and Stine's (1924; 31) Type *D* index.

[12] A similar lack of matching problem can occur if national holidays do not always appear in the same month of the year.

[13] If the missing product is missing in the previous year (for the same month), go backward in time to the last year (for the same month) when the product was present. If the product was not present (for the same month) in any previous year, go to the year when the product first appears in the month under consideration and use this price as a carry-backward price for the years that the product was missing.

[14] Thus for this section where we use year-over-year carry-forward (or carry-backward) prices for any strongly seasonal products that happen to be missing in one or more years, the set of "available" products in month m for any year in our sample is the set of products that appeared in at least one month m over all month m's in the sample of years.

Fisher, and *Törnqvist–Theil indices*[15] for month m in year y are defined as follows:

$$P_{LFB}^{y,m} \equiv \Sigma_{n\in S(m)} (p_{y,m,n}/p_{1,m,n})s_{1,m,n};\quad m = 1,\ldots,M;\ y = 1,\ldots,Y; \qquad (2)$$

$$P_{PFB}^{y,m} \equiv [\Sigma_{n\in S(m)} (p_{y,m,n}/p_{1,m,n})^{-1}s_{y,m,n}]^{-1};\quad m = 1,\ldots,M;\ y = 1,\ldots,Y; \qquad (3)$$

$$P_{FFB}^{y,m} \equiv [P_{LFB}^{y,m}\, P_{PFB}^{y,m}]^{1/2};\ m = 1,\ldots,M;\quad y = 1,\ldots,Y; \qquad (4)$$

$$P_{TFB}^{y,m} \equiv exp[\Sigma_{n\in S(m)} (½)(s_{1,m,n} + s_{y,m,n})ln(p_{y,m,n}/p_{1,m,n})];\quad m = 1,\ldots,M;\ y = 1,\ldots,Y. \qquad (5)$$

The expenditure shares, $s_{y,m,n}$, that appear in definitions (2)–(5) are defined by (1).

The chained versions of these four indices are defined in two stages. For the first stage, define the *chain link* for each of the aforementioned indices going from month m in year y–1 to month m in year y as follows:

$$P_{LLINK}^{y,m} \equiv \Sigma_{n\in S(m)} (p_{y,m,n}/p_{y-1,m,n})s_{y-1,m,n};\quad m = 1,\ldots,M;\ y = 2,\ldots,Y; \qquad (6)$$

$$P_{PLINK}^{y,m} \equiv [\Sigma_{n\in S(m)} (p_{y,m,n}/p_{y-1,m,n})^{-1}s_{y,m,n}]^{-1};\quad m = 1,\ldots,M;\ y = 2,\ldots,Y; \qquad (7)$$

$$P_{FLINK}^{y,m} \equiv [P_{LLINK}^{y,m}\, P_{PLINK}^{y,m}]^{1/2};\quad m = 1,\ldots,M;\ y = 2,\ldots,Y; \qquad (8)$$

$$P_{TLINK}^{y,m} \equiv exp[\Sigma_{n\in S(m)} (½)(s_{y-1,m,n} + s_{y,m,n})ln(p_{y,m,n}/p_{y-1,m,n})];\quad m = 1,\ldots,M;\ y = 2,\ldots,Y. \qquad (9)$$

Define the *chained Laspeyres, Paasche, Fisher*, and *Törnqvist–Theil indices* for month m in year 1 as unity:

$$P_{LCH}^{1,m} \equiv 1;\ P_{PCH}^{1,m} \equiv 1;\ P_{FCH}^{1,m} \equiv 1;\ P_{TCH}^{1,m} \equiv 1;\quad m = 1,\ldots,M. \qquad (10)$$

For years following year 1, these indices are defined by cumulating the corresponding chain links; that is, we have the following definitions:

$$P_{LCH}^{y,m} \equiv P_{LCH}^{y-1,m}\, P_{LLINK}^{y,m};\ m = 1,\ldots,M;\quad y = 2,\ldots,Y; \qquad (11)$$

$$P_{PCH}^{y,m} \equiv P_{PCH}^{y-1,m}\, P_{PLINK}^{y,m};\ m = 1,\ldots,M;\quad y = 2,\ldots,Y; \qquad (12)$$

$$P_{FCH}^{y,m} \equiv P_{FCH}^{y-1,m}\, P_{FLINK}^{y,m};\ m = 1,\ldots,M;\quad y = 2,\ldots,Y; \qquad (13)$$

$$P_{TCH}^{y,m} \equiv P_{TCH}^{y-1,m}\, P_{TLINK}^{y,m};\ m = 1,\ldots,M;\quad y = 2,\ldots,Y. \qquad (14)$$

[15]See Laspeyres (1871), Paasche (1874), Fisher (1922), Törnqvist (1936), Törnqvist and Törnqvist (1937), and Theil (1967).

For each month m, there are eight commonly used indices to choose from. From the viewpoint of the economic approach to index number theory, the two Laspeyres indices are subject to some upward substitution bias relative to a cost of living index, while the two Paasche indices are subject to some downward substitution bias. If there are smooth trends in prices and quantities, these substitution biases will be lower in magnitude if chained indices are used in place of their fixed-base counterparts; the opposite will be true if there is *price bouncing behavior*[16]—that is, if prices and quantities fluctuate erratically over time. Harvests of fresh fruits vary considerably for the same month of the year on a year-over-year basis, which leads to considerable fluctuations in prices and hence to price bouncing behavior. Thus, for our empirical example, we found that the year-over-year monthly Laspeyres fixed-base indices exhibited a considerable amount of upward substitution bias, and the chained Laspeyres indices exhibited even more upward bias. On the other hand, the year-over-year monthly Paasche fixed-base indices exhibited a considerable amount of downward substitution bias, and the chained Paasche indices exhibited even more downward bias. Thus, from the viewpoint of the economic approach to index number theory, the use of the Laspeyres and Paasche formulae is not recommended in the context of forming year-over-year monthly indices.

From the viewpoint of the economic approach to index number theory, the bilateral Fisher and Törnqvist–Theil indices have equally good properties; they are examples of *superlative index number formulae* and can deal adequately with substitution bias.[17] Moreover, they approximate each other numerically to the second order around any point that has equal price and quantity vectors in the two periods being compared.[18] Finally, the Fisher index has excellent properties from the viewpoint of the test approach to index number theory,[19] and the Törnqvist–Theil index has excellent properties from the viewpoint of the stochastic approach to index number theory.[20] Thus, these two indices have very desirable properties from the perspective of a variety of approaches to index number theory. For the year-over-year monthly indices listed in the annex, the fixed-base Fisher and Törnqvist–Theil indices approximated each other quite well for our empirical example.

From the viewpoint of the test approach to index number theory, the two fixed-base superlative indices have an advantage over their chained counterparts. They satisfy the following *multiperiod identity test*: If prices *and* quantities are the same in any two periods, the two fixed-base indices will register the same price level for these two periods. The two chained superlative indices do not satisfy this identity test if there are four or more periods in the set of comparisons.[21]

[16]This term was used by Szulc (1983) (1987) who also demonstrated empirically the chain drift problem for the Laspeyres index when prices bounce.

[17]See Diewert (1976), who defined a superlative index number formula as one which was consistent with a wide variety of consumer substitution responses to changes in relative prices.

[18]See Diewert (1978).

[19]See Diewert (1992).

[20]See Theil (1967).

[21]The corresponding strong identity test is as follows: If *prices* are the same in any two periods, the multilateral index will register the same price level for these two periods. For materials on the test approach to multilateral index number theory, see Diewert (1988) (1999b) (2021b),

These considerations suggest that the two fixed-base superlative indices are preferred indices in the previous menu of eight possible indices. However, there are two problems with the use of a fixed-base index:

- The prices and quantities of the base period may not be representative of prices and quantities in subsequent periods.
- New products may appear, and products present in the base period may disappear in subsequent periods, making comparisons between distant periods difficult.

The second set of difficulties could be regarded as a special case of the first set of difficulties. In the context of our fresh fruit empirical example, the problem of new and disappearing products is not present. However, fluctuations in harvests certainly occurred, and so there is the danger that the base period may not represent "typical" conditions, and thus the choice of a different base period could lead to very different indices. Indeed, for our empirical example, different choices of the base period do lead to very different indices.

In order to deal with the aforementioned first difficulty, we will turn to the use of *multilateral indices*. Fisher was the first index number theorist to suggest a solution to the problem of fixed-base price indices defined over three or more periods being dependent on the choice of the base period. He suggested taking the *arithmetic average* of the T fixed-base Fisher indices that used each observation as the base period, if there are T periods in the comparison.[22] The resulting index is independent of the choice of a base period, or put differently, it treats all possible choices of a base period in a symmetric manner.[23]

Gini (1924) (1931) soon picked up on Fisher's idea and applied it to calculating relative price levels for several Italian cities, but instead of taking an arithmetic average of the city-specific fixed-base Fisher indices, he suggested taking the *geometric average* of the individual fixed-base Fisher indices. Eltetö and Köves (1964) and Szulc (1964) showed how Gini's multilateral index could be derived as a solution to a least squares minimization problem, and so the index is now referred to as the GEKS index. It should be noted that Balk (1980a) (1980b) (1980c) (1981) was a pioneer in applying multilateral indices to seasonal data. However, he did not use the GEKS index in his empirical examples. Ivancic, Diewert, and Fox (2011) suggested the use of the GEKS index in the time series context.

Balk (1996) (2008), Zhang, Johansen and Nygaard (2019), and Diewert and Fox (2021).

22 "There remains the practical question: if we are not going to use all six, what single curve is the best one to use in their place, for the general purpose of all comparisons over a series of years? Doubtless the very best as to accuracy, were it practicable, is the blend or average of all six. . . . This is a compromise single series of six figures that can be substituted for the whole table of figures, for the purpose of blending all separate exact comparisons into one general *nearly* exact comparison" (Irving Fisher (1922; 304–305)). Fisher's T was equal to six.

23 However, there are two disadvantages to this multilateral approach to index number theory: (*i*) As new data become available, the multilateral indices have to be recomputed and the prior indices that applied to periods 1 to T are in general changed and (ii) not all bilateral comparisons between any two periods in the window of T observations are equally "good." These difficulties with the above multilateral methods can be overcome by using similarity linking which will be described later.

We now set up the notation that is required to describe how to calculate the year-over-year monthly GEKS indices. Recall that the Laspeyres and Paasche indices for month m in year y relative to year 1 were defined by definitions (2) and (3). In order to formally define the sequence of GEKS indices for each month, we need to define the *Laspeyres* and *Paasche indices* for month m in year y using month m in *year* z (instead of month m in *year 1*) as the base. These more general indices, $P_L^m(y/z)$ and $P_P^m(y/z)$, are defined as follows:

$$P_L^m(y/z) \equiv \Sigma_{n\in S(m)}\, p_{y,m,n}q_{z,m,n}/\Sigma_{n\in S(m)}\, p_{z,m,n}q_{z,m,n};\quad m = 1,\ldots,M;\; y = 1,\ldots,Y;\; z = 1,\ldots,Z; \tag{15}$$

$$P_P^m(y/z) \equiv \Sigma_{n\in S(m)}\, p_{y,m,n}q_{y,m,n}/\Sigma_{n\in S(m)}\, p_{z,m,n}q_{y,m,n};\quad m = 1,\ldots,M;\; y = 1,\ldots,Y;\; z = 1,\ldots,Z. \tag{16}$$

Thus, for each month m, $P_L^m(y/z)$ compares the prices of available products in month m of year y in the numerator using the corresponding available products in month m of year z as weights to the prices of available products in month m of year z in the denominator using the corresponding available products in month m of year z as weights. For each month m, $P_P^m(y/z)$ compares the prices of available products in month m of year y in the numerator using the corresponding available products in month m of year y as weights to the prices of available products in month m of year z in the denominator again using the corresponding available products in month m of year y as weights. The corresponding *Fisher index* for month m in year y using month m in year z as the base, $P_F^m(y/z)$, is defined as the geometric mean of Laspeyres and Paasche indices for month m in year y using month m in year z as the base period:

$$P_F^m(y/z) \equiv [P_L^m(y/z)P_P^m(y/z)]^{1/2};\quad m = 1,\ldots,M;\; y = 1,\ldots,Y;\; z = 1,\ldots,Z. \tag{17}$$

The Fisher fixed-base index for month m defined by (4) chose month m in year 1 as the base period and formed the following sequence of year-over-year price levels relative to year 1: $P_F^m(1/1) = 1$, $P_F^m(2/1)$, $P_F^m(3/1)$, . . ., $P_F^m(Y/1)$. But one could also use month m in year 2 as the base period and the following sequence of price levels to measure year-over-year inflation for each month m: $P_F^m(1/2)$, $P_F^m(2/2) = 1$, $P_F^m(3/2)$, . . ., $P_F^m(Y/2)$. Month m in each of Y years could be chosen as the base period, and thus we end up with Y alternative series of Fisher price levels for each month. Since each of these sequences of price levels is equally plausible, the *GEKS price levels*, $p_{GEKS}^{y,m}$, for each month m for years $y = 1,2,\ldots,Y$ are defined as the geometric mean of the separate indices we obtain by using each year as the base year:

$$p_{GEKS}^{y,m} \equiv [\prod_{z=1}^{Y} P_F^m(y/z)]^{1/Y};\quad m = 1,\ldots,M;\; y = 1,\ldots,Y. \tag{18}$$

Note that all time periods are treated in a symmetric manner in these definitions. The *GEKS price indices* $P_{GEKS}^{y,m}$ are obtained by normalizing these price levels so that the period 1 index is equal to 1 for each month. Thus, we have the following definitions for the month m year-over-year GEKS index for year y, $P_{GEKS}{}^{y,m}$:

$$P_{GEKS}{}^{y,m} \equiv p_{GEKS}{}^{y,m}/p_{GEKS}{}^{1,m};\ m = 1,\ldots,M;\quad y = 1,\ldots,Y. \tag{19}$$

If prices and quantities are the same in any two periods, then the resulting GEKS indices will be identical for those two periods, which is a desirable property.

There is a problem associated with the use of the GEKS index in a time series context: When an additional month of data becomes available, the GEKS indices need to be recomputed, and the existing historical pattern of price levels will change in general. This poses problems for nonrevisable indices like a CPI. A solution to this problem was proposed by Ivancic, Diewert, and Fox (2009) (2011). Their method added the price and quantity data for the most recent time period to a *window* of consecutive time periods, and they also dropped the price and quantity data for the oldest period from the previous window of observations in order to obtain a new window. The GEKS indices for the new window of observations were calculated in the usual way, and the ratio of the index value for the last month in the new window to the index value for the previous month in the new window was used as an *update factor* for the value of the index for the last period in the existing index. The resulting indices are called *Rolling Window GEKS indices*. Unfortunately, the resulting indices no longer satisfy the multiperiod identity test, and so they are not entirely free of chain drift. However, empirical studies have shown that the method does not generate a substantial amount of chain drift. There is also a problem associated with exactly how we should link the latest data in the rolling window to the previously calculated indices. Krsinich (2016; 383) called the aforementioned method for linking the new window to the previous window *the movement splice* method. Krsinich (2016; 383) also suggested that a better choice to link the results of the new window to the previous window is to link the new observation to the index value in the second time period in the previous window of observations. She called this the *window splice* method. Let T be the length of the window. De Haan (2015; 26) suggested that the link period t should be chosen to be in the middle of the first window time span; that is, choose $t = T/2$ if T is an even integer or $t = (T + 1)/2$ if T is an odd integer. The Australian Bureau of Statistics (2016; 12) called this the *half splice* method for linking the results of the two windows. Diewert and Fox (2021) suggested linking the last observation in the current window to all possible choices of periods that overlap in the two windows and taking the geometric mean of the resulting estimates for the price level in the final period of the current window. They termed this the *mean splice*, and they recommended it as perhaps being best since the result of choosing each of the possible linking periods is equally valid.[24]

For our empirical example, we simply implemented the GEKS method using the entire six years of data for each month; that is, we did not calculate rolling window GEKS indices. Thus, these estimated year-over-year GEKS indices listed in Table A.21 for each month are not practical real-time indices, but they are of interest so that the effects of changing the base year can be studied. We will discuss how $P_{GEKS}{}^{y,m}$ defined by (19) performed using our Israeli data set on strongly seasonal fresh fruits after we have defined some alternative multilateral indices.

The GEKS multilateral method treats each set of price indices using the prices of one period as the base period as being equally valid, and hence an averaging of the resulting parities seems to be appropriate under this hypothesis. Thus, the method is "democratic" in that each bilateral index number comparison between any two periods gets the same weight in the overall method. However, *it is not the case that all bilateral comparisons of price between two periods are equally accurate*: If the relative prices in periods r and t are very similar, then the Laspeyres and Paasche price indices will be very close to each other, and hence it is likely that the "true" price comparison between these two periods will be very close to the bilateral Fisher index between these two periods. In particular, if the two price vectors are exactly proportional, then we want the price index between these two periods to be equal to the factor of proportionality, and the direct Fisher index between these two periods satisfies this proportionality test. On the other hand, the GEKS index comparison between the two periods would not in general satisfy this proportionality test.[25] Furthermore if prices are identical between two periods but the quantity vectors are different, then the GEKS price index between the two periods would not equal unity in general.[26]

Linking observations that have the most *similar structure of relative prices* addresses these difficulties with the GEKS method. Hill (1997) (1999a) (1999b) (2009) and Diewert (2009) developed this multilateral similarity linking method in the context of making cross-country comparisons. In the time series context, this linking of observations with the most similar price structures was pioneered by Hill (2001) (2004).

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two observations. various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981), Kravis, Heston, and Summers (1982; 104–106), Hill (1997) (2009), Aten and Heston (2009), and Diewert (2009). However, Hill and

[24] This method for linking the two windows was also suggested by Ivancic, Diewert, and Fox (2011; 33) in a footnote. Later in this chapter when we study similarity linking, we will see that all links are not necessarily equally good.

[25] If both prices and quantities are proportional to each other for the two periods being compared, then the GEKS price index between the two periods will satisfy this (weak) proportionality test. However, we would like the GEKS price index between the two periods to satisfy the *strong proportionality test*; that is, if the two price vectors are proportional (and the two quantity vectors are not necessarily proportional to each other), then we would like the GEKS price index between the two periods to equal the factor of proportionality.

[26] See Zhang, Johansen, and Nygaard (2019; 689) for details on this point.

Timmer (2006) pointed out a problem with these measures of relative price dissimilarity: They do not take into account the *lack of matching problem*; that is, these measures fail to recognize that bilateral comparisons of prices made over a smaller number of products are not as reliable as comparisons made over a larger number of matched products.[27] This lack of matching problem is a big one in the context of constructing index numbers for a product category where many or most products are only available in some months of the year. In our empirical example, only about 60 percent of the seasonal products are available in a typical month.

For our empirical example, we will use the *predicted share measure of relative price dissimilarity*. In situations where carry-forward prices are not used, this method penalizes a lack of price matching between two observations.[28] In order to define this measure, it is useful to introduce some notation for the *vectors* of prices and quantities for month m in year y, $p^{y,m}$ and $q^{y,m}$. If product n in month m of year y is present, then define the price and quantity of that product to be $p_{y,m,n}$ and $q_{y,m,n}$, as usual. If product n in month m of year y is not present, then define the quantity of that product to be 0 so that $q_{y,m,n} \equiv 0$ and define $p_{y,m,n}$ to be the year-over-year carry-forward (or carry-backward) price. With these additional variables defined, the N dimensional price and quantity vectors for month m in year y are well defined as $p^{y,m} \equiv [p_{y,m,1},p_{y,m,2},\ldots,p_{y,m,N}]$ and $q^{y,m}$ ° $[q_{y,m,1},q_{y,m,2},\ldots,q_{y,m,N}]$ for $y = 1,\ldots,Y$ and $m = 1,\ldots,M$. With this new notation, prices and quantities are well defined for all N products for each year and month. Thus, *the expenditure share for product n in month m and year y*, $s_{y,m,n}$, can now be defined for all N products as

$$s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m};\; y = 1,\ldots,Y;\; m = 1,2,\ldots,M;\; n = 1,2,\ldots,N, \tag{20}$$

where $p^{y,m}\cdot q^{y,m} \equiv \Sigma_{n=1}^{N}\, p_{y,m,n}\, q_{y,m,n}$ is the inner product of the vectors $p^{y,m}$ and $q^{y,m}$. Note that even though these expenditure shares use imputed prices for missing products, they are equal to the actual expenditure shares for all products.

Now think of using the *prices* of month m in year z and the *quantities* of month m in year y to *predict* the actual month m, year y, product n expenditure share $s_{y,m,n}$ defined by (20) for $n = 1,\ldots,N$. Denote this *predicted share* by $s_{z,y,m,n}$, which is defined as follows:

$$s_{z,y,m,n} \equiv p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m};\; y = 1,\ldots,Y;\; z = 1,\ldots,Z;\; m = 1,2,\ldots,M;\; n = 1,2,\ldots,N. \tag{21}$$

If the prices in month m of year y are proportional to the prices of month m in year z so that $p^{z,m} = \lambda p^{y,m}$, where λ is a positive number, then it can verified that the predicted shares defined by (21) will be equal to the actual expenditure shares defined by (20) for month m in year y; that is, for the two months defined by y,m, and z,m, we will have $s_{y,m,n} = s_{z,y,m,n}$ for $n = 1,\ldots,N$. The following *predicted share measure of relative price dissimilarity* between the prices of month m in year y and the prices of month m in year z, $\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m})$, is well defined even if some product prices and shares in the two months being compared are equal to zero:

$$\begin{aligned}\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}) &\equiv \Sigma_{n=1}^{N}[s_{y,m,n} - s_{z,y,m,n}]^2 + \Sigma_{n=1}^{N}[s_{z.m,n} - s_{y,z,m,n}]^2 \\ &= \Sigma_{n=1}^{N}[(p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}) - (p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m})]^2 \\ &+ \Sigma_{n=1}^{N}[(p_{z,m,n}q_{z,m,n}/p^{z,m}\cdot q^{z,m}) - (p_{y,m,n}q_{z,m,n}/p^{y,m}\cdot q^{z,m})]^2. \end{aligned} \tag{22}$$

In general, $\Delta_{PS}(p^r,p^t,q^r,q^t)$ takes on values between 0 and 2. If $\Delta_{PS}(p^r,p^t,q^r,q^t) = 0$, then it must be the case that relative prices are the same in month m of years z and y; that is, we have $p^{z,m} = \lambda p^{y,m}$ for some $\lambda > 0$. A bigger value of $\Delta_{PS}(p^r,p^t,q^r,q^t)$ generally indicates bigger deviations from price proportionality.

To see how this predicted share measure of relative price dissimilarity turned out for our Israeli data on 14 classes of fresh fruits for the month of January, see Table 9.1. The month m is equal to 1 (January). The years y and z range from 1 to 6. Fruits 1, 2, 4, 5, 6, 12, and 13 were always available in January for each of the six years in our sample; the other seven fruits were always missing in January. Thus, there are no carry-forward imputed prices that are used for the January data. For a listing of the nonzero price $p_{y,1,n}$ and quantity $q_{y,1,n}$ data for January 2012–2017 (years 1–6), see Table A.1 in the annex.

The matrix of predicted share measures of relative price dissimilarity for the month of January for all pairs of years in our sample is nonnegative, symmetric, and has zeros down its main diagonal. The measure of relative price dissimilarity between years 1 and 2 is 0.00306; between years 1 and 3 is .00632; and so on.

This matrix is used to construct $P_S^{y,1}$, the *similarity-linked price index for January*. The *real-time version* of this index is constructed as follows. Set $P_S^{1,1} \equiv 1$. The year-over-year index for January in year 2 is set equal to the bilateral Fisher index $P_F^{m}(y/z)$, where $m = 1$, $y = 2$, and $z = 1$ (see definition (17)). Using our new vector notation, this Fisher index is equal to $[p^{2,1}\times q^{1,1}\; p^{2,1}\cdot q^{2,1}/p^{1,1}\cdot q^{1,1}\; p^{1,1}\cdot q^{2,1}]^{1/2}$. Thus, the year 2 similarity-linked index for January is $P_S^{2,1} \equiv P_F^{1}(2/1)$. Now look down the $y = 3$ column in Table 9.1. We need to link year 3 to either year 1 or year 2. The dissimilarity measures for these two years are 0.00632 and 0.00082, respectively. The degree of relative price dissimilarity is far smaller for the link to year 2 than it is to year 1 (year 3 January prices are much closer to being proportional to year 2 prices than

[27] "Although these measures perform well when there are few gaps in the data, they can generate highly misleading results when there are many gaps. This is because they fail to penalize bilateral comparisons made over a small number of matched headings" (Robert Hill and Marcel Timmer (2006; 366)). Hill and Timmer go on and propose a measure of relative price dissimilarity that penalizes a lack of price matching. Their measure is based on econometric considerations. The measure that we use also penalizes a lack of price matching but it has a different motivation.

[28] In this section where year-over-year carry-forward prices are used, all prices are matched, so there is no penalty for a lack of matching. However, in the next section, we will not use any form of imputed price so the predicted share measure of price dissimilarity will penalize a lack of matching.

Table 9.1 Predicted Share Measures of Price Dissimilarity for January for Years 1–6

$m = 1$	$y = 1$	$y = 2$	$y = 3$	$y = 4$	$y = 5$	$y = 6$
$z = 1$	0.00000	0.00306	0.00632	0.00062	0.00810	0.00363
$z = 2$	0.00306	0.00000	0.00082	0.00429	0.00325	0.00119
$z = 3$	0.00632	0.00082	0.00000	0.00696	0.00375	0.00233
$z = 4$	0.00062	0.00429	0.00696	0.00000	0.01019	0.00421
$z = 5$	0.00810	0.00325	0.00375	0.01019	0.00000	0.00171
$z = 6$	0.00363	0.00119	0.00233	0.00421	0.00171	0.00000

Table 9.2 Year-over-Year Alternative Indices for January

Year y	$P_{LFB}^{y,1}$	$P_{PFB}^{y,1}$	$P_{FFB}^{y,1}$	$P_{TFB}^{y,1}$	$P_{LCH}^{y,1}$	$P_{PCH}^{y,1}$	$\mathbf{P_{FCH}^{y,1}}$	$P_{TCH}^{y,1}$	$P_{GEKS}^{y,1}$	$P_S^{y,1}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99746	0.99881	0.99813	0.99817	0.99746	0.99881	0.99813	0.99817	0.99814	0.99813
3	1.03276	1.01894	1.02583	1.02591	1.02762	1.01799	1.02280	1.02261	1.02295	1.02280
4	1.01159	1.00992	1.01076	1.01072	1.01586	0.99872	1.00725	1.00700	1.00816	1.01076
5	1.12212	1.12896	1.12554	1.12582	1.14808	1.10989	1.12883	1.12854	1.12973	1.13415
6	1.07410	1.06543	1.06976	1.06889	1.09958	1.04827	1.07362	1.07252	1.07153	1.06944
Mean	1.03970	1.03700	1.03830	1.03830	1.04810	1.02890	1.03840	1.03810	1.03840	1.03920

to year 1 prices), so we use the Fisher link from period 2 to period 3, $P_F^1(3/2)$. Thus, the final year 3 similarity-linked index for January is $P_S^{3,1} \equiv P_S^{2,1} \times P_F^1(3/2)$. Now we need to link year 4 to either year 1, 2, or 3. Look down the $y = 4$ column in Table 9.1 to find the lowest dissimilarity measure above the main diagonal of the matrix. The smallest of the three numbers 0.00062, 0.00429, and 0.00696 is 0.00062. Thus, we link the year 4 January data to the year 1 January data using the Fisher January link from year 1 to year 4, $P_F^1(4/1)$, and the year 4 similarity-linked final index value is $P_S^{4,1} \equiv P_S^{1,1} \times P_F^1(4/1) = P_F^1(4/1)$. *Thus, for each year, as the new January data become available, we use the Fisher bilateral index that links the new period to the previous period that has the lowest measure of relative price dissimilarity.* The final two bilateral links are year 5 to year 2 and year 6 to year 2. The resulting year 5 and 6 similarity-linked index values are $P_S^{5,1} \equiv P_S^{2,1} \times P_F^1(5/2)$ and $P_S^{6,1} \equiv P_S^{2,1} \times P_F^1(6/2)$. The optimal set of *bilateral links* for the January year-over-year real-time similarity-linked indices can be summarized as follows:

```
    5
  |
1 – 2 – 3
|   |
4   6
```

Using our empirical data set, we calculated the 10 year-over-year alternative indices for January that are defined earlier. These indices are the fixed-base Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices, $P_{LFB}^{y,1}$, $P_{PFB}^{y,1}$, $P_{FFB}^{y,1}$, and $P_{TFB}^{y,1}$, the corresponding chained indices, $P_{LCH}^{y,1}$, $P_{PCH}^{y,1}$, $P_{FCH}^{y,1}$, and $P_{TCH}^{y,1}$, the GEKS index, $P_{GEKS}^{y,1}$, and the predicted share similarity-linked index, $P_S^{y,1}$. The year superscript y takes on the values 1–6. These indices are listed in Table 9.2.

Looking at Table 9.2, it can be seen that the fixed-base Laspeyres indices exceed the fixed-base Paasche indices by about 0.27 percentage points on average. The gap between the chained Laspeyres indices and the chained Paasche indices is much larger at 1.92 percentage points. These gaps indicate that the Laspeyres and Paasche indices suffer from some upward or downward substitution bias. The larger gap for the chained indices also indicates that the chained Laspeyres and Paasche indices are subject to a considerable amount of chain drift. The remaining six indices are all close to each other on average.

Our year-over-year data on January fresh fruit consumption for Israel for the six years in our sample did not have any missing products that changed from year to year; fruits 1, 2, 4, 5, 6, 12, and 13 were always available in January for each of the six years in our sample; the other seven fruits were always missing in January. Thus, no imputed prices were used for the January data. However, imputed carry-forward (or carry-backward) prices were used for other months.

For example, for our particular data set, the month of May has eight missing prices, which were imputed by six carry-forward prices and two carry-backward prices. Products 1, 2, 3, 5, 6, 7, and 10 were always present in May. Products 4, 11, 12, and 14 were always missing in May. The remaining products 8, 9, and 13 were sometimes present and sometimes absent in May. Thus, carry-forward or carry-backward prices were used to impute the missing prices for products 8, 9, and 13. The data for May are listed in Tables A.7 and A.8 in the annex. The eight imputed prices are listed in these tables using italics. To see how the predicted share measure of relative price dissimilarity defined by (22) turned out for our Israeli data for the month of May, see Table 9.3. The month m is equal to 5 (May). As usual, the years y and z range from 1 to 6.

The *real-time* set of bilateral links that minimize the predicted share measure of relative price dissimilarity for the May data for the current year with the May data for a prior year are as follows: link 2 to 1; 3 to 1; 4 to 3; 5 to 3; and 6 to 4. The optimal set of links can be summarized as follows:

1 – 2
|
3 – 4 – 6
|
5.

Using the price and quantity data for May that is listed in Tables A.7 and A.8 in the annex, we calculated the May year-over-year indices using the fixed-base Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices, $P_{LFB}{}^{y,5}$, $P_{PFB}{}^{y,5}$, $P_{FFB}{}^{y,5}$, and $P_{TFB}{}^{y,5}$, the corresponding chained indices, $P_{LCH}{}^{y,5}$, $P_{PCH}{}^{y,5}$, $P_{FCH}{}^{y,5}$, and $P_{TCH}{}^{y,5}$, the GEKS index, $P_{GEKS}{}^{y,5}$, and the predicted share similarity-linked index, $P_S{}^{y,5}$, for the years 1–6. These indices are listed in Table 9.4.

The results for the year-over-year May indices are similar to the results for the year-over-year January indices in some respects:

- The chained Laspeyres indices $P_{LCH}{}^{y,5}$ ended up at 1.36519, which is well above the final value for the chained Paasche indices $P_{PCH}{}^{y,5}$, which was 1.18589.
- The fixed-base Fisher and Törnqvist–Theil indices, the GEKS indices, and the similarity-linked indices, $P_{FFB}{}^{y,5}$, $P_{TFB}{}^{y,5}$, $P_{GEKS}{}^{y,5}$, and $P_S{}^{y,5}$, all ended up at much the same levels and, in general, were quite close to each other.

The big difference between the May results and the January results is that the chained Fisher and Törnqvist–Theil indices for May, $P_{FCH}{}^{y,5}$, $P_{TCH}{}^{y,5}$, ended up well below the other May superlative indices, $P_{FFB}{}^{y,5}$, $P_{TFB}{}^{y,5}$, $P_{GEKS}{}^{y,5}$, and $P_S{}^{y,5}$. This is due to the influence of the six carry-forward prices that are used in the May year-over-year data. There were no imputed prices for the January data, and hence there was no carry-forward bias for this month. Thus, if there is general inflation in the segment of the economy under consideration and carry-forward prices are used to replace missing prices, then the use of chained superlative indices will tend to lead to indices that are biased downward relative to their fixed-base counterparts.

The year-over-year indices for all 12 months are reported in Table A.21 in the annex. Table 9.5 reports the overall mean and variance of all 10 indices, where the index values are stacked into a single column with 72 rows for each of the 10 indices.

On average, the cumulated year-over-year fixed-base Laspeyres indices $P_{LFB}{}^{y,m}$ exceeded their cumulated fixed-base Paasche counterparts by 1.1365 – 1.1001 = 0.0364 or 3.64 percentage points. The average gap between the chained Laspeyres and Paasche indices was 1.1560 – 1.0817 = 0.0743 or 7.43 percentage points. These are substantial differences and indicate that the use of these indices should be avoided. The fixed-base Fisher, fixed-base Törnqvist–Theil, chained Fisher, and predicted share similarity-linked indices, $P_{FFB}{}^{y,m}$, $P_{TFB}{}^{y,m}$, $P_{FCH}{}^{y,m}$, and $P_S{}^{y,m}$, all had similar means and variances and performed equally well on

Table 9.3 Predicted Share Measures of Price Dissimilarity for May for Years 1–6

$m = 5$	$y = 1$	$y = 2$	$y = 3$	$y = 4$	$y = 5$	$y = 6$
$z = 1$	0.00000	0.00617	0.00250	0.02222	0.01103	0.01324
$z = 2$	0.00617	0.00000	0.00578	0.02768	0.00883	0.01908
$z = 3$	0.00250	0.00578	0.00000	0.01226	0.00409	0.00690
$z = 4$	0.02222	0.02768	0.01226	0.00000	0.01060	0.00175
$z = 5$	0.01103	0.00883	0.00409	0.01060	0.00000	0.00810
$z = 6$	0.01324	0.01908	0.00690	0.00175	0.00810	0.00000

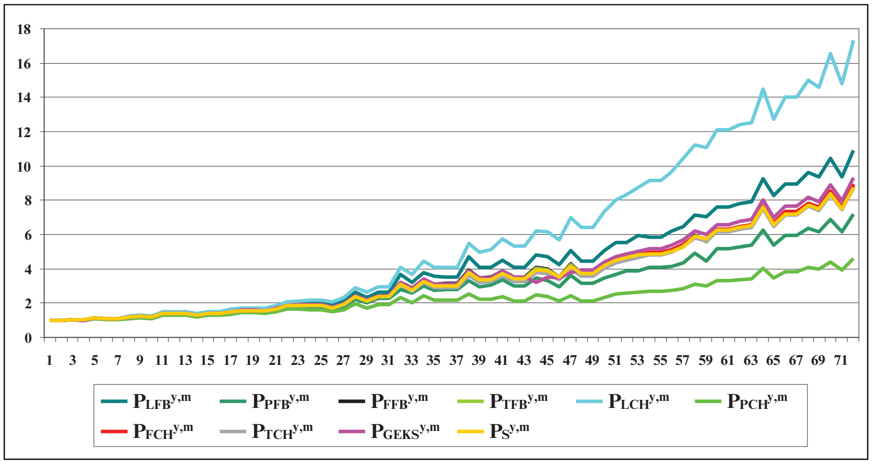
Table 9.4 Year-over-Year Alternative Indices for May

Year y	$P_{LFB}{}^{y,5}$	$P_{PFB}{}^{y,5}$	$P_{FFB}{}^{y,5}$	$P_{TFB}{}^{y,5}$	$P_{LCH}{}^{y,5}$	$P_{PCH}{}^{y,5}$	$P_{FCH}{}^{y,5}$	$P_{TCH}{}^{y,5}$	$P_{GEKS}{}^{y,5}$	$P_S{}^{y,5}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.95731	0.91814	0.93752	0.93708	0.95731	0.91814	0.93752	0.93708	0.93879	0.93752
3	1.04955	1.02931	1.03938	1.03929	1.07750	0.99674	1.03634	1.03544	1.04223	1.03938
4	1.29576	1.26861	1.28211	1.27958	1.34446	1.21671	1.27899	1.27733	1.28376	1.28275
5	1.15686	1.15394	1.15540	1.15718	1.22628	1.06571	1.14318	1.14348	1.15227	1.14281
6	1.29885	1.29900	1.29893	1.29611	1.36519	1.18589	1.27239	1.27244	1.29548	1.29399
Mean	1.12640	1.11150	1.11890	1.11820	1.16180	1.06390	1.11140	1.11100	1.11880	1.11610

Table 9.5 Year-over-Year Index Means and variances over All Months and Years for 10 indices Using Carry-Forward Prices

	$P_{LFB}{}^{y,m}$	$P_{PFB}{}^{y,m}$	$P_{FFB}{}^{y,m}$	$P_{TFB}{}^{y,m}$	$P_{LCH}{}^{y,m}$	$P_{PCH}{}^{y,m}$	$P_{FCH}{}^{y,m}$	$P_{TCH}{}^{y,m}$	$P_{GEKS}{}^{y,m}$	$P_S{}^{y,m}$
Mean	1.1365	1.1001	1.1180	1.1170	1.1560	1.0817	1.1176	1.1154	1.1111	1.1178
variance	0.0161	0.0101	0.0125	0.0123	0.0203	0.0079	0.0121	0.0117	0.0130	0.0122

Figure 9.1 Cumulated Year-over-Year Indices Using Carry Forward Prices



our particular data set with means between 1.117 and 1.118. The mean of the chained Törnqvist–Theil indices was a bit lower at 1.1154, and their variance was also lower. This may reflect the fact that chaining indices that use carry-forward prices in a period of high general inflation will tend to lower the average inflation rate and may also lower the variance. The mean of the GEKS indices was 1.111, which is below 1.117. On the other hand, the variance of the GEKS indices was 0.0130, which is above the range of the variances for the "best" indices, $P_{FFB}^{y,m}$, $P_{TFB}^{y,m}$, $P_{FCH}^{y,m}$, and $P_{S}^{y,m}$, which was between 0.0121 and 0.0125.

In order to illustrate the differences between the 10 different index number formulae, we cumulated the year-over-year indices listed in Table A.21 in the annex and plotted the resulting cumulated indices in Figure 9.1. Thus, the first six points for the series P_{LFB} are the January year-over-year fixed-base Laspeyres indices for years 1–6: $P_{LFB}^{1,1}$, $P_{LFB}^{2,1}$, . . ., $P_{LFB}^{6,1}$. The next six points for the P_{LFB} series are the February year-over-year fixed-base Laspeyres indices for years 1–6 times the final value for the January fixed-base Laspeyres series, $P_{LFB}^{6,1}$. Thus, the values for the listed P_{LFB} series in Figure 9.1 for observations 7–12 are the cumulated indices $P_{LFB}^{6,1}\times P_{LFB}^{1,2}$, $P_{LFB}^{6,1}\times P_{LFB}^{2,2}$, . . ., $P_{LFB}^{6,1}\times P_{LFB}^{6,2}$. The next six points for the P_{LFB} series are the March year-over-year fixed-base Laspeyres indices for years 1–6 times the cumulated value for observation 12 of the cumulated fixed-base Laspeyres series, which is $P_{LFB}^{6,1}\times P_{LFB}^{6,2}$. Thus, the values for the listed P_{LFB} series in Figure 9.1 for observations 13–18 are the cumulated indices $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{1,3}$, $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{2,3}$, . . ., $P_{LFB}^{6,1}\times P_{LFB}^{6,2}\times P_{LFB}^{6,3}$, and so on. The final six observations for the P_{LFB} series are defined as $P_{LFB}^{6,1\prime}P_{LFB}^{6,2\prime}P_{LFB}^{6,3\prime}\ldots\times P_{LFB}^{6,11}$ times the December year-over-year fixed-base Laspeyres indices for years 1–6, $P_{LFB}^{1,12}$, $P_{LFB}^{2,12}$, . . ., $P_{LFB}^{6,12}$. The remaining nine cumulated series were constructed in a similar manner.

The highest series is the cumulated chained Laspeyres index P_{LCH} followed by the cumulated fixed-base Laspeyres index, P_{LFB}. The lowest series is the cumulated chained Paasche index P_{PCH} followed by the cumulated fixed-base Paasche index, P_{PFB}. The remaining six indices are all clustered together in the middle of these outlier series, with the cumulated GEKS indices P_{GEKS} lying slightly above the remaining five clustered indices. The cumulated chained Törnqvist–Theil indices P_{TCH} are just a bit below the other four clustered indices.

The aforementioned series used carry-forward or carry-backward prices for seasonal products, which were at times not available in their "regular" seasonally available months. However, when there is general inflation (or deflation) in an economy, there is a risk of introducing a significant amount of bias when carry-forward prices are used to fill in for the missing prices. Hence, in the following section, we will calculate year-over-year indices without using carry-forward prices.

Once the Laspeyres and Paasche indices are eliminated from consideration, it can be seen that the remaining six year-over-year monthly indices are all fairly close to each other.

In the following section, we will construct the same 10 indices, but we will not use any imputed prices. Instead, we will use bilateral indices that are based on the common set of products that are actually present in both periods for each bilateral comparison. The resulting indices can then be compared with the indices that are plotted in Figure 9.1. The new indices that do not use carry-forward prices are listed in Table A.22 in the annex.

We conclude this section with a brief discussion on the use of carry-forward prices by statistical agencies. In many cases, a simple carry-forward price for a missing price is not used; instead, the price of a close substitute is used or an *inflation-adjusted carry-forward price* is used. In the latter case, the last available price is multiplied by an index of prices for related products that are available in the two periods that are being compared.[29] Depending on the price index concept that is being used, the use of inflation-adjusted

[29] Armknecht and Maitland-Smith (1999) had a good discussion on the various methods used by statistical agencies to construct some sort of inflation-adjusted carry-forward price. This discussion is very relevant in recent times when COVID problems substantially increased the frequency of missing prices.

carry-forward prices will be at least approximately equivalent to simply using the index that is restricted to the products that are available in the two periods under consideration. Thus, in the following section, we will look at the use of *maximum overlap bilateral indices*; that is, products that are not present in both periods being compared are simply dropped.[30] The problem with using the price of a close substitute to fill in a missing price is that the choice of the substitute product is necessarily somewhat arbitrary. To eliminate this arbitrariness, we will focus on the construction of maximum overlap indices (or matched model indices) in the following section.

3. Maximum Overlap Year-over-Year Monthly Indices

Recall the notation that was introduced in Section 1 where the set of products n which are present in the marketplace during month m of year y was denoted by $S(y,m)$. Data on prices and quantities are available for Y years and say $M = 12$ months. Again the price of product n in month m of year y is denoted by $p_{y,m,n}$, and the corresponding quantity is denoted by $q_{y,m,n}$. In this section, we do not use carry-forward prices, so if product n is missing in month m of year y, we set $p_{y,m,n} = 0$ and $q_{y,m,n} = 0$. Using these new prices and quantities, the *expenditure share for product n in month m and year y*, $s_{y,m,n}$, can now be defined for all N products as[31]

$$s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m};\ y = 1,\ldots,Y;\quad m = 1,2,\ldots,M;\ n = 1,2,\ldots,N. \tag{23}$$

In the previous section, the Laspeyres, Paasche, and Fisher indices that compared the prices of month m in year y to the prices of month m in year z were defined by (15)–(17). These definitions used carry-forward and carry-backward prices for prices of seasonal products which happened to be absent in some years. In this section, we want to avoid the use of any imputed prices, so these indices are redefined by definitions (24)–(26) for $m = 1,\ldots,M$; $y = 1,\ldots,Y$; $z = 1,\ldots,Y$:

$$P_L^{m*}(y/z) \equiv \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{y,m,n}q_{z,m,n} / \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{z,m,n}q_{z,m,n}; \tag{24}$$

$$P_P^{m*}(y/z) \equiv \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{y,m,n}q_{y,m,n} / \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{z,m,n}q_{y,m,n}; \tag{25}$$

$$P_F^{m*}(y/z) \equiv [P_L^{m*}(y/z)P_P^{m*}(y/z)]^{1/2}. \tag{26}$$

The indices defined by (24)–(26) are called *bilateral maximum overlap Laspeyres, Paasche, and Fisher indices*, respectively. The Laspeyres index that compares the prices of month m in year y to the prices of month m in year z, $P_L^{m*}(y/z)$, compares the prices of month m products that are available *in both year y and year z*. The jointly available product prices of year y appear in the numerator and are compared to the jointly available products of year z, which appear in the denominator. The quantities of jointly available products for year z appear as weights in both the numerator and the denominator. Similarly, the Paasche index that compares the prices of month m in year y to the prices of month m in year z, $P_P^{m*}(y/z)$, compares the prices of month m products that are available *in both year y and year z*. The jointly available product prices of year y appear in the numerator and are compared to the jointly available products of year z, which appear in the denominator. The quantities of jointly available products for year y appear as weights in both the numerator and the denominator. As usual, the corresponding Fisher index $P_F^{m*}(y/z)$ is the geometric mean of $P_L^{m*}(y/z)$ and $P_P^{m*}(y/z)$.

The sequence of maximum overlap year-over-year fixed-base Laspeyres indices for month m will be denoted by $P_{LFB}^{y,m*}$ for $y = 1,2,\ldots,Y$. For our empirical example, $Y = 6$ and the *year-over-year maximum overlap fixed-base Laspeyres indices* $P_{LFB}^{y,m*}$ for months $m = 1,\ldots,12$ are defined to be the indices $P_L^{m*}(1/1)$, $P_L^{m*}(2/1)$, $P_L^{m*}(3/1)$, $P_L^{m*}(4/1)$, $P_L^{m*}(5/1)$, and $P_L^{m*}(6/1)$, where the maximum overlap Laspeyres link indices $P_L^{m*}(y/z)$ are defined by (24). Similarly, the *year-over-year maximum overlap fixed-base Paasche indices* $P_{PFB}^{y,m*}$ for months $m = 1,\ldots,12$ are defined to be the indices $P_P^{m*}(1/1)$, $P_P^{m*}(2/1)$, $P_P^{m*}(3/1)$, $P_P^{m*}(4/1)$, $P_P^{m*}(5/1)$, and $P_P^{m*}(6/1)$, where the maximum overlap Paasche link indices $P_P^{m*}(y/z)$ are defined by (25). Finally, the *year-over-year maximum overlap fixed-base Fisher indices* $P_{FFB}^{y,m*}$ for months $m = 1,\ldots,12$ are defined to be the indices $P_F^{m*}(1/1)$, $P_F^{m*}(2/1)$, $P_F^{m*}(3/1)$, $P_F^{m*}(4/1)$, $P_F^{m*}(5/1)$, and $P_F^{m*}(6/1)$, where the maximum overlap Fisher bilateral link indices $P_F^{m*}(y/z)$ are defined by (26). These fixed-base maximum overlap Laspeyres, Paasche, and Fisher indices for our May data are listed in Table 9.6.[32]

Define the *year-over-year maximum overlap chained Laspeyres, Paasche*, and *Fisher indices* for month m in year 1, $P_{LCH}^{1,m*}$, $P_{PCH}^{1,m*}$, and $P_{FCH}^{1,m*}$, as unity:

$$P_{LCH}^{1,m*} \equiv 1;\ P_{PCH}^{1,m*} \equiv 1;\ P_{FCH}^{1,m*} \equiv 1;\quad m = 1,\ldots,M. \tag{27}$$

For years following year 1, these maximum overlap indices for the same month m are defined by cumulating the corresponding successive annual year-over-year links defined by (24)–(26); that is, we have the following definitions:

$$P_{LCH}^{y,m*} \equiv P_{LCH}^{y-1,m*}\, P_L^{m*}(y/(y-1));\quad m = 1,\ldots,M;\ y = 2,\ldots,Y; \tag{28}$$

$$P_{PCH}^{y,m*} \equiv P_{PCH}^{y-1,m*}\, P_P^{m*}(y/(y-1));\quad m = 1,\ldots,M;\ y = 2,\ldots,Y; \tag{29}$$

$$P_{FCH}^{y,m*} \equiv P_{FCH}^{y-1,m*}\, P_F^{m*}(y/(y-1));\quad m = 1,\ldots,M;\ y = 2,\ldots,Y. \tag{30}$$

[30] This type of index dates back to Marshall (1887). Keynes (1909) (1930; 94) called it the *highest common factor method*, while Triplett (2004; 18) called it the *overlapping link method*.

[31] These "new" expenditure shares turn out to be identical to the expenditure shares defined by (20) in the previous section.

[32] If the set of available seasonal products is the same every year for a particular month, then the maximum overlap indices for that month will coincide with the corresponding indices defined in the previous section, since there are no imputed prices for the year-over-year indices when the available products are the same every year for the given month.

The *maximum overlap GEKS price levels*, $p_{GEKS}^{y,m*}$, for each month m for years $y = 1,2,\ldots,Y$ is defined as the geometric mean of the separate indices we obtain by using each year as the base year:

$$p_{GEKS}^{y,m*} \equiv [\prod_{z=1}^{Y} P_F^{m*}(y/z)]^{1/Y};\ m = 1,\ldots,M;\quad y = 1,\ldots,Y, \tag{31}$$

where $P_F^{m*}(y/z)$ is defined by (26). The *maximum overlap GEKS price indices* $P_{GEKS}^{y,m*}$ are obtained by *normalizing* these price levels so that the period 1 index is equal to 1. Thus, we have the following definitions for the month m *year-over-year maximum overlap GEKS index* for year y, $P_{GEKS}^{y,m*}$:

$$P_{GEKS}^{y,m*} \equiv p_{GEKS}^{y,m*}/p_{GEKS}^{1,m*};\ m = 1,\ldots,M;\quad y = 1,\ldots,Y. \tag{32}$$

The maximum overlap GEKS indices along with the chained maximum overlap Laspeyres, Paasche, and Fisher indices for our May data are listed in Table 9.6.

Constructing the bilateral maximum overlap Törnqvist–Theil index between every pair of years using the data for month m is more complicated. It is necessary to construct *conditional expenditure shares*, which are expenditure shares for product n for month m in year y that are conditional on product n being purchased in both years y and z. First, we note that $q_{y,m,n}$ is well defined for all y,m, and n as actual expenditures on product n for month m in year y. If there is no expenditure on product n for month m in year y, $q_{y,m,n}$ is defined to be equal to 0. In the 0 expenditure case, define the corresponding price, $p_{y,m,n}$, to be 0 as well. In the case where $q_{y,m,n} > 0$, the corresponding price $p_{y,m,n}$ is defined to be the usual positive unit value price. With these conventions, $p_{y,m,n}$ and $q_{y,m,n}$ are defined for all y, m, and n. Now define the *expenditure for product n in month m of year y, conditional on positive month m, year z quantities*, $e_{y,z,m,n}$, as follows:[33]

$$e_{y,z,m,n} \equiv p_{y,m,n}q_{y,m,n}\ \textit{if}\ q_{z,m,n} > 0;\ y = 1,\ldots,Y;\ z = 1,\ldots,Z;\quad m = 1,\ldots,M;\ n = 1,\ldots,N; \equiv 0\ \textit{if}\ q_{z,m,n} = 0. \tag{33}$$

Thus, $e_{y,z,m,n}$ will be positive if and only if there are sales of product n in month m for years y and z. Define the *total expenditure on products sold in month m of year y conditional on positive year z expenditure on products sold in month m of year z*, $e_{y,z,m}$, as the sum over n of the $e_{y,z,m,n}$ defined by (33):

$$e_{y,z,m} \equiv \Sigma_{n=1}^{N} e_{y,z,m,n};\ y = 1,\ldots,Y;\ z = 1,\ldots,Z;\quad m = 1,\ldots,M. \tag{34}$$

Thus, $e_{y,z,m}$ is equal to total sales of products sold in month m of year y provided the products are also sold in month m of year z. Using definitions (33) and (34), the *expenditure share for product n in month m of year y, conditional on products being present in years y and z*, $s_{y,z,m,n}$, is defined as follows:

$$s_{y,z,m,n} \equiv e_{y,z,m,n}/e_{y,z,m};\ y = 1,\ldots,Y;\ z = 1,\ldots,Z;\quad m = 1,\ldots,M;\ n = 1,\ldots,N. \tag{35}$$

Note that if $y = z$, then the conditional shares $s_{y,y,m,n}$ defined by (35) collapse down to the actual expenditure shares on product n in month m of year y, $s_{y,m,n}$, defined by (23); that is, we have:

$$s_{y,y,m,n} = s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/\Sigma_{k=1}^{N} p_{y,m,k}q_{y,m,k};\quad y = 1,\ldots,Y;\ m = 1,\ldots,M;\ n = 1,\ldots,N. \tag{36}$$

The bilateral *maximum overlap Törnqvist–Theil index that compares the prices of month m in year y to the prices of month m in year z*, $P_T^{m*}(y/z)$, is defined as follows:

$$P_T^{m*}(y/z) \equiv \exp[\Sigma_{n\in S(y,m)\cap S(z,m)} (½)(s_{y,z,m,n} + s_{z,y,m,n})ln(p_{y,m,n}/p_{z,m,n})];\quad y = 1,\ldots,Y;\ z = 1,\ldots,Z;\ m = 1,\ldots,M. \tag{37}$$

Thus, only the product prices that are positive in month m of year y and in month m of year z appear in the summations on the right-hand side of definitions (37). $P_T^{m*}(y/z)$ compares the prices of month m products that are available *in both year y and year z*. The bilateral indices $P_T^{m*}(y/z)$ defined by (37) can be used to construct the maximum overlap fixed-base and chained Törnqvist–Theil indices.

The sequence of maximum overlap year-over-year fixed-base Törnqvist–Theil indices for month m will be denoted by $P_{TFB}^{y,m*}$ for $y = 1,2,\ldots,Y$. For our empirical example, $Y = 6$ and the *year-over-year maximum overlap fixed-base Törnqvist–Theil indices* $P_{TFB}^{y,m*}$ for months $m = 1,\ldots,12$ are defined to be the indices $P_T^{m*}(1/1)$, $P_T^{m*}(2/1)$, $P_T^{m*}(3/1)$, $P_T^{m*}(4/1)$, $P_T^{m*}(5/1)$, and $P_T^{m*}(6/1)$, where the maximum overlap link indices $P_T^{m*}(y/z)$ are defined by (37).

Define the *year-over-year maximum overlap chained Törnqvist–Theil index* for month m in year 1, $P_{TCH}^{1,m*}$, as unity:

$$P_{TCH}^{1,m*} \equiv 1;\ m = 1,\ldots,M. \tag{38}$$

For years following year 1, the *maximum overlap chained Törnqvist–Theil indices* for the month m in the years $y = 2,\ldots,Y$, $P_{TCH}^{y,m*}$, are defined by cumulating the corresponding successive annual year-over-year links for month m defined by (37); that is, we have the following definitions:

$$P_{TCH}^{y,m*} \equiv P_{TCH}^{y-1,m*}\ P_T^{m*}(y/(y-1));\ m = 1,\ldots,M;\quad y = 2,\ldots,Y. \tag{39}$$

The fixed-base and chained maximum overlap Törnqvist–Theil indices for our May data are listed in Table 9.6.

In order to define the year-over-year predicted share similarity-linked indices for a particular month, we need to define the relative price dissimilarity matrix for each month. It turns out that we can still use definitions (21) and (22) to define the new dissimilarity matrix using our "new" data that does not use carry-forward prices. For convenience, we

[33] When defining $e_{y,z,m,n}$ in a statistical programming package, it is useful to define the dummy variables, $\delta_{z,y,n} = \{1$ if $q_{z,y,n} > 0$; $\delta_{z,y,n} = 0$ if $q_{z,y,n} = 0\}$ and then define $e_{y,z,m,n}$ as $p_{y,m,n}q_{y,m,n}\delta_{z,y,n}$.

Table 9.6 May Predicted Share Measures of Price Dissimilarity Excluding Imputed Prices

m = 5	y = 1	y = 2	y = 3	y = 4	y = 5	y = 6
z = 1	0.00000	0.02471	0.02505	0.04297	0.03604	0.03227
z = 2	0.02471	0.00000	0.00988	0.02858	0.01565	0.01926
z = 3	0.02505	0.00988	0.00000	0.01226	0.00409	0.01042
z = 4	0.04297	0.02858	0.01226	0.00000	0.01060	0.00204
z = 5	0.03604	0.01565	0.00409	0.01060	0.00000	0.01445
z = 6	0.03227	0.01926	0.01042	0.00204	0.01445	0.00000

repeat these definitions. Thus, define the *predicted share for product n in month m of year y*, $s_{z,y,m,n}$, that uses the month m quantities of year y and the prices of month m in year z as follows:

$$s_{z,y,m,n} \equiv p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m};\ y = 1,\ldots,Y;\ z = 1,\ldots,Z;\ m = 1,2,\ldots,M;\ n = 1,2,\ldots,N. \quad (40)$$

Define the *predicted share measure of relative price dissimilarity* between the prices of month m in year y and the prices of month m in year z, $\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m})$, as follows:

$$\begin{aligned}\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}) &\equiv \Sigma_{n=1}^{N}[s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{n=1}^{N}[s_{z.m,n} - s_{y,z,m,n}]^2 \\ &= \Sigma_{n=1}^{N}[(p_{y,m,n}q_{y,m,n}/p^{y,m}\cdot q^{y,m}) - (p_{z,m,n}q_{y,m,n}/p^{z,m}\cdot q^{y,m})]^2 \\ &+ \Sigma_{n=1}^{N}[(p_{z,m,n}q_{z,m,n}/p^{z,m}\cdot q^{z,m}) - (p_{y,m,n}q_{z,m,n}/p^{y,m}\cdot q^{z,m})]^2. \quad (41)\end{aligned}$$

If the products that were purchased in month m of years y and z were identical, then the “new” measure of relative price dissimilarity defined by (41) will be identical to the “old” measure defined by (22). However, in the case where prices in years y and z are not matched, The measure of price dissimilarity defined by (41) is larger than the corresponding measure defined by (22); that is, there is now a *penalty for a lack of price matching* (which can be large if the difference between $s_{y.m,n}$ and $s_{z.m,n}$ is large for an unmatched product n).

To see how the predicted share measure of relative price dissimilarity defined by (41) turned out for our Israeli data for the month of May when we do not use imputed prices, see Table 9.6. The month m is equal to 5 (May). As usual, the years y and z range from 1 to 6.

The predicted share measures of relative price dissimilarity listed in Table 9.6 have a mean equal to 0.01601, whereas the measures listed in Table 9.3 in the previous section had a mean equal to 0.0089. Thus, *excluding* the use of imputed prices for the predicted share measures of dissimilarity for our May year-over-year data substantially *increased* the resulting measures of price dissimilarity. The predicted share measures of price dissimilarity grow in magnitude when imputed prices are replaced by zero prices because the measures impose a substantial penalty for a lack of price matching.[34]

[34]See the discussion of the predicted share multilateral method in Diewert (2021b).

The new real-time predicted share relative price similarity-linked price indices for May (that exclude the use of imputed prices), $P_S^{y,5*}$, are constructed as follows. Set $P_S^{1,5*} \equiv 1$. The year-over-year index for May in year 2 is set equal to the maximum overlap bilateral Fisher index $P_F^{m*}(y/z)$, where $m = 5$, $y = 2$, and $z = 1$ (see definition (26)). Thus, the year 2 similarity-linked index for May is $P_S^{2,5*} \equiv P_F^{5*}(2/1)$. Now look down the $y = 3$ column in Table 9.6. We need to link year 3 to either year 1 or year 2. The dissimilarity measures for these two years relative to year 3 are 0.02505 and 0.00988, respectively. The degree of relative price dissimilarity is far smaller for the link to year 2 than it is to year 1, so we use the maximum overlap Fisher link (for the month 5 data) from period 2 to period 3, $P_F^{5*}(3/2)$, to construct the year 3 similarity-linked index for May as $P_S^{3,5*} \equiv P_S^{2,5*}\times P_F^{5*}(3/2)$. Now we need to link year 4 to year 1, 2, or 3. Look down the $y = 4$ column in Table 9.6 to find the lowest dissimilarity measure above the main diagonal of the matrix. The smallest of the three numbers 0.04297, 0.02858, and 0.01226 is 0.01226. Thus, we link the year 4 May data to the year 3 May data using the maximum overlap Fisher May link from year 3 to year 4, $P_F^{5*}(4/3)$, and the year 4 similarity-linked index value is $P_S^{4,5*} \equiv P_S^{3,5*}\times P_F^{5*}(4/3)$. Thus, each year, as the new May data become available, we use the maximum overlap Fisher bilateral index that links the new period to the previous period that has the lowest measure of relative price dissimilarity. The final two bilateral links are year 5 to year 3 and year 6 to year 4. The resulting year 5 and 6 similarity-linked index values are $P_S^{5,5*} \equiv P_S^{3,5*}\times P_F^{5*}(5/3)$ and $P_S^{6,5*} \equiv P_S^{4,5*}\times P_F^{5}(6/4)$. The set of optimal real-time bilateral links for the May data can be summarized as follows:

```
1 – 2 – 3 – 4
    |   |
    5   6.
```

The new set of May bilateral links is different from the set of bilateral links for May that used carry-forward and carry-backward prices. To see the differences between the carry-forward indices for May listed in Table 9.4 in the previous section with the corresponding maximum overlap indices for May that are described earlier, see Table 9.7. The indices listed in Table 9.7 do not use any imputed prices in their construction.

As was the case for the carry-forward indices listed in Table 9.4, the maximum overlap fixed-base and chained Laspeyres indices for May, $P_{LFB}^{y,5*}$ and $P_{LCH}^{y,5*}$, listed in Table 9.7 end up well above the superlative indices and the maximum overlap fixed-base and the chained Paasche indices for May, $P_{PFB}^{y,5*}$ and $P_{PCH}^{y,5*}$, end up well below the superlative indices. The remaining six superlative indices (the fixed-base and chained Fisher indices, $P_{FFB}^{y,5*}$ and $P_{FCH}^{y,5*}$, the fixed-base and chained Törnqvist–Theil indices, $P_{TFB}^{y,5*}$ and $P_{TCH}^{y,5*}$, the GEKS indices $P_{GEKS}^{y,5*}$, and the predicted share similarity-linked indices $P_S^{y,5*}$) ended up in year 6 at 1.3204, 1.2729, 1.31917, 1.27122, 1.3123, and 1.2898, respectively. It appears that the chained Fisher and Törnqvist–Theil indices suffer from some downward chain drift since the other four superlative indices are free of chain drift and ended up (on average) about 3.77 percentage points above where the average of the two chained superlative indices ended. Thus, for our May data, it appears that *the use of carry-forward prices for missing product prices led to a substantial downward bias.*

Table 9.7 Year-over-Year Maximum Overlap Indices for May

Year y	$P_{LFB}^{y,5*}$	$P_{PFB}^{y,5*}$	$P_{FFB}^{y,5*}$	$P_{TFB}^{y,5*}$	$P_{LCH}^{y,5*}$	$P_{PCH}^{y,5*}$	$P_{FCH}^{y,5*}$	$P_{TCH}^{y,5*}$	$P_{GEKS}^{y,5*}$	$P_{S}^{y,5*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.95007	0.91814	0.93397	0.93252	0.95007	0.91814	0.93397	0.93252	0.94462	0.93397
3	1.05674	1.03102	1.04380	1.04354	1.06935	0.99802	1.03307	1.03104	1.05052	1.03307
4	1.33870	1.26554	1.30161	1.29967	1.33429	1.21827	1.27496	1.27191	1.29677	1.27496
5	1.17963	1.17093	1.17527	1.17658	1.21701	1.06707	1.13958	1.13863	1.16610	1.13587
6	1.34224	1.29900	1.32044	1.31917	1.36461	1.18740	1.27293	1.27122	1.31228	1.28980
Mean	1.14460	1.11410	1.12920	1.12860	1.15590	1.06480	1.10910	1.10760	1.12840	1.11130

Table 9.8 Year-over-Year Index Means and variances over All Months and Years for 10 Indices Using Maximum Overlap Bilateral Indices

	$P_{LFB}^{y,m*}$	$P_{PFB}^{y,m*}$	$P_{FFB}^{y,m*}$	$P_{TFB}^{y,m*}$	$P_{LCH}^{y,m*}$	$P_{PCH}^{y,m*}$	$P_{FCH}^{y,m*}$	$P_{TCH}^{y,m*}$	$P_{GEKS}^{y,m*}$	$P_{S}^{y,m*}$
Mean	1.1381	1.1003	1.1189	1.1177	1.1591	1.0765	1.1163	1.1136	1.1187	1.1184
variance	0.0167	0.0101	0.0128	0.0126	0.0209	0.0076	0.0119	0.0115	0.0125	0.0123

Thus, the use of carry-forward prices to replace missing prices is not recommended.

The year-over-year indices for all 12 months are reported in Table A.22 in the annex. The following table reports the overall mean and variance for all eight indices, where the index values are stacked into a single column with 72 rows for each of the eight indices. The averages reported in Table 9.8 use maximum overlap indices, whereas the corresponding averages reported in Table 9.5 used carry-forward prices, which will tend to give lower indices given that there was general fruit inflation in Israel for the six years in our sample. The averages reported in Table 9.8 are in fact higher than the corresponding averages in Table 9.5 with the exceptions of the chained Paasche and chained Fisher indices.

As usual, the fixed-base and chained Laspeyres maximum overlap indices, $P_{LFB}^{y,m*}$ and $P_{LCH}^{y,m*}$, and the fixed-base and chained Paasche maximum overlap indices, $P_{PFB}^{y,m*}$ and $P_{PCH}^{y,m*}$, have some considerable amounts of upward and downward substitution bias relative to the remaining superlative indices. The chained Fisher and chained Törnqvist–Theil indices, $P_{FFB}^{y,m*}$ and $P_{TCH}^{y,m*}$, appear to have some amount of downward chain drift bias relative to the remaining four superlative indices, which are free of chain drift bias by construction. The fixed-base Fisher, fixed-base Törnqvist–Theil, GEKS, and similarity-linked indices, $P_{FFB}^{y,m*}$, $P_{TFB}^{y,m*}$, $P_{GEKS}^{y,m*}$, and $P_{S}^{y,m*}$, all have about the same mean and variance and appear to be equally satisfactory for our particular empirical example. The means for these four maximum overlap indices over all months are 1.1189, 1.1177, 1.1187, and 1.1184 and the average of these four averages is 1.1184. The corresponding means for the carry-forward indices $P_{FFB}^{y,m}$, $P_{TFB}^{y,m}$, $P_{GEKS}^{y,m}$, and $P_{S}^{y,m}$ from Table 9.5 are 1.1180, 1.1170, 1.1111, and 1.1178 and the average of these four averages is 1.1160. Thus, the use of carry-forward prices leads to an average downward bias of about 0.24 percentage points compared to the corresponding maximum overlap indices for our best index number formulae for our particular empirical example. This is a significant downward bias.[35]

In order to illustrate the differences between the 10 different index number formulae, we cumulated the 10 year-over-year indices listed in Table A.22 in the annex and plotted the resulting cumulated indices on Figure 9.2. The construction of the cumulated series for each index formula follows the same as the process we used to construct Figure 9.1.

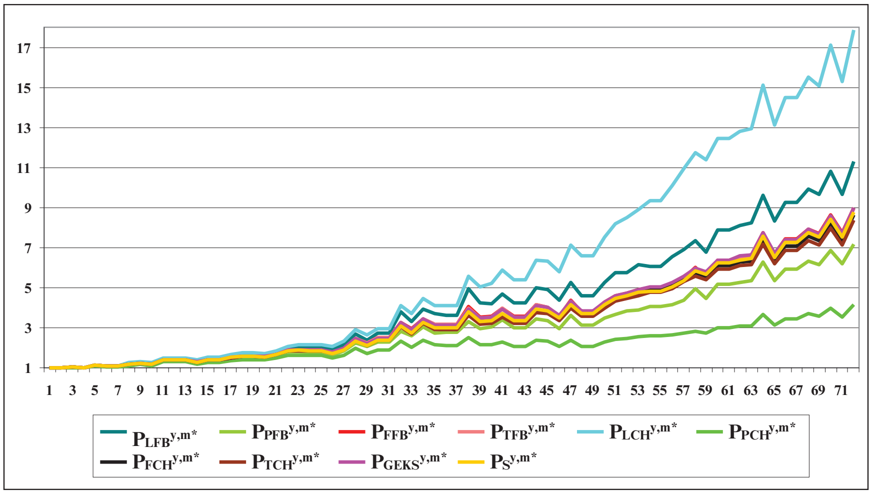
The indices plotted in Figure 9.2 are very close to their counterparts plotted in Figure 9.1. For the most part, the indices plotted in Figure 9.2 are a bit above their counterparts plotted in Figure 9.1 due to the fact that the Figure 9.1 indices used carry-forward prices, which tend to lower measured inflation in a period of general inflation. The highest series shown in Figure 9.2 is the cumulated chained Laspeyres index P_{LCH}^{*} followed by the cumulated fixed-base Laspeyres index, P_{LFB}^{*}. The lowest series is the cumulated chained Paasche index P_{PCH}^{*} followed by the cumulated fixed-base Paasche index, P_{PFB}^{*}. The remaining six indices are all clustered together in the middle of these outlier series.

Our conclusions regarding the use of year-over-year monthly indices at this point are as follows:

- The use of the Laspeyres and Paasche indices should be avoided. The fixed-base and chained Laspeyres indices tend to lie well above the clustered superlative indices, while the fixed-base and chained Paasche indices tend to lie well below the clustered superlative indices.
- The chained Fisher and Törnqvist–Theil indices may suffer from a small amount of chain drift.
- The fixed-base Fisher, Törnqvist–Theil, GEKS, and predicted share similarity-linked indices are all fairly close to each other in the present context where we are

[35] One-sixth of the indices listed in Tables A.21 and A.22 are equal to 1, so the actual bias is even larger.

Figure 9.2 Cumulated Year-over-Year Monthly Indices Using Maximum Overlap Indices



measuring year-over-year inflation for each month in the year.

- The use of carry-forward prices will tend to lead to indices that are biased downward if there is general inflation; so in order to avoid this potential bias, it is best to use the indices that use maximum overlap superlative bilateral indices as their basic building blocks. Thus, the maximum overlap fixed-base Fisher and fixed-base Törnqvist–Theil, GEKS, and predicted share similarity-linked indices, $P_{FFB}^{y,m*}$, $P_{TFB}^{y,m*}$, $P_{GEKS}^{y,m*}$, and $P_S^{y,m*}$, emerge as our "best" choices for year-over-year monthly indices.

In the following two sections, we turn our attention to annual price indices.

4. The Construction of Annual Indices Using Carry-Forward Prices

Assuming that each product in each season of the year is a separate "annual" product is the simplest and theoretically most satisfactory method for dealing with seasonal products when the goal is to construct annual price and quantity indices. This idea can be traced back to Mudgett in the consumer price context and to Stone in the producer price context:

> The basic index is a yearly index and as a price or quantity index is of the same sort as those about which books and pamphlets have been written in quantity over the years.
>
> Bruce *D*. Mudgett (1955; 97)

> The existence of a regular seasonal pattern in prices which more or less repeats itself year after year suggests very strongly that the varieties of a product available at different seasons cannot be transformed into one another without cost and that, accordingly, in all cases where seasonal variations in price are significant, the varieties available at different times of the year should be treated, in principle, as separate products.
>
> Richard Stone (1956; 74–75)

Using carry-forward prices for missing products and using the notation explained in Section 2, the N-dimensional price and quantity vectors for month m in year y are defined as $p^{y,m} \equiv [p_{y,m,1}, p_{y,m,2},. . .,p_{y,m,N}]$ and $q^{y,m} \circ [q_{y,m,1},q_{y,m,2},. . .,q_{y,m,N}]$ for $y = 1,. . .,Y$ and $m = 1,. . .,M$.[36] The *year y annual price and quantity vectors* are defined as the NM-dimensional vectors $p^y \equiv [p^{y,1}, p^{y,2},. . ., p^{y,M}]$ and $q^y \equiv [q^{y,1}, q^{y,2},. . ., q^{y,M}]$ respectively for $y = 1,. . .,Y$. Using this new notation, the year *y annual fixed-base Laspeyres price index* using carry-forward prices is defined as follows:

$$
\begin{aligned}
P_{LFB}^{\ y} &\equiv p^y \cdot q^1/p^1 \cdot q^1;\ y = 1,. . .,Y; \\
&= \Sigma_{m=1}^{M}\, p^{y,m} \cdot q^{1,m}/\Sigma_{m=1}^{M}\, p^{1,m} \cdot q^{1,m} \\
&= \Sigma_{m=1}^{M}\, [p^{y,m} \times q^{1,m}/p^{1,m} \cdot q^{1,m}][p^{1,m} \cdot q^{1,m}/\Sigma_{m=1}^{M}\, p^{1,m} \cdot q^{1,m}] \\
&= \Sigma_{m=1}^{M}\, S_{1,m}\, P_{LFB}^{\ y,m},
\end{aligned}
\qquad (42)
$$

where $S_{1,m} \equiv p^{1,m} \cdot q^{1,m}/\Sigma_{m=1}^{M}\, p^{1,m} \cdot q^{1,m}$ is the month m share of total year 1 expenditure on the seasonal products in scope, and $P_{LFB}^{\ y,m} \equiv p^{y,m} \cdot q^{1,m}/p^{1,m} \cdot q^{1,m}$ is the Laspeyres fixed-base price index for month m in year y, which was defined by (2) in Section 2.[37] Thus, *the annual fixed-base Laspeyres price index for year y*, $P_{LFB}^{\ y}$, is a year 1 monthly expenditure share-weighted *arithmetic* average of the M year-over-year fixed-base

[36] The quantity $q_{y,m,n}$ is the quantity of product n purchased in month m of year y; if no amount of this product was purchased in month m of year y, $q_{y,m,n} = 0$. If product n was never purchased in any month, $p_{y,m,n} = 0$. If some amount of product n was purchased in month m of any year $y = 1,. . .,Y$, then $p_{y,m,n}$ is the actual unit value price if product n was purchased in year y; otherwise $p_{y,m,n}$ is a carry-forward or carry-backward price. The share of product n in the monthly expenditure on all products in month m of year y is defined as $s_{y,m,n} \equiv p_{y,m,n}q_{y,m,n}/\Sigma_{k\in S(m)}\, p_{y,m,k}q_{y,m,k} = p_{y,m,n}q_{y,m,n}/p^{y,m} \cdot q^{y,m}$ for $y = 1,. . .,Y$; $m = 1,2,. . .,M$; $n = 1,. . .,N$.

[37] The new definition for $P_{LFB}^{\ y,m}$ is equivalent to definition (2).

Laspeyres monthly indices for year y. These annual fixed-base Laspeyres indices are listed in Table 9.10 for our Israeli data set.

The year *y annual fixed-base Paasche index* using carry-forward prices is defined as follows:

$$P_{PFB}^{\ y} \equiv p^y \cdot q^y / p^1 \cdot q^y;\ y = 1,\ldots,Y;$$
$$= 1/[p^1 \cdot q^y / p^y \cdot q^y]$$
$$= [\Sigma_{m=1}^{M} p^{1,m} \cdot q^{y,m} / / \Sigma_{m=1}^{M} p^{y,m} \cdot q^{y,m}]^{-1}$$
$$= [(\Sigma_{m=1}^{M} p^{1,m} \cdot q^{y,m} / p^{y,m} \cdot q^{y,m})/(p^{y,m} \cdot q^{y,m} / \Sigma_{m=1}^{M} p^{y,m} \cdot q^{y,m})]^{-1}$$
$$= [\Sigma_{m=1}^{M} S_{y,m} (P_{PFB}^{\ y,m})^{-1}]^{-1}, \tag{43}$$

where $P_{PFB}^{\ y,m} \equiv p^{y,m} \cdot q^{y,m} / p^{1,m} \cdot q^{y,m}$ is the Paasche fixed-base price index for month m in year y, which was defined by (3) in Section 2,[38] and the *month m shares of annual expenditures on the seasonal products in scope for year y*, $S_{y,m}$, is defined as follows:

$$S_{y,m} \equiv p^{y,m} \cdot q^{y,m} / \Sigma_{k=1}^{M} p^{y,k} \cdot q^{y,k};\ m = 1,\ldots,M;\ y = 1,\ldots,Y. \tag{44}$$

Thus, *the annual fixed-base Paasche price index for year y*, $P_{PFB}^{\ y}$, is a year y monthly expenditure share-weighted *harmonic* average of the M fixed-base year-over-year Paasche monthly indices for year y. These annual fixed-base Paasche indices are listed in Table 9.10 for our Israeli data set.

The year *y annual fixed-base Fisher index* is defined as the geometric mean of the annual Laspeyres and Paasche indices defined by (42) and (43):

$$P_{FFB}^{\ y} = [P_{LFB}^{\ y} P_{PFB}^{\ y}]^{1/2};\ y = 1,\ldots,Y. \tag{45}$$

In Section 2, recall that the fixed-base Törnqvist–Theil indices for month m in year y were defined as $P_{TFB}^{\ y,m} \equiv \exp[\Sigma_{n\in S(m)} (\frac{1}{2})(s_{1,m,n} + s_{y,m,n})\ln(p_{y,m,n}/p_{1,m,n})]$ for $m = 1,\ldots,12$; $y = 1,\ldots,Y$. The *fixed-base annual Törnqvist–Theil index* for year y using carry-forward prices is defined as follows:

$$P_{TFB}^{\ y} \equiv exp[\Sigma_{m=1}^{M} \Sigma_{n\in S(m)} (\tfrac{1}{2})(S_{1,m}s_{1,m,n} + S_{y,m}s_{y,m,n})\ ln(p_{y,m,n}/p_{1,m,n})];\ y = 1,\ldots,Y, \tag{46}$$

where the within-month expenditure shares $s_{y,m,n}$ are defined by (1) and the month m expenditure shares in year y, $S_{y,m}$, are defined by (44).

In order to define the annual chained Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices as well as the annual GEKS indices, it is necessary to define bilateral annual Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices for all pairs of years y and z. Thus, define these *bilateral annual indices* that compare the prices of year y relative to the base year z as follows:

$$P_L(y/z) \equiv p^y \cdot q^z / p^z \cdot q^z;\ z = 1,\ldots,Z;\ y = 1,\ldots,Y; \tag{47}$$

$$P_P(y/z) \equiv p^y \cdot q^y / p^z \cdot q^y;\ z = 1,\ldots,Z;\ y = 1,\ldots,Y; \tag{48}$$

$$P_F(y/z) \equiv [P_L(y/z)P_P(y/z)]^{1/2};\ z = 1,\ldots,Z;\ y = 1,\ldots,Y; \tag{49}$$

$$P_T(y/z) \equiv exp[\Sigma_{m=1}^{M} \Sigma_{n\in S(m)} (\tfrac{1}{2})(S_{z,m}s_{z,m,n} + S_{y,m}s_{y,m,n})\ ln(p_{y,m,n}/p_{z,m,n})];\ z = 1,\ldots,Z;\ y = 1,\ldots,Z. \tag{50}$$

The annual chained Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices are defined as follows for year 1:

$$P_{LCH}^{\ 1} \equiv 1;\ P_{PCH}^{\ 1} \equiv 1;\ P_{FCH}^{\ 1} \equiv 1;\ P_{TCH}^{\ 1} \equiv 1. \tag{51}$$

For years y following year 1, the aforementioned annual chained indices are defined recursively using the annual bilateral indices defined by (47)–(50) as follows:

$$P_{LCH}^{\ y} \equiv P_{LCH}^{\ y-1} P_L(y/(y-1));\ y = 2,\ldots,Y; \tag{52}$$
$$P_{PCH}^{\ y} \equiv P_{PCH}^{\ y-1} P_P(y/(y-1));\ y = 2,\ldots,Y; \tag{53}$$
$$P_{FCH}^{\ y} \equiv P_{FCH}^{\ y-1} P_F(y/(y-1));\ y = 2,\ldots,Y; \tag{54}$$
$$P_{TCH}^{\ y} \equiv P_{TCH}^{\ y-1} P_T(y/(y-1));\ y = 2,\ldots,Y. \tag{55}$$

The Fisher fixed-base index for year y, $P_{FFB}^{\ y}$, defined by (45) chose year 1 as the base period and formed the following sequence of year-over-year price levels relative to year 1: $P_F(1/1) = 1$, $P_F(2/1)$, $P_F(3/1)$, . . ., $P_F(Y/1)$. But one could also use year 2 as the base period and use the following sequence of price levels to measure annual inflation for each year y: $P_F(1/2)$, $P_F(2/2) = 1$, $P_F(3/2)$, . . ., $P_F(Y/2)$. Each year could be chosen as the base period, and thus we end up with Y alternative series of Fisher price levels for each year. Since each of these sequences of price levels is equally plausible, following Gini (1924) (1931), Eltetö and Köves (1964), and Szulc (1964), the *GEKS price level*s, $p_{GEKS}^{\ y}$, for years $y = 1,2,\ldots,Y$ are defined as the geometric mean of the separate indices we obtain by using each year as the base year:

$$p_{GEKS}^{\ y} \equiv [\Pi_{z=1}^{Y} P_F(y/z)]^{1/Y};\ y = 1,\ldots,Y. \tag{56}$$

Note that each choice of a base year z is treated in a symmetric manner in these definitions. The annual *GEKS price indices* $P_{GEKS}^{\ y}$ are obtained by normalizing these price levels so that the year 1 index is equal to 1. Thus, we have the following definitions for the annual GEKS index for year y (using carry-forward prices), $P_{GEKS}^{\ y}$:

$$P_{GEKS}^{\ y} \equiv p_{GEKS}^{\ y} / p_{GEKS}^{\ 1};\ y = 1,\ldots,Y. \tag{57}$$

The annual GEKS price indices using carry-forward prices are also listed in Table 9.10 using the data from our empirical example.

The basic building blocks used to form the GEKS multilateral index are the bilateral Fisher indices $P_F(y/z)$. It is not necessary to use the Fisher bilateral indices as the basic building blocks; instead, the bilateral Törnqvist–Theil

[38] The new definition for $P_{PFB}^{\ y,m}$ is equivalent to definition (3).

Table 9.9 Annual Predicted Share Measures of Price Dissimilarity Using Carry-Forward Prices

	$y = 1$	$y = 2$	$y = 3$	$y = 4$	$y = 5$	$y = 6$
$z = 1$	0.00000	0.00196	0.00198	0.00176	0.00173	0.00225
$z = 2$	0.00196	0.00000	0.00107	0.00207	0.00109	0.00264
$z = 3$	0.00198	0.00107	0.00000	0.00104	0.00068	0.00099
$z = 4$	0.00176	0.00207	0.00104	0.00000	0.00129	0.00055
$z = 5$	0.00173	0.00109	0.00068	0.00129	0.00000	0.00102
$z = 6$	0.00225	0.00264	0.00099	0.00055	0.00102	0.00000

Table 9.10 Alternative Annual Mudgett Stone Indices That Use Year-over-Year Carry-Forward Prices

Year	$P_{LFB}{}^y$	$P_{PFB}{}^y$	$P_{LCH}{}^y$	$P_{PCH}{}^y$	$P_{FFB}{}^y$	$P_{FCH}{}^y$	$P_{TFB}{}^y$	$P_{TCH}{}^y$	$P_{GEKS}{}^y$	$P_{CCDI}{}^y$	$P_S{}^y$
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1299	1.0611	1.1299	1.0611	1.0950	1.0950	1.0892	1.0892	1.0929	1.0900	1.0950
3	1.1224	1.0745	1.1470	1.0502	1.0982	1.0975	1.0963	1.0918	1.0966	1.0941	1.0975
4	1.1891	1.1411	1.2322	1.1083	1.1648	1.1686	1.1624	1.1626	1.1676	1.1647	1.1686
5	1.2241	1.1549	1.2729	1.1060	1.1890	1.1865	1.1871	1.1805	1.1884	1.1856	1.1903
6	1.2306	1.1752	1.3070	1.1102	1.2026	1.2046	1.2015	1.1988	1.2044	1.2020	1.2056
Mean	1.1494	1.1011	1.1815	1.0726	1.1249	1.1254	1.1227	1.1205	1.1250	1.1227	1.1262

indices $P_T(y/z)$ defined by (50) could be used.[39] Thus, following Caves, Christensen, and Diewert (1982) and Inklaar and Diewert (2016), the *CCDI price levels*, $p_{CCDI}{}^y$, for years $y = 1,2,\ldots,Y$ are defined as the geometric mean of the separate indices we obtain by using each year as the base year and $P_T(y/z)$ as the bilateral building blocks:

$$p_{CCDI}{}^y \equiv [\prod_{z=1}{}^Y P_T(y/z)]^{1/Y};\ y = 1,\ldots,Y. \qquad (58)$$

The annual CCDI index for year y, $P_{CCDI}{}^y$, is defined as the following normalization of the CCDI price levels:

$$P_{CCDI}{}^y \equiv p_{CCDI}{}^y / p_{CCID}{}^1;\ y = 1,\ldots,Y. \qquad (59)$$

The 10 annual indices $P_{LFB}{}^y$, $P_{PFB}{}^y$, $P_{FFB}{}^y$, $P_{TFB}{}^y$, $P_{LCH}{}^y$, $P_{PCH}{}^y$, $P_{FCH}{}^y$, $P_{TCH}{}^y$, $P_{GEKS}{}^y$, and $P_{CCDI}{}^y$ that use year-over-year carry-forward prices for our empirical example are listed in Table 9.10.

Our final annual Mudgett Stone annual index that uses year-over-year carry-forward prices for missing prices is the predicted share similarity-linked index $P_S{}^y$.

The year y, month m, product n actual expenditure share is $s_{y,m,n} \equiv p_{y,m,n}\, q_{y,m,n}/p^{y,m}\cdot q^{y,m}$. The prediction for this share using the price of product n of month m in year z, $p_{z,m,n}$, and the actual quantity of product n for month m in year y is the *predicted share* $s_{z,y,m,n} \equiv p_{z,m,n}\, q_{y,m,n}/p^{z,m}\cdot q^{y,m}$ for $n = 1,\ldots,N$, $m = 1,\ldots,M$, $z = 1,\ldots,Y$, and $y = 1,\ldots,Y$. The new annual measure of *Predicted Share Price Dissimilarity* between the prices of years z and y, $\Delta_{PSA}(p^z,p^y,q^z,q^y)$, is defined as follows:

$$\begin{aligned}\Delta_{PSA}(p^z,p^y,q^z,q^y) &\equiv \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N [s_{y.m,n} - s_{z,y,m,n}]^2 \\ &+ \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N [s_{z.m,n} - s_{y,z,m,n}]^2 \\ &= \Sigma_{m=1}{}^M \Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}),\end{aligned} \qquad (60)$$

where $\Delta_{PS}(p^{z,m},p^{y,m},q^{z,m},q^{y,m}) \equiv \Sigma_{n=1}{}^N [s_{y.m,n} - s_{z,y,m,n}]^2 + \Sigma_{n=1}{}^N [s_{z.m,n} - s_{y,z,m,n}]^2$ is the month m measure of *monthly* price dissimilarity between the product prices of month m in years z and y that was defined in Section 2 by (22). Thus, the new annual measure of price dissimilarity (using carry-forward prices) is equal to the sum over the M monthly product price dissimilarity measures for month m prices in years z and y using carry-forward prices.

Here is the table of the bilateral measures of annual predicted share price dissimilarity for our empirical example.

The *real-time* set of bilateral links that minimize the predicted share measures of relative price dissimilarity for the annual data are as follows: link 2 to 1; 3 to 2; 4 to 3; 5 to 3; and 6 to 4. The optimal set of bilateral links can be summarized as follows:

```
1 – 2 – 3 – 4
        |   |
        5   6
```

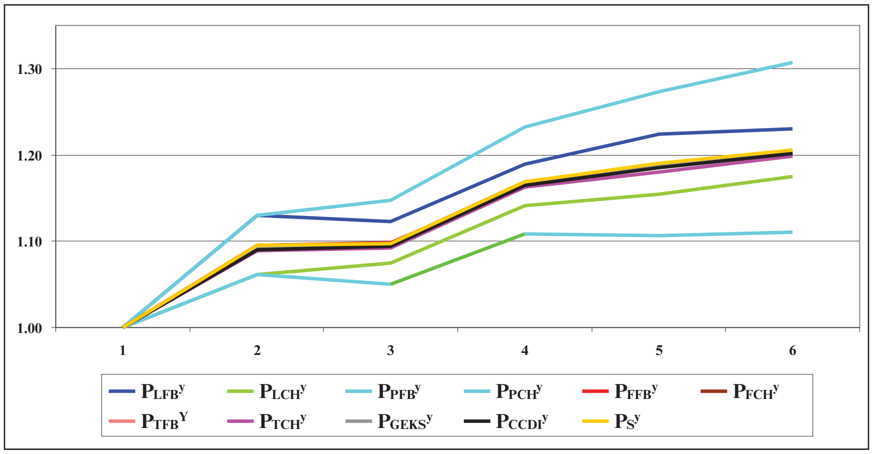
Thus, we define $P_S{}^1 \equiv 1$, $P_S{}^2 \equiv P_F(2/1)$, $P_S{}^3 \equiv P_S{}^2 \times P_F(3/2)$, $P_S{}^4 \equiv P_S{}^3 \times P_F(4/3)$, $P_S{}^5 \equiv P_S{}^3 \times P_F(5/3)$, and $P_S{}^6 \equiv P_S{}^4 \times P_F(6/4)$, where the bilateral annual Fisher indices $P_F(y/z)$ are defined by (49).

The 11 annual indices $P_{LFB}{}^y$, $P_{PFB}{}^y$, $P_{FFB}{}^y$, $P_{TFB}{}^y$, $P_{LCH}{}^y$, $P_{PCH}{}^y$, $P_{FCH}{}^y$, $P_{TCH}{}^y$, $P_{GEKS}{}^y$, $P_{CCDI}{}^y$, and $P_S{}^y$ that use year-over-year carry-forward prices for our empirical example are listed in Table 9.10 and plotted in Figure 9.3.

It can be seen that the annual fixed-base and chained Laspeyres indices, $P_{LFB}{}^y$ and $P_{LCH}{}^y$, lie well above the superlative indices and the annual fixed-base and chained Paasche indices, $P_{PFB}{}^y$ and $P_{PCH}{}^y$, lie well below the remaining indices.

[39] Caves, Christensen, and Diewert (1982) defined the quantity index counterpart to the price index defined by (58) using a different representation of the index. Inklaar and Diewert (2016) showed that the CCD definition was equivalent to the index defined by (58). Thus, the multilateral indices defined by (58) are called the CCDI indices. They are also called GEKS Törnqvist indices by statistical agencies.

Figure 9.3 Annual Indices Using Year-over-Year Carry-Forward Prices



The remaining indices are all tightly clustered together and cannot be easily distinguished in a figure. The 11 indices listed in Table 9.10 are plotted in Figure 9.3.

Thus, for our particular empirical example, all of the annual indices that are exact for a flexible functional form give much the same answer when we use year-over-year carry-forward prices. However, looking at the averages listed in Table 9.10, it can be seen that the three indices that use bilateral Törnqvist–Theil indices as building blocks, $P_{TFB}{}^{y}$, $P_{TCH}{}^{y}$, and $P_{CCDI}{}^{y}$, have slightly lower average index values than the indices that use bilateral Fisher indices as building blocks.

In Section 3, we saw that the use of year-over-year carry-forward prices for missing prices led to indices which were lower than the counterpart indices that did not use any imputed prices. We will see if the same tendency occurs when we compute annual Mudgett Stone indices using annual bilateral maximum overlap indices.

5. The Construction of Annual Indices Using Maximum Overlap Bilateral Indices

In order to define the annual Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices without using imputations for missing prices, it is necessary to define imputation-free bilateral annual Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices for all pairs of years y and z. Thus, define the following *maximum overlap bilateral annual indices* that compare the prices of year y relative to the base year z for products n that were available in years y and z as follows for $z = 1,\ldots,Z$; $y = 1,\ldots,Y$:

$$P_L{}^*(y/z) \equiv \Sigma_{m=1}{}^{M} \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{y,m,n}q_{z,m,n} / \Sigma_{m=1}{}^{M} \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{z,m,n}q_{z,m,n}; \quad (61)$$

$$P_P{}^*(y/z) \equiv \Sigma_{m=1}{}^{M} \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{y,m,n}q_{y,m,n} / \Sigma_{m=1}{}^{M} \Sigma_{n\in S(y,m)\cap S(z,m)}\, p_{z,m,n}q_{y,m,n}; \quad (62)$$

$$P_F{}^*(y/z) \equiv [P_L{}^*(y/z)P_P{}^*(y/z)]^{1/2}; \quad (63)$$

$$P_T{}^*(y/z) \equiv exp[\Sigma_{m=1}{}^{M} \Sigma_{n\in S(y,m)\cap S(z,m)}\, (½)(\Sigma_{y,z,m,n} + \Sigma_{z,y,m,n})ln(p_{y,m,n}/p_{z,m,n})] \quad (64)$$

where $S(y,m)\cap S(z,m)$ is the set of products n that are available in *both* years y and z for month m. The price of product n in month m of year y, $p_{y,m,n}$, is the unit value price for that product if it is purchased in month m of year y and it is set equal to 0 if the product is not available or not sold.[40] The corresponding quantity, $q_{y,m,n}$, is the actual quantity of product n that is sold in month m of year y (which will equal 0 if the product is not available or not sold). Thus carry-forward prices are not used in definitions (61)–(64). The *conditional expenditure shares*, $\Sigma_{y,z,m,n}$, which appear in definition (64), need some explanation, which is provided subsequently.

The *actual expenditure on product n in month m of year y* is equal to $e_{y,m,n}$ defined as follows:

$$e_{y,m,n} \equiv p_{y,m,n}q_{y,m,n};\ y = 1,\ldots,Y;\ m = 1,\ldots,M;\ n = 1,\ldots,N. \quad (65)$$

The *conditional year z expenditure on product n in month m of year y*, $e_{y,z,m,n}$, is defined as the actual expenditure on product n in month m of year y *if* the same product n is also sold in month m of year z and is defined to be 0 *if* product n is not sold in month m of year z. Thus the formal definition for $e_{y,z,m,n}$ is as follows:

$$e_{y,z,m,n} \equiv e_{y,m,n}\ if\ e_{z,m,n} > 0;\ y = 1,\ldots,Y;\ z = 1,\ldots,Z;\ m = 1,\ldots,M;\ n = 1,\ldots,N \equiv 0\ if\ e_{z,m,n} = 0. \quad (66)$$

Thus, $e_{y,z,m,n}$ will be positive only if product n is purchased in month m of years y and z. *The total year y expenditure on products that are available in both years y and z*, $E_{y,z}$, is defined as follows:

[40] If a product is available in month m of year y but not purchased, we treat it as if it were an unavailable product for that month.

$$E_{y,z} \equiv \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} e_{y,z,m,n};\ y = 1,...,Y;\ z = 1,...,Z. \qquad (67)$$

The *year y conditional on year z expenditure share on product n in month m of year y*, $\Sigma_{y,z,m,n}$, is defined as follows:

$$\Sigma_{y,z,m,n} \equiv e_{y,z,m,n}/E_{y,z};\ y = 1,...,Y;\ z = 1,...,Y;\ m = 1,...,M;\ n = 1,...,N. \qquad (68)$$

The conditional share $\Sigma_{y,z,m,n}$ is positive only if product n in month m is sold in both years y and z. These shares appear in definitions (64).

The *maximum overlap annual fixed-base Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices*, P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, and P_{TFB}^{y*}, are defined as follows:

$$P_{LFB}^{y*} \equiv P_L^*(y/1);\ P_{PFB}^{y*} \equiv P_P^*(y/1);\ P_{FFB}^{y*} \equiv P_F^*(y/1);\ P_{TFB}^{y*} \equiv P_T^*(y/1);\ y = 1,...,Y. \qquad (69)$$

The *maximum overlap annual chained Laspeyres, Paasche, Fisher, and Törnqvist–Theil indices* are defined as follows for year 1:

$$P_{LCH}^{1*} \equiv 1;\ P_{PCH}^{1*} \equiv 1;\ P_{FCH}^{1*} \equiv 1;\ P_{TCH}^{1*} \equiv 1. \qquad (70)$$

For years y following year 1, these indices are defined recursively using the bilateral maximum overlap annual indices defined by (55)–(58) as follows:

$$P_{LCH}^{y*} \equiv P_{LCH}^{y-1*}\, P_L^*(y/(y-1));\ y = 2,...,Y; \qquad (71)$$

$$P_{PCH}^{y*} \equiv P_{PCH}^{y-1*}\, P_P^*(y/(y-1));\ y = 2,...,Y; \qquad (72)$$

$$P_{FCH}^{y*} \equiv P_{FCH}^{y-1*}\, P_F^*(y/(y-1));\ y = 2,...,Y; \qquad (73)$$

$$P_{TCH}^{y*} \equiv P_{TCH}^{y-1*}\, P_T^*\,(y/(y-1));\ y = 2,...,Y. \qquad (74)$$

The *maximum overlap annual GEKS price levels*, p_{GEKS}^{y*}, are defined as follows:

$$p_{GEKS}^{y*} \equiv [\prod_{z=1}^{Y} P_F^*\,(y/z)]^{1/Y};\ y = 1,...,Y. \qquad (75)$$

The *maximum overlap annual GEKS price indices*, P_{GEKS}^{y*}, are defined as follows:

$$P_{GEKS}^{y*} \equiv p_{GEKS}^{y*}/p_{GEKS}^{1*};\ y = 1,...,Y. \qquad (76)$$

The *maximum overlap CCDI price levels*, p_{CCDI}^{y*}, for year y are defined as follows:

$$p_{CCDI}^{y*} \equiv [\prod_{z=1}^{Y} P_T^*(y/z)]^{1/Y};\ y = 1,...,Y. \qquad (77)$$

The *maximum overlap annual CCDI price indices*, P_{CCDI}^{y*}, are defined as follows:

$$P_{CCDI}^{y*} \equiv p_{CCDI}^{y*}/p_{CCDI}^{1*};\ y = 1,...,Y. \qquad (78)$$

The 10 maximum overlap annual indices P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{TFB}^{y*}, P_{LCH}^{y*}, P_{PCH}^{y*}, P_{FCH}^{y*}, P_{TCH}^{y*}, P_{GEKS}^{y*}, and P_{CCDI}^{y*} for our empirical example are listed in Table 9.12.

Table 9.11 Imputation-Free Annual Index-Predicted Share Measures of Price Dissimilarity

	y = 1	y = 2	y = 3	y = 4	y = 5	y = 6
z = 1	0.00000	0.00284	0.00272	0.00198	0.00272	0.00245
z = 2	0.00284	0.00000	0.00125	0.00275	0.00122	0.00305
z = 3	0.00272	0.00125	0.00000	0.00181	0.00086	0.00148
z = 4	0.00198	0.00275	0.00181	0.00000	0.00213	0.00056
z = 5	0.00272	0.00122	0.00086	0.00213	0.00000	0.00154
z = 6	0.00245	0.00305	0.00148	0.00056	0.00154	0.00000

Our final annual Mudgett Stone annual index that uses year-over-year maximum overlap prices is the 0-predicted share similarity-linked index P_S^{y*}.

Using our zero prices $p_{y,m,n}$ for products n that are not available in month m of year y, the year y, month m, product n actual expenditure share is $s_{y,m,n} \equiv p_{y,m,n}\, q_{y,m,n}/p^{y,m}\cdot q^{y,m}$. The *prediction* for this share using the price of product n of month m in year z, $p_{z,m,n}$, and the actual quantity of product n for month m in year y is the *predicted share* $s_{z,y,m,n} \equiv p_{z,m,n}\, q_{y,m,n}/p^{z,m}\cdot q^{y,m}$ for $n = 1,...,N$, $m = 1,...,M$, $z = 1,...,Y$, and $y = 1,...,Y$. Using these prices and shares, the new annual measure of *predicted share price dissimilarity* between the prices of years z and y, $\Delta_{PSA}^*(p^z,p^y,q^z,q^y)$, is defined as follows:

$$\Delta_{PSA}^*(p^z,p^y,q^z,q^y) \equiv \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} [s_{y,m,n} - s_{z,y,m,n}]^2 + \Sigma_{m=1}^{M} \Sigma_{n=1}^{N} [s_{z,m,n} - s_{y,z,m,n}]^2. \qquad (79)$$

Note that this measure of relative price dissimilarity does not use any imputed prices.[41]

The table of the new bilateral measures of annual predicted share price dissimilarity for our empirical example is Table 9.11.

A comparison of the entries in Tables 9.9 and 9.11 shows that the entries in Table 9.11 are always equal to or greater than the corresponding entries in Table 9.9. Many entries in Table 9.11 are substantially greater. This is due to the fact that the new measure of relative price dissimilarity that uses 0 values for missing prices instead of carry-forward prices substantially penalizes a lack of matching.

The *real-time* set of bilateral links which minimize the new predicted share measures of relative price dissimilarity for the annual data are as follows: link 2 to 1; 3 to 2; 4 to 3; 5 to 3; and 6 to 4. This is the same set of bilateral links that we used to construct the similarity-linked annual indices P_S^y that used carry-forward prices. Thus, we define $P_S^{1*} \equiv 1$, $P_S^{2*} \equiv P_F^*(2/1)$, $P_S^{3*} \equiv P_S^{2*} \times P_F^*\,(3/2)$, $P_S^{4*} \equiv P_S^{3*} \times P_F^*(4/3)$, $P_S^{5*} \equiv P_S^{3*} \times P_F^*(5/3)$, and $P_S^{6*} \equiv P_S^{4*} \times P_F^*(6/4)$ where the maximum

[41] However, one could argue that setting the price of a product that is not purchased in a period equal to 0 is also an imputation. Note that definition (79) is exactly the same as definition (60) in the previous section. But the previous definition used carry-forward (and carry-backward) prices for missing prices, whereas in this section, missing prices are set equal to 0. The actual shares of product n in month m of year y, $s_{y,m,n}$, are the same in definitions (60) and (79), but the predicted shares $s_{y,z,m,n} = p_{z,m,n} q_{y,m,n}/p^{z,m}\cdot q^{y,m}$ are now, in general, different due to the replacement of carry-forward prices by zero prices.

Table 9.12 Alternative Annual Mudgett Stone Indices Using Maximum Overlap Bilateral Indices

Year *y*	P_{LFB}^{y*}	P_{PFB}^{y*}	P_{LCH}^{y*}	P_{PCH}^{y*}	P_{FFB}^{y*}	P_{FCH}^{y*}	P_{TFB}^{y*}	P_{TCH}^{y*}	P_{GEKS}^{y*}	P_{CCDI}^{y*}	P_{S}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1373	1.0611	1.1373	1.0611	1.0986	1.0986	1.0921	1.0921	1.0964	1.0930	1.0986
3	1.1273	1.0750	1.1545	1.0468	1.1009	1.0994	1.0986	1.0929	1.0987	1.0958	1.0994
4	1.1919	1.1407	1.2419	1.0985	1.1660	1.1680	1.1633	1.1609	1.1683	1.1650	1.1680
5	1.2253	1.1565	1.2848	1.0962	1.1904	1.1868	1.1876	1.1792	1.1916	1.1881	1.1947
6	1.2344	1.1752	1.3194	1.0973	1.2044	1.2032	1.2031	1.1961	1.2056	1.2028	1.2053
Mean	1.1527	1.1014	1.1897	1.0666	1.1267	1.1260	1.1241	1.1202	1.1268	1.1242	1.1277

Table 9.13 Annual Mudgett Stone Indices Using Maximum Overlap Bilateral Indices and Their Year-over-Year Simple Approximations

Year *y*	P_{LFBA}^{y*}	P_{LFB}^{y*}	P_{PFBA}^{y*}	P_{PFB}^{y*}	P_{FFBA}^{y*}	P_{FFB}^{y*}	P_{GEKSA}^{y*}	P_{GEKS}^{y*}	P_{SA}^{y*}	P_{S}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.1053	1.1373	1.0538	1.0611	1.0789	1.0986	1.0785	1.0964	1.0789	1.0986
3	1.1141	1.1273	1.0706	1.0750	1.0920	1.1009	1.0902	1.0987	1.0896	1.0994
4	1.1802	1.1919	1.1438	1.1407	1.1617	1.1660	1.1612	1.1683	1.1614	1.1680
5	1.2012	1.2253	1.1520	1.1565	1.1761	1.1904	1.1785	1.1916	1.1788	1.1947
6	1.2279	1.2344	1.1817	1.1752	1.2045	1.2044	1.2040	1.2056	1.2014	1.2053
Mean	1.1381	1.1527	1.1003	1.1014	1.1189	1.1267	1.1187	1.1268	1.1184	1.1277

overlap bilateral annual Fisher indices $P_F^*(y/z)$ are defined by (63).

The 11 annual indices that use maximum overlap bilateral indices to link the months, P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{TFB}^{y*}, P_{LCH}^{y*}, P_{PCH}^{y*}, P_{FCH}^{y*}, P_{TCH}^{y*}, P_{GEKS}^{y*}, P_{CCDI}^{y*}, and P_S^{y*} are listed in Table 9.12.

As was the case for the Laspeyres and Paasche indices that used carry-forward prices, the new maximum overlap annual fixed-base and chained Laspeyres indices, P_{LFB}^{y*} and P_{LCH}^{y*}, are well above the superlative indices, and the new maximum overlap annual fixed-base and chained Paasche indices, P_{PFB}^{y*} and P_{PCH}^{y*}, are well below the superlative indices. Our five best indices are the fixed-base Fisher and Törnqvist–Theil indices and the multilateral GEKS, CCDI, and Predicted Share Price Similarity-linked indices. These five indices ended up at 1.2044, 1.2031, 1.2056, 1.2028, and 1.2053. The average of these five final values is 1.2048. The average of the five final values for the same indices listed in Table 9.10 is 1.2032. Thus, the differences between our best maximum overlap indices listed in Table 9.12 and the counterpart indices listed in Table 9.10 that used carry-forward prices are not large for our empirical example. The downward bias resulting from the use of carry-forward prices over the sample period is only about 0.16 percentage points. However, this bias is not negligible and can be avoided by using bilateral maximum overlap indices.

We conclude this section on annual indices by looking at some approximations to the "true" Mudgett Stone indices P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{GEKS}^{y*}, and P_S^{y*} that are listed in Table 9.12. In Section 3, year-over-year monthly indices were computed using bilateral maximum overlap indices as building blocks. In particular, the fixed-base Laspeyres, Paasche, and Fisher indices, $P_{LFB}^{y,m*}$, $P_{PFB}^{y,m,*}$ and $P_{FFB}^{y,m*}$, were computed along with the maximum overlap GEKS index and the predicted share similarity-linked indices, $P_{GEKS}^{y,m*}$ and $P_S^{y,m*}$. Some statistical agencies form annual indices by taking equally weighted averages of their month-to-month indices. In the previous section, we saw that the true Mudgett Stone annual Laspeyres index (using carry-forward prices for missing prices) could be computed as a share-weighted average of the monthly year-over-year indices. It is of interest to see how taking simple equally weighted averages of the monthly indices $P_{LFB}^{y,m*}$, $P_{PFB}^{y,m*}$, $P_{FFB}^{y,m*}$, $P_{GEKS}^{y,m*}$, and $P_S^{y,m*}$ can approximate the "true" Mudgett Stone indices P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{GEKS}^{y*}, and P_S^{y*}. Thus, the following *approximate annual indices* P_{LFBA}^{y*}, P_{PFBA}^{y*}, P_{FFBA}^{y*}, P_{GEKSA}^{y*}, and P_{SA}^{y*} for $y = 1,\ldots,Y$ are defined as follows:

$$P_{LFBA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{LFB}^{y,m*}; \tag{80}$$

$$P_{PFBA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{PFB}^{y,m*}; \tag{81}$$

$$P_{FFBA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{FFB}^{y,m*}; \tag{82}$$

$$P_{GEKSA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{GEKS}^{y,m*}; \tag{83}$$

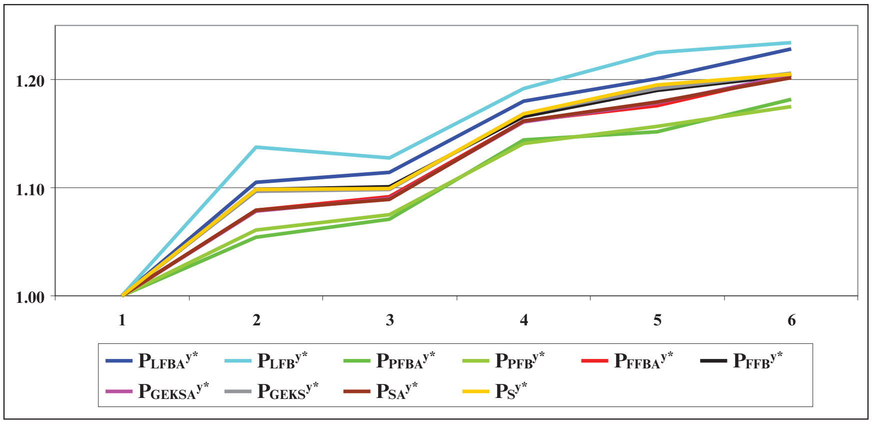
$$P_{SA}^{y*} \equiv (1/M)\Sigma_{m=1}^{M} P_{S}^{y,m*}. \tag{84}$$

The five "true" annual indices P_{LFB}^{y*}, P_{PFB}^{y*}, P_{FFB}^{y*}, P_{GEKS}^{y*}, and P_S^{y*} and their five approximations P_{LFBA}^{y*}, P_{PFBA}^{y*}, P_{FFBA}^{y*}, P_{GEKSA}^{y*}, and P_{SA}^{y*} evaluated using our Israeli data are listed in Table 9.13.

Figure 9.4 shows a plot of the above 10 indices.

As usual, the two fixed-base Laspeyres indices are well above the superlative indices and the two fixed-base Paasche indices are well below the superlative indices. What is interesting is that the approximate Laspeyres indices P_{LFBA}^{y*} lie well above their "true" counterparts P_{LFB}^{y*}. Moreover, there are some substantial differences in the average values for the

Figure 9.4 Annual Mudgett Stone Indices Using Maximum Overlap Bilateral Indices and Their Simple Approximations



"true" superlative indices and their approximations. The average for the true fixed-base Fisher annual indices $P_{FFB}{}^{y*}$ is 1.1267, which is well above the average for the approximate fixed-base Fisher indices $P_{FFBA}{}^{y*}$ of 1.1189. The average for the true similarity-linked Fisher indices $P_{S}{}^{y*}$ is 1.1277, which is well above the average for the approximate similarity-linked Fisher indices $P_{SA}{}^{y*}$ of 1.1184. The average for the true GEKS annual indices $P_{GEKS}{}^{y*}$ is 1.1268, which is also above the average for the GEKS approximate indices $P_{GEKSA}{}^{y*}$ of 1.1187.

Our conclusions regarding the construction of annual indices at this point are as follows:

- The use of the Laspeyres and Paasche Mudgett Stone indices should be avoided. The fixed-base and chained Laspeyres indices tend to lie well above the clustered superlative indices, while the fixed-base and chained Paasche indices tend to lie well below the clustered superlative indices.
- The amount of chain drift in the annual Fisher and Törnqvist–Theil indices was small for our empirical example. However, if one used the similarity-linked annual Mudgett Stone indices, there is no possibility of any chain drift.
- The Mudgett Stone fixed-base Fisher and Törnqvist–Theil indices and the GEKS and predicted share similarity-linked indices are all fairly close to each other in the present context where we are calculating annual indices.
- The use of carry-forward prices will tend to lead to annual indices which are biased downward if there is general inflation and so in order to avoid this potential bias, it is better to use the indices that use maximum overlap superlative bilateral indices as their basic building blocks. Thus, the maximum overlap annual fixed-base Fisher and fixed-base Törnqvist–Theil, GEKS, and predicted share similarity-linked indices, $P_{FFB}{}^{y,*}$, $P_{TFB}{}^{y*}$, $P_{GEKS}{}^{y*}$, and $P_{S}{}^{y*}$, emerge as our "best" choices for Mudgett Stone annual indices.
- Approximating "true" Mudgett Stone indices by taking a simple average of the year-over-year monthly indices discussed in Sections 2 and 3 can lead to substantial approximation errors. For our empirical example, the approximation error using the Laspeyres formula was substantial.

In the following sections, we turn our attention to month-to-month price indices.

6. Month-to-Month Indices using Carry-Forward Prices

Some new notation is required when constructing month-to-month indices for seasonal goods and services. Denote the *quantity purchased* of product n in month t as $q_{t,n}$, where $t = 1,2,\ldots,T$, where $T = MY$, where M denotes the number of months for the data set under consideration and Y denotes the number of years of seasonal product data. Thus t is now a monthly time indicator which runs from 1 to T. As usual, if no units of product n are purchased in month t, $q_{t,n} = 0$. If product n is purchased in month t, then denote the corresponding *unit value price* for this product by $p_{t,n} > 0$ for $n = 1,\ldots,N$ and $t = 1,\ldots,T$. In this section, if product n is missing in month t, then $p_{t,n}$ is set equal to the most recent previous month price for product n; that is, in this section, we replace missing prices by *month-to-month carry-forward prices*. If product n is missing in month 1, then $p_{1,n}$ is set equal to the price of product n in the next month when the product is sold; that is, in this case, we use a *month-to-month carry-backward* price for $p_{1,n}$. In general, these carry-forward and carry-backward prices will be substantially different from the carry-forward and carry-backward prices which were used in Sections 2 and 4. The frequency of imputed prices greatly increases when constructing price indices for strongly seasonal products. For our empirical example, there were 451 month-to-month carry-forward or carry-backward prices where the maximum number of available products over the months in our sample was 1008 = 72 months × 14 fresh fruit products. Tables A.23 and A.24 in the annex list the price and quantity data for fresh fruit purchased by households in Israel for the 72 months in the period 2012–2017. The sample probability that a price listed in Table A.23 is an imputed price is 0.447 = 451/1008. *Thus, the problem of missing prices can be a very big problem in the seasonal product context.*

In this section, we will set up the algebra for computing fixed-base and chained Laspeyres, Paasche, and Fisher month-to-month indices using carry-forward/carry-backward prices for unavailable products. The monthly price and quantity variables, $p_{t,n}$ and $q_{t,n}$ for product n in month t have been defined in the previous paragraph. Define the month t vectors of product prices and quantities, p^t and q^t as p^t ° $[p_{t,1},. . .,p_{t,N}]$ and $q^t \equiv [q_{t,1},. . .,q_{t,N}]$. For our empirical example, $T = 72$ and $N = 14$.

Denote the *bilateral Laspeyres, Paasche, and Fisher price indices* that compare the prices of month t relative to the prices of month r using carry-forward/carry-backward prices as $P_L(t/r)$, $P_P(t/r)$, and $P_F(t/r)$, respectively. These indices are defined as follows:

$$P_L(t/r) \equiv p^t{\cdot}q^r/p^r{\cdot}q^r;\ r = 1,. . .,T;\ t = 1,. . .,T; \tag{85}$$

$$P_P(t/r) \equiv p^t{\cdot}q^t/p^r{\cdot}q^t;\ r = 1,. . .,T;\ t = 1,. . .,T; \tag{86}$$

$$P_F(t/r) \equiv [P_L(t/r)P_P(t/r)]^{1/2};\ r = 1,. . .,T;\ t = 1,. . .,T. \tag{87}$$

The sequence of *T fixed-base Laspeyres indices* using carry-forward prices, $P_{LFB}{}^t$, is $P_L(1/1)$, $P_L(2/1)$, . . ., $P_L(T/1)$. The sequence of *T fixed-base Paasche indices* using carry-forward prices, $P_{PFB}{}^t$, is $P_P(1/1)$, $P_P(2/1)$, . . ., $P_P(T/1)$ and the sequence of *T fixed-base Fisher indices* using carry-forward prices, $P_{FFB}{}^t$, is $P_F(1/1)$, $P_F(2/1)$, . . ., $P_F(T/1)$. We use the data listed in Tables A.23 and A.24 in the annex to calculate these indices for our Israeli data set. These indices are listed in Table 9.15.

It should be noted that the month-to-month indices defined by (85)–(87) are not very reliable for our empirical example. Here is a list of the number of seasonal products that are actually available in months 1–12: 7, 8, 8, 7, 9, 10, 8, 7, 7, 10, 9, and 7. The maximum number of products is 14. Thus, for 5 out of the first 12 months, only one half of the seasonal fruits are available. When we look at matches for the products that are available in both month 1 and month m = 1,. . .,12, we find that the number of product matches is 7, 7, 7, 6, 5, 5, 3, 3, 4, 7, 7, and 7. We cannot expect any bilateral index number to be very reliable if the number of matched products is small.

Instead of choosing month 1 to be the fixed-base, we could choose any other month as the fixed-base. The resulting indices are called "star" indices. The 12 fixed-base Fisher star indices using months 1–12 as the base month are listed in Table A.25 of the annex and are plotted in Figure 9.5. These indices have been normalized to equal 1 in month 1.

A number of points emerge from the study of Figure 9.5:

- The seasonal fluctuations in prices are enormous;
- The choice of a base period matters;
- Any monthly index number is unlikely to be very reliable for our particular data set.

The problems associated with the reliability of month-to-month indices of strongly seasonal products are much bigger than the problem of finding reliable year-over-year monthly indices. As was seen in the previous sections, our best year-over-year monthly indices well behaved and approximated each other fairly well. This is not the case for month-to-month indices.

The *month-to-month chained Laspeyres, Paasche*, and *Fisher indices using carry-forward prices* for month 1 is defined as unity:

$$P_{LCH}{}^1 \equiv 1;\ P_{PCH}{}^1 \equiv 1;\ P_{FCH}{}^1 \equiv 1. \tag{88}$$

For months following month 1, these chained indices for month t are calculated by cumulating the corresponding successive month-to-month links using definitions (85)–(88); that is, we have the following definitions for $P_{LCH}{}^t$, $P_{PCH}{}^t$, and $P_{FCH}{}^t$:

$$P_{LCH}{}^t \equiv P_{LCH}{}^{t-1}\ P_L(t/(t-1));\ t = 2,3,. . .,T; \tag{89}$$

$$P_{PCH}{}^t \equiv P_{PCH}{}^{t-1}\ P_P(t/(t-1));\ t = 2,3,. . .,T; \tag{90}$$

$$P_{FCH}{}^t \equiv P_{FCH}{}^{t-1}\ P_F(t/(t-1));\ t = 2,3,. . .,T. \tag{91}$$

The *month-to-month GEKS price levels* using carry-forward prices, $p_{GEKS}{}^t$, for each month t is defined as the geometric

Figure 9.5 Fisher Star Indices Using Months 1–12 as the Base Month Using Carry-Forward Prices

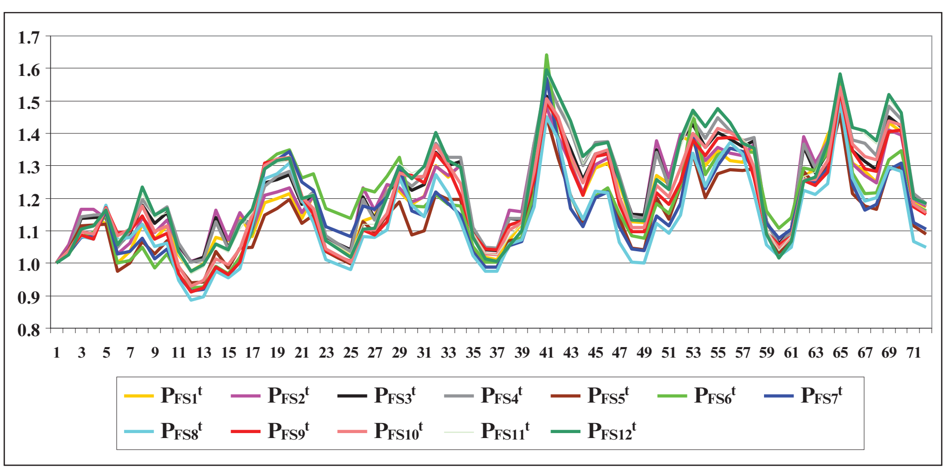


Table 9.14 Month-to-Month Predicted Share Measures of Price Dissimilarity Using Carry-Forward Prices

r, t	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0.0008	0.0015	0.0011	0.0020	0.0076	0.0086	0.0164	0.0181	0.0126	0.0022	0.0022
2	0.0008	0	0.0028	0.0018	0.0020	0.0070	0.0090	0.0180	0.0185	0.0110	0.0054	0.0074
3	0.0015	0.0028	0	0.0004	0.0043	0.0088	0.0122	0.0252	0.0208	0.0122	0.0057	0.0069
4	0.0011	0.0018	0.0004	0	0.0023	0.0075	0.0107	0.0215	0.0209	0.0119	0.0063	0.0073
5	0.0020	0.0020	0.0043	0.0023	0	0.0038	0.0083	0.0177	0.0241	0.0130	0.0055	0.0053
6	0.0076	0.0070	0.0088	0.0075	0.0038	0	0.0028	0.0094	0.0142	0.0076	0.0049	0.0056
7	0.0086	0.0090	0.0122	0.0107	0.0083	0.0028	0	0.0028	0.0049	0.0015	0.0028	0.0037
8	0.0164	0.0180	0.0252	0.0215	0.0177	0.0094	0.0028	0	0.0035	0.0045	0.0102	0.0122
9	0.0181	0.0185	0.0208	0.0209	0.0241	0.0142	0.0049	0.0035	0	0.0039	0.0073	0.0086
10	0.0126	0.0110	0.0122	0.0119	0.0130	0.0076	0.0015	0.0045	0.0039	0	0.0039	0.0054
11	0.0022	0.0054	0.0057	0.0063	0.0055	0.0049	0.0028	0.0102	0.0073	0.0039	0	0.0001
12	0.0022	0.0074	0.0069	0.0073	0.0053	0.0056	0.0037	0.0122	0.0086	0.0054	0.0001	0

mean of the separate indices we obtain by using each month as the base year:

$$p_{GEKS}{}^{t} \equiv [\prod_{r=1}{}^{T} P_F(t/r)]^{1/T};\ t = 1,2,...,T, \tag{92}$$

where $P_F(t/r)$ is defined by (87). The *month-to-month GEKS price indices* $P_{GEKS}{}^{t}$ are obtained by *normalizing* the above price levels so that the month 1 index is equal to 1. Thus, we have the following definitions for the *GEKS month-to-month index using carry-forward prices* for month t:

$$P_{GEKS}{}^{t} \equiv p_{GEKS}{}^{t}/p_{GEKS}{}^{1};\ t = 1,2,...,T. \tag{93}$$

The month-to-month GEKS indices using carry-forward prices along with the chained month-to-month Laspeyres, Paasche, and Fisher indices for our Israeli data are listed in Table 9.15.

The final month-to-month index that we define in this section is the *predicted share similarity-linked index*, $P_S{}^{t}$. The *month t, product n actual expenditure share* $s_{t,n}$ is defined as follows:

$$s_{t,n} \equiv p_{t,n}q_{t,n}/p^{t}\cdot q^{t};\ t = 1,...,T;\ n = 1,...,N. \tag{94}$$

The prediction for this share $s_{t,n}$ using the price of product n in month r, $p_{r,n}$, and the actual quantity of product n in month t is the *predicted share* $s_{r,t,n} \equiv p_{r,n}q_{t,n}/p^{r}\cdot q^{t}$ for $n = 1,...,N$, $r = 1,...,T$, and $t = 1,...,T$. The new measure of *month-to-month predicted share price dissimilarity* between the prices of months r and t, $\Delta_{PS}(p^r,p^t,q^r,q^t)$, is defined as follows:

$$\Delta_{PS}(p^r,p^t,q^r,q^t) \equiv \Sigma_{n=1}{}^{N} [s_{t,n} - s_{r,t,n}]^2 + \Sigma_{n=1}{}^{N} [s_{r,n} - s_{t,r,n}]^2;\quad r = 1,...,T;\ t = 1,...,T. \tag{95}$$

The entire set of predicted share dissimilarity measures for our empirical example is a 72 by 72 element (symmetric) matrix. Table 9.14 lists the first 12 rows and columns of the matrix of the bilateral measures of annual predicted share price dissimilarity for our empirical example.

The set of real-time links which minimize the above dissimilarity measures for the first 12 observations are as follows:

```
11 – 12
 |
1 – 2 – 5 – 6 – 7 – 8 – 9
|  |
3 – 4 10
```

It can be seen that there are substantial differences in the measures of relative price dissimilarity across pairs of observations. If any measure that is not on the main diagonal of the matrix of dissimilarity measures is equal to zero, then prices are proportional for the corresponding pair of months. It can be seen that for months 11 and 12, the dissimilarity measure is 0.0001 so that prices are "almost" proportional to each other for that pair of months.

The real-time *month-to-month predicted share indices* for months 1 to 12 are defined as follows: $P_S{}^{1} \equiv 1$; and $P_S{}^{2} \equiv P_F(2/1)$, where the bilateral Fisher indices $P_F(t/r)$ are defined by (87). $P_S{}^{3} \equiv P_F(3/1)P_S{}^{1}$; $P_S{}^{4} \equiv P_F(4/3)P_S{}^{3}$; $P_S{}^{5} \equiv P_F(5/2)P_S{}^{2}$; $P_S{}^{6} \equiv P_F(6/5)P_S{}^{5}$; $P_S{}^{7} \equiv P_F(7/6)P_S{}^{6}$; $P_S{}^{8} \equiv P_F(8/7)P_S{}^{7}$; $P_S{}^{9} \equiv P_F(9/8)P_S{}^{8}$; $P_S{}^{10} \equiv P_F(10/7)P_S{}^{7}$; and $P_S{}^{11} \equiv P_F(11/1)P_S{}^{1}$; $P_S{}^{12} \equiv P_F(12/11)P_S{}^{11}$.[42]

The predicted share indices (using carry-forward prices) $P_S{}^{t}$ along with the other seven indices defined in this section are listed in Table 9.15.

Just an observation about Table 9.15: All the indices listed in the table are indicated with period 1 = 1, and the mean in the last row is simply the mean value of the index series and is not the average monthly rate of change. For instance, P_{LFB} ends on 1.17122 in period 72 and P_{PFB} ends on 1.17533. These

[42] The optimal real-time bilateral Fisher index links for the next 12 months are as follows: 13/12, 14/13, 15/5, 16/7, 17/6, 18/17, 191/8, 20/18, 21/11, 22/12, 23/12, and 24/23. The optimal links are usually to an adjacent month or to the same (or almost the same) month in a previous year. Thus, the bilateral links for the relative price similarity-linked indices are a mixture of chain links and year-over-year links (or almost year-over-year links).

Table 9.15 Alternative Month-to-Month Price Indices Using Carry-Forward Prices

t	$P_{LFB}{}^{t}$	$P_{LCH}{}^{t}$	$P_{PFB}{}^{t}$	$P_{PCH}{}^{t}$	$P_{FCH}{}^{t}$	$P_{FFB}{}^{t}$	$P_{GEKS}{}^{t}$	$P_{S}{}^{t}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.07104	1.07104	1.04247	1.04247	1.05666	1.05666	1.04029	1.05666
3	1.12812	1.18503	1.14757	1.14754	1.16613	1.13780	1.11189	1.13780
4	1.14886	1.19038	1.14564	1.14800	1.16900	1.14725	1.11046	1.14060
5	1.18497	1.19577	1.06165	1.14848	1.17189	1.12162	1.17620	1.13804
6	1.13858	1.05378	0.88167	0.98633	1.01949	1.00192	1.01400	0.99005
7	1.21631	1.08372	0.88295	0.97245	1.02658	1.03631	1.06940	0.99693
8	1.42856	1.13888	0.88040	0.99986	1.06711	1.12148	1.17789	1.03629
9	1.30179	1.08934	0.88833	0.91995	1.00107	1.07537	1.08136	0.97215
10	1.23076	1.11710	1.00892	0.92437	1.01617	1.11433	1.10818	1.00552
11	1.03294	1.03012	1.02018	0.79126	0.90282	1.02654	0.99894	1.02654
12	0.97081	0.97490	0.98105	0.74151	0.85023	0.97592	0.94974	0.96674
13	0.99746	0.99246	0.99881	0.75995	0.86846	0.99813	0.96193	0.98747
14	1.04362	1.03468	1.11663	0.83848	0.93143	1.07951	1.04603	1.05907
15	1.08902	0.98580	1.05160	0.75233	0.86119	1.07015	1.01127	0.99942
16	1.16801	1.06661	1.12797	0.81803	0.93409	1.14781	1.07162	0.96551
17	1.22562	1.11251	0.95378	0.88975	0.99492	1.08119	1.15252	1.08213
18	1.31761	1.16525	1.06669	1.01038	1.08505	1.18553	1.25997	1.18016
19	1.42276	1.24559	1.01184	1.05267	1.14508	1.19983	1.30647	1.24545
20	1.43404	1.30259	1.02953	1.07895	1.18551	1.21507	1.31543	1.27774
21	1.25621	1.24446	1.03156	0.97653	1.10238	1.13836	1.18427	1.20512
22	1.20769	1.27816	1.20852	0.96247	1.10914	1.20811	1.17854	1.19813
23	1.07109	1.15892	1.06924	0.85334	0.99446	1.07017	1.06410	1.05901
24	1.05248	1.13311	1.04479	0.83222	0.97108	1.04863	1.04214	1.03411
25	1.03276	1.10994	1.01894	0.81540	0.95133	1.02583	1.02120	1.01308
26	1.07388	1.15260	1.19015	0.92508	1.03259	1.13052	1.13451	1.10356
27	1.14208	1.11185	1.14434	0.86585	0.98117	1.14321	1.12441	1.11025
28	1.26758	1.19400	1.21955	0.93474	1.05645	1.24333	1.18939	1.24205
29	1.34863	1.26883	1.10795	0.98096	1.11565	1.22238	1.28808	1.26070
30	1.40760	1.11401	0.99362	0.82832	0.96060	1.18263	1.19633	1.04240
31	1.58269	1.18055	0.92020	0.83100	0.99047	1.20681	1.24132	1.07482
32	1.65416	1.29680	1.02471	0.89232	1.07571	1.30193	1.33663	1.22615
33	1.41549	1.24967	1.13016	0.81043	1.00637	1.26481	1.25566	1.14710
34	1.33751	1.23520	1.26916	0.77779	0.98017	1.30289	1.22736	1.11724
35	1.08703	1.09589	1.08361	0.63866	0.83660	1.08532	1.06450	1.06552
36	1.02305	1.03395	1.01285	0.59973	0.78746	1.01793	1.00769	1.00292
37	1.01159	1.02930	1.00992	0.59594	0.78320	1.01076	1.00590	1.01076
38	1.02156	1.03700	1.12598	0.64186	0.81584	1.07250	1.08243	1.04927
39	1.10562	1.04371	1.13885	0.63579	0.81461	1.12211	1.10462	1.09715
40	1.37534	1.24541	1.35528	0.75386	0.96895	1.36527	1.28859	1.18982
41	1.62925	1.43926	1.34510	0.95365	1.17156	1.48037	1.56601	1.35919
42	1.68676	1.35068	1.14155	0.78643	1.03064	1.38763	1.43064	1.30049
43	1.86492	1.22188	0.92485	0.67190	0.90608	1.31331	1.32798	1.09659
44	1.67566	1.18164	0.87283	0.63377	0.86538	1.20937	1.25786	1.04733
45	1.45074	1.22157	1.15154	0.62550	0.87413	1.29251	1.28446	1.20540
46	1.39276	1.24584	1.23105	0.62949	0.88558	1.30941	1.29239	1.36846

t	$P_{LFB}{}^t$	$P_{LCH}{}^t$	$P_{PFB}{}^t$	$P_{PCH}{}^t$	$P_{FCH}{}^t$	$P_{FFB}{}^t$	$P_{GEKS}{}^t$	$P_S{}^t$
47	1.24213	1.17629	1.24786	0.56406	0.81455	1.24499	1.18302	1.25870
48	1.12808	1.07490	1.12696	0.50559	0.73719	1.12752	1.09100	1.12850
49	1.12212	1.07078	1.12896	0.50356	0.73431	1.12554	1.08518	1.12408
50	1.21916	1.16985	1.32416	0.58703	0.82869	1.27058	1.23697	1.24264
51	1.25881	1.09697	1.23041	0.52255	0.75712	1.24453	1.18846	0.99844
52	1.41954	1.21707	1.36379	0.58065	0.84065	1.39139	1.29862	1.10860
53	1.48016	1.26847	1.27100	0.68677	0.93335	1.37160	1.42021	1.30618
54	1.63803	1.18186	1.03243	0.58748	0.83326	1.30044	1.32237	1.09955
55	1.74314	1.30177	1.04150	0.64676	0.91757	1.34740	1.40911	1.21081
56	1.58174	1.37926	1.09335	0.65673	0.95174	1.31506	1.40016	1.33623
57	1.41498	1.40348	1.21309	0.63257	0.94223	1.31015	1.34033	1.39017
58	1.35851	1.39696	1.36991	0.61114	0.92398	1.36420	1.30895	1.36324
59	1.09904	1.14658	1.09780	0.48818	0.74815	1.09842	1.09469	1.08706
60	1.02215	1.06443	1.02087	0.45361	0.69487	1.02151	1.03615	1.00963
61	1.07410	1.12216	1.06543	0.47921	0.73331	1.06976	1.07690	1.06754
62	1.20643	1.28194	1.33332	0.57751	0.86042	1.26829	1.28266	1.24961
63	1.29331	1.23062	1.28941	0.53091	0.80830	1.29136	1.26451	1.12883
64	1.43622	1.29686	1.35392	0.55962	0.85191	1.39446	1.33515	1.17617
65	1.58284	1.41595	1.36272	0.64700	0.95714	1.46866	1.56457	1.32146
66	1.60835	1.17240	1.06355	0.51021	0.77341	1.30788	1.32005	1.11244
67	1.82150	1.18662	0.90346	0.49082	0.76316	1.28283	1.31651	1.09936
68	1.68998	1.21567	0.92494	0.49073	0.77237	1.25025	1.31291	1.10346
69	1.66533	1.22295	1.23161	0.48846	0.77289	1.43214	1.39315	1.26962
70	1.46701	1.24639	1.34549	0.47086	0.76608	1.40494	1.37485	1.46035
71	1.18124	1.09020	1.19949	0.37845	0.64233	1.19033	1.15473	1.19055
72	1.17122	1.08049	1.17533	0.37628	0.63763	1.17327	1.13682	1.18184
Mean	1.30070	1.17250	1.10940	0.74309	0.92385	1.19470	1.19600	1.13770

are pretty close, so the average monthly rate of change of these two is also very close, 0. 223 percent and 0.228 percent, respectively, while the means of the series are quite different. I don't know what this mean of the index series tells you other than this is the average value. It is calculated in the same way in many of the other tables. Maybe it tells you something, and not much to do about it now, so maybe just leave as is.

The Laspeyres, Paasche, and Fisher fixed-base indices end up at much the same level and the similarity-linked indices end up at a bit higher level. However, the seasonal fluctuations in $P_S{}^t$ are much smaller. The three chained indices are all subject to a large amount of downward chain drift. This is due to the fact that the strongly seasonal products come into season at relatively high prices and then trend down to relatively low prices at the end of their seasonal availability. They behave in the same manner as fashion goods, which are also subject to tremendous downward chain drift.[43] The first two of our three indices ($P_{FFB}{}^t$, $P_{GEKS}{}^t$, and $P_S{}^t$) have roughly the same mean but the similarity-linked index $P_S{}^t$ ends up well above $P_{FFB}{}^t$ and $P_{GEKS}{}^t$ for $t = 72$. The above six series are plotted in Figure 9.6.

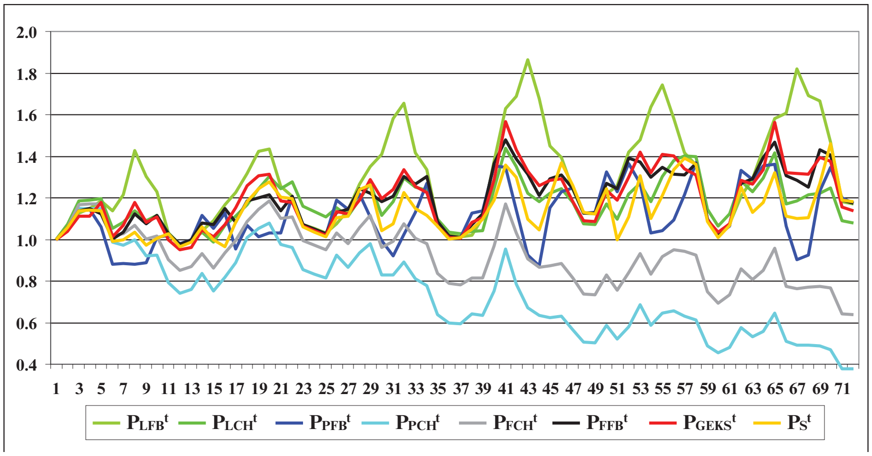
It can be seen that our three best indices, $P_{FFB}{}^t$, $P_{GEKS}{}^t$ and $P_S{}^t$, are much closer to each other than four of the other, five indices that suffer from substitution bias or chain drift bias.[44]

The use of carry-forward prices in the context of an elementary index category that includes many strongly seasonal products can lead to a large number of imputed prices, which in turn can lead to indices which are very different from their matched product counterpart indices. None of the above indices can be regarded as being very reliable since the proportion of carry-forward prices is so large. In the following section, we will compute the maximum overlap counterpart indices to the eight indices listed above. This will cure any carry-forward/carry-backward bias that probably is present in the above eight indices.

[43] Note that the year-over-year monthly indices did not suffer from this tremendous downward chain drift. Thus, year-over-year indices work well for both strongly seasonal goods and services and fashion goods.

[44] The chained Laspeyres index ends up reasonably close to the three superlative indices. It appears that the upward substitution bias (which a Laspeyres index is subject to) approximately offsets the downward chain drift bias that the chained indices are subject to in the present context when beginning of season prices are generally higher than the corresponding end-of-season prices.

Figure 9.6 Alternative Month-to-Month Indices Using Carry-Forward Prices



7. Month-to-Month Indices Using Maximum Overlap Bilateral Indices as Building Blocks

The month-to-month maximum overlap indices that are defined in this section are analogues to the eight indices that were defined in the previous section. The difference is that the building block bilateral indices between periods r and t use only the prices and quantities that are actually available in periods r and t. As in the previous section, the price and quantity of product n purchased in month t is p_{tn} and q_{tn}, respectively. If there are no purchases of product n in period t, set $p_{tn} = q_{tn} = 0$. Thus any missing prices are set equal to zero in this section. As usual, the set of available products in period t is denoted by $S(t)$ for $t = 1,. . .,T$.

Denote the *maximum overlap bilateral Laspeyres, Paasche, and Fisher price indices* that compare the prices of month t to the prices of month r as $P_L^*(t/r)$, $P_P^*(t/r)$, and $P_F^*(t/r)$, respectively. These indices are defined as follows:

$$P_L^*(t/r) \equiv \Sigma_{n\in S(t)\cap S(r)}\, p_{t,n}q_{r,n}/\Sigma_{n\in S(t)\cap S(r)}\, p_{r,n}q_{r,n};\quad r = 1,. . .,T;\ t = 1,. . .,T; \tag{96}$$

$$P_P^*(t/r) \equiv \Sigma_{n\in S(t)\cap S(r)}\, p_{t,n}q_{t,n}/\Sigma_{n\in S(t)\cap S(r)}\, p_{r,n}q_{t,n};\quad r = 1,. . .,T;\ t = 1,. . .,T; \tag{97}$$

$$P_F^*(t/r) \equiv [P_L^*(t/r)P_P^*(t/r)]^{1/2};\quad r = 1,. . .,T;\ t = 1,. . .,T. \tag{98}$$

The sequence of T *maximum overlap fixed-base Laspeyres indices*, P_{LFB}^{t*}, is $P_L^*(1/1)$, $P_L^*(2/1)$, . . ., $P_L^*(T/1)$. The sequence of T *maximum overlap fixed-base Paasche indices*, P_{PFB}^{t*}, is $P_P^*(1/1)$, $P_P^*(2/1)$, . . ., $P_P^*(T/1)$ and the sequence of T *maximum overlap fixed-base Fisher indices*, P_{FFB}^{t*}, is $P_F^*(1/1)$, $P_F^*(2/1)$, . . ., $P_F^*(T/1)$. We use the data listed in Tables A.23 and A.24 in the annex to calculate these indices for our Israeli data set. These indices are listed in Table 9.17.

As in the previous section, instead of choosing month 1 to be the fixed-base, we could choose any other month as the fixed-base. The 12 maximum overlap fixed-base Fisher star indices using months 1–12 as the base month are listed in Table A.26 of the annex and are plotted in Figure 9.7. These indices have been normalized to equal 1 in month 1.

A comparison of Figures 9.5 and 9.7 shows that the use of maximum overlap fixed-base Fisher indices has led to alternative fixed-base indices which are very close to each other for the months of December, January, and February but have *much larger seasonal fluctuations* than their fixed-base Fisher index carry-forward counterparts for other months of the year. For these alternative fixed-base Fisher indices, *the use of maximum overlap bilateral Fisher indices has led to index values in month 72 which are on average 2.68 percentage points above their carry-forward fixed-base Fisher index counterparts*. Thus we have a rough estimate of the cumulative amount of downward bias that the use of carry-forward prices induced for our empirical example over the six-year sample period.

The *maximum overlap month-to-month chained Laspeyres, Paasche*, and *Fisher indices* for month 1 is defined as unity:

$$P_{LCH}^{1*} \equiv 1;\ P_{PCH}^{1*} \equiv 1;\ P_{FCH}^{1*} \equiv 1. \tag{99}$$

For months following month 1, these chained indices for month t are calculated by cumulating the corresponding successive month-to-month links using definitions (96)–(98); that is, we have the following definitions for P_{LCH}^{t*}, P_{PCH}^{t*}, and P_{FCH}^{t*}:

$$P_{LCH}^{t*} \equiv P_{LCH}^{t-1*}\ P_L^*(t/(t-1));\ t = 2,3,. . .,T; \tag{100}$$

$$P_{PCH}^{t*} \equiv P_{PCH}^{t-1*}\ P_P^*(t/(t-1));\ t = 2,3,. . .,T; \tag{101}$$

$$P_{FCH}^{t*} \equiv P_{FCH}^{t-1*}\ P_F^*(t/(t-1));\ t = 2,3,. . .,T. \tag{102}$$

The *maximum overlap month-to-month GEKS price level*, p_{GEKS}^{t*}, for each month t is defined as the geometric mean of

Figure 9.7 Maximum Overlap Fisher Star Indices Using Months 1-12 as the Base

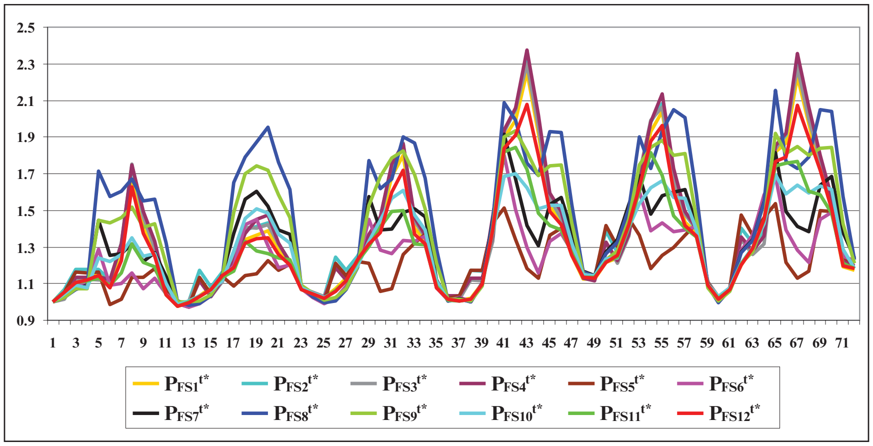


Table 9.16 Month-to-Month Predicted Share Measures of Price Dissimilarity Using Zeros for Missing Prices

r, t	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0.1029	0.1075	0.1115	0.4470	0.5477	0.6367	0.6410	0.3713	0.1441	0.0157	0.0022
2	0.1029	0	0.0028	0.0122	0.2387	0.6201	0.6924	0.7014	0.4901	0.2498	0.1198	0.1051
3	0.1075	0.0028	0	0.0062	0.2353	0.6254	0.6967	0.7089	0.4909	0.2562	0.1261	0.1111
4	0.1115	0.0122	0.0062	0	0.2097	0.5398	0.6203	0.6285	0.4593	0.2865	0.1359	0.1073
5	0.4470	0.2387	0.2353	0.2097	0	0.0539	0.0912	0.1017	0.3485	0.2456	0.3900	0.3686
6	0.5477	0.6201	0.6254	0.5398	0.0539	0	0.0250	0.0795	0.2432	0.2248	0.3883	0.4635
7	0.6367	0.6924	0.6967	0.6203	0.0912	0.0250	0	0.0204	0.1716	0.1974	0.3854	0.5560
8	0.6410	0.7014	0.7089	0.6285	0.1017	0.0795	0.0204	0	0.1224	0.1472	0.3619	0.5584
9	0.3713	0.4901	0.4909	0.4593	0.3485	0.2432	0.1716	0.1224	0	0.0148	0.1963	0.3671
10	0.1441	0.2498	0.2562	0.2865	0.2456	0.2248	0.1974	0.1472	0.0148	0	0.0956	0.1429
11	0.0157	0.1198	0.1261	0.1359	0.3900	0.3883	0.3854	0.3619	0.1963	0.0956	0	0.0123
12	0.0022	0.1051	0.1111	0.1073	0.3686	0.4635	0.5560	0.5584	0.3671	0.1429	0.0123	0

the separate maximum overlap indices we obtain by using each month as the base year:

$$p_{GEKS}^{t*} \equiv [\textstyle\prod_{r=1}^{T} P_F^{*}(t/r)]^{1/T};\ t = 1,2,\ldots,T, \qquad (103)$$

where $P_F^{*}(t/r)$ is defined by (98). The *maximum overlap month-to-month GEKS price indices* P_{GEKS}^{t*} are obtained by *normalizing* the above price levels so that the month 1 index is equal to 1. Thus we have the following definition for the month *t year-over-year maximum overlap GEKS index*, P_{GEKS}^{t*}:

$$P_{GEKS}^{t*} \equiv p_{GEKS}^{t*}/p_{GEKS}^{1*};\ t = 1,2,\ldots,T. \qquad (104)$$

The various month-to-month Laspeyres, Paasche, and Fisher fixed-base and chained indices as well as the GEKS index defined above in this section using maximum overlap bilateral indices as building blocks using our Israeli data are listed in Table 9.17.

The final month-to-month index that we define in this section is the predicted share similarity-linked index, P_S^{t*}. Definitions (105) and (106) are the same as definitions (94) and (95) in the previous section, but in this section, the price of an unavailable product is set to 0. For convenience, we repeat these definitions. The *month t, product n actual expenditure share*, $s_{t,n}$, is defined as follows:

$$s_{t,n} \equiv p_{t,n}q_{t,n}/p^t\cdot q^t;\ t = 1,\ldots,T;\ n = 1,\ldots,N. \qquad (105)$$

The prediction for this share $s_{t,n}$ using the price of product n in month r, $p_{r,n}$, and the actual quantity of product n in month t is the *predicted share* $s_{r,t,n} \equiv p_{r,n}q_{t,n}/p^r\cdot q^t$ for $n = 1,\ldots,N$, $r = 1,\ldots,T$, and $t = 1,\ldots,T$. The new measure of *predicted share price dissimilarity* between the prices of months r and t, $\Delta_{PS}(p^r,p^t,q^r,q^t)$, is defined as follows:

$$\Delta_{PS}(p^r,p^t,q^r,q^t) \equiv \textstyle\sum_{n=1}^{N} [s_{t,n} - s_{r,t,n}]^2 + \sum_{n=1}^{N} [s_{r,n} - s_{t,r,n}]^2;\ r = 1,\ldots,T;\ t = 1,\ldots,T. \qquad (106)$$

The entire set of predicted share dissimilarity measures for our empirical example is a 72 by 72 element (symmetric) matrix. Table 9.16 lists the first 12 rows and columns of the matrix of the bilateral measures of predicted share price dissimilarity for our empirical example.

The set of real-time links that minimize the above dissimilarity measures for the first 12 observations are as follows:

11
|
1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10
|
12

It can be seen that the new set of bilateral links is the set of links that generates chained Fisher indices for months 1 to 10. However, months 11 and 12 are linked directly to month 1. It can also be seen that the measures of price dissimilarity in Table 9.16 are much larger than the corresponding measures in Table 9.14, which used artificial carry-forward/carry-backward prices for the missing prices. It turns out that the set of bilateral links for the first 12 months basically determines the seasonal fluctuations for the similarity-linked indices P_S^{t*} for the remainder of the sample.[45]

The predicted share indices (using maximum overlap bilateral Fisher indices as the basic building blocks) P_S^{t*} along with the other seven indices defined in this section are listed in Table 9.17.

The maximum overlap fixed-base Laspeyres and Paasche indices, P_{LFB}^{t*} and P_{PFB}^{t*}, end up at much the same place (1.17122 and 1.17533) and have similar means (1.35950 and 1.35160). The chained Laspeyres and Paasche indices, P_{LCH}^{t*} and P_{PCH}^{t*}, suffer from some downward chain drift and end up far apart at 1.11995 and 0.21988, respectively. The downward chain drift problem carries over to the maximum overlap chained Fisher index, P_{FCH}^{t*}, which ends up at 0.49624. Our three best indices from the viewpoint of controlling substitution bias and chain drift bias, P_{FFB}^{t*}, P_{GEKS}^{t*}, and P_S^{t*}, end up at 1.17327, 1.18952[46], and 1.19115. respectively. The means of P_{FFB}^{t*} and P_{GEKS}^{t*} are similar at 1.3552 and 1.3468. These means are far above the mean of the similarity-linked indices P_S^{t*} which is 1.1892. It turns

Table 9.17 Alternative Month-to-Month Price Indices Using Maximum Overlap Bilateral Indices as Building Blocks

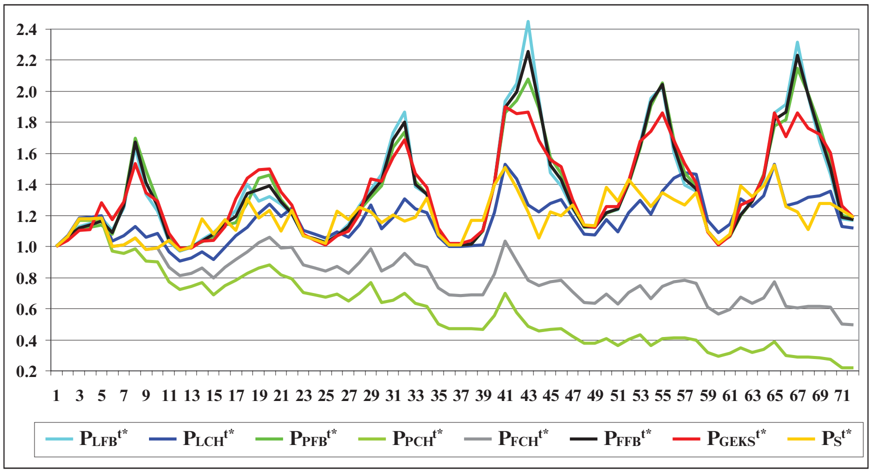
t	P_{LFB}^{t*}	P_{LCH}^{t*}	P_{PFB}^{t*}	P_{PCH}^{t*}	P_{FCH}^{t*}	P_{FFB}^{t*}	P_{GEKS}^{t*}	P_S^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.07104	1.07104	1.06104	1.06104	1.06603	1.06603	1.03802	1.06603
3	1.12812	1.18503	1.11303	1.16798	1.17647	1.12055	1.10386	1.17647
4	1.15044	1.19078	1.12373	1.16845	1.17956	1.13701	1.11167	1.17956
5	1.18406	1.19694	1.14104	1.16942	1.18310	1.16235	1.28331	1.18310
6	1.10502	1.03417	1.07887	0.97269	1.00296	1.09186	1.17550	1.00296
7	1.24566	1.06832	1.28386	0.95860	1.01198	1.26462	1.28536	1.01198
8	1.64472	1.13041	1.69981	0.98562	1.05554	1.67204	1.53539	1.05554
9	1.33555	1.05897	1.48835	0.90641	0.97973	1.40988	1.34806	0.97973
10	1.23076	1.08596	1.29420	0.90374	0.99067	1.26208	1.29133	0.99067
11	1.03294	0.96785	1.04925	0.77360	0.86529	1.04107	1.08720	1.04107
12	0.97081	0.90818	0.98105	0.72496	0.81141	0.97592	0.99061	0.97592
13	0.99746	0.92454	0.99881	0.74299	0.82881	0.99813	0.99017	0.99684
14	1.04362	0.96387	1.02824	0.76965	0.86130	1.03590	1.03346	1.17902
15	1.08902	0.91833	1.06632	0.69057	0.79635	1.07761	1.04121	1.08056
16	1.15867	0.99822	1.13743	0.75088	0.86576	1.14800	1.12905	1.17474
17	1.23204	1.07256	1.15330	0.78529	0.91776	1.19202	1.30850	1.10498
18	1.40150	1.12341	1.28409	0.82835	0.96466	1.34151	1.44174	1.30841
19	1.29310	1.21533	1.44276	0.86212	1.02360	1.36588	1.49357	1.18142
20	1.32299	1.27417	1.45909	0.88364	1.06109	1.38937	1.50199	1.23391
21	1.27093	1.19560	1.30112	0.81994	0.99011	1.28593	1.35277	1.09986
22	1.20769	1.24737	1.21638	0.79472	0.99564	1.21203	1.26983	1.23179
23	1.07109	1.10288	1.06924	0.70461	0.88153	1.07017	1.07993	1.06906
24	1.05248	1.07832	1.04479	0.68717	0.86081	1.04863	1.04214	1.04392
25	1.03276	1.05627	1.01894	0.67328	0.84331	1.02583	1.01037	1.02270
26	1.07388	1.09687	1.06790	0.69442	0.87275	1.07089	1.07080	1.22856

[45] The remainder of the real-time maximum overlap bilateral Fisher index links for the next 60 months are as follows: 13/12, 14/3, 15/2, 16/15, 17/5, 18/6, 19/8, 20/8, 21/9, 22/11, 23/12 and 24/23, 25/24, 26/14, 27/15, 28/4, 29/17, 30/18, 31/30, 32/19, 33/21, 34/11, 35/23, 36/23, 37/1, 38/26, 39/27, 40/28, 41/29, 42/30, 43/31, 44/32, 45/21, 46/10, 47/11, 48/25, 49/48, 50/38, 51/16, 52/51, 53/29, 54/30, 55/43, 56/20, 57/21, 58/35, 59/25, 60/59, 61/59, 62/50, 63/2, 64/40, 65/41, 66/54, 67/43, 68/44, 69/9, 70/46, 71/22, and 72/49.

[46] From Table 9.15, the carry-forward GEKS index ended up at 1.13682. Using maximum overlap bilateral Fisher indices, the resulting GEKS index ended up at 1.18952. Thus, the use of carry-forward prices led to a downward bias of 5.27 percentage points over the six-year sample period.

t	P_{LFB}^{t*}	P_{LCH}^{t*}	P_{PFB}^{t*}	P_{PCH}^{t*}	P_{FCH}^{t*}	P_{FFB}^{t*}	P_{GEKS}^{t*}	P_{S}^{t*}
27	1.14208	1.05809	1.12088	0.64996	0.82929	1.13143	1.09977	1.17215
28	1.26148	1.14177	1.23415	0.70168	0.89507	1.24774	1.20934	1.25327
29	1.36612	1.26826	1.31684	0.76739	0.98654	1.34125	1.43467	1.22223
30	1.46661	1.11351	1.39075	0.63881	0.84340	1.42818	1.42150	1.15449
31	1.73296	1.18814	1.64024	0.65250	0.88049	1.68596	1.57446	1.20526
32	1.86733	1.30512	1.73601	0.70065	0.95626	1.80047	1.68844	1.16278
33	1.38675	1.24237	1.41926	0.63598	0.88889	1.40291	1.47063	1.18929
34	1.33751	1.21976	1.33703	0.61354	0.86509	1.33727	1.38167	1.31066
35	1.08703	1.06732	1.07810	0.50379	0.73329	1.08256	1.11771	1.07810
36	1.02305	1.00314	1.01285	0.47308	0.68889	1.01793	1.01873	1.01195
37	1.01159	0.99864	1.00992	0.47009	0.68516	1.01076	1.01965	1.01076
38	1.02156	1.00610	1.01478	0.47200	0.68912	1.01816	1.03899	1.16812
39	1.10562	1.01261	1.10047	0.46755	0.68807	1.10305	1.10616	1.17108
40	1.40435	1.21657	1.40366	0.55437	0.82124	1.40401	1.40571	1.39663
41	1.93372	1.52949	1.86189	0.70095	1.03542	1.89746	1.90004	1.50841
42	2.04948	1.43492	1.93935	0.57426	0.90775	1.99365	1.85555	1.37756
43	2.44964	1.26995	2.08106	0.48573	0.78540	2.25784	1.86789	1.22151
44	1.94826	1.22130	1.88870	0.45816	0.74803	1.91825	1.68235	1.05506
45	1.47755	1.27915	1.57473	0.46934	0.77483	1.52537	1.55868	1.22173
46	1.39276	1.30456	1.46928	0.47089	0.78377	1.43051	1.51589	1.19999
47	1.24213	1.18988	1.26300	0.42194	0.70856	1.25252	1.30463	1.26828
48	1.12808	1.08051	1.12696	0.37820	0.63926	1.12752	1.13747	1.13921
49	1.12212	1.07637	1.12896	0.37668	0.63675	1.12554	1.12370	1.13475
50	1.21916	1.17596	1.21377	0.40957	0.69400	1.21646	1.25767	1.38339
51	1.23969	1.09662	1.24530	0.36458	0.63230	1.24249	1.25967	1.29063
52	1.42259	1.21667	1.39905	0.40512	0.70206	1.41077	1.42183	1.43303
53	1.67018	1.29789	1.63196	0.43211	0.74889	1.65096	1.67706	1.34386
54	1.95352	1.20862	1.90595	0.36369	0.66299	1.92959	1.74172	1.25757
55	2.03052	1.35442	2.05315	0.40619	0.74173	2.04180	1.85986	1.34547
56	1.60294	1.44271	1.68914	0.41245	0.77140	1.64547	1.68088	1.30412
57	1.39502	1.47625	1.47791	0.41505	0.78276	1.43587	1.53684	1.26875
58	1.35851	1.46359	1.38700	0.39760	0.76284	1.37268	1.41013	1.34737
59	1.09904	1.17136	1.09780	0.31761	0.60994	1.09842	1.09276	1.09738
60	1.02215	1.08744	1.02087	0.29512	0.56650	1.02151	1.01005	1.01922
61	1.07410	1.14641	1.06543	0.31177	0.59784	1.06976	1.06802	1.07767
62	1.20643	1.30964	1.19934	0.34893	0.67600	1.20288	1.26692	1.39115
63	1.29331	1.25722	1.29424	0.32078	0.63505	1.29377	1.30348	1.32072
64	1.43515	1.32893	1.46351	0.33813	0.67033	1.44926	1.45723	1.39001
65	1.86123	1.52850	1.77723	0.38896	0.77106	1.81874	1.86177	1.52597
66	1.91700	1.26434	1.81516	0.30051	0.61640	1.86539	1.70997	1.25740
67	2.31683	1.28283	2.14817	0.28713	0.60691	2.23091	1.86260	1.22459
68	1.96840	1.31863	1.98236	0.28708	0.61526	1.97537	1.76427	1.11160
69	1.67463	1.32951	1.77584	0.28527	0.61585	1.72450	1.72233	1.27951
70	1.46701	1.35499	1.54608	0.27515	0.61059	1.50602	1.60516	1.27885
71	1.18124	1.13072	1.20127	0.22115	0.50006	1.19121	1.26341	1.23088
72	1.17122	1.11995	1.17533	0.21988	0.49624	1.17327	1.18952	1.19115
Mean	1.35950	1.17720	1.35160	0.59613	0.81450	1.35520	1.34680	1.18920

Figure 9.8 Alternative Maximum Overlap Month-to-Month Price Indices



out that the seasonal fluctuations in the maximum overlap fixed-base Fisher indices and the GEKS indices are very much larger than the seasonal fluctuations in the predicted share similarity-linked indices P_S^{t*} as can be seen in Figure 9.8.

The chained Paasche and Fisher indices suffer from a massive amount of downward chain drift. The remaining six indices end up in much the same place. However, the seasonal peaks in four of the remaining indices (the fixed-base Laspeyres and Paasche indices, the fixed-base Fisher and the GEKS indices) are huge. The maximum overlap predicted share similarity-linked index P_S^{t*} has the best axiomatic properties (no chain drift and little or no substitution bias) and it has limited seasonal fluctuations for our empirical example so *it emerges as our best index*. From Figure 9.8, it can be seen that the chained Maximum Overlap Laspeyres index P_{LCH}^{t*} turns out to be fairly close to our similarity-linked indices and thus for this empirical example, it provides an adequate approximation to our preferred indices. For our example, the downward chain drift bias in P_{LCH}^{t*} just nicely counterbalances the upward substitution bias that is inherent in the Laspeyres formula.

Figure 9.8 also reveals another interesting property of our empirical example. For the months of December, January, and February, the three superlative indices, P_{FFB}^{t*}, $P_{GEKS}^{t,*}$ and P_S^{t*}, and the two fixed-base Laspeyres and Paasche indices, P_{LFB}^{t*} and P_{PFB}^{t*}, all exhibit similar values. Thus these five indices do capture the overall *trend* in the prices of the seasonal products in our example.

The similarity-linked indices P_S^{t*} perform the best in terms of reducing the size of the month-to-month seasonal fluctuations and they also have the best axiomatic properties in terms of being free from chain drift. However, a weakness associated with the use of these indices is that our real-time linking methodology means that the seasonal pattern in these indices for the first year will basically determine the pattern of seasonality for the entire sample. This weakness can be overcome by using the first year or the first two years of data as "training data" for the linking methodology. Instead of using real-time linking for say the first two years of data, use Robert Hill's symmetric method for linking the months in the first two years.[47] Then starting at month 1 of year 3, real-time similarity linking of the current month with a prior month could be used.

In the following two sections, we turn our attention to indices that are based only on price information for strongly seasonal products. Section 8 looks at alternative price indices that use the month-to-month carry-forward prices that were used in Section 6 while Section 9 constructs month-to-month maximum overlap price indices using only price data.

8. Month-to-Month Unweighted Price Indices Using Carry-Forward Prices

For many categories of consumer spending, statistical agencies will not have access to price and quantity (or expenditure) data pertaining to the category under consideration: only information on prices will be available. In this section, we will assume that carry-forward prices are used as estimates for missing prices and in the subsequent section, we will consider price indices that do not use carry-forward prices.

For our empirical example, we will use the monthly price data that are listed in Table A.23 in the annex. The price data in that table include month-to-month carry-forward/carry-backward prices.

As usual, define the period t price for product n as $p_{t,n}$ for $t = 1,\ldots,T$ and $n = 1,\ldots,N$. Define the month t price vector as $p^t \equiv [p_{t,1},p_{t,2},\ldots,p_{t,n}]$ for $t = 1,\ldots,T$.

Price indices for a category of products that depend only on prices are called *elementary price indices*. The three most commonly used elementary indices that measure the price level of month t relative to month r are the *Dutot* (1738), *Carli* (1764), and *Jevons* (1865) *indices* defined by (107)–(109):

$$P_D(t/r) \equiv (1/N)\Sigma_{n=1}^{N} p_{t,n}/(1/N)\Sigma_{n=1}^{N} p_{r,n}; \quad r = 1,\ldots,T;\ t = 1,\ldots,T; \qquad (107)$$

[47] See Hill (1999a) (1999b) (2001) (2004).

$$P_C(t/r) \equiv (1/N)\Sigma_{n=1}^{N} p_{t,n}/p_{r,n};\ r = 1,\ldots,T;\ t = 1,\ldots,T; \quad (108)$$

$$P_J(t/r) \equiv (\prod_{n=1}^{N} p_{t,n})^{1/N}/(\prod_{n=1}^{N} p_{r,n})^{1/N}\ r = 1,\ldots,T;\ t = 1,\ldots,T; = (\prod_{n=1}^{N} p_{t,n}/p_{r,n})^{1/N}. \quad (109)$$

Thus, the Dutot bilateral price index between the prices of month t relative to the prices of month r is equal to the arithmetic mean of the month t prices divided by the arithmetic mean of the month r prices; the Carli bilateral price index is equal to the arithmetic mean of the month t relative to month r price ratios $p_{t,n}/p_{r,n}$ and the Jevons bilateral index is equal to the geometric mean of the month t prices divided by the geometric mean of the month r prices, which in turn is equal to the geometric mean of the month t relative to month r price ratios $p_{t,n}/p_{r,n}$.

The sequence of T *fixed-base Dutot indices* using carry-forward prices, $P_D^{\,t}$, is $P_D(1/1)$, $P_D(2/1)$, . . ., $P_D(T/1)$. The sequence of T *fixed-base Carli indices* using carry-forward prices, $P_{CFB}^{\,t}$, is $P_C(1/1)$, $P_C(2/1)$, . . ., $P_C(T/1)$, and the sequence of T *fixed-base Jevons indices* using carry-forward prices, $P_J^{\,t}$, is $P_J(1/1)$, $P_J(2/1)$, . . ., $P_J(T/1)$. We use the data listed in Table A.23 in the annex to calculate these indices for our Israeli data set. These indices are listed in Table 9.18.

The *month-to-month chained Carli index using carry-forward prices* for month 1 is defined as unity:

$$P_{CCH}^{\,1} \equiv 1. \quad (110)$$

For months following month 1, the *chained Carli indices* are calculated by cumulating the corresponding successive month-to-month links using definition (108); that is, we have the following definition for $P_{CCH}^{\,t}$:

$$P_{CCH}^{\,t} \equiv P_{CCH}^{\,t-1} P_C(t/(t\text{-}1));\ t = 2,3,\ldots,T. \quad (111)$$

It is easy to show that the chained Dutot and Jevons indices are equal to their fixed-base counterpart indices when there are no missing prices, as is the case in this section. This explains why we labeled the fixed-base Dutot and Jevons indices for month t as $P_D^{\,t}$ and $P_J^{\,t}$ instead of $P_{DFB}^{\,t}$ and $P_{JFB}^{\,t}$ or $P_{DCH}^{\,t}$ and $P_{JCH}^{\,t}$: When there are no missing prices, $P_{DFB}^{\,t} = P_{DCH}^{\,t} \equiv P_D^{\,t}$ and $P_{JFB}^{\,t} = P_{JCH}^{\,t} \equiv P_J^{\,t}$. The chained Carli indices $P_{CCH}^{\,t}$ are also listed in Table 9.18.

The problem with the chained Carli indices is that they do not satisfy the time reversal test; that is, we have the following inequality:

$$P_C(2/1)P_C(1/2) \geq 1. \quad (112)$$

The inequality in (112) will be strict unless the prices in month 1 are proportional to the prices in month 2. Thus, the Carli index is subject to some upward bias whenever the base period is changed.

The problem with the Dutot index is that it is not invariant to changes in the units of measurement. This makes use of the Dutot index problematic.[48]

[48] One might try to eliminate the problem of a lack of invariance of the Dutot index to changes in the units of measurement by using normalized

Table 9.18 The Jevons, Dutot, Fixed-Base, and Chained Carli Indices Using Carry-Forward Prices, the Maximum Overlap GEKS Index, and the Maximum Overlap Similarity-Linked Index

t	$P_J^{\,t}$	$P_D^{\,t}$	$P_{CFB}^{\,t}$	$P_{CCH}^{\,t}$	$P_{GEKS}^{\,t*}$	$P_S^{\,t*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.01888	1.01878	1.02003	1.02003	1.03802	1.06603
3	1.05599	1.06351	1.05935	1.05882	1.10386	1.17647
4	1.06514	1.06928	1.06866	1.06833	1.11167	1.17956
5	1.07792	1.07601	1.08327	1.08376	1.28331	1.18310
6	1.02557	1.03396	1.04024	1.03776	1.17550	1.00296
7	1.04735	1.05805	1.07226	1.06616	1.28536	1.01198
8	1.10182	1.11760	1.16524	1.13348	1.53539	1.05554
9	1.05621	1.05865	1.09245	1.09429	1.34806	0.97973
10	1.06156	1.06464	1.09046	1.10334	1.29133	0.99067
11	0.97955	0.98459	0.99489	1.02529	1.08720	1.04107
12	0.94967	0.96230	0.96136	0.99468	0.99061	0.97592
13	0.95821	0.97314	0.96927	1.00444	0.99017	0.99684
14	0.99804	1.04025	1.01816	1.05076	1.03346	1.17902
15	0.97844	0.98930	0.99075	1.03619	1.04121	1.08056
16	1.02347	1.03179	1.04208	1.08578	1.12905	1.17474
17	1.07370	1.08282	1.09925	1.14344	1.30850	1.10498
18	1.22714	1.35453	1.29257	1.34373	1.44174	1.30841
19	1.28602	1.46450	1.38169	1.41759	1.49357	1.18142
20	1.29322	1.46203	1.38325	1.42715	1.50199	1.23391
21	1.22418	1.38662	1.29066	1.36227	1.35277	1.09986
22	1.25629	1.42261	1.31927	1.40476	1.26983	1.23179
23	1.18656	1.36777	1.25147	1.32963	1.07993	1.06906
24	1.17082	1.36179	1.23601	1.31297	1.04214	1.04392
25	1.16260	1.36044	1.22894	1.30423	1.01037	1.02270
26	1.20522	1.43031	1.27490	1.36157	1.07080	1.22856
27	1.20074	1.39904	1.26447	1.36227	1.09977	1.17215
28	1.24473	1.43129	1.30773	1.41426	1.20934	1.25327
29	1.28659	1.39762	1.32394	1.47150	1.43467	1.22223
30	1.26504	1.38902	1.32241	1.46683	1.42150	1.15449
31	1.29298	1.44004	1.37273	1.50979	1.57446	1.20526
32	1.29515	1.36149	1.36210	1.53599	1.68844	1.16278
33	1.22631	1.28810	1.26650	1.46550	1.47063	1.18929
34	1.19160	1.21688	1.20932	1.43452	1.38167	1.31066
35	1.11357	1.15022	1.12376	1.35288	1.11771	1.07810
36	1.08118	1.12598	1.09222	1.31427	1.01873	1.01195
37	1.07156	1.12269	1.08250	1.30328	1.01965	1.01076
38	1.10322	1.17581	1.11907	1.34595	1.03899	1.16812
39	1.11574	1.16616	1.12620	1.36516	1.10616	1.17108
40	1.20195	1.23057	1.21958	1.48180	1.40571	1.39663
41	1.42570	1.63141	1.51539	1.83477	1.90004	1.50841
42	1.34350	1.42732	1.40515	1.76198	1.85555	1.37756
43	1.29980	1.39500	1.39812	1.72162	1.86789	1.22151
44	1.25314	1.34720	1.33798	1.66456	1.68235	1.05506

(*Continued*)

Table 9.18 (Continued)

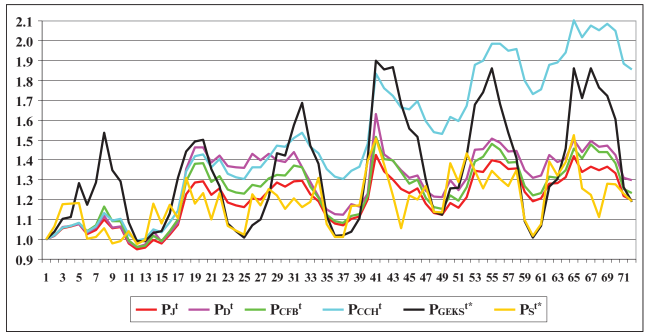
t	P_J^t	P_D^t	P_{CFB}^t	P_{CCH}^t	P_{GEKS}^{t*}	P_S^{t*}
45	1.23276	1.31106	1.28143	1.65494	1.55868	1.22173
46	1.25742	1.32236	1.30070	1.69701	1.51589	1.19999
47	1.17922	1.24531	1.21036	1.59692	1.30463	1.26828
48	1.13279	1.21433	1.16173	1.53877	1.13747	1.13921
49	1.12548	1.21284	1.15338	1.52977	1.12370	1.13475
50	1.18213	1.29924	1.22050	1.61642	1.25767	1.38339
51	1.16087	1.24792	1.19460	1.59629	1.25967	1.29063
52	1.21285	1.30717	1.26221	1.67242	1.42183	1.43303
53	1.34489	1.45066	1.39368	1.87848	1.67706	1.34386
54	1.33983	1.45410	1.41573	1.89868	1.74172	1.25757
55	1.39688	1.50685	1.48163	1.98599	1.85986	1.34547
56	1.38893	1.48612	1.45195	1.98514	1.68088	1.30412
57	1.35399	1.44161	1.38760	1.95032	1.53684	1.26875
58	1.35824	1.44685	1.38897	1.95951	1.41013	1.34737
59	1.24006	1.34832	1.26886	1.79840	1.09276	1.09738
60	1.19193	1.30860	1.22095	1.73033	1.01005	1.01922
61	1.20749	1.32199	1.23448	1.75437	1.06802	1.07767
62	1.28133	1.42560	1.31699	1.87821	1.26692	1.39115
63	1.28275	1.38939	1.31091	1.89150	1.30348	1.32072
64	1.31298	1.40690	1.34676	1.94097	1.45723	1.39001
65	1.41872	1.50138	1.45670	2.10304	1.86177	1.52597
66	1.33997	1.43862	1.40790	2.01611	1.70997	1.25740
67	1.36710	1.49540	1.47694	2.07715	1.86260	1.22459
68	1.34716	1.46562	1.44058	2.05237	1.76427	1.11160
69	1.36463	1.47183	1.44090	2.08666	1.72233	1.27951
70	1.33366	1.42500	1.38455	2.04992	1.60516	1.27885
71	1.21783	1.31114	1.25511	1.88353	1.26341	1.23088
72	1.19735	1.29857	1.23319	1.85468	1.18952	1.19115
Mean	1.19810	1.28450	1.24130	1.51050	1.34680	1.18920

We cannot apply the economic approach to index number theory in the present context since the economic approach depends on the availability of price and quantity (or expenditure) data.

From the perspective of the test or axiomatic approach to index number theory when only price data are available, the Jevons index seems to be the best choice since it satisfies the most "reasonable" tests.[49]

Table 9.18 lists the Jevons, Dutot, fixed-base, and chained Carli indices using carry-forward prices along with our two multilateral indices from the previous section that used bilateral maximum overlap Fisher indices as their basic building blocks, the GEKS, and predicted share similarity-linked indices, P_{GEKS}^{t*} and P_S^{t*}.

Figure 9.9 Carry-Forward Jevons, Dutot, and Carli Indices and Maximum Overlap GEKS and Similarity-Linked Indices



Our "best" index from the previous section, the maximum overlap predicted share index, P_S^{t*}, finished up at 1.19115, which is close to where the maximum overlap GEKS index, P_{GEKS}^{t*}, finished at 1.18952. We preferred P_S^{t*} over P_{GEKS}^{t*} because the similarity-linked index had better axiomatic properties and the seasonal fluctuations in P_{GEKS}^{t*} were very large. The carry-forward Jevons index P_J^t performed pretty well compared to P_S^{t*}: P_J^t ended up at 1.19735 (compared to 1.19115 for P_S^{72*}) and the mean of P_J^t was 1.1981 compared to the mean of P_S^{t*}, which was 1.1892. The next best-performing unweighted index is the fixed-base Carli index which finished up at 1.23319 (mean was 1.2413), which is 4.2 percentage points above P_S^{72*} = 1.19115. The Dutot index ended up at 1.29857, which is 10.7 percentage points above P_S^{72*}. Finally, the chained Carli index, P_{CCH}^t, exhibited tremendous upward chain drift, ending up at 1.85468, which is 66.4 percentage points above P_S^{72*}. Figure 9.9 shows a plot of these indices.

It can be seen from Figure 9.9 that the Jevons index P_J^t approximates our "best" index P_S^{t*} fairly well; the two indices end up in much the same place with P_S^{t*}, and the indices are always close to each other for the months of December, January, and February. For mid-year months, P_S^{t*} is generally below P_J^t. The Carli fixed-base and Dutot indices are in general close to each other and tend to lie above their Jevons index counterparts. The seasonal fluctuations in the GEKS and chained Carli indices are very large indeed. Finally, the substantial upward chain drift in the chained Carli index is evident by looking at Figure 9.9.

The Jevons index that is listed in Table 9.18 uses carry-forward prices. In previous sections, we have seen that the use of carry-forward prices leads to a downward bias for our empirical example as compared to indices that do not use carry-forward prices. In the following section, we will compute additional elementary indices that do not use quantity or expenditure weights, but instead of using carry-forward prices, we will use maximum overlap unweighted bilateral indices.

prices; that is, prices divided by the price of each product at the beginning of the sample period. In this case, the normalized fixed-base Dutot index of prices in period t relative to prices in period 1 becomes $P_{DN}(t/1) \equiv (1/N)\Sigma_{n=1}^N (p_{tn}/p_{1n})/(1/N)\Sigma_{n=1}^N (p_{1n}) = (1/N)\Sigma_{n=1}^N (p_{tn}/p_{1n}) = P_{CFB}(t/1)$. Thus, the normalized fixed-base Dutot index becomes the fixed-base Carli index.

[49] For materials on the test approach to bilateral index number theory when only price information is available, see Eichhorn (1978; 152–160), Dalén (1992), and Diewert (1995; 5–17) (2021a).

9. Month-to-Month Unweighted Price Indices Using Maximum Overlap Bilateral Indices

In this section, for our empirical example, we again use the monthly price data that are listed in Table A.23 in the annex. However, the carry-forward/carry-backward prices that are listed in italics in Table A.23 are set equal to 0 in this section.

The new period t price for product n (that is equal to 0 if the product is not available) is defined as $p_{t,n}$ for $t = 1,. . .,T$ and $n = 1,. . .,N$. The month t price vector is defined as $p^t \equiv [p_{t,1},p_{t,2},. . .,p_{t,n}]$ for $t = 1,. . .,T$. As usual, the set of prices n of products that are purchased in month t is defined as $S(t)$ for $t = 1,. . .,T$. The number of products that are purchased in month t is $N(t) \le N$. The set of products that are purchased in both months r and t is the intersection set $S(r)\cap S(t)$ and the number of matched products that are purchased in both months r and t is $N(r,t)$.

The bilateral *maximum overlap Jevons, Dutot, and Carli indices* that measure the level of prices in month t relative to the prices in month r, $P_J^*(t/r)$, $P_D^*(t/r)$, and $P_C^*(t/r)$ are defined as follows:

$$P_J^*(t/r) \equiv [\prod_{n\in S(r)\cap S(t)} (p_{t,n}/p_{r,n})]^{1/N(r,t)};\quad r = 1,. . .,T;\ t = 1,. . .,T; \tag{113}$$

$$P_D^*(t/r) \equiv \Sigma_{n\in S(r)\cap S(t)} (p_{t,n}/N(r,t))/\Sigma_{n\in S(r)\cap S(t)} (p_{r,n}/N(r,t)) = \Sigma_{n\in S(r)\cap S(t)}\, p_{t,n}/S_{n\in S(r)\cap S(t)}\, p_{r,n};\quad r = 1,. . .,T;\ t = 1,. . .,T; \tag{114}$$

$$P_C^*(t/r) \equiv [1/N(r,t)]\, \Sigma_{n\in S(r)\cap S(t)} (p_{t,n}/p_{r,n});\quad r = 1,. . .,T;\ t = 1,. . .,T. \tag{115}$$

The maximum overlap Jevons index $P_J^*(t/r)$ is equal to the geometric mean of the price ratios $p_{t,n}/p_{r,n}$ of the products that are present in both months r and t. The maximum overlap Dutot index $P_D^*(t/r)$ is equal to the arithmetic mean of the month t prices $p_{t,n}$ divided by the arithmetic mean of the month r prices $p_{r,n}$ where both averages include only the products that are present in both months r and t. The maximum overlap Carli index $P_C^*(t/r)$ is equal to the arithmetic average of the price ratios $p_{t,n}/p_{r,n}$ of the products that are present in both months r and t.

The sequence of monthly maximum overlap *fixed-base Jevons indices*, P_{JFB}^{t*}, is $P_J^*(2/1)$, $P_J^*(2/1)$, . . ., $P_J^*(T/1)$. The sequence of maximum overlap monthly *fixed-base Dutot indices*, P_{DFB}^{t*}, is $P_D^*(1/1)$, $P_D^*(2/1)$, . . ., $P_D^*(T/1)$. Finally, the sequence of maximum overlap monthly *fixed-base Carli indices*, P_{CFB}^{t*}, is $P_C^*(1/1)$, $P_C^*(2/1)$, . . ., $P_C^*(T/1)$. We use the data listed in Table A.23 in the annex (with the carry-forward and carry-backward prices replaced by zeros) to calculate these indices for our Israeli data set. These indices are listed in Table 9.19.

The maximum overlap bilateral Jevons, Dutot, and Carli indices, $P_J^*(t/r)$, $P_D^*(t/r)$, and $P_C^*(t/r)$, defined by (113)–(115) are used to define the *maximum overlap Jevons, Dutot, and Carli chained indices*, P_{JCH}^{t*}, P_{DCH}^{t*}, and P_{CCH}^{t*}, as follows:

$$P_{JCH}^{1*} \equiv 1;\ P_{DCH}^{1*} \equiv 1;\ P_{CCH}^{1*} \equiv 1. \tag{116}$$

$$P_{JCH}^{t*} \equiv P_{JCH}^{t-1*} P_J^*(t/[t-1]);\ t = 2,3,. . .,T; \tag{117}$$

$$P_{DCH}^{t*} \equiv P_{DCH}^{t-1*} P_D^*(t/[t-1]);\ t = 2,3,. . .,T; \tag{118}$$

$$P_{CCH}^{t*} \equiv P_{CCH}^{t-1*} P_C^*(t/[t-1]);\ t = 2,3,. . .,T. \tag{119}$$

The maximum overlap Jevons and Dutot indices are not necessarily equal to the corresponding fixed-base Jevons and Dutot indices as was the case in the previous section

Table 9.19 The Jevons, Dutot, Fixed-Base, and Chained Carli Indices Using Carry-Forward Prices and the Maximum Overlap Fixed-Base and Chained Jevons, Dutot, and Carli Indices

t	P_J^t	P_D^t	P_{CFB}^t	P_{CCH}^t	P_{JFB}^{t*}	P_{JCH}^{t*}	P_{DFB}^{t*}	P_{DCH}^{t*}	P_{CFB}^{t*}	P_{CCH}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.01888	1.01878	1.02003	1.02003	1.0381	1.0381	1.0484	1.0484	1.0401	1.0401
3	1.05599	1.06351	1.05935	1.05882	1.0816	1.1052	1.0944	1.1386	1.0847	1.1093
4	1.06514	1.06928	1.06866	1.06833	1.0986	1.1244	1.1131	1.1522	1.1024	1.1292
5	1.07792	1.07601	1.08327	1.08376	1.1350	1.1561	1.1655	1.1699	1.1431	1.1673
6	1.02557	1.03396	1.04024	1.03776	1.1352	1.0597	1.1666	1.0766	1.1458	1.0806
7	1.04735	1.05805	1.07226	1.06616	1.3194	1.1052	1.3589	1.1298	1.3359	1.1397
8	1.10182	1.11760	1.16524	1.13348	1.7883	1.2231	1.8620	1.2828	1.8289	1.2836
9	1.05621	1.05865	1.09245	1.09429	1.4071	1.1117	1.4128	1.1261	1.4194	1.1840
10	1.06156	1.06464	1.09046	1.10334	1.2798	1.0769	1.2758	1.0937	1.2862	1.1509
11	0.97955	0.98459	0.99489	1.02529	1.0719	0.9503	1.0493	0.9548	1.0811	1.0242
12	0.94967	0.96230	0.96136	0.99468	1.0075	0.8932	0.9919	0.9025	1.0141	0.9631
13	0.95821	0.97314	0.96927	1.00444	1.0257	0.9094	1.0199	0.9280	1.0299	0.9819
14	0.99804	1.04025	1.01816	1.05076	1.0609	0.9406	1.0653	0.9693	1.0651	1.0169

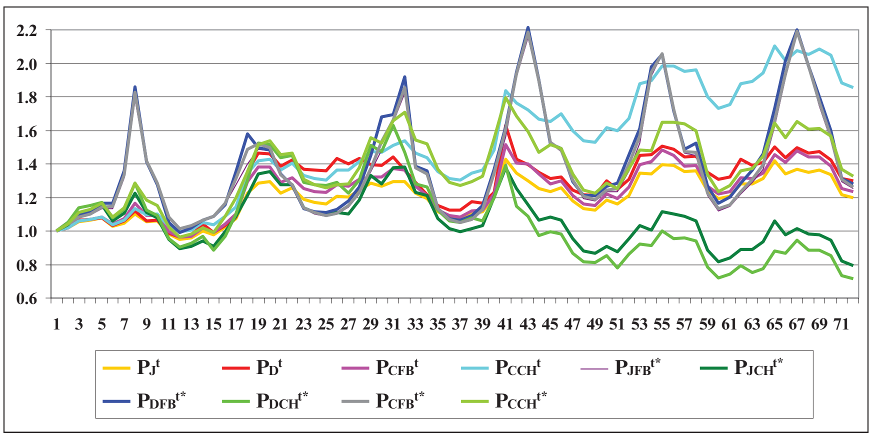
(Continued)

Table 9.19 (Continued)

t	P_J^t	P_D^t	P_{CFB}^t	P_{CCH}^t	P_{JFB}^{t*}	P_{JCH}^{t*}	P_{DFB}^{t*}	P_{DCH}^{t*}	P_{CFB}^{t*}	P_{CCH}^{t*}
15	0.97844	0.98930	0.99075	1.03619	1.0826	0.9085	1.0879	0.8853	1.0859	0.9922
16	1.02347	1.03179	1.04208	1.08578	1.1603	0.9941	1.1579	0.9690	1.1662	1.0872
17	1.07370	1.08282	1.09925	1.14344	1.2780	1.0851	1.3548	1.1026	1.3106	1.1959
18	1.22714	1.35453	1.29257	1.34373	1.4017	1.2113	1.5774	1.3043	1.4876	1.3538
19	1.28602	1.46450	1.38169	1.41759	1.4860	1.3386	1.4979	1.5260	1.5202	1.5143
20	1.29322	1.46203	1.38325	1.42715	1.4774	1.3561	1.4807	1.5167	1.4979	1.5381
21	1.22418	1.38662	1.29066	1.36227	1.3293	1.2780	1.3405	1.4379	1.3356	1.4550
22	1.25629	1.42261	1.31927	1.40476	1.2698	1.2779	1.2723	1.4526	1.2748	1.4651
23	1.18656	1.36777	1.25147	1.32963	1.1328	1.1399	1.1311	1.2913	1.1392	1.3084
24	1.17082	1.36179	1.23601	1.31297	1.1029	1.1099	1.1156	1.2737	1.1083	1.2756
25	1.16260	1.36044	1.22894	1.30423	1.0875	1.0944	1.1122	1.2697	1.0942	1.2586
26	1.20522	1.43031	1.27490	1.36157	1.1018	1.1087	1.1285	1.2884	1.1057	1.2774
27	1.20074	1.39904	1.26447	1.36227	1.1448	1.1015	1.1806	1.2241	1.1500	1.2786
28	1.24473	1.43129	1.30773	1.41426	1.2219	1.1837	1.2630	1.3027	1.2287	1.3762
29	1.28659	1.39762	1.32394	1.47150	1.3529	1.3327	1.4509	1.5089	1.3740	1.5572
30	1.26504	1.38902	1.32241	1.46683	1.4822	1.2796	1.6819	1.4740	1.5345	1.5255
31	1.29298	1.44004	1.37273	1.50979	1.6476	1.3771	1.6923	1.6285	1.6602	1.6558
32	1.29515	1.36149	1.36210	1.53599	1.8228	1.3811	1.9192	1.4708	1.8547	1.7061
33	1.22631	1.28810	1.26650	1.46550	1.3730	1.2301	1.3871	1.2818	1.3770	1.5425
34	1.19160	1.21688	1.20932	1.43452	1.3381	1.2101	1.3565	1.2633	1.3402	1.5208
35	1.11357	1.15022	1.12376	1.35288	1.1296	1.0749	1.1349	1.1227	1.1336	1.3694
36	1.08118	1.12598	1.09222	1.31427	1.0649	1.0133	1.0725	1.0609	1.0705	1.2912
37	1.07156	1.12269	1.08250	1.30328	1.0460	0.9953	1.0640	1.0525	1.0511	1.2696
38	1.10322	1.17581	1.11907	1.34595	1.0657	1.0140	1.0912	1.0794	1.0703	1.2948
39	1.11574	1.16616	1.12620	1.36516	1.1240	1.0342	1.1538	1.0620	1.1276	1.3271
40	1.20195	1.23057	1.21958	1.48180	1.3363	1.2002	1.3684	1.1969	1.3459	1.5539
41	1.42570	1.63141	1.51539	1.83477	1.5631	1.3763	1.6218	1.3930	1.6052	1.7942
42	1.34350	1.42732	1.40515	1.76198	1.9200	1.2509	1.9490	1.1476	1.9374	1.6841
43	1.29980	1.39500	1.39812	1.72162	2.1559	1.1571	2.2132	1.0877	2.1874	1.5928
44	1.25314	1.34720	1.33798	1.66456	1.8955	1.0625	1.9020	0.9740	1.8969	1.4696
45	1.23276	1.31106	1.28143	1.65494	1.5229	1.0827	1.5208	0.9938	1.5291	1.5144
46	1.25742	1.32236	1.30070	1.69701	1.4739	1.0656	1.4939	0.9823	1.4839	1.4932
47	1.17922	1.24531	1.21036	1.59692	1.2947	0.9523	1.2935	0.8687	1.3019	1.3391
48	1.13279	1.21433	1.16173	1.53877	1.1947	0.8788	1.2137	0.8152	1.2046	1.2416
49	1.12548	1.21284	1.15338	1.52977	1.1794	0.8675	1.2099	0.8126	1.1879	1.2271
50	1.18213	1.29924	1.22050	1.61642	1.2322	0.9063	1.2685	0.8519	1.2416	1.2848
51	1.16087	1.24792	1.19460	1.59629	1.2356	0.8740	1.2854	0.7784	1.2524	1.2528
52	1.21285	1.30717	1.26221	1.67242	1.3595	0.9541	1.4546	0.8633	1.4020	1.3723
53	1.34489	1.45066	1.39368	1.87848	1.5225	1.0301	1.6111	0.9237	1.5652	1.4824
54	1.33983	1.45410	1.41573	1.89868	1.9247	1.0039	1.9786	0.9129	1.9472	1.4795
55	1.39688	1.50685	1.48163	1.98599	2.0574	1.1150	2.0550	0.9991	2.0577	1.6493
56	1.38893	1.48612	1.45195	1.98514	1.7187	1.1002	1.7154	0.9544	1.7277	1.6476
57	1.35399	1.44161	1.38760	1.95032	1.4657	1.0867	1.4854	0.9566	1.4751	1.6370
58	1.35824	1.44685	1.38897	1.95951	1.4518	1.0590	1.5221	0.9389	1.4673	1.5992
59	1.24006	1.34832	1.26886	1.79840	1.2102	0.8828	1.2683	0.7824	1.2271	1.3362
60	1.19193	1.30860	1.22095	1.73033	1.1181	0.8156	1.1659	0.7192	1.1313	1.2351
61	1.20749	1.32199	1.23448	1.75437	1.1474	0.8370	1.2004	0.7405	1.1584	1.2694

t	$P_J^{\,t}$	$P_D^{\,t}$	$P_{CFB}^{\,t}$	$P_{CCH}^{\,t}$	$P_{JFB}^{\,t*}$	$P_{JCH}^{\,t*}$	$P_{DFB}^{\,t*}$	$P_{DCH}^{\,t*}$	$P_{CFB}^{\,t*}$	$P_{CCH}^{\,t*}$
62	1.28133	1.42560	1.31699	1.87821	1.2190	0.8892	1.2850	0.7927	1.2338	1.3574
63	1.28275	1.38939	1.31091	1.89150	1.2905	0.8909	1.3652	0.7523	1.3069	1.3742
64	1.31298	1.40690	1.34676	1.94097	1.3674	0.9334	1.4596	0.7753	1.3943	1.4460
65	1.41872	1.50138	1.45670	2.10304	1.6272	1.0597	1.7302	0.8827	1.6586	1.6431
66	1.33997	1.43862	1.40790	2.01611	1.9131	0.9761	2.0145	0.8679	1.9373	1.5561
67	1.36710	1.49540	1.47694	2.07715	2.1811	1.0150	2.2012	0.9464	2.1886	1.6537
68	1.34716	1.46562	1.44058	2.05237	1.9852	0.9808	1.9837	0.8861	1.9853	1.6077
69	1.36463	1.47183	1.44090	2.08666	1.7458	0.9781	1.8014	0.8853	1.7684	1.6100
70	1.33366	1.42500	1.38455	2.04992	1.5445	0.9430	1.5974	0.8549	1.5606	1.5638
71	1.21783	1.31114	1.25511	1.88353	1.2941	0.8187	1.3124	0.7349	1.3074	1.3663
72	1.19735	1.29857	1.23319	1.85468	1.2509	0.7914	1.2800	0.7168	1.2636	1.3245
Mean	1.19810	1.28450	1.24130	1.51050	1.3690	1.0647	1.4049	1.0640	1.3835	1.3662

Figure 9.10 Jevons, Dutot, and Carli Carry-Forward and Maximum Overlap Fixed-Base and Chained Indices



when carry-forward prices were used as imputations for the missing prices. Thus, in general, $P_{JCH}^{\,t*} \neq P_{JFB}^{\,t*}$ and $P_{DCH}^{\,t*} \neq P_{DFB}^{\,t*}$. The six elementary indices using bilateral maximum overlap price indices as basic building blocks, $P_{JFB}^{\,t*}$, $P_{JCH}^{\,t*}$, $P_{DFB}^{\,t*}$, $P_{DCH}^{\,t*}$, $P_{CFB}^{\,t*}$, and $P_{CCH}^{\,t*}$, are listed in Table 9.19 along with the four elementary indices that used carry-forward prices from the previous section, $P_J^{\,t}$, $P_D^{\,t}$, $P_{CFB}^{\,t}$, and $P_{CCH}^{\,t}$ for comparison purposes.

The 10 indices listed in Table 9.19 are plotted in Figure 9.10.

The four chained indices all seem to suffer from some form of chain drift: The maximum overlap chained Carli $P_{CCH}^{\,t*}$ ends up high at 1.3245, while the carry-forward chained Carli index $P_{CCH}^{\,t}$ ends up much too high at 1.855. The chained maximum overlap Jevons and Dutot indices, $P_{JCH}^{\,t*}$ and $P_{DCH}^{\,t*}$, suffer from severe downward chain drift and end up at 0.7914 and 0.7168, respectively. The carry-forward Dutot index $P_D^{\,t}$ ended up at 1.2986, and its maximum overlap fixed-base counterpart $P_{DFB}^{\,t*}$ ended up at 1.2800. Our "best" index using price and expenditure information was the maximum overlap similarity-linked index $P_S^{\,t*}$, which ended up at 1.1911. Thus, the Dutot indices $P_D^{\,t}$ and $P_{DFB}^{\,t*}$ have a considerable amount of upward bias relative to our preferred index. In general, the fixed-base maximum overlap Jevons, Dutot, and Carli indices, $P_{JFB}^{\,t*}$, $P_{DFB}^{\,t*}$, and $P_{CFB}^{\,t*}$, are fairly close to each other, but they end up at 1.2509, 1.2800, and 1.2636, respectively, which is well above where the similarity-linked maximum overlap index ended (1.19115). Also, $P_{JFB}^{\,t*}$, $P_{DFB}^{\,t*}$, and $P_{CFB}^{\,t*}$ have large seasonal fluctuations relative to $P_S^{\,t*}$. These three maximum overlap fixed-base indices cannot be readily distinguished from each other in Figure 9.10. The index that provides the closest approximation to $P_S^{\,t*}$ is the Jevons index $P_J^{\,t}$, which uses carry-forward prices.

However, as we have seen in previous sections, the use of carry-forward prices can lead to significant bias as compared to the same index, which uses maximum overlap indices. From Table 9.19, the mean of the fixed-base Jevons indices using carry-forward prices (the $P_J^{\,t}$) is 1.1981, while the mean of the fixed-base maximum overlap indices $P_{JFB}^{\,t*}$ is 1.3690. Thus on average, the downward bias in the use of the carry-forward indices using the Jevons formula is 1.3690 – 1.1981 or 17.09 percentage points. Similarly, the downward bias in the use of carry-forward prices using fixed-base Dutot indices is 1.4049 – 1.2845 or 12.04 percentage points, and the downward bias in the use of carry-forward prices using fixed-base Carli indices is 1.3835 – 1.2413 or 14.22 percentage points. *Thus, the use of carry-forward*

Table 9.20 Year-over-Year Monthly Maximum Overlap Fixed-Base Jevons Indices

y	P_{JF1}^{y*}	P_{JF2}^{y*}	P_{JF3}^{y*}	P_{JF4}^{y*}	P_{JF5}^{y*}	P_{JF6}^{y*}	P_{JF7}^{y*}	P_{JF8}^{y*}	P_{JF9}^{y*}	P_{JF10}^{y*}	P_{JF11}^{y*}	P_{JF12}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0257	1.0761	0.9763	1.0341	1.0051	1.3229	1.3156	1.1127	1.0596	1.0734	1.0568	1.0947
3	1.0875	1.1228	1.0478	1.0916	1.1310	1.2719	1.2127	1.1121	1.1080	1.1008	1.0698	1.0570
4	1.0460	1.0815	1.0361	1.1838	1.2999	1.4799	1.2876	1.0803	1.1763	1.1953	1.2225	1.1859
5	1.1794	1.2487	1.1236	1.2055	1.2587	1.4438	1.3801	1.2180	1.1986	1.1544	1.1290	1.1097
6	1.1474	1.2449	1.1716	1.1927	1.3593	1.4416	1.3280	1.1518	1.2860	1.2645	1.2320	1.2416

Table 9.21 Year-over-Year Monthly Maximum Overlap Chained Jevons Indices

y	P_{JC1}^{y*}	P_{JC2}^{y*}	P_{JC3}^{y*}	P_{JC4}^{y*}	P_{JC5}^{y*}	P_{JC6}^{y*}	P_{JC7}^{y*}	P_{JC8}^{y*}	P_{JC9}^{y*}	P_{JC10}^{y*}	P_{JC11}^{y*}	P_{JC12}^{y*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0257	1.0761	0.9763	1.0341	1.0051	1.3229	1.3156	1.1127	1.0596	1.0734	1.0568	1.0947
3	1.0875	1.1228	1.0478	1.0916	1.0909	1.2106	1.2166	1.1200	1.1080	1.1008	1.0539	1.0570
4	1.0460	1.0815	1.0361	1.1838	1.2919	1.3731	1.2876	1.0803	1.1763	1.1842	1.2079	1.1859
5	1.1794	1.2487	1.1218	1.2055	1.1940	1.3397	1.3801	1.2180	1.1986	1.1804	1.1290	1.1097
6	1.1474	1.2449	1.1563	1.1927	1.2893	1.3376	1.3280	1.1518	1.2860	1.2414	1.2073	1.2416

prices for elementary indices in situations where there is general inflation cannot be recommended due to the potentially large downward bias that the use of carry-forward prices can generate.

Instead of using maximum overlap bilateral Jevons, Dutot and Carli indices as basic inputs into fixed-base and chained indices of prices (without quantity or expenditure weights), it is possible to use multilateral methods to form elementary indices. We conclude this section by considering two such multilateral methods that just use price information for many periods: the time product dummy method and a similarity based linking method.

The *time product dummy method* assumes that the price of product n in month t, $p_{t,n}$, is approximately equal to the product of two factors: a time factor $\pi_t > 0$ that represents the *price level in month t* and a product factor, $\alpha_n > 0$, that represents the utility of product n relative to all products in scope. It is convenient to take logarithms of both sides of the approximate equations $p_{t,n} \approx \pi_t\alpha_n$ in order to obtain the approximate equations $lnp_{t,n} \approx ln\pi_t + ln\alpha_n = \rho_t + \beta_n$, where $\rho_t \equiv ln\pi_t$ and $\beta_n \equiv ln\alpha_n$. Estimates ρ_t^* and β_n^* for the parameters ρ_t and β_n can be obtained by solving the following least squares minimization problem:

$$min_{\rho,\,\beta}\ \{\Sigma_{t=1}^{T}\,\Sigma_{n\in S(t)}\,[lnp_{t,n} - \rho_t - \beta_n]^2\}, \qquad (120)$$

where we set $\rho_1 \equiv 0$ in order to prevent multicollinearity problems. Denote the solution to (120) by ρ_t^* for $t = 2,3,\ldots,T$ and β_n^* for $n = 1,\ldots,N$. Define $\rho_1^* \equiv 0$ and define $\pi_t^* \equiv \exp[\rho_t^*]$ for $t = 1,\ldots,T$. The *Time Product Dummy index* for month t, P_{TPD}^t, is defined to be π_t^*; that is, we have $P_{TPD}^t \equiv \pi_t^*$ for $t = 1,\ldots,T$.[50] If there are no missing observations so that all N products are present in all N periods, then the Time Product Dummy price indices are equal to the fixed-base (and chained) Jevons index $P_J(t/1) = P_J^t$.[51] Thus the Time Product Dummy index P_{TPD}^t is a natural generalization of the Jevons index to the case of missing observations. This standard Time Product Dummy index P_{TPD}^t is listed in Table 9.22 and plotted in Figure 9.11.

It is of interest to calculate year-over-year maximum overlap fixed-base and chained Jevons indices for each month. Denote the sequence of year-over-year fixed-base and chained maximum overlap Jevons indices for month m and year y as P_{JFm}^{y*} and P_{JCm}^{y*} respectively for $m = 1,\ldots,M$ and $y = 1,\ldots,Y$. These month-over-month Jevons indices are listed in Tables 9.20 and 9.21 for our empirical example.

It can be seen that, for the most part, the fixed-base Jevons indices in Table 9.20 approximated their chained counterparts in Table 9.21 fairly well. For the months m, where the list of available products is the same for all years, the fixed-base and chained maximum overlap indices for those months will be the same; that is, we have $P_{JFm}^{y*} = P_{JCm}^{y*}$ for $y = 1,\ldots,6$ for months m, where the available products are always the same year-over-year.

A possible disadvantage of using the Time Product Dummy indices P_{TPD}^t is that every month when there is a new observation, the indices have to be recomputed and there is the problem of linking the new index for the latest month with the prior indices. A possible solution to this problem is the following one. (i) Compute the Time Product Dummy indices for a historical data set that consists of 12 consecutive months. Call the resulting indices P_{TPD}^t for $m = 1,\ldots,12$. (ii) Set the *Mixed TPD and Jevons index*, P_{TPDJ}^t, for the first 12 months equal to the corresponding Time Product Dummy indices so that $P_{TPDJ}^t = P_{TPD}^t$ for $t = 1,\ldots,12$. (ii) For

[50] In the statistics literature, this model is known as the fixed effects model. In the economics literature, the method is due to Court (1939; 109–111) in the hedonic regression context and to Summers (1973) in the international comparison context where it is known as the country product dummy regression model. See Diewert (2021c) for more on the history of this multilateral method and its interpretation from the perspective of the economic approach to index number theory.

[51] This result is a special case of a more general result obtained by Triplett and McDonald (1977; 150). See also Diewert (2021c; 51).

subsequent months, use the year-over-year fixed-base maximum overlap Jevons indices P_{JFm}^{y*} to link month m in year $y \geq 2$ to P_{TPDJ}^{m}. Thus for our empirical example, for year $y = 2$, we have $P_{TPDJ}^{12+m} = P_{TPDJ}^{m} \times P_{JFm}^{2*}$ for $m = 1,\ldots,12$; for year $y = 3$, we have $P_{TPDJ}^{24+m} = P_{TPDJ}^{m} \times P_{JFm}^{3*}$ for $m = 1,\ldots,12$; . . . and for year $y = 6$, we have $P_{TPDJ}^{60+m} = P_{TPDJ}^{m} \times P_{JFm}^{6*}$ for $m = 1,\ldots,12$. The reason for using the fixed-base monthly maximum overlap year-over-year Jevons indices listed in Table 9.20 instead of the chained indices listed in Table 9.21 is that the resulting mixed TPD and Jevons indices, P_{TPDJ}^{t} satisfy Walsh's multiperiod identity test; that is, if prices in months r and t are the same, then $P_{TPDJ}^{r} = P_{TPDJ}^{t}$.[52] If an index satisfies this test, then it is free from chain drift.

An advantage of the mixed TPD and Jevons index is that it can be implemented in real time without revision or linking problems. However, a disadvantage of P_{TPDJ}^{t} is that the seasonal pattern of prices that occurred in the first year of "training" data will persist in subsequent periods. If there are changing seasonal patterns, then this property of the method may be problematic. It could be addressed by periodically changing the base year of training data and then starting a new set of indices. Furthermore, the seasonal pattern of prices could be subject to more or less random fluctuations. In order to address this randomness problem, the time period dummy method could be implemented using two or more years of training data rather than just using a single-year data. This could lead to a more representative set of seasonal factors. We implemented this modification using our empirical data set.

The final *blended TPD and Jevons index*, P_{TPDJ}^{t*}, is defined as follows: (i) Compute the TPD indices for a historical data set that consists of 24 consecutive months. Call the resulting indices P_{TPD}^{t*} for $m = 1,\ldots,24$. (ii) Set P_{TPDJ}^{t*} for the first 12 months equal to the corresponding TPD indices so that $P_{TPDJ}^{t*} = P_{TPD}^{t*}$ for $t = 1,\ldots,12$. (ii) For subsequent months, use the year-over-year fixed-base maximum overlap Jevons indices P_{JFm}^{y*} to link month m in year $y \geq 2$ to P_{TPDJ}^{m}. Thus repeating our earlier description, for our empirical example, for year $y = 2$, we have $P_{TPDJ}^{12+m*} = P_{TPDJ}^{m*} \times P_{JFm}^{2*}$ for $m = 1,\ldots,12$; for year $y = 3$, we have $P_{TPDJ}^{24+m*} = P_{TPDJ}^{m*} \times P_{JFm}^{3*}$ for $m = 1,\ldots,12$; . . . and for year $y = 6$, we have $P_{TPDJ}^{60+m*} = P_{TPDJ}^{m*} \times P_{JFm}^{6*}$ for $m = 1,\ldots,12$. The blended indices P_{TPDJ}^{t*} are listed in Table 9.22 and are plotted in Figure 9.11.

The final elementary index that we consider in this section is an adaptation of the predicted share multilateral index P_S^{t*} that was defined in the previous section. Since in the present section, we are considering price indices that depend solely on price information, in place of a maximum overlap bilateral Fisher index to link the prices of two months, the maximum overlap bilateral Jevons index $P_J^{*}(t/r)$ defined by (113) will be used to relate the prices of the current month to a previous month that has the lowest measure of relative price dissimilarity. In the previous section, the predicted share measure of relative price dissimilarity between the prices of two months was defined by (105). This definition depended on the availability of quantity (or expenditure) information, but in the present context, only price information is available. When quantity and expenditure information is not available, it is natural to assume that either quantities purchased in a month or expenditures on available products are equal. assumption of equal quantities depends on units of product measurement, which are to some extent arbitrary and so we will make the assumption of equal expenditures on available products in each month. This assumption is equivalent to an assumption that expenditure shares on available products in a month are equal.

Recall that the price of product n in month t is denoted by $p_{t,n}$ for $n = 1,\ldots,N$ and $t = 1,\ldots,T$, where $T = YM$, Y is the number of years in the sample and M is the number of months in a year. If product n in month t was not available (that is, not purchased by the households in scope), then $p_{t,n}$ is set equal to 0. The vector of month t prices is $p^t \equiv [p_{t,1},\ldots,p_{t,N}]$ for $t = 1,\ldots,T$. The set of available products in month t is $S(t)$ and the number of available products in month t is $N(t)$ for $t = 1,\ldots,T$. The set of products that are available in both months t and r is the intersection of the sets $S(t)$ and $S(r)$, denoted by $S(t) \cap S(r)$. The number of matched products that are available in both months t and r is $N(t,r)$. If there are no unmatched products in months t and r, then $N(t) = N(r) = N(t,r)$. We assume that there is at least one matched product between every pair of months in the sample.

The *imputed quantities*, $q_{t,n}$, that will generate equal expenditure shares for products n that are present in month t are defined as follows for $t = 1,\ldots,T$:

$$\begin{aligned} q_{t,n} &\equiv 1/p_{t,n}N(t) \text{ if } n \in S(t); \\ &\equiv 0 \text{ if } n \notin S(t). \end{aligned} \qquad (121)$$

The imputed expenditure share for product n in month t is $s_{t,n} \equiv p_{t,n}q_{t,n}/p^t \cdot q^t$ for $t = 1,\ldots,T$ and $n = 1,\ldots,N$. Using the $q_{t,n}$ defined by (121), these expenditure shares are equal to the following expressions for $t = 1,\ldots,T$:

$$s_{t,n} = 1/N(t) \text{ if } n \in S(t); = 0 \text{ if } n \notin S(t). \qquad (122)$$

To form month t predicted shares, use the prices of month r and the (imputed) quantities of month t to form the following *predicted shares*, $s_{t,r,n}$:

$$s_{t,r,n} \equiv p_{r,n}q_{t,n}/p^r \cdot q^t;\ t = 1,\ldots,T;\ r = 1,\ldots,T;\ n = 1,\ldots,N. \qquad (123)$$

If product n is not available in month t, so that $n \notin S(t)$, then $q_{t,n} = 0$ and the following equations hold:

$$s_{t,n} = s_{t,r,n} = 0;\ t = 1,\ldots,T;\ r = 1,\ldots,T;\ n \notin S(t). \qquad (124)$$

If product n is available in month t, so that $n \in S(t)$, then $q_{t,n} > 0$, $p_{t,n} > 0$, $s_{t,n} = 1/N(t)$ and the predicted shares $s_{t,r,n}$ satisfy the following equations:

$$\begin{aligned} s_{t,r,n} &\equiv p_{r,n}q_{t,n}/p^r \cdot q^t;\ t = 1,\ldots,T;\ r = 1,\ldots,T;\ n \in S(t) \\ &= [p_{r,n}/p_{t,n}N(t)]/\Sigma_{k \in S(t)} [p_{r,k}/p_{t,k}N(t)] \text{ using (121)} \\ &= [p_{r,n}/p_{t,n}]/\Sigma_{k \in S(t)} [p_{r,k}/p_{t,k}] \\ &= [p_{r,n}/p_{t,n}]/\Sigma_{k \in S(t) \cap S(r)} [p_{r,k}/p_{t,k}], \end{aligned} \qquad (125)$$

[52] See Walsh (1901; 389), (1921; 540).

where the last equality follows from the fact that $p_{r,k} = 0$ if k does not belong to $S(r)$; that is, if $k \notin S(r)$.

The *predicted share error* $e_{t,r,n}$ in using $s_{t,r,n}$ to predict $s_{t,n}$ is defined as follows:

$$e_{t,r,n} \equiv s_{t,n} - s_{t,r,n};\ t = 1,\ldots,T;\ r = 1,\ldots,T;\ n = 1,\ldots,N. \quad (126)$$

Using definitions (122) and (123), it is straightforward to show that the sum over products n of the prediction errors $e_{t,r,n}$ is equal to 0 for each pair of months, r and t; that is, we have:

$$\Sigma_{n=1}^{N} e_{t,r,n} = 0;\ t = 1,\ldots,T;\ r = 1,\ldots,T. \quad (127)$$

Note that if product n is not available in month t, then using (122) and (124), it can be seen that the predicted error $e_{t,r,n}$ will equal 0; that is, we have the following equalities:

$$e_{t,r,n} = 0;\ t = 1,\ldots,T;\ r = 1,\ldots,T;\ n \notin S(t). \quad (128)$$

Using (127) and (128), it can be seen that the mean of the predicted errors $e_{t,r,n}$ over all products that are available in month t is equal to zero; that is, we have:

$$[1/N(t)]\Sigma_{n\in S(t)}\, e_{t,r,n} = 0;\ t = 1,\ldots,T;\ r = 1,\ldots,T. \quad (129)$$

Using only price information, the *predicted share measure of relative price dissimilarity* between months t and r is $\Delta_{PS}(p^t,p^r)$ defined as follows:

$$\Delta_{PS}(p^t,p^r) \equiv \Sigma_{t,r} + \Sigma_{r,t};\ t = 1,\ldots,T;\ r = 1,\ldots,T, \quad (130)$$

where $\Sigma_{t,r}$ is defined as[53]

$$\begin{aligned}\Sigma_{t,r} &\equiv \Sigma_{n=1}^{N} e_{t,r,n}^{2};\ t = 1,\ldots,T;\ r = 1,\ldots,T \quad (131)\\ &= \Sigma_{n\in S(t)}\, e_{t,r,n}^{2} \text{ using (128)}\\ &= \Sigma_{n\in S(t)}\, \{s_{t,n} - s_{t,r,n}\}^2 \text{ using (126)}\\ &= \Sigma_{n\in S(t), n\notin S(r)}\, \{[1/N(t)] - 0\}^2 + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - x_{t,r,n}\}^2 \text{ using (122) and (125)}\\ &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - x_{t,r,n}\}^2.\end{aligned}$$

The $x_{t,r,n}$ are *normalized price ratios for matched products* present in periods t and r and are defined as follows:

$$x_{t,r,n} \equiv (p_{r,n}/p_{t,n})/\Sigma_{k\in S(t)\cap S(r)}\, (p_{r,k}/p_{t,k});\ t = 1,\ldots,T;\ r = 1,\ldots,T;\ n \in S(t)\cap S(r). \quad (132)$$

If prices in months t and r are equal (or proportional so that $p^t = \lambda p^r$ for some scalar $\lambda > 0$), then $N(t) = N(r) = N(t,r) = N(r,t)$ and $x_{t,r,n} = 1/N(t,r) = 1/N(t)$. In this case, $\Sigma_{t,r} = \Sigma_{r,t} = 0$ and thus $\Delta_{PS}(p^t,p^r) = 0$. In general, $\Delta_{PS}(p^t,p^r) \geq 0$, $\Delta_{PS}(p^t,p^r) = \Delta_{PS}(p^r,p^t)$ (symmetry property) and $\Delta_{PS}(p^t,p^r) \leq 2$.

Note that $[N(t) - N(t,r)]/N(t)^2 \geq 0$ is a *penalty term* that is positive if available products purchased in months t and r are not matched. If the list of available products in months t and r is identical, then $N(t) = N(r) = N(t,r) = N(r,t)$, and this penalty term is equal to 0. If products are matched in months t and r, then $S(t) = S(r) = S(t)\cap S(r)$ and the second set of terms in the last equality of (131) becomes $\Sigma_{n\in S(t)}\, \{[1/N(t)] - x_{t,r,n}\}^2$, which, using (129), is proportional to the *variance* of the normalized price ratios, $x_{t,r,n}$. Next, we will obtain a decomposition of this second set of terms in the case where products are not matched between months t and r.

In the general case where prices in month t are not necessarily proportional to prices in month r, we calculate the mean and variance of the $x_{t,r,n}$ over products n that are present in months t and r. It turns out that the sum of the $x_{t,r,n}$ over products n that are present in both months t and r is always equal to 1; that is, we have the following equalities using definitions (132):

$$\Sigma_{n\in S(t)\cap S(r)}\, x_{t,r,n} = \Sigma_{n\in S(t), n\notin S(r)}\, (p_{r,n}/p_{t,n})/\Sigma_{k\in S(t)\cap S(r)}\, (p_{r,k}/p_{t,k});\ t = 1,\ldots,T;\ r = 1,\ldots,T = 1. \quad (133)$$

The number of common products that are present in months t and r is $N(t,r)$. Thus the *mean* of $x_{t,r,n}$ over the common products n that are present in months t and r is $\mu_{t,r}$ defined as follows:

$$\begin{aligned}\mu_{t,r} &\equiv \Sigma_{n\in S(t)\cap S(r)}\, x_{t,r,n}/N(t,r);\ t = 1,\ldots,T;\ r = 1,\ldots,T \quad (134)\\ &= 1/N(t,r) \text{ using (133).}\end{aligned}$$

Note that (134) implies that

$$\Sigma_{n\in S(t)\cap S(r)}\, [x_{t,r,n} - \mu_{t,r}] = 0;\ t = 1,\ldots,T;\ r = 1,\ldots,T. \quad (135)$$

A *measure of relative price dissimilarity* between the common product prices of months t and r is $\delta_{t,r}$ defined as follows:

$$\delta_{t,r} \equiv \Sigma_{n\in S(t)\cap S(r)}\, [x_{t,r,n} - \mu_{t,r}]^2;\ t = 1,\ldots,T;\ r = 1,\ldots,T. \quad (136)$$

It can be seen that $\delta_{t,r}$ is proportional to the *variance* of the normalized price ratios $x_{t,r,n}$ over products that are present in both months t and r. If the prices of products that are present in both months t and r are identical or proportional to each other, it can be verified that $\delta_{t,r}$ is equal to 0.

Using equations (131), we have the following decomposition for $\Sigma_{t,r}$:

$$\begin{aligned}\Sigma_{t,r} &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - x_{t,r,n}\}^2;\ t = 1,\ldots,T;\ r = 1,\ldots,T \quad (137)\\ &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - \mu_{t,r} - [x_{t,r,n} - \mu_{t,r}]\}^2\\ &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - \mu_{t,r}\}^2 + \Sigma_{n\in S(t)\cap S(r)}\, \{x_{t,r,n} - \mu_{t,r}\}^2 \text{ using (135)}\\ &= \{[N(t) - N(t,r)]/N(t)^2\} + \Sigma_{n\in S(t)\cap S(r)}\, \{[1/N(t)] - [1/N(t,r)]\}^2 + \delta_{t,r} \text{ using (134) and (136)}\\ &= \{[N(t) - N(t,r)]/N(t)^2\} + N(t,r)\{[1/N(t)] - [1/N(t,r)]\}^2 + \delta_{t,r}\\ &= [1/N(t,r)] - [1/N(t)] + \delta_{t,r}.\end{aligned}$$

[53] $\Sigma_{r,t}$ are defined by (131) and (132) by simply interchanging t and r.

Table 9.22 Month-to-Month Modified Predicted Share Measures of Price Dissimilarity Using Zeros for Missing Prices

r, t	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0.0189	0.0195	0.0499	0.1517	0.1644	0.4155	0.4118	0.2229	0.0458	0.0366	0.0037
2	0.0189	0	0.0009	0.0186	0.1004	0.1852	0.4470	0.4488	0.2515	0.0676	0.0597	0.0263
3	0.0195	0.0009	0	0.0182	0.1026	0.1875	0.4532	0.4565	0.2537	0.0686	0.0610	0.0275
4	0.0499	0.0186	0.0182	0	0.0828	0.1666	0.4291	0.4312	0.2395	0.1019	0.0950	0.0612
5	0.1517	0.1004	0.1026	0.0828	0	0.0439	0.1845	0.2001	0.3021	0.1368	0.2001	0.1656
6	0.1644	0.1852	0.1875	0.1666	0.0439	0	0.0673	0.1246	0.1956	0.0989	0.1358	0.1716
7	0.4155	0.4470	0.4532	0.4291	0.1845	0.0673	0	0.0276	0.0739	0.1122	0.1723	0.4144
8	0.4118	0.4488	0.4565	0.4312	0.2001	0.1246	0.0276	0	0.0582	0.1033	0.1836	0.4012
9	0.2229	0.2515	0.2537	0.2395	0.3021	0.1956	0.0739	0.0582	0	0.0447	0.0871	0.2173
10	0.0458	0.0676	0.0686	0.1019	0.1368	0.0989	0.1122	0.1033	0.0447	0	0.0149	0.0451
11	0.0366	0.0597	0.0610	0.0950	0.2001	0.1358	0.1723	0.1836	0.0871	0.0149	0	0.0320
12	0.0037	0.0263	0.0275	0.0612	0.1656	0.1716	0.4144	0.4012	0.2173	0.0451	0.0320	0

The following decomoition for $\Sigma_{r,t}$ can be derived in an analogous fashion:

$$\Sigma_{r,t} = [1/N(r,t)] - [1/N(r)] + \delta_{r,t};\quad t = 1,\ldots,T;\ r = 1,\ldots,T. \tag{138}$$

Using (137), (138), and definitions (130), the *predicted share measure of relative price dissimilarity* between months t and r has the following decomposition:

$$\begin{aligned}\Delta_{PS}(p^t,p^r) &\equiv \Sigma_{t,r} + \Sigma_{r,t};\ t = 1,\ldots,T;\ r = 1,\ldots,T\\ &= [1/N(t,r)] - [1/N(t)] + \delta_{t,r} + [1/N(r,t)] - [1/N(r)] + \delta_{r,t}\\ &= [2/N(t,r)] - [1/N(t)] - [1/N(r)] + \delta_{t,r} + \delta_{r,t},\end{aligned} \tag{139}$$

where the last equality follows from the fact that $N(t,r) = N(r,t)$. It can be seen that the term $[2/N(t,r)] - [1/N(t)] - [1/N(r)] \geq 0$ is the *total penalty for a possible lack of matching of the overlap prices* in months t and r. If the products that are present in month t are also present in month r, then $N(t) = N(r) = N(t,r)$ and there is no penalty for a lack of matching. The term $d_{t,r}$ is proportional to the variance of the normalized matched relative prices $p_{r,n}/p_{t,n}$ and the term $\delta_{r,t}$ is proportional to the variance of the normalized matched reciprocal relative prices $p_{t,n}/p_{r,n}$. If prices in months r and t are equal or proportional, then the penalty for a lack of matching is 0 and the two matched relative price dissimilarity terms $\delta_{t,r}$ and $\delta_{r,t}$ are also equal to 0. It should be noted that the *actual unmatched prices* in periods t and *s do not play a role* in the measure of relative price dissimilarity between the prices of months t and r. However, the *number of unmatched prices* does play a role.

The problem of trading off a lack of matching of prices between two periods and the dispersion in the matched prices is a difficult one.[54] The modified predicted share methodology explained earlier does accomplish this tradeoff, but further research may find more direct methods for making this tradeoff. What is clear is that a method for linking bilateral indices needs to take into account the lack of matching of prices and the method should be based on some principles. The principle used in the aforementioned method was to use the prices of month r and the imputed quantities of month t to predict the imputed expenditure shares of period t. In the future, "better" principles may be found.[55]

The entire set of modified predicted share price dissimilarity measures for our empirical example is a 72 by 72 element (symmetric) matrix. Table 9.22 lists the first 12 rows and columns of the matrix of the bilateral measures of modified predicted share price dissimilarity for our empirical example.

The set of real-time links which minimize these dissimilarity measures for the first 12 observations are as follows:[56]

1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10 – 11
|
12.

The maximum overlap bilateral Jevons indices $P_J^*(t/r)$ defined by (113) are used to link the prices of month t to a prior month r. It can be seen that the new set of bilateral links is the set of links that generates the chained maximum overlap Jevons indices for months 1 to 11. Thus the *similarity-linked maximum overlap Jevons index*, P_{SJ}^{t*}, equals P_{JCH}^{t*}, the maximum overlap chained Jevons index defined by (116) and (117), for $t = 1,\ldots,11$. However, month 12 is linked directly to month 1. Thus, $P_{SJ}^{12*} = P_{SJ}^{1*}\times P_J^*(12/1)$. The remainder of the similarity-linked maximum overlap Jevons indices are

[54] For additional discussions of this issue see Hill and Timmer (2006). For additional measures of relative price dissimilarity for matched prices, see Diewert (2009).

[55] In general, matching of product prices can be increased by increasing the length of the period. However, for many purposes, a monthly or quarterly price index is required. Another way of increasing product matches over two periods is to broaden the definition of a product. For example, instead of matching the price of a specific brand of chocolate bar purchased in a particular location, many types of chocolate bar purchased in a local area could be aggregated together to form a single unit value price over the two periods in scope. Thus, our monthly Israeli fruit data has aggregated over all locations in Israel and all types of fruit. This means that our indices may suffer from some unit value bias. For a discussion on the tradeoff between increased product matching and unit value bias, see Chessa (2019).

[56] This set of bilateral links is almost the same as the set of links that were used to link the first 12 observations of the predicted share indices P_S^{t*}; see the links listed in Table 9.16 in Section 7.

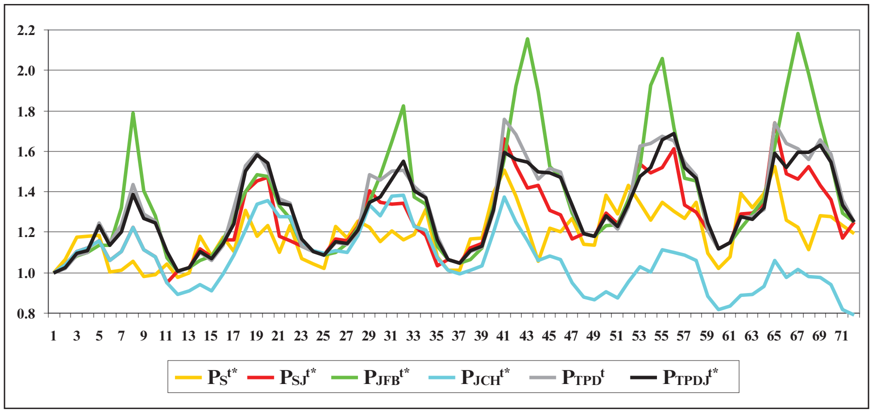
constructed in the same manner that was used to construct the predicted share similarity-linked indices P_S^{t*} except that the new matrix of dissimilarity measures $\Delta_{PS}(p^t,p^r)$ is used in place of the previous matrix of predicted share dissimilarity measures $\Delta_{PS}(p^r,p^t,q^r,q^t)$ defined by (106) and the maximum overlap bilateral Fisher indices that were used to link the prices of months in real time to the prices of previous months are replaced by maximum overlap bilateral Jevons indices. Again, it turns out that the set of bilateral links for the first 12 months basically determines the seasonal fluctuations for the similarity-linked indices P_S^{t*} for the remainder of the sample.[57] The similarity-linked maximum overlap Jevons indices, P_{SJ}^{t*}, are listed in Table 9.23.

Table 9.23 Similarity-Linked Indices, Maximum Overlap Jevons Fixed-Base and Chained Indices, and Time Product Dummy Indices

t	P_S^{t*}	P_{SJ}^{t*}	P_{JFB}^{t*}	P_{JCH}^{t*}	P_{TPD}^{t}	P_{TPDJ}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.06603	1.03812	1.03812	1.03812	1.02229	1.02743
3	1.17647	1.10519	1.08161	1.10519	1.08834	1.09381
4	1.17956	1.12441	1.09864	1.12441	1.10026	1.10881
5	1.18310	1.15614	1.13498	1.15614	1.24733	1.23499
6	1.00296	1.05968	1.13521	1.05968	1.14674	1.13568
7	1.01198	1.10518	1.31939	1.10518	1.23890	1.20161
8	1.05554	1.22312	1.78827	1.22312	1.43385	1.38662
9	0.97973	1.11168	1.40705	1.11168	1.28879	1.26808
10	0.99067	1.07692	1.27978	1.07692	1.25377	1.24421
11	1.04107	0.95033	1.07188	0.95033	1.10326	1.10361
12	0.97592	1.00749	1.00749	0.89324	1.00749	1.00749
13	0.99684	1.02568	1.02568	0.90936	1.02568	1.02568
14	1.17902	1.11710	1.06090	0.94059	1.10007	1.10560
15	1.08056	1.07900	1.08262	0.90851	1.06255	1.06789
16	1.17474	1.16270	1.16025	0.99405	1.13772	1.14657
17	1.10498	1.16209	1.27798	1.08509	1.32010	1.24134
18	1.30841	1.40187	1.40174	1.21125	1.52851	1.50241
19	1.18142	1.45397	1.48595	1.33856	1.59591	1.58084
20	1.23391	1.47303	1.47735	1.35611	1.50812	1.54295
21	1.09986	1.17793	1.32928	1.27802	1.36559	1.34364
22	1.23179	1.15592	1.26980	1.27785	1.34194	1.33548
23	1.06906	1.13275	1.13275	1.13993	1.13275	1.16629
24	1.04392	1.10289	1.10289	1.10988	1.10289	1.10289
25	1.02270	1.08747	1.08747	1.09436	1.08747	1.08747
26	1.22856	1.16556	1.10175	1.10873	1.14779	1.15355
27	1.17215	1.15798	1.14475	1.10153	1.14033	1.14606
28	1.25327	1.22744	1.22192	1.18373	1.20107	1.21042
29	1.22223	1.40403	1.35287	1.33273	1.48657	1.34725
30	1.15449	1.34776	1.48215	1.27956	1.45849	1.37485
31	1.20526	1.34029	1.64760	1.37706	1.50247	1.46191
32	1.16278	1.34423	1.82280	1.38112	1.50689	1.55295
33	1.18929	1.23171	1.37302	1.23008	1.42795	1.40500
34	1.31066	1.18542	1.33810	1.21013	1.37619	1.36957
35	1.07810	1.03496	1.12963	1.07487	1.17947	1.16308
36	1.01195	1.06487	1.06487	1.01325	1.06487	1.06487
37	1.01076	1.04600	1.04600	0.99529	1.04600	1.04600
38	1.16812	1.12268	1.06566	1.01400	1.10557	1.11112
39	1.17108	1.14508	1.12395	1.03423	1.12762	1.13329
40	1.39663	1.33105	1.33629	1.20021	1.30246	1.31258
41	1.50841	1.66272	1.56307	1.37630	1.76046	1.59547
42	1.37756	1.52866	1.92004	1.25089	1.68208	1.55939
43	1.22151	1.41849	2.15585	1.15706	1.56196	1.54720
44	1.05506	1.43013	1.89554	1.06246	1.46420	1.49802
45	1.22173	1.30768	1.52287	1.08274	1.51602	1.49165
46	1.19999	1.28729	1.47388	1.06558	1.49868	1.47337
47	1.26828	1.16789	1.29467	0.95233	1.31599	1.33300
48	1.13921	1.19473	1.19473	0.87882	1.19473	1.19473
49	1.13475	1.17935	1.17935	0.86751	1.17935	1.17935
50	1.38339	1.29628	1.23215	0.90634	1.27652	1.28293
51	1.29063	1.24180	1.23563	0.87403	1.21512	1.22698
52	1.43303	1.35551	1.35948	0.95406	1.32639	1.33670
53	1.34386	1.53671	1.52251	1.03014	1.62705	1.47457
54	1.25757	1.49143	1.92473	1.00391	1.64111	1.52141
55	1.34547	1.52037	2.05735	1.11495	1.67415	1.65834
56	1.30412	1.61240	1.71865	1.10020	1.65081	1.68895
57	1.26875	1.33240	1.46566	1.08668	1.54468	1.51985
58	1.34737	1.29692	1.45182	1.05901	1.47800	1.46867
59	1.09738	1.21017	1.21017	0.88275	1.21017	1.24599
60	1.01922	1.11805	1.11805	0.81555	1.11805	1.11804
61	1.07767	1.14744	1.14744	0.83699	1.14744	1.14744
62	1.39115	1.29232	1.21897	0.88916	1.27262	1.27901
63	1.32072	1.29483	1.29046	0.89089	1.27510	1.26479
64	1.39001	1.34113	1.36739	0.93338	1.31231	1.32252
65	1.52597	1.73864	1.62717	1.05969	1.73957	1.59232
66	1.25740	1.48917	1.91313	0.97614	1.63863	1.51910
67	1.22459	1.46295	2.18107	1.01502	1.61092	1.59570
68	1.11160	1.52475	1.98518	0.98081	1.56108	1.59714
69	1.27951	1.42959	1.74583	0.97806	1.65734	1.63071
70	1.27885	1.36176	1.54445	0.94301	1.58539	1.54454
71	1.23088	1.17084	1.29409	0.81873	1.35927	1.33240
72	1.19115	1.25093	1.25093	0.79142	1.25093	1.25093
Mean	1.18920	1.25460	1.36900	1.06470	1.32860	1.31120

[57] The remainder of the real-time maximum overlap predicted share bilateral Jevons index links for the next 60 months are as follows: 13/12, 14/3, 15/2, 16/4, 17/5, 18/6, 19/7, 20/19, 21/9, 22/11, 23/12 and 24/23, 25/24, 26/14, 27/2, 28/4, 29/6, 30/6, 31/7, 32/7, 33/21, 34/22, 35/22, 36/24, 37/36, 38/26, 39/27, 40/28, 41/29, 42/30, 43/31, 44/20, 45/21, 46/10, 47/22, 48/25, 49/48, 50/26, 51/40, 52/51, 53/29, 54/42, 55/43, 56/20, 57/21, 58/35, 59/25, 60/59, 61/59, 62/38, 63/39, 64/40, 65/41, 66/54, 67/43, 68/44, 69/9, 70/46, 71/22, and 72/48. Most of these bilateral links link the same months as were used to construct P_S^{t*}.

Figure 9.11 Similarity-Linked Indices and Five Indices That Use Only Price Information



From Table 9.23, it can be seen that the chained Jevons index, $P_{JCH}{}^{t*}$, that uses maximum overlap bilateral Jevons indices suffers from a considerable amount of downward chain drift so it cannot be recommended for use. The "best" index that uses price and quantity information, the predicted share similarity-linked index that uses maximum overlap bilateral Fisher indices as basic building blocks, $P_S{}^{t*}$, finished about 6 percentage points below the remaining four indices, $P_{SJ}{}^{t*}$, $P_{JFB}{}^{t*}$, $P_{TPD}{}^{t*}$ and $P_{TPDJ}{}^{t*}$. These four indices cannot control adequately for substitution bias (since they depend only on price information), whereas $P_S{}^{t*}$ does deal adequately with substitution bias. Thus, if a statistical agency is forced to rely on price data alone for an elementary index that has strongly seasonal products, then there is a strong likelihood that some substitution bias will occur which can be substantial.

The four indices, $P_{SJ}{}^{t*}$, $P_{JFB}{}^{t*}$, $P_{TPD}{}^{t*}$ and $P_{TPDJ}{}^{t*}$, all finished at exactly the same value in month 72. However, their means were substantially different in some cases. The standard time product dummy indices, $P_{TPD}{}^{t*}$, and the Modified TPD indices that used year-over-year maximum overlap Jevons indices as well, $P_{TPDJ}{}^{t*}$, had means of 1.3286 and 1.3112 respectively. As can be seen in Figure 9.11, these two indices approximated each other rather well. The mean of the Jevons maximum overlap fixed-base indices, $P_{JFB}{}^{t*}$, was the highest of the six indices at 1.3690. Figure 9.11 shows that this high mean is due to very large upward seasonal fluctuations for months in the middle of the year, where product matches with the products available in January were very low. The price index (over the five indices that used only price information) that best approximated the Predicted Share index $P_S{}^{t*}$ is the Similarity-Linked Maximum Overlap Jevons index, $P_{SJ}{}^{t*}$.

Figure 9.11 shows that $P_S{}^{t*}$ has by far the smallest seasonal variations. Relative to this preferred index, the chained Jevons index, $P_{JCH}{}^{t*}$, has a large downward bias and the fixed-base Jevons index, $P_{JFB}{}^{t*}$, has a large upward bias on average due to its huge seasonal fluctuations. The remaining three indices, $P_{SJ}{}^{t*}$, $P_{TPD}{}^{t}$ and $P_{TPDJ}{}^{t*}$, finish at the same point which is 6 percentage points above $P_S{}^{72*}$. These three indices are fairly close to each other, but the similarity-linked Jevons index, $P_{SJ}{}^{t*}$, tends to lie below the two Time Product Dummy indices, $P_{TPD}{}^{t}$ and $P_{TPDJ}{}^{t*}$, and $P_{SJ}{}^{t*}$ has smaller seasonal fluctuations. Overall, for our particular example, $P_{SJ}{}^{t*}$ provides the best approximation to our preferred index, $P_S{}^{t*}$.[58]

In the following section, indices which use annual baskets or annual expenditure shares will be studied.

10. Annual Basket Lowe Indices and Annual Share-Weighted Young Indices

For many consumer expenditure categories, national statistical agencies are not able to collect price and quantity information for many if not most expenditure categories. Instead, they collect a sample of prices in real time and collect annual household expenditures by broad category on a delayed basis using a consumer expenditure survey. Thus for many expenditure categories, statistical agencies construct either a Lowe (1823) or Young (1812) index. These indices use a combination of annual expenditures on a past year and current month information on prices. In this section, we will construct versions of these indices using our seasonal products data for Israel.

When constructing a price index for a category of household goods and services using annual expenditure data from a past period, statistical agencies have to deal with missing prices for strongly seasonal products (products that are not available for all seasons of the year). Agencies solve this problem by using carry-forward prices for the missing products.[59] Thus, in this section, for our empirical example, we again use the monthly price data that are listed in Table A.23 in the annex. This table has the carry-forward/carry-backward prices for products that are missing in any month.

The month t price for product n is defined as $p_{t,n}$ for $t = 1,\ldots,T$ and $n = 1,\ldots,N$, where $T = MY$ and M is the number of months in the year and Y is the number of years in the sample of prices. The notation for quantities is the same as was used in Section 2: $q_{y,m,n}$ is the quantity of product n that

[58] The correlation coefficients between $P_S{}^{t*}$ and $P_{SJ}{}^{t*}$, $P_{JFB}{}^{t*}$, $P_{JCH}{}^{t*}$, $P_{TPD}{}^{t*}$, $P_{TPDJ}{}^{t*}$ are 0.730, 0.392, 0.137, 0.620, and 0.602 respectively. The correlation coefficients between $P_{SJ}{}^{t*}$ and $P_{TPD}{}^{t*}$, $P_{TPDJ}{}^{t*}$ are 0.913, 0.898 respectively. Thus, these three indices approximate each other reasonably well.

[59] Other methods for imputing the missing prices are also used.

is purchased by households in month m of year y, where $y = 1,\ldots,Y$; $m = 1,\ldots,M$ and $n = 1,\ldots,N$. *Annual quantities of product n* purchased in year y, $q_{A,y,n}$, are obtained by summing purchases of product n in year y over the months in the year; that is, we have the following definitions:

$$q_{A,y,n} \equiv \Sigma_{m=1}^{M} q_{y,m,n};\ y = 1,\ldots,Y;\ n = 1,\ldots,N. \quad (140)$$

Using the annual basket weights for year 1 in our sample ($q_{a,1,n}$ defined by (140)), and the prices of month t, $p_{t,n}$, the *fixed-base Lowe index* for month t using the weights of year 1, $P_{LO1}{}^{t}$, is defined as follows:[60]

$$P_{LO1}{}^{t} \equiv \Sigma_{n=1}^{N} p_{t,n}q_{A,1,n}/\Sigma_{n=1}^{N} p_{1,n}q_{A,1,n};\ t = 1,\ldots,T. \quad (141)$$

Using the data listed in Tables A.23 and A.24, these fixed-base Lowe indices are listed in Table A.24 and plotted in Figure 9.12.[61]

Some statistical agencies use the annual weights of the prior year and use a new Lowe index for 12 months to update the prior year's indices. To approximate this type of index, we construct a *lagged two year chained Lowe index* for month t, $P_{LO2}{}^{t}$, as follows. For $t = 1,\ldots,24$, define $P_{LO2}{}^{t} \equiv P_{LO1}{}^{t}$. Thus, we use the annual quantities for year 1 for the first 24 months of data. For a month $t = 25,\ldots,36$ in the third year, define a new *link Lowe index* using the weights of year 2 and the prices of year 3 relative to December of year 2, $[\Sigma_{n=1}^{N} p_{t,n}q_{A,2,n}/\Sigma_{n=1}^{N} p_{24,n}q_{A,2,n}]$, and then link this index to the index value for $P_{LO2}{}^{t}$ at $t = 24$. Thus for $t = 25,\ldots,36$, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{24}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,2,n}/\Sigma_{n=1}^{N} p_{24,n}q_{A,2,n}]$. For year 4, use the quantity weights of year 3 and the new Lowe link index that compares the prices of month t in year 4 to month 12 in year 3 to extend the definition of $P_{LO2}{}^{t}$. Thus for $t = 37,\ldots,48$, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{36}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,3,n}/\Sigma_{n=1}^{N} p_{36,n}q_{A,3,n}]$. In a similar manner, for $t = 49,\ldots,60$, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{48}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,4,n}/\Sigma_{n=1}^{N} p_{48,n}q_{A,4,n}]$ and for $t = 61,\ldots,72$, define $P_{LO2}{}^{t} \equiv P_{LO2}{}^{60}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,5,n}/\Sigma_{n=1}^{N} p_{60,n}q_{A,5,n}]$. These chained Lowe indices using annual weights lagged one year, $P_{LO2}{}^{t}$, are listed in Table A.24 and plotted in Figure 9.12.

Some countries use annual weights that are lagged two years. To approximate this type of index, we construct a *Lagged two year chained Lowe index* for month t, $P_{LO3}{}^{t}$, as follows. For $t = 1,\ldots,36$, define $P_{LO3}{}^{t} \equiv P_{LO1}{}^{t}$. Thus, we use the annual quantities for year 1 for the first 36 months of data to construct this alternative Lowe index which will be equal to the fixed-base Lowe index, $P_{LO1}{}^{t}$, for the first three years of data. For the fourth year of data, define a new *link Lowe index* using the weights of year 2 and the prices of year 4 relative to December of year 3, $[\Sigma_{n=1}^{N} p_{t,n}q_{A,2,n}/\Sigma_{n=1}^{N} p_{36,n}q_{A,2,n}]$, and then link this index to the index value for $P_{LO3}{}^{t}$ at $t = 36$. Thus for $t = 37,\ldots,48$, define $P_{LO3}{}^{t} \equiv P_{LO3}{}^{36}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,2,n}/\Sigma_{n=1}^{N} p_{36,n}q_{A,2,n}]$. In a similar manner, for $t = 49,\ldots,60$, define $P_{LO3}{}^{t} \equiv P_{LO3}{}^{48}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,3,n}/\Sigma_{n=1}^{N} p_{48,n}q_{A,3,n}]$ and for $t = 61,\ldots,72$, define $P_{LO3}{}^{t} \equiv P_{LO3}{}^{60}\times[\Sigma_{n=1}^{N} p_{t,n}q_{A,4,n}/\Sigma_{n=1}^{N} p_{60,n}q_{A,4,n}]$. These partially chained Lowe indices using annual weights lagged two years, $P_{LO3}{}^{t}$, are listed in Table A.24 and plotted in Figure 9.12.

Recall the notation for prices and quantities that was used in Section 2: $q_{y,m,n}$ is the quantity of product n that is purchased by households in month m of year y, and $p_{y,m,n}$ is the corresponding price where $y = 1,\ldots,Y$; $m = 1,\ldots,M$ and $n = 1,\ldots,N$.[62] *Annual expenditures for product n* purchased in year y, $e_{A,y,n}$, are obtained by summing expenditures on product n in year y over the months in the year; that is, we have the following definitions:

$$e_{A,y,n} \equiv \Sigma_{m=1}^{M} p_{y,m,n}q_{y,m,n};\ y = 1,\ldots,Y;\ n = 1,\ldots,N; \quad (142)$$

$$e_{y} \equiv \Sigma_{n=1}^{N} e_{A,y,m};\ y = 1,\ldots,Y; \quad (143)$$

$$s_{A,y,n} \equiv e_{A,y,n}/e_{y;}\ y = 1,\ldots,Y;\ n = 1,\ldots,N, \quad (144)$$

where e_y is the total expenditure on all products in year y and $s_{A,y,n}$ is the annual expenditure share of product n in year y. We will use the annual expenditure shares on products for year 1, $s_{A,1,n}$, in order to define our next index.

The *fixed-base Young* (1812) *index* for month t using the annual weights of year 1, $P_{Y1}{}^{t}$, is defined as follows:

$$P_{Y1}{}^{t} \equiv \Sigma_{n=1}^{N} s_{A,1,n}(p_{t,n}/p_{1,n});\ t = 1,\ldots,T. \quad (145)$$

Using the data listed in Tables A.23 and A.24, this fixed-base Young index $P_{Y1}{}^{t}$ is listed in Table 9.24 and is plotted in Figure 9.12.[63]

The aforementioned Young index $P_{Y1}{}^{t}$ is not a real-time Young index. Many statistical agencies use the annual expenditure share weights of the prior year and construct a real-time Young index by using these lagged annual weights for one year and then they update their lagged annual weights for the following year. To approximate this type of index, we construct a (partially) *chained Young index* for month t, $P_{Y2}{}^{t}$, as follows. For $t = 1,\ldots,24$, define $P_{Y2}{}^{t} \equiv P_{Y1}{}^{t}$. Thus, we use the annual expenditure shares for year 1 for the first 24 months of data. For a month $t = 25,\ldots,36$ in the third year, define a new *link Young index* using the expenditure share weights of year 2 and the prices of year 3 relative to December of year 2, $\Sigma_{n=1}^{N} s_{A,2,n}(p_{t,n}/p_{24,n})$, and then link this index to the index value for $P_{Y2}{}^{t}$ at $t = 24$. Thus for $t = 25,\ldots,36$, define $P_{Y2}{}^{t} \equiv P_{Y2}{}^{24}\times\Sigma_{n=1}^{N} s_{A,2,n}(p_{t,n}/p_{24,n})$. For year 4, use the expenditure share weights of year 3 and the new Young link index that compares the prices of month t in year 4 to month 12 in year 3 to extend the definition of $P_{Y2}{}^{t}$. Thus for $t = 37,\ldots,48$, define $P_{Y2}{}^{t} \equiv P_{Y2}{}^{36}\times\Sigma_{n=1}^{N} s_{A,3,n}(p_{t,n}/p_{36,n})$. In a similar manner, for $t = 49,\ldots,60$, define $P_{Y2}{}^{t} \equiv P_{Y2}{}^{48}\times\Sigma_{n=1}^{N} s_{A,4,n}(p_{t,n}/p_{48,n})$ and for $t = 61,\ldots,72$, define $P_{Y2}{}^{t} \equiv P_{Y2}{}^{60}\times\Sigma_{n=1}^{N} s_{A,5,n}(p_{t,n}/p_{60,n})$. These partially chained Young indices using annual weights lagged one year, $P_{Y2}{}^{t}$, are listed in Table 9.24 and plotted in Figure 9.12.

As was the case with Lowe indices, some countries that produce Young indices use annual expenditure share weights that are lagged two years. To approximate this

[60] In the context of seasonal price indices, this type of index corresponds to Bean and Stine's (1924; 31) Type A index.

[61] The year 1 annual quantity weights $q_{A,1,n}$ are as follows for our sample of 14 types of fruits: 7.968, 7.159, 27.106, 2.285, 0.966, 8.805, 10.069, 2.266, 0.664, 0.884, 3.560, 9.528, 0.782, and 2.168.

[62] Carry-forward/carry-backward prices are used for missing products in this section.

[63] The year 1 annual expenditure shares $s_{A,1,n}$ are as follows for our sample of 14 types of fruits: 0.07688, 0.09895, 0.12712, 0.04061, 0.00644, 0.18375, 0.14648, 0.07842, 0.03345, 0.02055, 0.05667, 0.07869, 0.01647, 0.03552.

type of index, we construct a *lagged two year chained Young index* for month t, $P_{Y3}{}^{t}$, as follows. For $t = 1,\ldots,36$, define $P_{Y3}{}^{t} \equiv P_{Y1}{}^{t}$. Thus, we use the annual expenditure shares for year 1 for the first 36 months of data to construct this alternative Young index which will be equal to our initial fixed-base Young index, $P_{Y1}{}^{t}$, for the first three years of data. For the fourth year of data, define a new *link Young index* using the expenditure share weights of year 2 and the prices of year 4 relative to December of year 3, $\Sigma_{n=1}{}^{N}\, s_{A,2,n}(p_{t,n}/p_{36,n})$, and then link this index to the index value for $P_{Y3}{}^{t}$ at $t = 36$. Thus for $t = 37,\ldots,48$, define $P_{Y3}{}^{t} \equiv P_{Y3}{}^{36}\times\Sigma_{n=1}{}^{N}\, s_{A,2,n}(p_{t,n}/p_{36,n})$. In a similar manner, for $t = 49,\ldots,60$, define $P_{Y3}{}^{t} \equiv P_{Y3}{}^{48}\times\Sigma_{n=1}{}^{N}\, s_{A,3,n}(p_{t,n}/p_{48,n})$ and for $t = 61,\ldots,72$, define $P_{Y3}{}^{t} \equiv P_{Y3}{}^{60}\times\Sigma_{n=1}{}^{N}\, s_{A,4,n}(p_{t,n}/p_{60,n})$. These partially chained Young indices using annual expenditure share weights lagged two years $P_{Y3}{}^{t}$, are listed in Table 9.24 and plotted in Figure 9.12.

For comparison purposes, the maximum overlap predicted share similarity-linked indices $P_{S}{}^{t*}$ are also listed in Table 9.24 and are plotted in Figure 9.12. These indices were

Table 9.24 Alternative Lowe and Young Indices and Maximum Overlap Predicted Share Indices

t	$P_{LO1}{}^{t}$	$P_{LO2}{}^{t}$	$P_{LO3}{}^{t}$	$P_{Y1}{}^{t}$	$P_{Y2}{}^{t}$	$P_{Y3}{}^{t}$	$P_{S}{}^{t*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.04326	1.04326	1.04326	1.05268	1.05268	1.05268	1.06603
3	1.08548	1.08548	1.08548	1.10507	1.10507	1.10507	1.17647
4	1.08876	1.08876	1.08876	1.10788	1.10788	1.10788	1.17956
5	1.13936	1.13936	1.13936	1.15868	1.15868	1.15868	1.18310
6	1.03900	1.03900	1.03900	1.06992	1.06992	1.06992	1.00296
7	1.06729	1.06729	1.06729	1.10912	1.10912	1.10912	1.01198
8	1.19403	1.19403	1.19403	1.25591	1.25591	1.25591	1.05554
9	1.10376	1.10376	1.10376	1.15229	1.15229	1.15229	0.97973
10	1.08384	1.08384	1.08384	1.12758	1.12758	1.12758	0.99067
11	0.98815	0.98815	0.98815	1.02055	1.02055	1.02055	1.04107
12	0.95707	0.95707	0.95707	0.98609	0.98609	0.98609	0.97592
13	0.96145	0.96145	0.96145	0.99008	0.99144	0.99144	0.99684
14	1.00464	1.00464	1.00464	1.04855	1.03587	1.03587	1.17902
15	0.98637	0.98637	0.98637	1.01337	1.02120	1.02120	1.08056
16	1.03456	1.03456	1.03456	1.07048	1.07208	1.07208	1.17474
17	1.10996	1.10996	1.10996	1.15028	1.15016	1.15016	1.10498
18	1.23903	1.23903	1.23903	1.27051	1.27978	1.27978	1.30841
19	1.32716	1.32716	1.32716	1.36592	1.37230	1.37230	1.18142
20	1.34389	1.34389	1.34389	1.37751	1.39097	1.39097	1.23391
21	1.21519	1.21519	1.21519	1.23145	1.25047	1.25047	1.09986
22	1.20236	1.20236	1.20236	1.21261	1.23257	1.23257	1.23179
23	1.14380	1.14380	1.14380	1.14866	1.16844	1.16844	1.06906
24	1.12858	1.12858	1.12858	1.13117	1.15207	1.15207	1.04392
25	1.11756	1.11376	1.11756	1.11914	1.13660	1.15403	1.02270
26	1.17002	1.16392	1.17002	1.19040	1.19004	1.21753	1.22856
27	1.16512	1.17153	1.16512	1.17183	1.19817	1.21480	1.17215
28	1.21869	1.23906	1.21869	1.22908	1.26991	1.28190	1.25327
29	1.26146	1.30297	1.26146	1.28205	1.34101	1.35073	1.22223
30	1.21967	1.26375	1.21967	1.25523	1.30879	1.32380	1.15449
31	1.31282	1.37439	1.31282	1.36083	1.43366	1.44250	1.20526
32	1.36885	1.45695	1.36885	1.42380	1.52351	1.52542	1.16278
33	1.23566	1.30444	1.23566	1.27315	1.35170	1.35787	1.18929
34	1.20359	1.26686	1.20359	1.23562	1.31157	1.31230	1.31066
35	1.09640	1.13257	1.09640	1.11793	1.16504	1.17490	1.07810
36	1.06780	1.09851	1.06780	1.08641	1.12818	1.13999	1.01195

(Continued)

Table 9.24 (Continued)

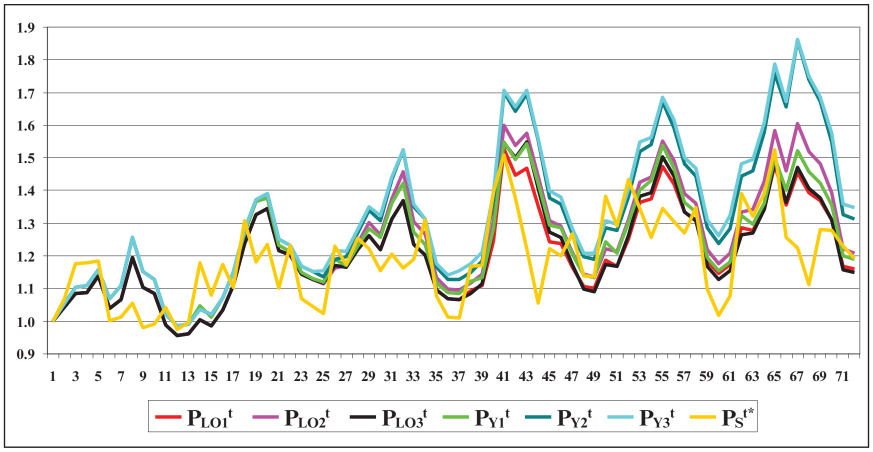
t	$P_{LO1}{}^{t}$	$P_{LO2}{}^{t}$	$P_{LO3}{}^{t}$	$P_{Y1}{}^{t}$	$P_{Y2}{}^{t}$	$P_{Y3}{}^{t}$	$P_{S}{}^{t*}$
37	1.06587	1.09628	1.06579	1.08420	1.12718	1.15462	1.01076
38	1.09187	1.11622	1.08513	1.12263	1.14963	1.17557	1.16812
39	1.11013	1.14497	1.11553	1.13424	1.18354	1.21131	1.17108
40	1.24637	1.30601	1.28028	1.28318	1.36839	1.39454	1.39663
41	1.52833	1.59991	1.54600	1.54905	1.70292	1.70586	1.50841
42	1.44653	1.53923	1.49936	1.49624	1.64162	1.65526	1.37756
43	1.46704	1.57449	1.55011	1.54226	1.69526	1.70704	1.22151
44	1.35281	1.44427	1.40749	1.42269	1.55393	1.56055	1.05506
45	1.24425	1.30625	1.27269	1.29453	1.37818	1.39849	1.22173
46	1.23748	1.29271	1.25707	1.28513	1.35919	1.38018	1.19999
47	1.16552	1.21047	1.17278	1.20599	1.26673	1.28470	1.26828
48	1.10676	1.14443	1.09805	1.14252	1.19705	1.20779	1.13921
49	1.10034	1.13626	1.09053	1.13441	1.18973	1.20886	1.13475
50	1.18691	1.22131	1.17300	1.24418	1.28716	1.30691	1.38339
51	1.16776	1.21323	1.16760	1.20785	1.27810	1.29792	1.29063
52	1.25206	1.30889	1.26315	1.30331	1.38020	1.40329	1.43303
53	1.36482	1.42634	1.38322	1.40161	1.52043	1.54805	1.34386
54	1.37422	1.44057	1.39343	1.43167	1.54143	1.56224	1.25757
55	1.47429	1.55230	1.50203	1.53753	1.67156	1.68595	1.34547
56	1.41979	1.49149	1.44386	1.46712	1.59174	1.61501	1.30412
57	1.33528	1.39119	1.33533	1.36367	1.48060	1.50056	1.26875
58	1.30946	1.36033	1.30726	1.33383	1.44496	1.46756	1.34737
59	1.18387	1.21857	1.16801	1.19556	1.28643	1.30911	1.09738
60	1.14666	1.17665	1.12767	1.15470	1.23912	1.26212	1.01922
61	1.17220	1.20921	1.15517	1.18209	1.28337	1.32271	1.07767
62	1.28637	1.33420	1.26463	1.32312	1.44306	1.48080	1.39115
63	1.27916	1.34273	1.26912	1.29807	1.46002	1.49454	1.32072
64	1.34213	1.42892	1.34161	1.36226	1.57761	1.60731	1.39001
65	1.47799	1.58492	1.48634	1.49954	1.75811	1.78672	1.52597
66	1.35495	1.46134	1.36328	1.40213	1.65693	1.67184	1.25740
67	1.45383	1.60427	1.47019	1.52241	1.85953	1.86366	1.22459
68	1.39418	1.52032	1.40637	1.45631	1.73987	1.74864	1.11160
69	1.36826	1.48091	1.37725	1.42296	1.67215	1.68685	1.27951
70	1.31139	1.39452	1.31040	1.35463	1.54773	1.57445	1.27885
71	1.16938	1.22085	1.15803	1.19906	1.32578	1.35928	1.23088
72	1.16123	1.20830	1.14936	1.18982	1.31397	1.34640	1.19115
Mean	1.20940	1.24830	1.21360	1.24240	1.31660	1.32920	1.18920

defined in Section 7. Recall that these indices had the "best" axiomatic and economic properties.

The fixed-base Young index $P_{Y1}{}^{t}$ ends up very close to our preferred index $P_{S}{}^{t*}$ at $t = 72$. However, the mean of $P_{Y1}{}^{t}$ is 5.3 percentage points above the mean of $P_{S}{}^{t*}$. It is interesting that the partially chained Young indices, $P_{Y2}{}^{t}$ and $P_{Y3}{}^{t}$, appear to have some upward chain drift since they finished 12.3 and 15.5 percentage points above $P_{S}{}^{72*}$. Note that $P_{LO1}{}^{t}$, $P_{Y1}{}^{t}$ and $P_{S}{}^{t*}$, all satisfy the multiperiod identity test so these indices are not subject to chain drift.[64] Turning to the Lowe indices, the partially chained Lowe index that used annual quantity weights lagged one year, $P_{LO2}{}^{t}$, ended up 1.7 percentage points above our preferred

[64] However, as soon as the base period annual quantities or annual expenditure shares are updated, the resulting Lowe and Young indices will be subject to potential chain drift. The similarity-linked indices $P_{S}{}^{t*}$ always satisfy the multiperiod identity test and hence are not subject to chain drift.

Figure 9.12 Lowe, Young, and Maximum Overlap Predicted Share Indices



similarity-linked index P_S^{t*}. However, on average, P_{LO2}^t was 9.4 percentage points above P_S^{t*}. The fixed-base Lowe index, P_{LO1}^t, and the partially chained Lowe index that used annual quantity weights lagged two years, P_{LO3}^t, ended up 3.0 and 4.2 percentage points below P_S^{72*}.

From Figure 9.12, it can be seen that none of the annual basket or annual expenditure share indices provide an adequate approximation to our preferred similarity-linked index P_S^{t*}. The large upward seasonal fluctuations in the two partially chained Young indices, P_{Y2}^t and P_{Y3}^t, are a particular cause for concern. In general, the indices that use annual quantities or expenditure shares as weights have an upward bias which is interesting since these indices use carry-forward prices so they should have a downward bias relative to P_S^{t*} since this similarity-linked index does not use carry-forward prices.

There is a conceptual problem with using annual basket indices along with carry-forward prices in the strongly seasonal products context. *The problem is that these indices have no theoretical justification*. To see the problem clearly, think of an extreme case of strong seasonality for an elementary category where each product is available in only one month of the year. It is simply impossible to construct a meaningful price (or quantity) index for this category of goods or services. There is no basis for comparing the prices or quantities of one month or quarter with the corresponding prices or quantities of a different month or quarter of the same year since the product categories do not overlap.[65] In the strongly seasonal products context where there is some product overlap for months in the same year, we can construct meaningful price indices between months in the same year for the set of overlap products between any two months, and this is exactly what was done to create the similarity-linked indices P_S^{t*}.

Our conclusions for this section are as follows:

- In the strongly seasonal products context, Lowe and Young indices using carry-forward prices for missing products are subject to both carry-forward bias and substitution bias and are unlikely to approximate alternative indices that have better axiomatic and economic properties.

It was noted earlier that Mudgett Stone indices, which compare the prices of the current year with the prices of a previous year, provide meaningful measures of annual inflation even in the case where each product in scope is only available in one month of the year. This type of index was studied in Sections 4 and 5. In the following section, this type of index will be generalized to provide a measure of annual inflation that is updated each month. The resulting measures of price change can be compared to smoothed measures of month-to-month price change.

Just a comment on footnote 66: *I* don'*t* think Baldwin's characteristic of the annual basket (AB) is helpful at all. Instead of his formulation, *I* would say that the AB shows the changing price from month-to-month of buying the same annual average basket of goods and services. This is a clear and simple concept and can be explained to users. There are numerous measurement problems, including for strongly seasonal products: prices of out-of-season products have to be estimated or imputed, but you can do this in a reasonable ways and imputations are used in a lot of economic statistics, including the national accounts. The alternative is to allow the expenditure weights to change during the year would solve this problem. The weights of out-of-season products would be zero and there would be no need to estimate/impute artificial prices. But allowing weights to

[65] Andrew Baldwin (1990; 258) noted that there is a problem with using annual basket (AB) indices in the seasonal context even if there is no strong seasonality: "For seasonal goods, the AB index is best considered an index partially adjusted for seasonal variation. It is based on annual quantities, which do not reflect the seasonal fluctuations in the volume of purchases, and on raw monthly prices, which do incorporate seasonal price fluctuations. Zarnowitz (1961; 256–257) calls it an index of 'a hybrid sort.' Being neither of sea nor land, it does not provide an appropriate measure either of monthly or 12 month price change. The question that an AB index answers with respect to price change from January to February say, or January of one year to January of the next, is 'What would the change in consumer prices have been if there were no seasonality in purchases in the months in question but prices nonetheless retained their own seasonal behaviour?' It is hard to believe that this is a question that anyone would be interested in asking."

change from month to month during the year would create other problems, in particular that one would leave the fixed basket approach and the index would no longer show only the effect of price changes; it would be difficult to interpret the changes in such an index since these might be caused by both price changes and changes in weights. It is my impression that users in general are happy with this approach and the stability and ease of interpretation this give to the CPI. Sure, you can criticize AB and Lowe/Young for many things, but they also have advantages, which are completely disregarded in this chapter.

11. Rolling Year Measures of Annual Inflation and Measures of Trend Inflation

In Sections 4 and 5, the price and quantity data pertaining to the 12 months of a calendar year were compared to the 12 months of a base calendar year. However, there is no need to restrict attention to calendar year comparisons: any 12 consecutive months of price and quantity data could be compared to the price and quantity data of the base year, provided that the January data in the non-calendar year are compared to the January data of the base year, the February data of the non-calendar year are compared to the February data of the base year, . . ., and the December data of the non-calendar year are compared to the December data of the base year.[66] Alterman, Diewert, and Feenstra (1999; 70) called the resulting indices *rolling year* indices.[67] This approach to the measurement of price change is consistent with three of the four main approaches to index number theory: (i) the comparison of purchases of products in the two periods using the base period consumption basket, the current period consumption basket, or an average of the two[68]; (ii) the test approach; and (iii) the stochastic approach.[69]

It is easy to explain how the rolling year indices work in principle: The prices of the 12 months in the current rolling year are compared to the corresponding monthly prices in the base year, where January prices are matched up with January prices, February prices with February prices, and so on. However, setting up the algebra for the maximum overlap Laspeyres and Paasche indices is somewhat complex, as will be seen subsequently.

Recall that p^t and q^t are the month t vectors of dimension 14 for our example for $t = 1,\ldots,72$. Treat these vectors as column vectors in what follows. The inner product of p^t and q^r is defined as $p^t \cdot q^r \equiv \Sigma_{n=1}^{14} p_{t,n} q_{r,n}$. If product n is not purchased in month t, then the nth components of p^t and q^t, $p_{t,n}$, and $q_{t,n}$, are set equal to 0. For $t = 1,\ldots,72$, define the diagonal 14 by 14 matrix Δ^t as follows: If $q_{t,n} > 0$, then the element in the nth row and nth column of Δ^t is set equal to 1; if $q_{t,n} = 0$, then the element in the nth row and nth column of Δ^t is set equal to 0. The remaining components of Δ^t are set equal to 0.

The rolling year indices cannot be defined until 12 months of data are available. Thus for our example data set which consists of 72 months of data, the rolling year indices will run from $t = 1$ to $t = 72$. When $t = 1$, the first 12 months of data are compared with the first 12 months of data and the resulting rolling year index will equal 1. When $t = 61$, the rolling year index compares the last 12 months of data with the first 12 months of data.

The algebra for the *rolling year fixed-base maximum overlap Laspeyres indices* is set out here. This index for period t is denoted by $P_{LRY}^{t*} = \text{Num}^t/\text{Den}^t$. The numerators, Num^t, and the denominators, Den^t, for P_{LRY}^{t*} are defined as follows:

$\text{Num}^1 \equiv \Sigma_{t=1}^{12} p^t \cdot q^t$; $\quad \text{Den}^1 \equiv \Sigma_{t=1}^{12} p^t \cdot q^t$;

$\text{Num}^2 \equiv \text{Num}^1 - p^1 \cdot q^1 + p^{13} \cdot q^1$; $\quad \text{Den}^2 \equiv \text{Den}^1 - p^1 \cdot q^1 + p^1 \cdot \Delta^{13} q^1$;

$\text{Num}^3 \equiv \text{Num}^2 - p^2 \cdot q^2 + p^{14} \cdot q^2$; $\quad \text{Den}^3 \equiv \text{Den}^2 - p^2 \cdot q^2 + p^2 \cdot \Delta^{14} q^2$;

$\text{Num}^4 \equiv \text{Num}^3 - p^3 \cdot q^3 + p^{15} \cdot q^3$; $\quad \text{Den}^4 \equiv \text{Den}^3 - p^3 \cdot q^3 + p^3 \cdot \Delta^{15} q^3$;

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$\text{Num}^{13} \equiv \text{Num}^{12} - p^{12} \cdot q^{12} + p^{24} \cdot q^{12}$; $\quad \text{Den}^{13} \equiv \text{Den}^{12} - p^{12} \cdot q^{12} + p^{12} \cdot \Delta^{24} q^{12}$;

$\text{Num}^{14} \equiv \text{Num}^{13} - p^{13} \cdot q^1 + p^{25} \cdot q^1$; $\quad \text{Den}^{14} \equiv \text{Den}^{13} - p^1 \cdot \Delta^{13} q^1 + p^1 \cdot \Delta^{25} q^1$;

$\text{Num}^{15} \equiv \text{Num}^{14} - p^{14} \cdot q^2 + p^{26} \cdot q^2$; $\quad \text{Den}^{15} \equiv \text{Den}^{14} - p^2 \cdot \Delta^{14} q^2 + p^2 \cdot \Delta^{26} q^2$;

••• •••

$\text{Num}^{25} \equiv \text{Num}^{24} - p^{24} \cdot q^{12} + p^{36} \cdot q^{12}$; $\quad \text{Den}^{25} \equiv \text{Den}^{24} - p^{12} \cdot \Delta^{24} q^{12} + p^{12} \cdot \Delta^{36} q^{12}$;

$\text{Num}^{26} \equiv \text{Num}^{25} - p^{25} \cdot q^1 + p^{37} \cdot q^1$; $\quad \text{Den}^{26} \equiv \text{Den}^{25} - p^1 \cdot \Delta^{25} q^1 + p^1 \cdot \Delta^{37} q^1$;

$\text{Num}^{27} \equiv \text{Num}^{26} - p^{26} \cdot q^2 + p^{38} \cdot q^2$; $\quad \text{Den}^{27} \equiv \text{Den}^{26} - p^2 \cdot \Delta^{26} q^2 + p^2 \cdot \Delta^{38} q^2$;

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$\text{Num}^{37} \equiv \text{Num}^{36} - p^{36} \cdot q^{12} + p^{48} \cdot q^{12}$; $\quad \text{Den}^{37} \equiv \text{Den}^{36} - p^{12} \cdot \Delta^{36} q^{12} + p^{12} \cdot \Delta^{48} q^{12}$;

$\text{Num}^{38} \equiv \text{Num}^{37} - p^{37} \cdot q^1 + p^{49} \cdot q^1$; $\quad \text{Den}^{38} \equiv \text{Den}^{37} - p^1 \cdot \Delta^{37} q^1 + p^1 \cdot \Delta^{49} q^1$;

$\text{Num}^{39} \equiv \text{Num}^{38} - p^{38} \cdot q^2 + p^{50} \cdot q^2$; $\quad \text{Den}^{39} \equiv \text{Den}^{38} - p^2 \cdot \Delta^{38} q^2 + p^2 \cdot \Delta^{50} q^2$;

••• •••

$\text{Num}^{49} \equiv \text{Num}^{48} - p^{48} \cdot q^{12} + p^{60} \cdot q^{12}$; $\quad \text{Den}^{49} \equiv \text{Den}^{48} - p^{12} \cdot \Delta^{48} q^{12} + p^{12} \cdot \Delta^{60} q^{12}$;

$\text{Num}^{50} \equiv \text{Num}^{49} - p^{49} \cdot q^1 + p^{61} \cdot q^1$; $\quad \text{Den}^{50} \equiv \text{Den}^{49} - p^1 \cdot \Delta^{49} q^1 + p^1 \cdot \Delta^{61} q^1$;

[66] Diewert (1983) suggested this type of comparison and termed the resulting index a "split year" comparison.
[67] Crump (1924; 185) used this term in the context of various seasonal adjustment procedures. Mendershausen (1937; 245) used the term "moving year." The term "rolling year" seems to be well established in the business literature in the UK.
[68] This leads to maximum overlap Laspeyres, Paasche, and Fisher indices.
[69] In order to rigorously justify rolling year indices from the viewpoint of the economic approach to index number theory, some restrictions on preferences are required. The details of these assumptions can be found in Diewert (1999a; 56–61). The problems associated with forming annual indices from monthly or quarterly indices have not been completely resolved from the viewpoint of the economic approach to index number theory.

$\text{Num}^{51} \equiv \text{Num}^{50} - p^{50}\cdot q^{2} + p^{62}\cdot q^{2};$ $\quad$ $\text{Den}^{51} \equiv \text{Den}^{50} - p^{2}\cdot\Delta^{50} q^{2} + p^{2}\cdot\Delta^{62} q^{2};$

•••

$\text{Num}^{61} \equiv \text{Num}^{60} - p^{60}\cdot q^{12} + p^{72}\cdot q^{12};$ $\quad$ $\text{Den}^{61} \equiv \text{Den}^{60} - p^{12}\cdot\Delta^{60} q^{12} + p^{12}\cdot\Delta^{72} q^{12};$

The period *t rolling year* (fixed-base maximum overlap) *Laspeyres index* is defined as

$$P_{LRY}^{t*} \equiv Num^t/Den^t;\ t = 1,\ldots,61. \qquad (146)$$

These rolling year Laspeyres indices are listed in Table 9.25 and are plotted in Figure 9.13.

Recall that in Section 5, the maximum overlap annual fixed-base Laspeyres indices, P_{LFB}^{y*} were defined by (69) for years $y = 1,\ldots,6$. It can be verified that the Rolling Year Laspeyres indices P_{LRY}^{t*} coincide with the earlier annual indices P_{LFB}^{y*} for t = 1, 13, 25, 37, 49, and 61 and $y = 1,\ldots,6$; that is, we have $P_{LRY}^{1*} = P_{LFB}^{1*} = 1$, $P_{LRY}^{13*} = P_{LFB}^{2*}$, $P_{LRY}^{25*} = P_{LFB}^{3*}$, $P_{LRY}^{37*} = P_{LFB}^{4*}$, $P_{LRY}^{49*} = P_{LFB}^{5*}$, and $P_{LRY}^{61*} = P_{LFB}^{6*}$. Thus, the new rolling year Laspeyres indices defined in this section are a natural extension of the fixed-base Mudgett Stone maximum overlap Laspeyres annual indices defined in Section 5. The new indices provide a seasonally adjusted measure of annual inflation for the current split year that consists of the last consecutive 12 months relative to the corresponding seasonal prices prevailing in a base year.

The aforementioned algebra is modified to define the *rolling year fixed-base maximum overlap Paasche indices*, P_{PRY}^{t*}. The numerators, Num^t, and the denominators, Den^t, for P_{PRY}^{t*} are defined as follows:

$\text{Num}^{1} \equiv \Sigma_{t=1}^{12}\, p^{t}\cdot q^{t};$ $\quad$ $\text{Den}^{1} \equiv \Sigma_{t=1}^{12}\, p^{t}\cdot q^{t};$

$\text{Num}^{2} \equiv \text{Num}^{1} - p^{1}\cdot q^{1} + p^{13}\cdot\Delta^{1} q^{13};$ $\quad$ $\text{Den}^{2} \equiv \text{Den}^{1} - p^{1}\cdot q^{1} + p^{1}\cdot q^{13};$

$\text{Num}^{3} \equiv \text{Num}^{2} - p^{2}\cdot q^{2} + p^{14}\cdot\Delta^{2} q^{14};$ $\quad$ $\text{Den}^{3} \equiv \text{Den}^{2} - p^{2}\cdot q^{2} + p^{2}\cdot q^{14};$

$\text{Num}^{4} \equiv \text{Num}^{3} - p^{3}\cdot q^{3} + p^{15}\cdot\Delta^{3} q^{15};$ $\quad$ $\text{Den}^{4} \equiv \text{Den}^{3} - p^{3}\cdot q^{3} + p^{3}\cdot q^{15};$

•••

$\text{Num}^{13} \equiv \text{Num}^{12} - p^{12}\cdot q^{12} + p^{24}\cdot\Delta^{12} q^{24};$ $\quad$ $\text{Den}^{13} \equiv \text{Den}^{12} - p^{12}\cdot q^{12} + p^{12}\cdot q^{24};$

$\text{Num}^{14} \equiv \text{Num}^{13} - p^{13}\cdot\Delta^{1} q^{13} + p^{25}\cdot\Delta^{1} q^{25};$ $\quad$ $\text{Den}^{14} \equiv \text{Den}^{13} - p^{1}\cdot q^{13} + p^{1}\cdot q^{25};$

$\text{Num}^{15} \equiv \text{Num}^{14} - p^{14}\cdot\Delta^{2}q^{14} + p^{26}\cdot\Delta^{2}q^{26};$ $\quad$ $\text{Den}^{15} \equiv \text{Den}^{14} - p^{2}\cdot q^{14} + p^{2}\cdot q^{26};$

•••

$\text{Num}^{25} \equiv \text{Num}^{24} - p^{24}\cdot\Delta^{12}q^{24} + p^{36}\cdot\Delta^{12}q^{36};$ $\quad$ $\text{Den}^{25} \equiv \text{Den}^{24} - p^{12}\cdot q^{24} + p^{12}\cdot q^{36};$

$\text{Num}^{26} \equiv \text{Num}^{25} - p^{25}\cdot\Delta^{1}q^{25} + p^{37}\cdot\Delta^{1}q^{37};$ $\quad$ $\text{Den}^{26} \equiv \text{Den}^{25} - p^{1}\cdot q^{25} + p^{1}\cdot q^{37};$

$\text{Num}^{27} \equiv \text{Num}^{26} - p^{26}\cdot\Delta^{2}q^{26} + p^{38}\cdot\Delta^{2}q^{38};$ $\quad$ $\text{Den}^{27} \equiv \text{Den}^{26} - p^{2}\cdot q^{26} + p^{2}\cdot q^{38};$

•••

$\text{Num}^{37} \equiv \text{Num}^{36} - p^{36}\cdot\Delta^{12}q^{36} + p^{48}\cdot\Delta^{12}q^{48};$ $\quad$ $\text{Den}^{37} \equiv \text{Den}^{36} - p^{12}\cdot q^{36} + p^{12}\cdot q^{48};$

$\text{Num}^{38} \equiv \text{Num}^{37} - p^{25}\cdot\Delta^{1}q^{37} + p^{37}\cdot\Delta^{1}q^{49};$ $\quad$ $\text{Den}^{38} \equiv \text{Den}^{37} - p^{1}\cdot q^{37} + p^{1}\cdot q^{49};$

$\text{Num}^{39} \equiv \text{Num}^{38} - p^{26}\cdot\Delta^{2}q^{38} + p^{38}\cdot\Delta^{2}q^{50};$ $\quad$ $\text{Den}^{39} \equiv \text{Den}^{38} - p^{2}\cdot q^{38} + p^{2}\cdot q^{50};$

•••

$\text{Num}^{49} \equiv \text{Num}^{48} - p^{48}\cdot\Delta^{12}q^{48} + p^{60}\cdot\Delta^{12}q^{60};$ $\quad$ $\text{Den}^{49} \equiv \text{Den}^{48} - p^{12}\cdot q^{48} + p^{12}\cdot q^{60};$

$\text{Num}^{50} \equiv \text{Num}^{49} - p^{49}\cdot\Delta^{1}q^{49} + p^{61}\cdot\Delta^{1}q^{61};$ $\quad$ $\text{Den}^{50} \equiv \text{Den}^{49} - p^{1}\cdot q^{49} + p^{1}\cdot q^{61};$

$\text{Num}^{51} \equiv \text{Num}^{50} - p^{50}\cdot\Delta^{2}q^{50} + p^{61}\cdot\Delta^{2}q^{62};$ $\quad$ $\text{Den}^{51} \equiv \text{Den}^{50} - p^{2}\cdot q^{50} + p^{2}\cdot q^{62};$

•••

$\text{Num}^{61} \equiv \text{Num}^{60} - p^{60}\cdot\Delta^{12}q^{60} + p^{72}\cdot\Delta^{12}q^{72};$ $\quad$ $\text{Den}^{61} \equiv \text{Den}^{60} - p^{12}\cdot q^{60} + p^{12}\cdot q^{72}.$

The period *t rolling year* (fixed-base maximum overlap) *Paasche Index* is defined as

$$P_{PRY}^{t*} \equiv Num^t/Den^t;\ t = 1,\ldots,61. \qquad (147)$$

The period *t rolling year* (fixed-base maximum overlap) *Fisher Index*, P_{FRY}^{t*} is defined as the geometric mean of the period t Rolling year Laspeyres and Paasche indices:

$$P_{FRY}^{t*} \equiv (P_{LRY}^{t*}\, P_{PRY}^{t*})^{1/2};\ t = 1,\ldots,61. \qquad (148)$$

These rolling year Paasche and Fisher indices are listed in Table 9.25 and are plotted in Figure 9.13.[70]

Table 9.25 Rolling Year Maximum Overlap Laspeyres, Paasche, and Fisher Price Indices and Some Moving Average Approximations

t	P_{LRY}^{t*}	P_{PRY}^{t*}	P_{FRY}^{t*}	P_{SMA}^{t*}	P_{FMMA}^{t*}	P_{SMMA}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99986	0.99993	0.99990	0.99975	0.99984	0.99984
3	1.01275	1.00477	1.00875	1.00867	1.00831	1.00831
4	1.00672	0.99713	1.00192	1.00110	1.00158	1.00158
5	1.00599	0.99599	1.00097	1.00072	1.00037	1.00037
6	1.00217	0.98795	0.99504	0.99455	0.99487	0.99487
7	1.05211	1.01266	1.03220	1.01867	1.02025	1.02025
8	1.09356	1.03034	1.06148	1.03205	1.04268	1.04268
9	1.11096	1.04550	1.07774	1.04614	1.05676	1.05676
10	1.12245	1.05347	1.08741	1.05562	1.06698	1.06698
11	1.12983	1.05505	1.09180	1.07467	1.07090	1.07090
12	1.13288	1.05680	1.09418	1.07688	1.07315	1.07315
13	1.13733	1.06109	1.09855	1.08225	1.07893	1.07893
14	1.13936	1.06215	1.10008	1.08429	1.08124	1.08099
15	1.14418	1.06902	1.10596	1.08820	1.08665	1.08485

(Continued)

[70]The indices defined in the last three columns of Table 9.25 will be defined later.

Table 9.25 (Continued)

t	P_{LRY}^{t*}	P_{PRY}^{t*}	P_{FRY}^{t*}	P_{SMA}^{t*}	P_{FMMA}^{t*}	P_{SMMA}^{t*}
16	1.15089	1.07688	1.11327	1.09543	1.09337	1.09134
17	1.15589	1.08247	1.11858	1.10163	1.09979	1.09776
18	1.16438	1.09490	1.12910	1.11089	1.10894	1.10602
19	1.14176	1.08425	1.11264	1.09874	1.09814	1.09323
20	1.11904	1.07086	1.09469	1.10062	1.08460	1.08040
21	1.11661	1.06762	1.09184	1.09500	1.08185	1.07840
22	1.12505	1.07248	1.09845	1.10207	1.08856	1.08601
23	1.12847	1.07606	1.10195	1.10829	1.09327	1.09149
24	1.12908	1.07732	1.10289	1.10901	1.09458	1.09221
25	1.12732	1.07502	1.10086	1.10648	1.09196	1.08956
26	1.12612	1.07412	1.09981	1.10554	1.09071	1.08856
27	1.12030	1.06836	1.09403	1.10077	1.08528	1.08385
28	1.11999	1.06776	1.09356	1.10068	1.08496	1.08377
29	1.12900	1.07624	1.10231	1.11200	1.09596	1.09390
30	1.15203	1.09503	1.12317	1.13460	1.11745	1.11406
31	1.18096	1.11449	1.14724	1.15222	1.13439	1.13259
32	1.18007	1.11904	1.14915	1.15350	1.13552	1.13385
33	1.16794	1.10638	1.13674	1.14500	1.12513	1.12501
34	1.16815	1.10961	1.13850	1.14756	1.12787	1.12777
35	1.17315	1.12032	1.14643	1.13882	1.13684	1.13597
36	1.18452	1.13220	1.15807	1.15384	1.15148	1.15118
37	1.19194	1.14068	1.16603	1.16389	1.16170	1.16144
38	1.19810	1.14830	1.17294	1.17368	1.17126	1.17173
39	1.21756	1.16467	1.19082	1.19068	1.18805	1.18849
40	1.22668	1.17024	1.19813	1.20012	1.19509	1.19647
41	1.22795	1.17032	1.19879	1.20299	1.19598	1.19801
42	1.21506	1.16311	1.18880	1.19000	1.18545	1.18642
43	1.20435	1.14967	1.17670	1.18052	1.17641	1.17645
44	1.21910	1.16329	1.19087	1.19031	1.18707	1.18601
45	1.23970	1.18457	1.21182	1.20998	1.20506	1.20239
46	1.24684	1.18607	1.21608	1.21369	1.20894	1.20639
47	1.24310	1.17770	1.20996	1.22533	1.20009	1.20336
48	1.23297	1.16529	1.19866	1.21184	1.18601	1.18911
49	1.22527	1.15652	1.19040	1.20236	1.17608	1.17882
50	1.22248	1.15126	1.18633	1.19785	1.17143	1.17343
51	1.22365	1.15275	1.18767	1.19847	1.17242	1.17404
52	1.22568	1.15747	1.19108	1.20084	1.17603	1.17667
53	1.22548	1.15675	1.19062	1.19745	1.17531	1.17467
54	1.23906	1.16849	1.20326	1.21183	1.18741	1.18749
55	1.23623	1.17260	1.20399	1.21181	1.18786	1.18748
56	1.22222	1.15521	1.18824	1.20227	1.17682	1.17816
57	1.20681	1.14061	1.17325	1.18706	1.16390	1.16641
58	1.20534	1.14562	1.17510	1.18791	1.16585	1.16733
59	1.21683	1.15703	1.18655	1.18250	1.18044	1.17700
60	1.22373	1.16429	1.19364	1.19305	1.19014	1.18686
61	1.23439	1.17520	1.20443	1.20662	1.20447	1.20137
Mean	1.15940	1.10380	1.13120	1.13060	1.12120	1.12050

Using the means listed in Table 9.25, it can be seen that the rolling year Laspeyres indices, P_{LRY}^{t*}, are on average 2.8 percentage points above the corresponding rolling year Fisher indices, P_{FRY}^{t*}, while the rolling year Paasche indices, P_{PRY}^{t*}, are on average 2.7 percentage points below the corresponding rolling year Fisher indices. This indicates that the Laspeyres and Paasche indices suffer from a considerable amount of substitution bias.

It can be verified that the rolling year Laspeyres, Paasche, and Fisher indices, P_{LRY}^{t*}, P_{PRY}^{t*}, and P_{FRY}^{t*}, coincide with the corresponding annual indices, listed in Table 9.12 in Section 5, P_{LFB}^{y*}, P_{PFB}^{y*}, and P_{FFB}^{y*}, for $t = 1, 13, 25, 37, 49$, and 61 and $y = 1,\ldots,6$.

The indices listed in Table 9.25 are plotted in Figure 9.13.

A comparison of the Laspeyres, Paasche, and Fisher rolling year indices shown in Figure 9.13 with the month-to-month indices that are plotted in Figures 9.7 to 9.12 indicates that the Rolling Year indices are much less variable than any of the month-to-month indices. Thus the rolling year indices both capture the trend in inflation and eliminate the seasonal fluctuations in the month-to-month measures of inflation. The upward bias in the rolling year Laspeyres index and the downward bias in the rolling year Paasche index are apparent in Figure 9.13.

At this point, our conclusions in this section are as follows:

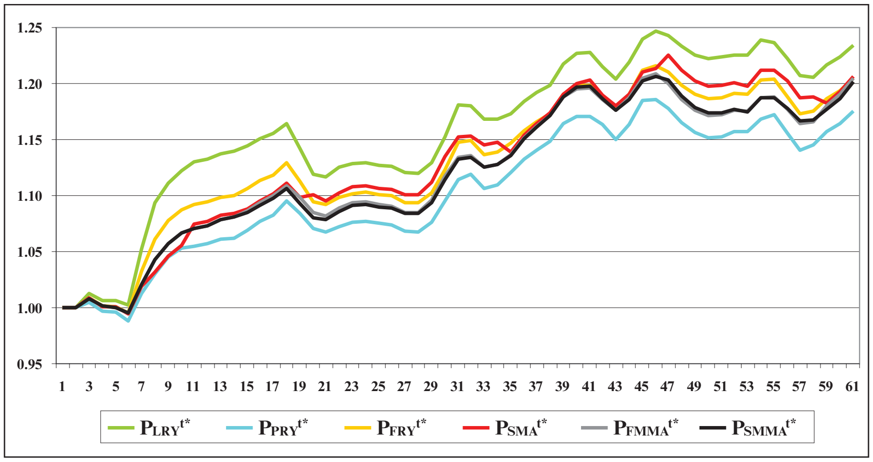
- Rolling year maximum overlap Laspeyres, Paasche, and Fisher indices can readily be calculated provided monthly information on prices and quantities is available.
- These indices are a natural generalization of the annual Mudgett Stone indices defined in Section 5 to provide annual index numbers for non-calendar years. They have the advantage that they can provide a new measure of trend inflation each month; that is, one does not have to wait until the end of a calendar year to get a current measure of inflation.
- These indices can be regarded as seasonally adjusted measures of trend inflation that is centered in the middle of the current non-calendar year that consists of the last string of 12 consecutive months. In the strongly seasonal products context, these indices provide the most accurate measures of inflation.
- As usual, the rolling year maximum overlap Fisher index of annualized inflation is preferred over the counterpart rolling year Laspeyres and Paasche indices which suffer from substitution bias.

In addition to the rolling year fixed-base Laspeyres, Paasche, and Fisher indices that are listed in Table 9.25 and plotted in Figure 9.13, there are three indices that are listed in Table 9.25. These additional indices are approximations to other indices and hence are not of primary importance, but they are of interest.

The first additional index of interest is a moving average of our "best" month-to-month maximum overlap similarity-linked indices P_S^{t*} defined in Section 9. How well can such an index approximate the rolling year Fisher index P_{FRY}^{t*} defined earlier? The 12-month moving average of P_S^{t*}, P_{MA}^{t}, is defined as follows:

$$P_{MA}^{1} \equiv (1/12)\Sigma_{m=1}^{12} P_S^{m*};\ P_{MA}^{t} \equiv P_{MA}^{t-1} + (1/12)P_S^{t+11*} - (1/12)P_S^{t-1*};\ t = 2,3,\ldots,61. \quad (149)$$

Figure 9.13 Rolling Year Laspeyres, Paasche, and Fisher Indices and Some Approximations



To make a price index out of these series of moving averages, divide $P_{MA}{}^{t}$ by $P_{MA}{}^{1}$. Thus, the *smoothed version of the month-to-month similarity-linked indices* $P_S{}^{t*}$ is the 12-month moving average series $P_{SMA}{}^{t*}$ defined as follows;

$$P_{SMA}{}^{t*} \equiv P_{MA}{}^{t}/P_{MA}{}^{1};\ t = 1,2,\ldots,61. \tag{150}$$

The smoothed month-to-month similarity indices $P_{SMA}{}^{t*}$ represent estimates of the *trend* in the month-to-month relative price similarity-linked indices $P_S{}^{t*}$. Thus, $P_{SMA}{}^{1}$ represents the trend in $P_S{}^{1*}$– $P_S{}^{12*}$ centered in the middle of year 1 of our sample; $P_{SMA}{}^{2}$ represents the trend in $P_S{}^{2*}$– $P_S{}^{13*}$ centered in the middle of the split year consisting of months 2–12 in year 1, January in year 2, and so on. Table 9.25 and Figure 9.13 show that the trend indices $P_{SMA}{}^{t*}$ are fairly close to the rolling year fixed-base maximum overlap Fisher indices $P_{FRY}{}^{t*}$; the two indices end up at 1.2066 and 1.2044, respectively and their means are 1.1306 and 1.1312, respectively. Thus for our particular data set, the rolling year Fisher indices not only have an explicit annual index number interpretation, but they also provide an estimate for the trend in the month-to-month similarity-linked Fisher indices, $P_S{}^{t*}$.[71]

We conclude this section by describing the last two additional indices of interest that are shown in Figure 9.13.

In Section 4, we saw that the true Mudgett Stone annual Laspeyres index could be computed as a share-weighted average of the monthly year-over-year indices. In Section 5, we took a simple equally weighted average of the maximum overlap fixed-base year-over-year monthly Fisher indices $P_{FFB}{}^{y,m*}$ and showed that the resulting index could provide an approximation to the "true" annual Mudgett Stone fixed-base Fisher indices $P_{FFB}{}^{y*}$. The resulting annual Mudgett Stone indices were defined by (82) and denoted by $P_{FFBA}{}^{y*}$ for $y = 1,\ldots,Y$. The same type of approximation can be made for the rolling year Fisher indices, $P_{FRY}{}^{t*}$. We indicate how these approximate rolling year Fisher indices $P_{FMMA}{}^{t*}$ can be defined.

$P_{FFB}{}^{y,m*}$ is the year-over-year monthly fixed-base maximum overlap Fisher price index for month m in year y. These indices were defined in Section 3 and are listed in Table A.22 in the annex. We simplify the notation and define $P(y,m)$ as follows:

$$P(y,m) \equiv P_{FFB}{}^{y,m*};\ y = 1,\ldots,6;\ m = 1,\ldots,12. \tag{151}$$

The 12-month moving averages of these indices, P^t for $t = 1,\ldots,61$, are defined as follows:

$$P^1 \equiv (1/12)\Sigma_{m=1}{}^{12}\ P(1,m)$$
$$P^2 \equiv P^1 + (1/12)P(2,1) - (1/12)P(1,1)$$
$$P^3 \equiv P^2 + (1/12)P(2,2) - (1/12)P(1,2)$$
$$P^4 \equiv P^3 + (1/12)P(2,3) - (1/12)P(1,3)$$
•••
$$P^{13} \equiv P^{12} + (1/12)P(2,12) - (1/12)P(1,12)$$
$$P^{14} \equiv P^{13} + (1/12)P(3,1) - (1/12)P(2,1)$$
$$P^{15} \equiv P^{14} + (1/12)P(3,2) - (1/12)P(2,2)$$
•••
$$P^{25} \equiv P^{24} + (1/12)P(3,12) - (1/12)P(2,12)$$
$$P^{26} \equiv P^{25} + (1/12)P(4,1) - (1/12)P(3,1)$$
$$P^{27} \equiv P^{24} + (1/12)P(4,2) - (1/12)P(3,2)$$
•••
$$P^{37} \equiv P^{36} + (1/12)P(4,12) - (1/12)P(3,12)$$
$$P^{38} \equiv P^{36} + (1/12)P(5,1) - (1/12)P(4,1)$$
$$P^{39} \equiv P^{37} + (1/12)P(5,2) - (1/12)P(4,2)$$
•••

[71] However, it should be kept in mind that the similarity-linked month-to-month indices $P_S{}^{t*}$ are conceptually quite different from the Rolling Year Fisher indices $P_{FRY}{}^{t*}$. In the case where all products are strongly seasonal and appear in only one month of the year, the Rolling Year Mudgett Stone indices are still well defined and meaningful from an economic perspective, whereas month-to-month indices maximum overlap indices cannot even be defined in this case. For a review of the early history of time series methods for measuring trends and providing seasonally adjusted series, see Diewert, Alterman, and Feenstra (2012). Oskar Anderson (1927; 552–554) provided a very clear statement of the arbitrariness of existing methods for decomposing time series into trend, seasonal, and erratic components.

$P^{49} \equiv P^{48} + (1/12)P(5,12) - (1/12)P(4,12)$

$P^{50} \equiv P^{49} + (1/12)P(6,1) - (1/12)P(5,1)$

$P^{51} \equiv P^{50} + (1/12)P(6,2) - (1/12)P(5,2)$

...

$P^{61} \equiv P^{60} + (1/12)P(6,12) - (1/12)P(5,12).$

Normalize this 12-month moving averages into an index which equals 1 in the base period. Define the *moving average index of the year-over-year monthly fixed-base maximum overlap Fisher indices*, $P_{FMMA}{}^{t*}$, as follows:

$$P_{FMMA}{}^{t*} \equiv P^t/P^1;\ t = 1,...,61, \qquad (152)$$

where P^t are defined by (151). $P_{FMMA}{}^{t*}$ are listed in Table 9.25 and are plotted in Figure 9.13.

Instead of using the year-over-year monthly fixed-base maximum overlap Fisher indices as basic building blocks to form the approximate rolling year index $P_{FMMA}{}^{t*}$, other year-over-year indices could be used as basic monthly building blocks, such as the maximum overlap similarity-linked monthly year-over-year monthly indices $P_S{}^{y,m*}$ listed in Table 9.6 in Section 3. These indices are also listed in Table A.22 in the annex. To construct the resulting approximate similarity-linked rolling year index $P_{SMMA}{}^{t*}$, redefine $P(y,m)$ as follows:

$$P(y,m) \equiv P_S{}^{y,m*};\ y = 1,...,6;\ m = 1,...,12. \qquad (153)$$

The 12-month moving averages of these indices, P^t for t = 1,. . .,61, can be defined using the algebra listed in (151) but using definitions (153) for $P(y,m)$. Define the *moving average index of the year-over-year monthly similarity-linked indices*, $P_{SMMA}{}^{t*}$, as follows:

$$P_{SMMA}{}^{t*} \equiv P^t/P^1;\ t = 1,...,61, \qquad (154)$$

where the P^t are defined by the algebra following (151) . $P_{SMMA}{}^{t*}$ are listed in Table 9.25 and are plotted in Figure 9.13.

From Table 9.25 and Figure 9.13, it can be seen that the indices $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ (which are normalized 12-month moving average series of the Fisher fixed-base and similarity-linked maximum overlap year-over-year monthly indices $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$) closely approximate each other and can barely be distinguished in Figure 9.13. This is to be expected since the underlying year-over-year monthly series, $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$, closely approximate each other.

In Section 5, two approximate annual maximum overlap Mudgett Stone indices, $P_{FFB}{}^{y*}$ and $P_{SA}{}^{y*}$, were defined and listed in Table 9.13 for y = 1,. . .,6. The indices $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ are extensions of these indices to rolling years. Thus, we have $P_{FFB}{}^{2*} = P_{FMMA}{}^{13*}$, $P_{FFB}{}^{3*} = P_{FMMA}{}^{25*}$, $P_{FFB}{}^{4*} = P_{FMMA}{}^{37*}$, $P_{FFB}{}^{5*} = P_{FMMA}{}^{49*}$, and $P_{FFB}{}^{6*} = P_{FMMA}{}^{61*}$. Similarly, comparing entries in Tables 9.13 and 9.25, we have $P_{SA}{}^{2*} = P_{SMMA}{}^{13*}$, $P_{SA}{}^{3*} = P_{SMMA}{}^{25*}$, $P_{SA}{}^{4*} = P_{SMMA}{}^{37*}$, $P_{SA}{}^{5*} = P_{SMMA}{}^{49*}$, and $P_{SA}{}^{6*} = P_{SMMA}{}^{61*}$. Thus, the indices $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ are natural extensions of the approximate calendar year annual Mudgett Stone indices $P_{FFB}{}^{y*}$ and $P_{SA}{}^{y*}$ to split years.

Our preferred rolling year index is the rolling year maximum overlap fixed-base Fisher index $P_{FRY}{}^{t*}$. This index and the two approximate indices $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ ended up at 1.20433, 1.20447, and 1.20137, respectively, for our empirical example, which is more or less the same place. However, the means of the three indices were 1.1312, 1.1212, and 1.1205, respectively. Thus, the two approximate indices were on average about 1 percentage point below the mean of the Fisher rolling year index, $P_{FRY}{}^{t*}$. The two approximate rolling year indices capture the trend quite well, but they give equal weights to each of the 12 months in the rolling year and thus are not as accurate (from the viewpoint of the economic approach to index number theory) as the rolling year Fisher index which weights the 12 year-over-year monthly indices according to their economic importance.

From Table 9.25 and Figure 9.13, it can be seen that the indices $P_{FMMA}{}^{t*}$ and $P_{SMMA}{}^{t*}$ (which are normalized 12-month moving average series of the Fisher fixed-base and similarity-linked maximum overlap year-over-year monthly indices $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$) closely approximate each other and can barely be distinguished in Figure 9.13. This is to be expected since the underlying year-over-year monthly series, $P_{FFB}{}^{y,m*}$ and $P_S{}^{y,m*}$, closely approximate each other.

12. Conclusion

The existence of strongly seasonal products raises a number of problems that national statistical offices face when attempting to construct CPIs that include strongly seasonal product categories.

This chapter has considered *four main classes of alternative price indices* that could be constructed for a strongly seasonal class of products:

- Year-over-year monthly indices (see Sections 2 and 3);
- Annual indices (see Sections 4,5, and 11);
- Month-to-month indices that measure consumer price inflation going from one month to the next month (see Sections 6 and 7 for indices that make use of price and quantity information and Sections 8 and 9 for indices that use only price information); and
- Month-to-month annual basket indices (or annual share indices) that make use of annual quantities or annual expenditure shares for a base year and monthly prices (see Section 10 for the Lowe and Young indices).

As was discussed in Section 10, in the strongly seasonal products context, Lowe or Young indices have little intuitive appeal. Consumers do not purchase an annual basket of strongly seasonal products in each month nor do they face carry-forward prices each month for this hypothetical annual basket of products.

The other three types of index have strong justifications. Month-to-month indices are required by central banks and others to monitor short-run movements in inflation. Annual indices are needed as deflators to produce annual constant dollar national accounts. It turns out that in the strongly seasonal products context, year-over-year monthly indices are far more accurate measures of inflation than month-to-month indices. Moreover, the year-over-year monthly indices are basic building blocks for accurate annual indices. *Thus in the strongly seasonal products context, all three types of index serve a useful purpose.* This is our first important conclusion.

There are five other more technical issues that proved to be important in producing price indices for strongly seasonal product groups:

- *Should carry-forward/*carry-*backward prices be used for missing prices in constructing a price index* or should maximum overlap indices be produced (which is roughly equivalent to using inflation-adjusted carry-forward prices for missing prices)? Common sense and our computations show that the use of carry-forward prices in the strongly seasonal products context will lead to a downward bias in the index if there is general inflation (and vice versa if there is general deflation as has occurred in Japan at times). Thus the use of carry-forward prices is not recommended.
- *Are monthly price and quantity (or expenditure) data available or are just monthly price data available*? The type of index that can be produced depends on data availability. Of course, indices that make use of price and quantity information are preferred, but statistical offices usually do not have price and quantity information, so the issue of which index to use in the prices only situation is important. Our results in Section 9 show that, for our empirical example, it is possible to come up with a prices only index that can provide a fairly satisfactory approximation to our "best" index that makes use of price and quantity information.
- *What is the "best" bilateral index number formula to use when making price comparisons between two periods*? When price and quantity information are available for the two periods under consideration, the Fisher price index is a good candidate for the "best" index. It has very good properties from the perspectives of both the economic approach to index number theory[72] and the test approach.[73] When only price information is available, the choice of a "best" functional form for a bilateral index is not so clear. If there are no missing prices (or if prices are completely matched across the two periods), then the Jevons index has the best axiomatic properties.[74] In the case where prices are not matched across the two periods, the best approach at this stage of our knowledge is probably the maximum overlap Jevons index.
- However, the choice of a "best" bilateral index number formula is not the end of the story. In making index comparisons across multiple time periods using bilateral indices as basic building blocks to link the prices of any pair of periods, one has to choose a path of bilateral links in order to link all of the periods. For example, one can choose the first period as the base period and link all subsequent periods to this base period, generating a sequence of fixed-base indices. Or one can calculate a chained index where the prices of period t are linked to the prices of period $t - 1$ and this chain link index is used to update the period t index level. The problem with fixed-base indices in the strongly seasonal context when producing month-to-month indices is that the choice of base period matters to a very significant degree; see Figure 9.5 in Section 6 and Figure 9.7 in Section 7. The problem with chained indices is that they are subject to *chain drift*; that is, if prices are identical in any two periods, it is desirable that the price index register the same index level for those two periods. Fixed base indices will satisfy this test but chained indices will in general not satisfy this multiperiod identity test. There are numerous examples in this chapter that show that chain drift can be a very significant problem when one uses chained indices. *Thus there is the problem of choosing a "best" path to link bilateral price indices into a single index*. Our suggested solution to this problem is to use a *measure of relative price dissimilarity* between the prices of any two periods and choose a path of bilateral links that minimizes the measure of price dissimilarity between the prices of the current period and the prices of all previous periods (up to some specified limit on how far back we want to go with the bilateral relative price comparisons). The price dissimilarity measure determines the path of bilateral links. If price and quantity information is available, then bilateral maximum overlap Fisher indices are used to make the bilateral links in the chosen path. If only price information is available, then maximum overlap Jevons indices are used to make the bilateral links. The resulting indices satisfy the multiperiod identity test and hence are free from chain drift. The main problem with this methodology is this: what is the "best" dissimilarity measure that could be used? We do not provide a definitive answer to this question but *the predicted share measure of relative price dissimilarity* suggested by Diewert (2021b) seems to work well for our empirical example when price and quantity information is available. When only price information is available, we adapted the predicted share measure of relative price dissimilarity to deal with this case; see definitions (131) and (139) in Section 9. For our particular example, this *modified predicted share* method (P_{SJ}^{t*}) that used maximum overlap Jevons indices for the bilateral links provided the closest approximation to our preferred predicted share similarity-linked indices (P_S^{t*}) that used price and quantity information and maximum overlap Fisher indices for the bilateral links.[75]
- *How to trade off a lack of matching of prices over two periods with a lack of price proportionality in the matched prices for the two periods*? In Section 9, we showed how the modified predicted share measure of relative price dissimilarity traded off a lack of matching of product prices over the two periods under consideration with a measure of relative price dissimilarity of the matched prices for the two periods. In the strongly seasonal products context, it is important to have a penalty for a lack of matching of prices between the two periods being compared. Consider an extreme case where we are matching the prices of a current period with the prices of two prior periods. For period 1, there is only one matched product and so if we look at only matched product prices, the matched prices of period 1 and 3 are proportional and any reasonable measure of relative price dissimilarity defined over matched prices will register a value of 0. On the other hand, there are 10 matched prices for periods 2 and 3, but the resulting matched prices are not quite proportional so the measure of relative price dissimilarity over matched products registers a positive value. Is it "best" to link the prices of period 3 with the single price of period 1 rather than to link the

[72] See Diewert (1976).
[73] See Diewert (1992; 221).
[74] See Diewert (1995). The economic approach to index number theory that relies on exact index number formulae cannot be implemented if only price information is available.
[75] See Figure 9.11 in Section 9.

prices of period 3 with the prices of period 2? Probably not. Thus we think it is important for a bilateral measure of relative price dissimilarity to have a penalty for a lack of matching of prices between the two periods. The Predicted Share and Modified Predicted Share measures of relative price dissimilarity do have a penalty for a lack of matching. Further research is required to see if "better" measures of relative price dissimilarity can be found.[76]

Another area that requires further research is the problem of integrating an elementary index for a strongly seasonal class of products with indices for other elementary categories where the problems associated with missing prices are not as severe. Thus different elementary categories of a national CPI may use different methods for constructing the various subindices. As a result, it may become difficult to explain and interpret the resulting national index.

For our data set, the year-over-year monthly indices (January data compared across years, February data compared across years, and so on) performed well. Thus for National Statistical Offices that use Lowe or Young indices, we suggest the use of monthly baskets for strongly seasonal products so that reasonably accurate year-over-year monthly Lowe indices could be computed. However, the problem is how to link these indices for a base year so that the year-over-year indices could be aligned to provide some indication of January to February inflation, February to March inflation, and so on for a base year. This could be done for the base year using some form of relative price similarity linking, or one could choose a base month in the base year which had the highest number of available products and use fixed-base maximum overlap Lowe or Fisher indices to link the months in the base year. Then going forward, year-over-year monthly Lowe type indices could be used to calculate the index.[77] Our general advice to National Statistical Officies is to explore the use of monthly baskets and similarity linking, particularly for periods with large shifts in consumer expenditure patterns, such as during the COVID-19 pandemic.[78]

Finally, we note that we have listed the complete data set that we used in the annex so that our results can be replicated by statistical agencies.[79] Moreover, this data set could be used by other researchers to construct alternative indices, which may have superior properties.[80]

[76] Research on this topic is sparse, but see Hill and Timmer (2006) for an alternative approach to these issues.

[77] The problems associated with reconciling the year-over-year estimates of inflation for each month with month-to-month estimates of inflation within a given year are similar to the problems associated with reconciling year-over-year annual CPI country inflation estimates with estimates of inflation across countries for the same year. The annual CPI inflation rates for a given country are very likely to be much more accurate than a measure of relative inflation across countries due to better matching of product prices within a country, which is analogous to the better matching of product prices across years for the same month in the strongly seasonal context. For a discussion of alternative approaches to reconciling the conflicting estimates of inflation, see Diewert and Fox (2017) (2018).

[78] Statistics Canada has used the predicted share linking methodology in its adjusted CPI; see *O*' Donnell and Yélou (2021).

[79] We have also taken care to carefully explain exactly how the various indices listed in this chapter were constructed.

[80] Turvey's (1979) artificial data set on seasonal products filled this role for many years.

Annex: Listing of the Data and Supplementary Tables

1. Year-over-Year Monthly Indices Using Year-over-Year Carry-Forward Prices

In order to illustrate the variation in the various seasonal product indices using actual country data, we tabulate the various indices described in the main text for Israel for 14 fresh fruit household consumption categories over the six years (2012–2017) which we relabel as years 1–6. The 14 fresh fruit categories are as follows:

- 1 = Lemons
- 2 = Avocados
- 3 = Watermelon
- 4 = Persimmon
- 5 = Grapefruit
- 6 = Bananas
- 7 = Peaches
- 8 = Strawberries
- 9 = Cherries
- 10 = Apricots
- 11 = Plums
- 12 = Clementines
- 13 = Kiwi fruit
- 14 = Mangos

The price and quantity data for the available products in each month are listed in Tables A.1–A.20. The price and quantity for product *n* in month *m* in year *t* are denoted by $p_{y,m,n}$ and $q_{y,m,n}$, respectively.

Fruits 1, 2, 4, 5, 6, 12, and 13 were always available in January for each of the six years in our sample, but the other fruits were always missing in January. The price and quantity data for February and the remaining months follow. Prices and quantities for products that were missing in a given month for all six years are not listed in the tables.

Fruits 1, 2, 4, 5, 6, 8, 12, and 13 were always available in February for each of the six years in our sample, but the other fruits were always missing in February.

Note that product 4 was missing in March of year 5; that is, $q_{5,3,4} = 0$. The corresponding price, $p_{5,3,4} = 11.02$, is an *imputed carry-forward price* from March of year 4. In Table A.4, this imputed price is printed in *italics* to distinguish it from observed prices. Products 1, 2, 5, 6, 8, 12, and 13 were present in every April. Products 3, 7, 9, 10, and 11 were missing in every March.

Fruits 1, 2, 5, 6, 8, 12, and 13 were always available in April for each of the six years in our sample, but the other fruits were always missing in April.

Products 1, 2, 3, 5, 6, and 10 were always present in May. However, product 8 was only present in year 1 of our sample so the price for product 8 in year 1 was *carried forward* for years 2–6; thus $p_{y,5,8}$ is set equal to $p_{y',5,8} = 16.68$ for $y = 2,3,4,5,6$. Product 9 was missing in years 1 and 2 and so the price for product 9 in years 1 and 2 was set equal to the *carry-backward*

Table A.1 Year-over-Year Price and Quantity Data for Month 1 (January)

y	$p_{y,1,1}$	$p_{y,1,2}$	$p_{y,1,4}$	$p_{y,1,5}$	$p_{y,1,6}$	$p_{y,1,12}$	$p_{y,1,13}$	$q_{y,1,1}$	$q_{y,1,2}$	$q_{y,1,4}$	$q_{y,1,5}$	$q_{y,1,6}$	$q_{y,1,12}$	$q_{y,1,13}$
1	5.41	8.29	9.46	4.88	6.22	5.81	11.82	0.370	0.676	0.465	0.082	1.881	1.824	0.135
2	6.28	8.70	10.55	5.21	5.57	5.89	10.72	0.430	0.897	0.417	0.058	1.957	1.579	0.103
3	6.63	8.88	10.49	5.03	5.44	6.30	14.94	0.513	0.890	0.486	0.298	2.261	1.175	0.067
4	6.20	8.00	8.94	4.99	6.27	5.83	14.98	0.645	0.975	0.559	0.160	2.281	1.492	0.100
5	7.07	11.13	12.59	5.35	6.12	5.93	14.59	0.552	1.006	0.485	0.093	2.647	1.737	0.206
6	6.51	9.64	11.11	5.25	6.07	5.83	17.88	0.906	1.172	0.630	0.057	3.262	2.093	0.056

Table A.2 Year-over-Year Price Data for Month 2 (February)

y	$p_{y,2,1}$	$p_{y,2,2}$	$p_{y,2,4}$	$p_{y,2,5}$	$p_{y,2,6}$	$p_{y,2,8}$	$p_{y,2,12}$	$p_{y,2,13}$
1	4.99	8.37	10.20	4.84	6.90	15.08	6.32	12.78
2	5.93	9.15	11.41	5.21	5.57	23.29	6.43	11.58
3	5.97	8.84	11.32	5.03	5.98	25.11	6.50	14.92
4	5.97	8.15	9.95	5.14	6.06	23.49	5.94	15.41
5	6.99	12.27	13.22	5.09	7.22	26.86	6.15	14.88
6	6.39	10.59	11.85	5.00	8.23	28.26	5.65	18.97

Table A.3 Year-over-Year Quantity Data for Month 2 (February)

y	$q_{y,2,1}$	$q_{y,2,2}$	$q_{y,2,4}$	$q_{y,2,5}$	$q_{y,2,6}$	$q_{y,2,8}$	$q_{y,2,12}$	$q_{y,2,13}$
1	0.701	0.920	0.510	0.103	2.087	1.134	1.408	0.102
2	0.624	0.831	0.412	0.269	2.621	0.593	1.664	0.155
3	0.754	1.075	0.486	0.119	2.308	0.737	1.492	0.168
4	0.553	1.031	0.412	0.156	2.591	0.766	1.919	0.117
5	0.658	0.717	0.386	0.157	2.299	0.648	1.886	0.108
6	0.657	0.859	0.447	0.260	2.211	0.711	2.071	0.111

Table A.4 Year-over-Year Price Data for Month 3 (March)

y	$p_{y,3,1}$	$p_{y,3,2}$	$p_{y,3,4}$	$p_{y,3,5}$	$p_{y,3,6}$	$p_{y,3,8}$	$p_{y,3,12}$	$p_{y,3,13}$
1	5.14	8.59	10.76	4.92	7.42	18.67	6.62	13.34
2	5.70	9.43	11.69	5.16	6.11	15.31	6.64	11.72
3	5.72	9.47	12.41	4.97	6.51	18.23	6.82	15.36
4	6.08	9.06	11.02	4.98	6.83	18.95	6.17	15.73
5	6.78	13.98	*11.02*	5.13	7.51	18.06	6.03	15.11
6	6.32	11.05	13.66	5.24	8.85	19.26	6.06	19.66

Table A.5 Year-over-Year Quantity Data for Month 3 (March)

y	$q_{y,3,1}$	$q_{y,3,2}$	$q_{y,3,4}$	$q_{y,3,5}$	$q_{y,3,6}$	$q_{y,3,8}$	$q_{y,3,12}$	$q_{y,3,13}$
1	0.661	0.908	0.362	0.081	1.819	0.884	1.269	0.112
2	0.684	0.732	0.257	0.116	2.242	0.947	1.160	0.154
3	0.822	0.612	0.290	0.121	2.012	0.845	1.496	0.085
4	0.658	0.828	0.209	0.181	2.255	0.813	1.556	0.108
5	0.708	0.694	0.000	0.234	2.490	1.107	1.509	0.152
6	0.759	0.787	0.293	0.095	2.395	1.038	1.650	0.102

Table A.6 Year-over-Year Price and Quantity Data for Month 4 (April)

y	$p_{y,4,1}$	$p_{y,4,2}$	$p_{y,4,5}$	$p_{y,4,6}$	$p_{y,4,8}$	$p_{y,4,12}$	$p_{y,4,13}$	$q_{y,4,1}$	$q_{y,4,2}$	$q_{y,4,5}$	$q_{y,4,6}$	$q_{y,4,8}$	$q_{y,4,12}$	$q_{y,4,13}$
1	5.08	9.06	5.13	7.25	18.24	7.01	13.70	0.689	0.585	0.156	1.876	0.609	0.728	0.131
2	6.84	11.00	5.43	6.26	16.62	7.18	12.42	0.760	0.591	0.092	2.141	0.698	0.766	0.129
3	6.00	10.27	5.09	7.60	17.80	7.72	16.91	0.617	0.662	0.157	1.737	0.663	0.997	0.053
4	7.04	12.60	5.41	9.68	18.35	7.03	16.30	0.895	0.683	0.129	1.550	0.687	1.252	0.135
5	7.05	18.26	5.07	8.40	18.80	6.58	16.36	0.766	0.460	0.079	1.988	0.585	1.231	0.122
6	6.47	12.59	5.45	10.75	16.85	6.28	20.39	0.773	0.627	0.037	2.047	0.926	1.210	0.069

Table A.7 Year-over-Year Price Data for Month 5 (May)

y	$p_{y,5,1}$	$p_{y,5,2}$	$p_{y,5,3}$	$p_{y,5,5}$	$p_{y,5,6}$	$p_{y,5,7}$	$p_{y,5,8}$	$p_{y,5,9}$	$p_{y,5,10}$	$p_{y,5,13}$
1	5.19	11.48	4.14	5.27	7.05	11.50	16.68	*40.84*	12.16	13.69
2	7.35	14.62	3.49	5.67	5.96	11.08	*16.68*	*40.84*	9.46	*13.69*
3	6.60	13.66	4.10	5.34	7.60	10.62	*16.68*	40.84	14.79	19.93
4	7.73	15.92	4.56	5.39	13.19	11.75	*16.68*	61.43	17.78	17.16
5	7.52	19.36	4.07	5.81	8.98	11.27	*16.68*	39.10	18.31	17.33
6	7.00	15.34	4.77	6.16	12.30	12.95	*16.68*	39.10	18.03	22.56

Table A.8 Year-over-Year Quantity Data for Month 5 (May)

y	$q_{y,5,1}$	$q_{y,5,2}$	$q_{y,5,3}$	$q_{y,5,5}$	$q_{y,5,6}$	$q_{y,5,7}$	$q_{y,5,8}$	$q_{y,5,9}$	$q_{y,5,10}$	$q_{y,5,13}$
1	0.751	0.409	4.106	0.076	1.730	0.922	0.456	0.000	0.206	0.080
2	0.626	0.321	6.504	0.053	1.913	1.273	0.000	0.000	0.370	0.000
3	0.682	0.417	5.244	0.075	1.526	1.525	0.000	0.088	0.176	0.045
4	0.660	0.528	4.211	0.056	1.054	1.183	0.000	0.016	0.107	0.041
5	0.785	0.584	5.430	0.103	1.726	1.287	0.000	0.138	0.284	0.069
6	0.814	0.587	5.891	0.065	1.504	1.243	0.000	0.000	0.322	0.040

Table A.9 Year-over-Year Price Data for Month 6 (June)

y	$p_{y,6,1}$	$p_{y,6,2}$	$p_{y,6,3}$	$p_{y,6,5}$	$p_{y,6,6}$	$p_{y,6,7}$	$p_{y,6,9}$	$p_{y,6,10}$	$p_{y,6,11}$	$p_{y,6,13}$
1	5.66	11.83	3.24	5.57	5.92	10.08	17.44	8.82	11.05	13.74
2	7.83	19.58	3.36	5.86	5.85	11.25	42.05	14.44	12.61	*13.74*
3	6.64	14.00	2.55	5.63	8.07	10.42	32.81	13.25	14.03	27.25
4	8.62	18.98	3.68	*5.63*	12.71	10.80	34.48	12.36	13.56	21.55
5	9.01	20.42	2.67	*5.63*	10.99	9.73	34.21	15.05	13.62	22.38
6	8.20	18.56	2.93	*5.63*	11.15	9.81	31.08	14.71	14.01	26.03

Table A.10 Year-over-Year Quantity Data for Month 6 (June)

y	$q_{y,6,1}$	$q_{y,6,2}$	$q_{y,6,3}$	$q_{y,6,5}$	$q_{y,6,6}$	$q_{y,6,7}$	$q_{y,6,9}$	$q_{y,6,10}$	$q_{y,6,11}$	$q_{y,6,13}$
1	0.724	0.440	6.698	0.036	1.486	1.657	0.717	1.270	0.290	0.022
2	0.766	0.266	7.738	0.051	1.419	1.831	0.228	0.616	0.523	0.000
3	0.678	0.450	8.118	0.036	1.016	1.910	0.466	0.694	0.335	0.011
4	0.673	0.295	7.038	0.000	0.653	1.880	0.299	0.777	0.354	0.014
5	0.599	0.318	8.876	0.000	0.792	1.922	0.406	0.472	0.382	0.031
6	0.915	0.436	9.693	0.000	0.969	2.487	0.560	0.727	0.378	0.019

price for product 9 in year 3; that is, $p_{1,5,9}$ and $p_{2,5,9}$ were set equal to $p_{3,5,9}$ = 40.84, the price of product 9 in year 3. The price of product 13 was missing in year 2, so this missing price was set equal to the price of product 13 in year 1; that is, we have $p_{2,5,13} = p_{1,5,13}$. Thus, Table A.7 shows eight imputed prices (which are in italics): six imputed carry-forward prices and two imputed carry-backward prices. Products 4, 11, 12, and 14 were missing in May for every year in our sample.

There were four missing prices for the products that were available for one or more months in June. Product 5 was missing in years 4, 5, and 6 and product 13 was missing in year 2. These four missing prices were replaced by carry-forward prices (in italics) in Table A.9. Products 4, 8, 12, and 14 were always missing in June.

Product 2 was missing in years 2, 4, 5, and 6 and so carry-forward prices (in italics) appear for these four prices in Table A.11. Fruits 1, 2, 3, 6, 7, 9, 11, and 14 were available in at least one July; the remaining six fruits were not available in July.

Product 2 was missing in years 2, 4, 5, and 6 and so carry-forward prices (in italics) appear in these four prices in Table A.11. Product 9 was missing in years 1 and 2 (use carry-backward prices) and years 4, 5, and 6

Table A.11 Year-over-Year Price Data for Month 7 (July)

y	$p_{y,7,1}$	$p_{y,7,2}$	$p_{y,7,3}$	$p_{y,7,6}$	$p_{y,7,7}$	$p_{y,7,9}$	$p_{y,7,11}$	$p_{y,7,14}$
1	7.40	13.02	3.18	6.65	9.27	20.10	8.82	10.41
2	9.96	*13.02*	3.30	7.46	11.91	53.77	10.65	10.92
3	7.51	15.44	2.29	10.76	11.23	37.98	12.59	10.63
4	9.83	*15.44*	2.51	15.91	10.24	30.62	10.36	12.32
5	11.34	*15.44*	3.09	12.56	10.66	37.31	12.85	11.35
6	10.86	*15.44*	2.32	14.74	9.87	35.14	11.23	13.48

Table A.12 Year-over-Year Quantity Data for Month 7 (July)

y	$q_{y,7,1}$	$q_{y,7,2}$	$q_{y,7,3}$	$q_{y,7,6}$	$q_{y,7,7}$	$q_{y,7,9}$	$q_{y,7,11}$	$q_{y,7,14}$
1	0.595	0.292	8.145	0.722	2.093	0.488	0.964	0.221
2	0.612	0.000	7.394	0.871	1.520	0.073	0.761	0.421
3	0.746	0.389	9.869	0.539	1.915	0.179	0.667	0.546
4	0.600	0.000	8.486	0.289	2.129	0.349	0.685	0.390
5	0.635	0.000	8.188	0.701	2.073	0.198	0.545	0.643
6	0.847	0.000	12.845	0.468	2.837	0.361	0.784	0.593

Table A.13 Year-over-Year Price Data for Month 8 (August)

y	$p_{y,8,1}$	$p_{y,8,2}$	$p_{y,8,3}$	$p_{y,8,6}$	$p_{y,8,7}$	$p_{y,8,9}$	$p_{y,8,11}$	$p_{y,8,14}$
1	10.62	18.23	3.28	8.24	9.06	*22.50*	8.13	9.15
2	9.44	*18.23*	3.83	7.78	11.53	*22.50*	10.94	10.35
3	8.23	19.44	3.12	10.56	11.84	22.50	13.30	8.94
4	9.87	*19.44*	2.51	12.25	10.14	*22.50*	9.61	10.40
5	10.30	*19.44*	4.01	9.65	10.73	*22.50*	13.20	11.19
6	10.87	*19.44*	2.60	12.20	10.39	*22.50*	11.09	11.37

Table A.14 Year-over-Year Quantity Data for Month 8 (August)

y	$q_{y,8,1}$	$q_{y,8,2}$	$q_{y,8,3}$	$q_{y,8,6}$	$q_{y,8,7}$	$q_{y,8,9}$	$q_{y,8,11}$	$q_{y,8,14}$
1	0.452	0.159	6.159	0.558	1.932	0.000	1.009	0.721
2	0.625	0.000	5.065	0.746	1.761	0.000	0.914	0.850
3	0.656	0.180	5.577	0.616	1.791	0.031	0.759	1.040
4	0.719	0.000	7.371	0.498	1.765	0.000	0.832	0.673
5	0.718	0.000	4.963	0.974	2.171	0.000	0.750	1.028
6	0.690	0.000	8.423	0.770	2.348	0.000	0.748	0.730

(use carry-forward prices). Thus, there were nine missing prices for the August data. Fruits 1, 2, 3, 6, 7, 9, 11, and 14 were available in at least one August; the remaining six fruits were not available in August.

Fruits 1, 2, 6, 7 11, 12, and 14 were present in every September for the six years in our sample. The remaining seven products were absent in all September months.

Product 7 was missing in years 2, 3, and 5 and product 11 was missing in year 5. These four missing prices were replaced by carry-forward prices. Products 3, 8, 9, and 10 were missing in every October.

Product 11 was missing in years 2, 3, and 5 and product 14 was missing in years 2, 4, and 5. These six missing prices were replaced by carry-forward prices. Products 3, 7, 8, 9, and 10 were missing in every November.

Fruits 1, 2, 4, 5, 6, 12, and 13 were always available in December for each of the six years in our sample; the remaining seven fruits were always missing in December.

Table A.15 Year-over-Year Price and Quantity Data for Month 9 (September)

y	$p_{y,9,1}$	$p_{y,9,2}$	$p_{y,9,6}$	$p_{y,9,7}$	$p_{y,9,11}$	$p_{y,9,12}$	$p_{y,9,14}$	$q_{y,9,1}$	$q_{y,9,2}$	$q_{y,9,6}$	$q_{y,9,7}$	$q_{y,9,11}$	$q_{y,9,12}$	$q_{y,9,14}$
1	9.27	11.65	8.55	8.03	8.15	6.88	10.03	0.647	0.335	0.643	2.379	1.104	0.058	0.857
2	8.00	11.92	7.21	10.07	11.31	7.36	10.85	0.650	0.235	0.957	1.927	0.716	0.231	0.710
3	7.12	12.03	9.34	11.43	13.44	7.20	9.66	0.758	0.532	0.921	1.899	0.543	0.181	0.932
4	9.42	12.23	9.59	10.85	11.18	7.89	12.29	0.594	0.278	0.792	1.604	0.689	0.114	0.667
5	8.91	13.52	8.39	11.77	14.61	7.40	11.52	0.831	0.473	1.335	2.022	0.568	0.243	0.972
6	10.11	17.74	10.52	10.72	12.03	7.98	12.49	0.752	0.282	1.502	2.136	0.948	0.188	0.945

Table A.16 Year-over-Year Price Data for Month 10 (October)

y	$p_{y,10,1}$	$p_{y,10,2}$	$p_{y,10,4}$	$p_{y,10,5}$	$p_{y,10,6}$	$p_{y,10,7}$	$p_{y,10,11}$	$p_{y,10,12}$	$p_{y,10,13}$	$p_{y,10,14}$
1	8.15	11.26	11.45	6.59	7.93	9.18	8.18	6.19	14.63	9.87
2	8.03	9.8	12.3	6.43	6.8	*9.18*	12.32	7.19	15.47	13.09
3	7.14	10.51	12.98	6.21	8.79	*9.18*	14.5	7.33	17.43	9.67
4	9.53	11.54	12.85	7.28	8.91	11.94	11.58	7.22	20.19	11.88
5	8.35	12.65	13.61	6.57	7.88	*11.94*	*11.58*	7.07	22.85	12.87
6	9.62	14.86	13.85	6.81	8.92	12.67	13.13	7.09	21.74	12.49

Table A.17 Year-over-Year Quantity Data for Month 10 (October)

y	$q_{y,10,1}$	$q_{y,10,2}$	$q_{y,10,3}$	$q_{y,10,5}$	$q_{y,10,6}$	$q_{y,10,7}$	$q_{y,10,9}$	$q_{y,10,10}$	$q_{y,10,11}$	$q_{y,10,13}$
1	0.724	0.409	0.428	0.030	1.021	1.569	0.648	0.420	0.055	0.395
2	0.635	0.673	0.537	0.078	1.721	0.000	0.373	0.612	0.045	0.306
3	0.742	0.666	0.108	0.048	1.365	0.000	0.269	0.641	0.040	0.486
4	0.724	0.537	0.117	0.041	1.459	1.508	0.717	0.402	0.064	0.438
5	1.018	0.735	0.272	0.046	2.183	0.000	0.000	0.863	0.101	0.420
6	0.811	0.505	0.159	0.044	1.996	1.294	0.457	0.367	0.055	0.456

Table A.18 Year-over-Year Price Data for Month 11 (November)

y	$p_{y,11,1}$	$p_{y,11,2}$	$p_{y,11,4}$	$p_{y,11,5}$	$p_{y,11,6}$	$p_{y,11,11}$	$p_{y,11,12}$	$p_{y,11,13}$	$p_{y,11,14}$
1	7.30	8.89	9.80	5.96	6.09	8.78	5.96	10.45	10.32
2	7.46	8.90	9.82	5.89	6.07	*8.78*	6.43	14.12	*10.32*
3	6.73	8.58	9.80	5.78	6.40	*8.78*	6.62	14.98	12.26
4	8.82	10.13	12.40	6.27	7.75	11.68	6.62	15.13	12.26
5	7.26	9.58	11.31	6.00	5.89	*11.68*	6.17	19.60	*12.26*
6	8.49	11.51	11.06	6.95	6.57	12.93	6.34	17.18	*12.26*

Table A.19 Year-over-Year Quantity Data for Month 11 (November)

y	$q_{y,11,1}$	$q_{y,11,2}$	$q_{y,11,4}$	$q_{y,11,5}$	$q_{y,11,6}$	$q_{y,11,11}$	$q_{y,11,12}$	$q_{y,11,13}$	$q_{y,11,14}$
1	0.712	0.765	0.510	0.101	2.069	0.410	1.309	0.124	0.223
2	0.603	1.000	0.601	0.085	2.405	0.000	1.726	0.064	0.000
3	0.594	0.897	0.510	0.087	2.328	0.000	1.344	0.080	0.220
4	0.612	1.066	0.435	0.080	2.400	0.283	1.148	0.099	0.000
5	0.992	1.075	0.557	0.150	2.920	0.000	1.313	0.087	0.000
6	0.836	0.990	0.443	0.158	3.014	0.116	1.167	0.076	0.179

Table A.20 Year-over-Year Price and Quantity Data for Month 12 (December)

y	$p_{y,12,1}$	$p_{y,12,2}$	$p_{y,12,4}$	$p_{y,12,5}$	$p_{y,12,6}$	$p_{y,12,12}$	$p_{y,12,13}$	$q_{y,12,1}$	$q_{y,12,2}$	$q_{y,12,4}$	$q_{y,12,5}$	$q_{y,12,6}$	$q_{y,12,12}$	$q_{y,12,13}$
1	5.41	8.29	9.46	4.88	6.22	5.81	11.82	0.370	0.676	0.465	0.082	1.881	1.824	0.135
2	6.28	8.70	10.55	5.21	5.57	5.89	10.72	0.430	0.897	0.417	0.058	1.957	1.579	0.103
3	6.63	8.88	10.49	5.03	5.44	6.30	14.94	0.513	0.890	0.486	0.298	2.261	1.175	0.067
4	6.20	8.00	8.94	4.99	6.27	5.83	14.98	0.645	0.975	0.559	0.160	2.281	1.492	0.100
5	7.07	11.13	12.59	5.35	6.12	5.93	14.59	0.552	1.006	0.485	0.093	2.647	1.737	0.206
6	6.51	9.64	11.11	5.25	6.07	5.83	17.88	0.906	1.172	0.630	0.057	3.262	2.093	0.056

Over all 12 months, there were 34 missing prices that were imputed. Thirty of the imputed prices were carry-forward prices and four of the imputed prices were carry-backward prices.

These data series were used to compute all of the year-over-year monthly indices that are listed in Table A.21:

Table A.21 Year-over-Year Indices for Months Using Carry-Forward Prices

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
1	1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1	0.99746	0.99881	0.99813	0.99817	0.99746	0.99881	0.99813	0.99817	0.99814	0.99813
3	1	1.03276	1.01894	1.02583	1.02591	1.02762	1.01799	1.02280	1.02261	1.02295	1.02280
4	1	1.01159	1.00992	1.01076	1.01072	1.01586	0.99872	1.00725	1.00700	1.00816	1.01076
5	1	1.12212	1.12896	1.12554	1.12582	1.14808	1.10989	1.12883	1.12854	1.12973	1.13415
6	1	1.07410	1.06543	1.06976	1.06889	1.09958	1.04827	1.07362	1.07252	1.07153	1.06944
1	2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	2	1.14673	1.05830	1.10163	1.09970	1.14673	1.05830	1.10163	1.09970	1.10937	1.10163
3	2	1.19856	1.13544	1.16657	1.16430	1.19530	1.10240	1.14791	1.14597	1.15856	1.14791
4	2	1.13489	1.06908	1.10149	1.09983	1.13690	1.04779	1.09144	1.08957	1.10156	1.09144
5	2	1.35079	1.25687	1.30298	1.30238	1.35316	1.23472	1.29259	1.29006	1.30486	1.29259
6	2	1.36333	1.26804	1.31482	1.31429	1.36508	1.23771	1.29984	1.29727	1.31271	1.29984
1	3	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	3	0.92742	0.91116	0.91925	0.91910	0.92742	0.91116	0.91925	0.91910	0.91727	0.91925
3	3	1.00396	0.99578	0.99986	0.99981	1.00995	0.98455	0.99717	0.99686	0.99912	0.99717
4	3	1.00033	0.99176	0.99603	0.99611	1.00911	0.98358	0.99626	0.99588	0.99714	0.99626
5	3	1.09322	1.06264	1.07782	1.07646	1.09945	1.05318	1.07607	1.07519	1.07794	1.08539
6	3	1.13016	1.11723	1.12368	1.12351	1.15073	1.10109	1.12564	1.12495	1.12558	1.12368
1	4	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	4	0.98803	0.98284	0.98543	0.98569	0.98803	0.98284	0.98543	0.98569	0.98766	0.98543
3	4	1.06459	1.06038	1.06248	1.06235	1.06796	1.04900	1.05844	1.05817	1.06550	1.06248
4	4	1.20496	1.18402	1.19444	1.19482	1.19860	1.16073	1.17951	1.17928	1.19142	1.18402

(Continued)

Table A.21 (Continued)

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
5	4	1.22481	1.18576	1.20513	1.20454	1.23398	1.15532	1.19400	1.19293	1.20392	1.20245
6	4	1.22173	1.17182	1.19652	1.19732	1.24896	1.13499	1.19061	1.18951	1.19466	1.17841
1	5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	5	0.95731	0.91814	0.93752	0.93708	0.95731	0.91814	0.93752	0.93708	0.93879	0.93752
3	5	1.04955	1.02931	1.03938	1.03929	1.07750	0.99674	1.03634	1.03544	1.04223	1.03938
4	5	1.29576	1.26861	1.28211	1.27958	1.34446	1.21671	1.27899	1.27733	1.28376	1.28275
5	5	1.15686	1.15394	1.15540	1.15718	1.22628	1.06571	1.14318	1.14348	1.15227	1.14281
6	5	1.29885	1.29900	1.29893	1.29611	1.36519	1.18589	1.27239	1.27244	1.29548	1.29399
1	6	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	6	1.39164	1.22166	1.30388	1.29242	1.39164	1.22166	1.30388	1.29242	1.31098	1.30388
3	6	1.22178	1.12981	1.17489	1.17396	1.25257	1.05876	1.15159	1.14046	1.16554	1.15159
4	6	1.44251	1.31595	1.37778	1.37391	1.50699	1.25245	1.37384	1.36073	1.39106	1.37384
5	6	1.36006	1.18481	1.26941	1.26930	1.38245	1.12163	1.24523	1.23252	1.26646	1.25428
6	6	1.33890	1.21385	1.27484	1.27390	1.38772	1.11708	1.24507	1.23232	1.26886	1.25412
1	7	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	7	1.34108	1.18579	1.26105	1.24052	1.34108	1.18579	1.26105	1.24052	1.24998	1.26105
3	7	1.16473	1.05154	1.10669	1.10140	1.21160	1.03449	1.11955	1.09931	1.10632	1.11955
4	7	1.15777	1.08526	1.12093	1.11635	1.25418	1.02377	1.13313	1.11271	1.12257	1.13313
5	7	1.27857	1.21396	1.24585	1.24441	1.40919	1.10546	1.24812	1.22775	1.24618	1.24812
6	7	1.16724	1.06722	1.11611	1.11371	1.29886	0.98785	1.13273	1.11312	1.12442	1.13599
1	8	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	8	1.17083	1.15923	1.16501	1.16556	1.17083	1.15923	1.16501	1.16556	0.15685	1.16501
3	8	1.15211	1.11885	1.13536	1.13510	1.15792	1.12961	1.14367	1.14349	1.13068	1.14367
4	8	1.02823	0.99631	1.01215	1.01276	1.07022	1.01120	1.04029	1.04008	1.02243	1.04029
5	8	1.23369	1.21298	1.22329	1.22287	1.31264	1.15293	1.23020	1.22990	1.21444	1.22329
6	8	1.08462	1.05909	1.07178	1.07241	1.21200	1.00477	1.10353	1.10473	1.08073	1.09604
1	9	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	9	1.14248	1.10310	1.12262	1.12333	1.14248	1.10310	1.12262	1.12333	1.11489	1.12262
3	9	1.24526	1.16240	1.20312	1.20144	1.24737	1.18134	1.21391	1.21481	1.20187	1.21391
4	9	1.24783	1.22435	1.23603	1.23613	1.29599	1.20422	1.24927	1.25078	1.23805	1.24701
5	9	1.33579	1.23154	1.28261	1.28165	1.35658	1.23101	1.29227	1.29321	1.28087	1.29501
6	9	1.31824	1.29386	1.30599	1.30586	1.42107	1.25797	1.33704	1.33840	1.31605	1.30599
1	10	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	10	1.05802	1.01422	1.03589	1.03679	1.05802	1.01422	1.03589	1.03679	1.03745	1.03589
3	10	1.10122	1.06718	1.08407	1.08219	1.14481	1.05881	1.10097	1.10408	1.09587	1.08407
4	10	1.21299	1.20960	1.21129	1.21150	1.23038	1.16216	1.19578	1.20230	1.19357	1.21129
5	10	1.19970	1.09432	1.14580	1.14499	1.20798	1.12170	1.16404	1.17143	1.15553	1.17915
6	10	1.29717	1.26337	1.28016	1.28015	1.32255	1.22509	1.27289	1.28047	1.27244	1.28940
1	11	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	11	1.02552	1.02452	1.02502	1.02479	1.02552	1.02452	1.02502	1.02479	1.03130	1.02502
3	11	1.04068	1.04095	1.04081	1.04058	1.03426	1.04261	1.03842	1.03856	1.04307	1.03842
4	11	1.21879	1.21606	1.21742	1.21737	1.19912	1.23241	1.21565	1.21542	1.21782	1.21742
5	11	1.08609	1.04101	1.06331	1.06158	1.05063	1.05983	1.05522	1.05462	1.05707	1.04532
6	11	1.17818	1.15325	1.16565	1.16445	1.15826	1.16658	1.16241	1.16159	1.16249	1.15150
1	12	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	12	1.07104	1.06784	1.06944	1.06922	1.07104	1.06784	1.06944	1.06922	1.06907	1.06944

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
3	12	1.04248	1.03372	1.03809	1.03803	1.04200	1.03319	1.03759	1.03744	1.03607	1.03759
4	12	1.16428	1.15713	1.16070	1.16065	1.18173	1.14757	1.16453	1.16408	1.16277	1.16070
5	12	1.04311	1.03999	1.04155	1.04147	1.05685	1.02477	1.04069	1.04050	1.03999	1.03727
6	12	1.21874	1.20829	1.21350	1.21311	1.23554	1.20465	1.22000	1.21948	1.21593	1.21140
Mean		1.13650	1.10010	1.11800	1.11700	1.15600	1.08170	1.11760	1.11540	1.11110	1.11780

2. Year-Over-Year Monthly Indices Using Maximum Overlap Bilateral Indices

The data listed in Tables A.1 to A.20 were used to compute all of the maximum overlap indices that are listed in Table A.22. However, the imputed prices (in italics) listed in Tables A.1 to A.20 were set equal to 0 when computing the year-over-year maximum overlap indices that are listed in Table A.22. Thus, the year-over-year maximum overlap indices do not use any imputed prices. The indices listed in Table A.22 are discussed in Section 3 of the main text.

In order to fit all 10 maximum overlap indices in one row, the index titles have omitted the asterisk; that is, $P_{LFB}^{y,m}$ in row 1 of Table A.22 should be listed as $P_{LFB}^{y,m*}$, $P_{PFB}^{y,m}$ should be listed as $P_{PFB}^{y,m*}$, and so on.

Our best indices are the fixed-base Fisher and Törnqvist–Theil indices, $P_{FFB}^{y,m*}$ and $P_{TFT}^{y,m*}$, the GEKS indices, $P_{GEKS}^{y,m*}$, and the predicted share price similarity-linked indices, $P_{S}^{y,m*}$. The average index value over all observations for these four maximum overlap indices is 1.1184. The average index value for the corresponding four carry-forward indices is 1.1160. Thus, the use of carry-forward prices led to a downward bias for our best indices of about 0.24 percentage points per observation.

Table A.22 Year-over-Year Alternative Indices Using Maximum Overlap Price Indices

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
1	1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1	0.99746	0.99881	0.99813	0.99817	0.99746	0.99881	0.99813	0.99817	0.99814	0.99813
3	1	1.03276	1.01894	1.02583	1.02591	1.02762	1.01799	1.02280	1.02261	1.02295	1.02280
4	1	1.01159	1.00992	1.01076	1.01072	1.01586	0.99872	1.00725	1.00700	1.00816	1.01076
5	1	1.12212	1.12896	1.12554	1.12582	1.14808	1.10989	1.12883	1.12854	1.12973	1.13415
6	1	1.07410	1.06543	1.06976	1.06889	1.09958	1.04827	1.07362	1.07252	1.07153	1.06944
1	2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	2	1.14673	1.05830	1.10163	1.09970	1.14673	1.05830	1.10163	1.09970	1.10937	1.10163
3	2	1.19856	1.13544	1.16657	1.16430	1.19530	1.10240	1.14791	1.14597	1.15856	1.14791
4	2	1.13489	1.06908	1.10149	1.09983	1.13690	1.04779	1.09144	1.08957	1.10156	1.09144
5	2	1.35079	1.25687	1.30298	1.30238	1.35316	1.23472	1.29259	1.29006	1.30486	1.29259
6	2	1.36333	1.26804	1.31482	1.31429	1.36508	1.23771	1.29984	1.29727	1.31271	1.29984
1	3	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	3	0.92742	0.91116	0.91925	0.91910	0.92742	0.91116	0.91925	0.91910	0.91669	0.91925
3	3	1.00396	0.99578	0.99986	0.99981	1.00995	0.98455	0.99717	0.99686	0.99852	0.99717
4	3	1.00033	0.99176	0.99603	0.99611	1.00911	0.98358	0.99626	0.99588	0.99724	0.99626
5	3	1.09845	1.06264	1.08040	1.07833	1.10327	1.05318	1.07793	1.07685	1.08213	1.09208
6	3	1.13016	1.11723	1.12368	1.12351	1.15473	1.09091	1.12237	1.12130	1.12515	1.12368
1	4	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	4	0.98803	0.98284	0.98543	0.98569	0.98803	0.98284	0.98543	0.98569	0.98766	0.98543
3	4	1.06459	1.06038	1.06248	1.06235	1.06796	1.04900	1.05844	1.05817	1.06550	1.06248
4	4	1.20496	1.18402	1.19444	1.19482	1.19860	1.16073	1.17951	1.17928	1.19142	1.18402
5	4	1.22481	1.18576	1.20513	1.20454	1.23398	1.15532	1.19400	1.19293	1.20392	1.20245
6	4	1.22173	1.17182	1.19652	1.19732	1.24896	1.13499	1.19061	1.18951	1.19466	1.17841
1	5	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	5	0.95007	0.91814	0.93397	0.93252	0.95007	0.91814	0.93397	0.93252	0.94462	0.93397
3	5	1.05674	1.03102	1.04380	1.04354	1.06935	0.99802	1.03307	1.03104	1.05052	1.03307

(Continued)

Table A.22 (Continued)

y	m	$P_{LFB}^{y,m}$	$P_{PFB}^{y,m}$	$P_{FFB}^{y,m}$	$P_{TFB}^{y,m}$	$P_{LCH}^{y,m}$	$P_{PCH}^{y,m}$	$P_{FCH}^{y,m}$	$P_{TCH}^{y,m}$	$P_{GEKS}^{y,m}$	$P_{S}^{y,m}$
4	5	1.33870	1.26554	1.30161	1.29967	1.33429	1.21827	1.27496	1.27191	1.29677	1.27496
5	5	1.17963	1.17093	1.17527	1.17658	1.21701	1.06707	1.13958	1.13863	1.16610	1.13587
6	5	1.34224	1.29900	1.32044	1.31917	1.36461	1.18740	1.27293	1.27122	1.31228	1.28980
1	6	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	6	1.39305	1.22166	1.30455	1.29307	1.39305	1.22166	1.30455	1.29307	1.31228	1.30455
3	6	1.22178	1.12981	1.17489	1.17396	1.25384	1.05675	1.15109	1.13948	1.16557	1.15109
4	6	1.44355	1.31595	1.37827	1.37444	1.50911	1.25008	1.37350	1.35982	1.39145	1.37350
5	6	1.36089	1.18481	1.26980	1.26966	1.38439	1.11951	1.24492	1.23170	1.26652	1.25386
6	6	1.33968	1.21385	1.27521	1.27426	1.38967	1.11496	1.24476	1.23149	1.26903	1.25369
1	7	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	7	1.35835	1.18579	1.26914	1.24812	1.35835	1.18579	1.26914	1.24812	1.25497	1.26914
3	7	1.16473	1.05154	1.10669	1.10140	1.22720	1.01345	1.11522	1.09393	1.10445	1.11522
4	7	1.15635	1.08526	1.12024	1.11432	1.27371	1.00296	1.13025	1.10813	1.12406	1.13025
5	7	1.28326	1.21396	1.24813	1.24583	1.43113	1.08299	1.24495	1.22269	1.24902	1.24495
6	7	1.16630	1.06722	1.11566	1.11153	1.31909	0.96776	1.12985	1.10854	1.12591	1.13310
1	8	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	8	1.17882	1.15923	1.16899	1.16969	1.17882	1.15923	1.16899	1.16969	1.15923	1.16899
3	8	1.15211	1.12012	1.13600	1.13583	1.16583	1.12447	1.14496	1.14526	1.13180	1.14496
4	8	1.02645	0.99631	1.01127	1.01139	1.07223	1.00660	1.03889	1.03874	1.02300	1.03889
5	8	1.24152	1.21298	1.22717	1.22682	1.31511	1.14769	1.22855	1.22832	1.21689	1.23551
6	8	1.08548	1.05909	1.07220	1.07226	1.21428	1.00020	1.10205	1.10331	1.08186	1.09457
1	9	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	9	1.14248	1.10310	1.12262	1.12333	1.14248	1.10310	1.12262	1.12333	1.11489	1.12262
3	9	1.24526	1.16240	1.20312	1.20144	1.24737	1.18134	1.21391	1.21481	1.20187	1.21391
4	9	1.24783	1.22435	1.23603	1.23613	1.29599	1.20422	1.24927	1.25078	1.23805	1.24701
5	9	1.33579	1.23154	1.28261	1.28165	1.35658	1.23101	1.29227	1.29321	1.28087	1.29501
6	9	1.31824	1.29386	1.30599	1.30586	1.42107	1.25797	1.33704	1.33840	1.31605	1.30599
1	10	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	10	1.08104	1.01422	1.04710	1.04526	1.08104	1.01422	1.04710	1.04526	1.04292	1.04710
3	10	1.14138	1.06718	1.10366	1.09872	1.16972	1.05881	1.11288	1.11310	1.10523	1.11288
4	10	1.21299	1.20960	1.21129	1.21150	1.25714	1.09468	1.17310	1.17560	1.18322	1.21129
5	10	1.11588	1.09432	1.10505	1.10521	1.21812	1.05657	1.13448	1.13676	1.14941	1.17489
6	10	1.29717	1.26337	1.28016	1.28015	1.33365	1.16062	1.24413	1.24654	1.26320	1.29090
1	11	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	11	1.02936	1.02452	1.02694	1.02660	1.02936	1.02452	1.02694	1.02660	1.03219	1.02694
3	11	1.04420	1.04095	1.04257	1.04220	1.03812	1.03314	1.03563	1.03532	1.04139	1.03563
4	11	1.22044	1.21606	1.21825	1.21813	1.21418	1.21225	1.21322	1.21259	1.21672	1.21825
5	11	1.05775	1.04101	1.04934	1.04823	1.05385	1.04249	1.04816	1.04730	1.05206	1.04727
6	11	1.17818	1.15325	1.16565	1.16445	1.16180	1.15211	1.15694	1.15584	1.15950	1.16565
1	12	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	12	1.07104	1.06784	1.06944	1.06922	1.07104	1.06784	1.06944	1.06922	1.06907	1.06944
3	12	1.04248	1.03372	1.03809	1.03803	1.04200	1.03319	1.03759	1.03744	1.03607	1.03759
4	12	1.16428	1.15713	1.16070	1.16065	1.18173	1.14757	1.16453	1.16408	1.16277	1.16070
5	12	1.04311	1.03999	1.04155	1.04147	1.05685	1.02477	1.04069	1.04050	1.03999	1.03727
6	12	1.21874	1.20829	1.21350	1.21311	1.23554	1.20465	1.22000	1.21948	1.21593	1.21140
Mean		1.13810	1.10030	1.11890	1.11770	1.15910	1.07650	1.11630	1.11360	1.11870	1.11840

3. Listing of the Data Using Month-to-Month Carry-Forward and Carry-Backward Prices

Tables A.23 and A.24 list the price and quantity data for fresh fruit purchased by households in Israel for the 72 months in the years 2012–2017. Carry-forward (and carry-backward) unit value prices are used for the prices which were not sold in month t. Note that these new carry-forward prices are different from the *year-over-year carry-forward prices*, which were listed earlier in various tables. The new carry-forward prices are *month-to-month carry-forward prices*. In Table A.23, these carry-forward prices are listed in italics. For example if a product n is present in month t but missing for the subsequent three months, then the last existing price $p_{t,n}$ is carried forward for the next four months; that is, we have $p_{t+1,n} \equiv p_{t+2,n} \equiv p_{t+3,n} \equiv p_{t,n}$. There are 451 carry-forward prices listed in Table A.23. The maximum number of monthly product prices is 1008 = 72×14. Thus, the sample probability that a price listed in Table A.23 is an imputed price is 0.447 = 451/1008. The earlier year-over-year carry-forward/carry-backward prices do not coincide with the month-to-month carry-forward/carry-backward prices listed here in italics.

The monthly quantity data are listed in Table A.24. The quantity data listed in Table A.24 are the same as the quantity data that were listed earlier in various tables in this annex but the earlier data were listed as year-over-year data for each month. The data listed here are month-to-month data that start at January 2012 and end at December 2017.

Table A.23 Month-to-Month Price Data Using Carry-Forward and Carry-Backward Prices

t	$p_{t,1}$	$p_{t,2}$	$p_{t,3}$	$p_{t,4}$	$p_{t,5}$	$p_{t,6}$	$p_{t,7}$	$p_{t,8}$	$p_{t,9}$	$p_{t,10}$	$p_{t,11}$	$p_{t,12}$	$p_{t,13}$	$p_{t,14}$
1	5.41	8.29	*4.14*	9.46	4.88	6.22	*11.50*	*15.08*	*17.44*	*12.16*	*11.05*	5.81	11.82	*10.41*
2	4.99	8.37	*4.14*	10.20	4.84	6.90	*11.50*	15.08	*17.44*	*12.16*	*11.05*	6.32	12.78	*10.41*
3	5.14	8.59	*4.14*	10.76	4.92	7.42	*11.50*	18.67	*17.44*	*12.16*	*11.05*	6.62	13.34	*10.41*
4	5.08	9.06	*4.14*	*10.76*	5.13	7.25	*11.50*	18.24	*17.44*	*12.16*	*11.05*	7.01	13.70	*10.41*
5	5.19	11.48	4.14	*10.76*	5.27	7.05	11.50	16.68	*17.44*	12.16	*11.05*	*7.01*	13.69	*10.41*
6	5.66	11.83	3.24	*10.76*	5.57	5.92	10.08	*16.68*	17.44	8.82	11.05	*7.01*	13.74	*10.41*
7	7.40	13.02	3.18	*10.76*	*5.57*	6.65	9.27	*16.68*	20.10	*8.82*	8.82	*7.01*	*13.74*	10.41
8	10.62	18.23	3.28	*10.76*	*5.57*	8.24	9.06	*16.68*	*20.10*	*8.82*	8.13	*7.01*	*13.74*	9.15
9	9.27	11.65	*3.28*	*10.76*	*5.57*	8.55	8.03	*16.68*	*20.10*	*8.82*	8.15	6.88	*13.74*	10.03
10	8.15	11.26	*3.28*	11.45	6.59	7.93	9.18	*16.68*	*20.10*	*8.82*	8.18	6.19	14.63	9.87
11	7.30	8.89	*3.28*	9.80	5.96	6.09	*9.18*	*16.68*	*20.10*	*8.82*	8.78	5.96	10.45	10.32
12	6.78	8.34	*3.28*	9.48	5.34	5.63	*9.18*	*16.68*	*20.10*	*8.82*	*8.78*	5.63	10.27	*10.32*
13	6.28	8.70	*3.28*	10.55	5.21	5.57	*9.18*	*16.68*	*20.10*	*8.82*	*8.78*	5.89	10.72	*10.32*
14	5.93	9.15	*3.28*	11.41	5.21	*5.57*	*9.18*	23.29	*20.10*	*8.82*	*8.78*	6.43	11.58	*10.32*
15	5.70	9.43	*3.28*	11.69	5.16	6.11	*9.18*	15.31	*20.10*	*8.82*	*8.78*	6.64	11.72	*10.32*
16	6.84	11.00	*3.28*	*11.69*	5.43	6.26	*9.18*	16.62	*20.10*	*8.82*	*8.78*	7.18	12.42	*10.32*
17	7.35	14.62	3.49	*11.69*	5.67	5.96	11.08	*16.62*	*20.10*	9.46	*8.78*	*7.18*	*12.42*	*10.32*
18	7.83	19.58	3.36	*11.69*	5.86	5.85	11.25	*16.62*	42.05	14.44	12.61	*7.18*	*12.42*	*10.32*
19	9.96	*19.58*	3.30	*11.69*	*5.86*	7.46	11.91	*16.62*	53.77	*14.44*	10.65	*7.18*	*12.42*	10.92
20	9.44	*19.58*	3.83	*11.69*	*5.86*	7.78	11.53	*16.62*	*53.77*	*14.44*	10.94	*7.18*	*12.42*	10.35
21	8.00	11.92	*3.83*	*11.69*	*5.86*	7.21	10.07	*16.62*	*53.77*	*14.44*	11.31	7.36	*12.42*	10.85
22	8.03	9.80	*3.83*	12.30	6.43	6.80	*10.07*	*16.62*	*53.77*	*14.44*	12.32	7.19	15.47	13.09
23	7.46	8.90	*3.83*	9.82	5.89	6.07	*10.07*	*16.62*	*53.77*	*14.44*	*12.32*	6.43	14.12	*13.09*
24	6.92	8.61	*3.83*	10.19	5.28	5.79	*10.07*	*16.62*	*53.77*	*14.44*	*12.32*	6.46	14.64	*13.09*
25	6.63	8.88	*3.83*	10.49	5.03	5.44	*10.07*	*16.62*	*53.77*	*14.44*	*12.32*	6.30	14.94	*13.09*
26	5.97	8.84	*3.83*	11.32	5.03	5.98	*10.07*	25.11	*53.77*	*14.44*	*12.32*	6.50	14.92	*13.09*
27	5.72	9.47	*3.83*	12.41	4.97	6.51	*10.07*	18.23	*53.77*	*14.44*	*12.32*	6.82	15.36	*13.09*
28	6.00	10.27	*3.83*	*12.41*	5.09	7.60	*10.07*	17.80	*53.77*	*14.44*	*12.32*	7.72	16.91	*13.09*
29	6.60	13.66	4.10	*12.41*	5.34	*7.60*	10.62	*17.80*	40.84	14.79	*12.32*	*7.72*	19.93	*13.09*
30	6.64	14.00	2.55	*12.41*	5.63	8.07	10.42	*17.80*	32.81	13.25	14.03	*7.72*	27.25	*13.09*
31	7.51	15.44	2.29	*12.41*	*5.63*	10.76	11.23	*17.80*	37.98	*13.25*	12.59	*7.72*	*27.25*	10.63
32	8.23	19.44	3.12	*12.41*	*5.63*	10.56	11.84	*17.80*	22.50	*13.25*	13.30	*7.72*	*27.25*	8.94

(*Continued*)

Table A.23 (Continued)

t	$p_{t,1}$	$p_{t,2}$	$p_{t,3}$	$p_{t,4}$	$p_{t,5}$	$p_{t,6}$	$p_{t,7}$	$p_{t,8}$	$p_{t,9}$	$p_{t,10}$	$p_{t,11}$	$p_{t,12}$	$p_{t,13}$	$p_{t,14}$
33	7.12	12.03	*3.12*	*12.41*	*5.63*	9.34	11.43	*17.80*	*22.50*	*13.25*	13.44	7.20	*27.25*	9.66
34	7.14	10.51	*3.12*	12.98	6.21	8.79	*11.43*	*17.80*	*22.50*	*13.25*	14.50	7.33	17.43	9.67
35	6.73	8.58	*3.12*	9.80	5.78	6.40	*11.43*	*17.80*	*22.50*	*13.25*	*14.50*	6.62	14.98	12.26
36	6.47	7.80	*3.12*	8.98	5.44	6.09	*11.43*	*17.80*	*22.50*	*13.25*	*14.50*	6.26	14.61	*12.26*
37	6.20	8.00	*3.12*	8.94	4.99	6.27	*11.43*	*17.80*	*22.50*	*13.25*	*14.50*	5.83	14.98	*12.26*
38	5.97	8.15	*3.12*	9.95	5.14	6.06	*11.43*	23.49	*22.50*	*13.25*	*14.50*	5.94	15.41	*12.26*
39	6.08	9.06	*3.12*	11.02	4.98	6.83	*11.43*	18.95	*22.50*	*13.25*	*14.50*	6.17	15.73	*12.26*
40	7.04	12.60	*3.12*	*11.02*	5.41	9.68	*11.43*	18.35	*22.50*	*13.25*	*14.50*	7.03	16.30	*12.26*
41	7.73	15.92	4.56	*11.02*	5.39	13.19	11.75	*18.35*	61.43	17.78	*14.50*	*7.03*	17.16	*12.26*
42	8.62	18.98	3.68	*11.02*	*5.39*	12.71	10.80	*18.35*	34.48	12.36	13.56	*7.03*	21.55	*12.26*
43	9.83	*18.98*	2.51	*11.02*	*5.39*	15.91	10.24	*18.35*	30.62	*12.36*	10.36	*7.03*	*21.55*	12.32
44	9.87	*18.98*	2.51	*11.02*	*5.39*	12.25	10.14	*18.35*	*30.62*	*12.36*	9.61	*7.03*	*21.55*	10.40
45	9.42	12.23	*2.51*	*11.02*	*5.39*	9.59	10.85	*18.35*	*30.62*	*12.36*	11.18	7.89	*21.55*	12.29
46	9.53	11.54	*2.51*	12.85	7.28	8.91	11.94	*18.35*	*30.62*	*12.36*	11.58	7.22	20.19	11.88
47	8.82	10.13	*2.51*	12.40	6.27	7.75	*11.94*	*18.35*	*30.62*	*12.36*	11.68	6.62	15.13	*11.88*
48	7.87	11.09	*2.51*	11.94	5.52	6.00	*11.94*	*18.35*	*30.62*	*12.36*	*11.68*	6.20	14.36	*11.88*
49	7.07	11.13	*2.51*	12.59	5.35	6.12	*11.94*	*18.35*	*30.62*	*12.36*	*11.68*	5.93	14.59	*11.88*
50	6.99	12.27	*2.51*	13.22	5.09	7.22	*11.94*	26.86	*30.62*	*12.36*	*11.68*	6.15	14.88	*11.88*
51	6.78	13.98	*2.51*	*13.22*	5.13	7.51	*11.94*	18.06	*30.62*	*12.36*	*11.68*	6.03	15.11	*11.88*
52	7.05	18.26	*2.51*	*13.22*	5.07	8.40	*11.94*	18.80	*30.62*	*12.36*	*11.68*	6.58	16.36	*11.88*
53	7.52	19.36	4.07	*13.22*	5.81	8.98	11.27	*18.80*	39.10	18.31	*11.68*	*6.58*	17.33	*11.88*
54	9.01	20.42	2.67	*13.22*	*5.81*	10.99	9.73	*18.80*	34.21	15.05	13.62	*6.58*	22.38	*11.88*
55	11.34	*20.42*	3.09	*13.22*	*5.81*	12.56	10.66	*18.80*	37.31	*15.05*	12.85	*6.58*	*22.38*	11.35
56	10.30	*20.42*	4.01	*13.22*	*5.81*	9.65	10.73	*18.80*	*37.31*	*15.05*	13.20	*6.58*	*22.38*	11.19
57	8.91	13.52	*4.01*	*13.22*	*5.81*	8.39	11.77	*18.80*	*37.31*	*15.05*	14.61	7.40	*22.38*	11.52
58	8.35	12.65	*4.01*	13.61	6.57	7.88	*11.77*	*18.80*	*37.31*	*15.05*	*14.61*	7.07	22.85	12.87
59	7.26	9.58	*4.01*	11.31	6.00	5.89	*11.77*	*18.80*	*37.31*	*15.05*	*14.61*	6.17	19.60	*12.87*
60	6.70	9.15	*4.01*	10.85	5.45	5.31	*11.77*	*18.80*	*37.31*	*15.05*	*14.61*	5.83	17.21	*12.87*
61	6.51	9.64	*4.01*	11.11	5.25	6.07	*11.77*	*18.80*	*37.31*	*15.05*	*14.61*	5.83	17.88	*12.87*
62	6.39	10.59	*4.01*	11.85	5.00	8.23	*11.77*	28.26	*37.31*	*15.05*	*14.61*	5.65	18.97	*12.87*
63	6.32	11.05	*4.01*	13.66	5.24	8.85	*11.77*	19.26	*37.31*	*15.05*	*14.61*	6.06	19.66	*12.87*
64	6.47	12.59	*4.01*	*13.66*	5.45	10.75	*11.77*	16.85	*37.31*	*15.05*	*14.61*	6.28	20.39	*12.87*
65	7.00	15.34	4.77	*13.66*	6.16	12.30	12.95	*16.85*	*37.31*	18.03	*14.61*	*6.28*	22.56	*12.87*
66	8.20	18.56	2.93	*13.66*	*6.16*	11.15	9.81	*16.85*	31.08	14.71	14.01	*6.28*	26.03	*12.87*
67	10.86	*18.56*	2.32	*13.66*	*6.16*	14.74	9.87	*16.85*	35.14	*14.71*	11.23	*6.28*	*26.03*	13.48
68	10.87	*18.56*	2.60	*13.66*	*6.16*	12.20	10.39	*16.85*	*35.14*	*14.71*	11.09	*6.28*	*26.03*	11.37
69	10.11	17.74	*2.60*	*13.66*	*6.16*	10.52	10.72	*16.85*	*35.14*	*14.71*	12.03	7.98	*26.03*	12.49
70	9.62	14.86	*2.60*	13.85	6.81	8.92	12.67	*16.85*	*35.14*	*14.71*	13.13	7.09	21.74	12.49
71	8.49	11.51	*2.60*	11.06	6.95	6.57	*12.67*	*16.85*	*35.14*	*14.71*	12.93	6.34	17.18	12.26
72	7.38	12.96	*2.60*	10.94	6.35	6.38	*12.67*	*16.85*	*35.14*	*14.71*	*12.93*	6.15	16.26	*12.26*

Table A.24 Monthly Quantity Data for Household Fresh Fruit Consumption

t	$q_{t,1}$	$q_{t,2}$	$q_{t,3}$	$q_{t,4}$	$q_{t,5}$	$q_{t,6}$	$q_{t,7}$	$q_{t,8}$	$q_{t,9}$	$q_{t,10}$	$q_{t,11}$	$q_{t,12}$	$q_{t,13}$	$q_{t,14}$
1	0.370	0.676	0.000	0.465	0.082	1.881	0.000	0.000	0.000	0.000	0.000	1.824	0.135	0.000
2	0.701	0.920	0.000	0.510	0.103	2.087	0.000	1.134	0.000	0.000	0.000	1.408	0.102	0.000
3	0.661	0.908	0.000	0.362	0.081	1.819	0.000	0.884	0.000	0.000	0.000	1.269	0.112	0.000

t	$q_{t,1}$	$q_{t,2}$	$q_{t,3}$	$q_{t,4}$	$q_{t,5}$	$q_{t,6}$	$q_{t,7}$	$q_{t,8}$	$q_{t,9}$	$q_{t,10}$	$q_{t,11}$	$q_{t,12}$	$q_{t,13}$	$q_{t,14}$
4	0.689	0.585	0.000	0.000	0.156	1.876	0.000	0.609	0.000	0.000	0.000	0.728	0.131	0.000
5	0.751	0.409	4.106	0.000	0.076	1.730	0.922	0.456	0.000	0.206	0.000	0.000	0.080	0.000
6	0.724	0.440	6.698	0.000	0.036	1.486	1.657	0.000	0.717	1.270	0.290	0.000	0.022	0.000
7	0.595	0.292	8.145	0.000	0.000	0.722	2.093	0.000	0.488	0.000	0.964	0.000	0.000	0.221
8	0.452	0.159	6.159	0.000	0.000	0.558	1.932	0.000	0.000	0.000	1.009	0.000	0.000	0.721
9	0.647	0.335	0.000	0.000	0.000	0.643	2.379	0.000	0.000	0.000	1.104	0.058	0.000	0.857
10	0.724	0.409	0.000	0.428	0.030	1.021	1.569	0.000	0.000	0.000	0.648	0.420	0.055	0.395
11	0.712	0.765	0.000	0.510	0.101	2.069	0.000	0.000	0.000	0.000	0.410	1.309	0.124	0.223
12	0.678	0.923	0.000	0.390	0.150	2.274	0.000	0.000	0.000	0.000	0.000	1.545	0.127	0.000
13	0.430	0.897	0.000	0.417	0.058	1.957	0.000	0.000	0.000	0.000	0.000	1.579	0.103	0.000
14	0.624	0.831	0.000	0.412	0.269	2.621	0.000	0.593	0.000	0.000	0.000	1.664	0.155	0.000
15	0.684	0.732	0.000	0.257	0.116	2.242	0.000	0.947	0.000	0.000	0.000	1.160	0.154	0.000
16	0.760	0.591	0.000	0.000	0.092	2.141	0.000	0.698	0.000	0.000	0.000	0.766	0.129	0.000
17	0.626	0.321	6.504	0.000	0.053	1.913	1.273	0.000	0.000	0.370	0.000	0.000	0.000	0.000
18	0.766	0.266	7.738	0.000	0.051	1.419	1.831	0.000	0.228	0.616	0.523	0.000	0.000	0.000
19	0.612	0.000	7.394	0.000	0.000	0.871	1.520	0.000	0.073	0.000	0.761	0.000	0.000	0.421
20	0.625	0.000	5.065	0.000	0.000	0.746	1.761	0.000	0.000	0.000	0.914	0.000	0.000	0.850
21	0.650	0.235	0.000	0.000	0.000	0.957	1.927	0.000	0.000	0.000	0.716	0.231	0.000	0.710
22	0.635	0.673	0.000	0.537	0.078	1.721	0.000	0.000	0.000	0.000	0.373	0.612	0.045	0.306
23	0.603	1.000	0.000	0.601	0.085	2.405	0.000	0.000	0.000	0.000	0.000	1.726	0.064	0.000
24	0.621	1.010	0.000	0.481	0.170	2.522	0.000	0.000	0.000	0.000	0.000	1.718	0.089	0.000
25	0.513	0.890	0.000	0.486	0.298	2.261	0.000	0.000	0.000	0.000	0.000	1.175	0.067	0.000
26	0.754	1.075	0.000	0.486	0.119	2.308	0.000	0.737	0.000	0.000	0.000	1.492	0.168	0.000
27	0.822	0.612	0.000	0.290	0.121	2.012	0.000	0.845	0.000	0.000	0.000	1.496	0.085	0.000
28	0.617	0.662	0.000	0.000	0.157	1.737	0.000	0.663	0.000	0.000	0.000	0.997	0.053	0.000
29	0.682	0.417	5.244	0.000	0.075	1.526	1.525	0.000	0.088	0.176	0.000	0.000	0.045	0.000
30	0.678	0.450	8.118	0.000	0.036	1.016	1.910	0.000	0.466	0.694	0.335	0.000	0.011	0.000
31	0.746	0.389	9.869	0.000	0.000	0.539	1.915	0.000	0.179	0.000	0.667	0.000	0.000	0.546
32	0.656	0.180	5.577	0.000	0.000	0.616	1.791	0.000	0.031	0.000	0.759	0.000	0.000	1.040
33	0.758	0.532	0.000	0.000	0.000	0.921	1.899	0.000	0.000	0.000	0.543	0.181	0.000	0.932
34	0.742	0.666	0.000	0.108	0.048	1.365	0.000	0.000	0.000	0.000	0.269	0.641	0.040	0.486
35	0.594	0.897	0.000	0.510	0.087	2.328	0.000	0.000	0.000	0.000	0.000	1.344	0.080	0.220
36	0.649	1.077	0.000	0.657	0.092	2.463	0.000	0.000	0.000	0.000	0.000	1.534	0.110	0.000
37	0.645	0.975	0.000	0.559	0.160	2.281	0.000	0.000	0.000	0.000	0.000	1.492	0.100	0.000
38	0.553	1.031	0.000	0.412	0.156	2.591	0.000	0.766	0.000	0.000	0.000	1.919	0.117	0.000
39	0.658	0.828	0.000	0.209	0.181	2.255	0.000	0.813	0.000	0.000	0.000	1.556	0.108	0.000
40	0.895	0.683	0.000	0.000	0.129	1.550	0.000	0.687	0.000	0.000	0.000	1.252	0.135	0.000
41	0.660	0.528	4.211	0.000	0.056	1.054	1.183	0.000	0.016	0.107	0.000	0.000	0.041	0.000
42	0.673	0.295	7.038	0.000	0.000	0.653	1.880	0.000	0.299	0.777	0.354	0.000	0.014	0.000
43	0.600	0.000	8.486	0.000	0.000	0.289	2.129	0.000	0.349	0.000	0.685	0.000	0.000	0.390
44	0.719	0.000	7.371	0.000	0.000	0.498	1.765	0.000	0.000	0.000	0.832	0.000	0.000	0.673
45	0.594	0.278	0.000	0.000	0.000	0.792	1.604	0.000	0.000	0.000	0.689	0.114	0.000	0.667
46	0.724	0.537	0.000	0.117	0.041	1.459	1.508	0.000	0.000	0.000	0.717	0.402	0.064	0.438
47	0.612	1.066	0.000	0.435	0.080	2.400	0.000	0.000	0.000	0.000	0.283	1.148	0.099	0.000
48	0.635	0.938	0.000	0.553	0.163	2.983	0.000	0.000	0.000	0.000	0.000	1.645	0.104	0.000
49	0.552	1.006	0.000	0.485	0.093	2.647	0.000	0.000	0.000	0.000	0.000	1.737	0.206	0.000

(*Continued*)

Table A.24 (Continued)

t	$q_{t,1}$	$q_{t,2}$	$q_{t,3}$	$q_{t,4}$	$q_{t,5}$	$q_{t,6}$	$q_{t,7}$	$q_{t,8}$	$q_{t,9}$	$q_{t,10}$	$q_{t,11}$	$q_{t,12}$	$q_{t,13}$	$q_{t,14}$
50	0.658	0.717	0.000	0.386	0.157	2.299	0.000	0.648	0.000	0.000	0.000	1.886	0.108	0.000
51	0.708	0.694	0.000	0.000	0.234	2.490	0.000	1.107	0.000	0.000	0.000	1.509	0.152	0.000
52	0.766	0.460	0.000	0.000	0.079	1.988	0.000	0.585	0.000	0.000	0.000	1.231	0.122	0.000
53	0.785	0.584	5.430	0.000	0.103	1.726	1.287	0.000	0.138	0.284	0.000	0.000	0.069	0.000
54	0.599	0.318	8.876	0.000	0.000	0.792	1.922	0.000	0.406	0.472	0.382	0.000	0.031	0.000
55	0.635	0.000	8.188	0.000	0.000	0.701	2.073	0.000	0.198	0.000	0.545	0.000	0.000	0.643
56	0.718	0.000	4.963	0.000	0.000	0.974	2.171	0.000	0.000	0.000	0.750	0.000	0.000	1.028
57	0.831	0.473	0.000	0.000	0.000	1.335	2.022	0.000	0.000	0.000	0.568	0.243	0.000	0.972
58	1.018	0.735	0.000	0.272	0.046	2.183	0.000	0.000	0.000	0.000	0.000	0.863	0.101	0.420
59	0.992	1.075	0.000	0.557	0.150	2.920	0.000	0.000	0.000	0.000	0.000	1.313	0.087	0.000
60	0.896	1.191	0.000	0.562	0.128	2.976	0.000	0.000	0.000	0.000	0.000	1.955	0.134	0.000
61	0.906	1.172	0.000	0.630	0.057	3.262	0.000	0.000	0.000	0.000	0.000	2.093	0.056	0.000
62	0.657	0.859	0.000	0.447	0.260	2.211	0.000	0.711	0.000	0.000	0.000	2.071	0.111	0.000
63	0.759	0.787	0.000	0.293	0.095	2.395	0.000	1.038	0.000	0.000	0.000	1.650	0.102	0.000
64	0.773	0.627	0.000	0.000	0.037	2.047	0.000	0.926	0.000	0.000	0.000	1.210	0.069	0.000
65	0.814	0.587	5.891	0.000	0.065	1.504	1.243	0.000	0.000	0.322	0.000	0.000	0.040	0.000
66	0.915	0.436	9.693	0.000	0.000	0.969	2.487	0.000	0.560	0.727	0.378	0.000	0.019	0.000
67	0.847	0.000	12.845	0.000	0.000	0.468	2.837	0.000	0.361	0.000	0.784	0.000	0.000	0.593
68	0.690	0.000	8.423	0.000	0.000	0.770	2.348	0.000	0.000	0.000	0.748	0.000	0.000	0.730
69	0.752	0.282	0.000	0.000	0.000	1.502	2.136	0.000	0.000	0.000	0.948	0.188	0.000	0.945
70	0.811	0.505	0.000	0.159	0.044	1.996	1.294	0.000	0.000	0.000	0.457	0.367	0.055	0.456
71	0.836	0.990	0.000	0.443	0.158	3.014	0.000	0.000	0.000	0.000	0.116	1.167	0.076	0.179
72	0.623	0.980	0.000	0.558	0.079	3.119	0.000	0.000	0.000	0.000	0.000	1.447	0.062	0.000

4. Month-to-Month Fixed-Base Fisher Indices Using Carry-Forward Prices

Here is a listing of the 12 Fisher fixed-base "star" indices, $P_{F1}{}^{t} - P_{F12}{}^{t}$, that are plotted in Figure 9.5 in Section 6 of the main text.

For the final month, the lowest index value is 1.0487 and the highest is 1.1843. The lowest average value for the 12 indices is 1.1382 and the highest is 1.2354. Thus, the choice of a base month matters a lot for our particular empirical example.

5. Maximum Overlap Month-to-Month Fixed-Base Fisher Indices

Here is a listing of the 12 Fisher fixed-base "star" indices, $P_{F1}{}^{t*} - P_{F12}{}^{t*}$, that are plotted in Figure 9.7 in Section 7 of the main text.

For the final month 72, the lowest index value, as shown in Table A.26, is 1.1733 and the highest is 1.2386. From Table A.25, the lowest index value for month 72 is 1.0487 and the highest is 1.1843. Thus, using maximum overlap bilateral Fisher indices in place of carry-forward Fisher indices has led to a narrower spread of final index values when different base periods are used. The arithmetic mean of the month 72 means recorded in the last row of Table A.25 is 1.1947 and the corresponding arithmetic mean of the month 72 means recorded in the last row of Table A.26 is 1.3420. Thus, for these alternative fixed-base Fisher indices, the use of carry-forward prices led to final index values which on average are 14.7 percentage points below the corresponding maximum overlap indices. This is a very large downward bias over the six-year period.

The use of maximum overlap indices in place of their carry-forward counterparts has greatly increased seasonal fluctuations and led to fixed-base Fisher indices which have much larger seasonal peaks. Thus, the use of carry-forward prices led to Fisher fixed-base indices which have a lower trend and much lower seasonal fluctuations than their maximum overlap counterpart indices.

6. Computation of the Geary–Khamis Indices

The GK multilateral method was introduced by Geary (1958) in the context of making international comparisons of prices. Khamis (1970) showed that the equations that define the method have a positive solution under certain conditions. A modification of this method has been adapted to the time series context and is being used to construct some components of the Dutch CPI; see Chessa (2016). The GK index was the multilateral index chosen by the Dutch to avoid the chain drift problem for the segments of their CPI

Table A.25 Fisher Star Month-to-Month Indices Using Carry-Forward Prices and Using Months 1 to 12 as the Base Month

t	P_{FS1}^{t}	P_{FS2}^{t}	P_{FS3}^{t}	P_{FS4}^{t}	P_{FS5}^{t}	P_{FS6}^{t}	P_{FS7}^{t}	P_{FS8}^{t}	P_{FS9}^{t}	P_{FS10}^{t}	P_{FS11}^{t}	P_{FS12}^{t}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0567	1.0567	1.0310	1.0391	1.0414	1.0295	1.0307	1.0397	1.0376	1.0381	1.0306	1.0258
3	1.1378	1.1661	1.1378	1.1444	1.1163	1.0828	1.0794	1.0824	1.0885	1.0979	1.1057	1.1056
4	1.1473	1.1666	1.1406	1.1473	1.1189	1.0945	1.0828	1.0745	1.0744	1.0894	1.1097	1.1148
5	1.1216	1.1380	1.1433	1.1501	1.1216	1.1517	1.1615	1.1784	1.1720	1.1687	1.1658	1.1638
6	1.0019	1.0283	1.0528	1.0502	0.9758	1.0019	1.0292	1.0713	1.0918	1.0841	1.0693	1.0575
7	1.0363	1.0624	1.0924	1.0980	1.0007	1.0089	1.0363	1.0789	1.0991	1.1048	1.1073	1.1075
8	1.1215	1.1398	1.1789	1.1974	1.0674	1.0488	1.0772	1.1215	1.1463	1.1765	1.2144	1.2342
9	1.0754	1.0951	1.1241	1.1482	1.0291	0.9868	1.0139	1.0521	1.0754	1.0978	1.1396	1.1494
10	1.1143	1.1342	1.1548	1.1735	1.0694	1.0298	1.0452	1.0622	1.0916	1.1143	1.1554	1.1687
11	1.0265	1.0525	1.0564	1.0613	0.9876	0.9618	0.9607	0.9480	0.9687	0.9900	1.0265	1.0363
12	0.9759	1.0053	1.0043	1.0043	0.9405	0.9246	0.9132	0.8867	0.9131	0.9305	0.9667	0.9759
13	0.9981	1.0195	1.0181	1.0099	0.9419	0.9266	0.9190	0.8960	0.9245	0.9470	0.9878	0.9968
14	1.0795	1.1640	1.1403	1.1282	1.0376	0.9911	0.9869	0.9763	0.9891	1.0146	1.0531	1.0603
15	1.0701	1.0711	1.0459	1.0429	0.9850	0.9684	0.9626	0.9543	0.9664	0.9952	1.0349	1.0413
16	1.1478	1.1547	1.1310	1.1305	1.0470	1.0198	1.0037	0.9821	1.0022	1.0434	1.1051	1.1211
17	1.0812	1.1010	1.1262	1.1224	1.0504	1.0951	1.1105	1.1197	1.1408	1.1510	1.1568	1.1669
18	1.1855	1.2092	1.2468	1.2377	1.1476	1.3062	1.2908	1.2592	1.3078	1.3012	1.2914	1.2898
19	1.1998	1.2183	1.2588	1.2676	1.1687	1.3376	1.3186	1.2743	1.3188	1.3217	1.3073	1.3150
20	1.2151	1.2320	1.2731	1.2827	1.1976	1.3490	1.3436	1.3084	1.3210	1.3251	1.3159	1.3245
21	1.1384	1.1558	1.1819	1.1900	1.1240	1.2629	1.2509	1.1961	1.2072	1.2051	1.2051	1.1994
22	1.2081	1.2079	1.2154	1.2205	1.1480	1.2755	1.2238	1.1243	1.1511	1.1693	1.2146	1.2095
23	1.0702	1.0803	1.0815	1.0855	1.0371	1.1692	1.1127	1.0112	1.0402	1.0436	1.0755	1.0691
24	1.0486	1.0593	1.0597	1.0586	1.0164	1.1526	1.0970	0.9953	1.0214	1.0241	1.0523	1.0437
25	1.0258	1.0403	1.0405	1.0362	0.9986	1.1375	1.0830	0.9809	1.0059	1.0036	1.0293	1.0192
26	1.1305	1.2327	1.2043	1.1968	1.1269	1.2292	1.1767	1.0815	1.1029	1.1042	1.1211	1.1039
27	1.1432	1.1648	1.1376	1.1339	1.0937	1.2199	1.1666	1.0809	1.0882	1.1017	1.1212	1.1077
28	1.2433	1.2420	1.2158	1.2189	1.1479	1.2680	1.2041	1.1036	1.1274	1.1543	1.2065	1.2054
29	1.2224	1.2332	1.2610	1.2671	1.1895	1.3273	1.3018	1.2629	1.2784	1.2777	1.3055	1.2981
30	1.1826	1.1895	1.2235	1.2364	1.0882	1.1772	1.1622	1.1787	1.2691	1.2651	1.2775	1.2595
31	1.2068	1.2033	1.2413	1.2706	1.1000	1.1725	1.1469	1.1443	1.2482	1.2677	1.2952	1.2985
32	1.3019	1.2976	1.3420	1.3697	1.2148	1.2121	1.2202	1.2740	1.3390	1.3632	1.3907	1.4024
33	1.2648	1.2703	1.3005	1.3269	1.1960	1.1809	1.1849	1.2139	1.2938	1.3039	1.3192	1.3164
34	1.3029	1.3018	1.3127	1.3251	1.1978	1.1775	1.1489	1.1277	1.2065	1.2456	1.2924	1.2963
35	1.0853	1.1035	1.1044	1.1100	1.0370	1.0508	1.0341	1.0248	1.0907	1.0843	1.0947	1.0803
36	1.0179	1.0436	1.0426	1.0485	0.9869	1.0036	0.9890	0.9757	1.0434	1.0287	1.0310	1.0131
37	1.0108	1.0403	1.0396	1.0465	0.9881	1.0034	0.9880	0.9751	1.0413	1.0259	1.0253	1.0070
38	1.0725	1.1639	1.1377	1.1390	1.0692	1.0604	1.0542	1.0599	1.1184	1.1026	1.0823	1.0549
39	1.1221	1.1583	1.1333	1.1390	1.0770	1.0790	1.0673	1.0716	1.1308	1.1265	1.1205	1.0991
40	1.3653	1.3688	1.3462	1.3703	1.2488	1.2142	1.1805	1.1739	1.2581	1.2819	1.3341	1.3388
41	1.4804	1.4785	1.5155	1.5668	1.4504	1.6421	1.5691	1.4516	1.4957	1.5063	1.5830	1.5946
42	1.3876	1.3904	1.4365	1.4829	1.3227	1.3804	1.3708	1.3821	1.4386	1.4521	1.5118	1.5202
43	1.3133	1.3054	1.3512	1.4131	1.2089	1.2042	1.1682	1.2066	1.2977	1.3343	1.4143	1.4374
44	1.2094	1.2165	1.2582	1.3000	1.1224	1.1362	1.1127	1.1345	1.2105	1.2481	1.3054	1.3284
45	1.2925	1.3041	1.3368	1.3709	1.2028	1.2072	1.2017	1.2224	1.3292	1.3358	1.3597	1.3636
46	1.3094	1.3234	1.3472	1.3735	1.2208	1.2313	1.2203	1.2177	1.3374	1.3498	1.3685	1.3716

(*Continued*)

Table A.25 (Continued)

t	P_{FS1}^{t}	P_{FS2}^{t}	P_{FS3}^{t}	P_{FS4}^{t}	P_{FS5}^{t}	P_{FS6}^{t}	P_{FS7}^{t}	P_{FS8}^{t}	P_{FS9}^{t}	P_{FS10}^{t}	P_{FS11}^{t}	P_{FS12}^{t}
47	1.2450	1.2572	1.2602	1.2706	1.1377	1.1542	1.1126	1.0647	1.1810	1.2053	1.2488	1.2578
48	1.1275	1.1509	1.1508	1.1461	1.0469	1.0841	1.0452	1.0042	1.0975	1.1095	1.1296	1.1327
49	1.1255	1.1493	1.1488	1.1392	1.0407	1.0762	1.0397	1.0007	1.0976	1.1100	1.1302	1.1309
50	1.2706	1.3768	1.3484	1.3414	1.2020	1.1842	1.1467	1.1236	1.2171	1.2389	1.2548	1.2569
51	1.2445	1.2632	1.2344	1.2366	1.1352	1.1533	1.1146	1.0925	1.1821	1.2047	1.2230	1.2270
52	1.3914	1.3980	1.3800	1.3826	1.2372	1.2391	1.1836	1.1472	1.2501	1.2938	1.3568	1.3816
53	1.3716	1.3947	1.4292	1.4400	1.3357	1.4644	1.3933	1.3401	1.3791	1.4007	1.4570	1.4715
54	1.3004	1.3149	1.3578	1.3857	1.2020	1.2719	1.2288	1.2361	1.3319	1.3565	1.4178	1.4203
55	1.3474	1.3579	1.4038	1.4477	1.2765	1.3329	1.3004	1.3176	1.3852	1.4164	1.4676	1.4771
56	1.3151	1.3380	1.3813	1.4064	1.2884	1.3544	1.3538	1.3732	1.3862	1.4036	1.4296	1.4323
57	1.3102	1.3306	1.3603	1.3775	1.2848	1.3436	1.3437	1.3362	1.3793	1.3797	1.3815	1.3695
58	1.3642	1.3681	1.3763	1.3862	1.2915	1.3406	1.2832	1.2153	1.2615	1.2945	1.3522	1.3539
59	1.0984	1.1290	1.1288	1.1305	1.0902	1.1618	1.1223	1.0572	1.0959	1.0947	1.1063	1.0908
60	1.0215	1.0615	1.0601	1.0547	1.0317	1.1072	1.0771	1.0217	1.0561	1.0469	1.0387	1.0165
61	1.0698	1.1014	1.1001	1.0990	1.0672	1.1403	1.1061	1.0492	1.0828	1.0804	1.0837	1.0664
62	1.2683	1.3893	1.3594	1.3669	1.2824	1.2924	1.2604	1.2254	1.2554	1.2584	1.2686	1.2536
63	1.2914	1.3091	1.2785	1.2927	1.2403	1.2847	1.2485	1.2127	1.2394	1.2571	1.2797	1.2665
64	1.3945	1.3735	1.3425	1.3727	1.3029	1.3408	1.2902	1.2451	1.2798	1.3149	1.3685	1.3685
65	1.4687	1.4550	1.4901	1.5336	1.4576	1.5815	1.5379	1.5020	1.5309	1.5422	1.5838	1.5841
66	1.3079	1.3050	1.3475	1.3801	1.2143	1.2774	1.2420	1.2622	1.3468	1.3668	1.4220	1.4176
67	1.2828	1.2699	1.3145	1.3704	1.1760	1.2134	1.1623	1.1908	1.2910	1.3300	1.3984	1.4088
68	1.2502	1.2462	1.2885	1.3319	1.1650	1.2162	1.1824	1.2008	1.2822	1.3179	1.3666	1.3777
69	1.4321	1.4109	1.4510	1.4835	1.2912	1.3188	1.2881	1.2955	1.4044	1.4386	1.5051	1.5197
70	1.4049	1.3955	1.4213	1.4444	1.2955	1.3475	1.3086	1.2838	1.4097	1.4265	1.4589	1.4645
71	1.1903	1.1973	1.2030	1.2136	1.1149	1.1814	1.1242	1.0677	1.1729	1.1801	1.1966	1.1989
72	1.1733	1.1765	1.1806	1.1834	1.0889	1.1616	1.1050	1.0487	1.1516	1.1567	1.1790	1.1843
Mean	1.1947	1.2126	1.2225	1.2354	1.1383	1.1821	1.1584	1.1382	1.1896	1.2031	1.2303	1.2309

Table A.26 Fisher Star Maximum Overlap Month-to-Month Indices Using Months 1 to 12 as the Base Month

t	P_{FS1}^{t*}	P_{FS2}^{t*}	P_{FS3}^{t*}	P_{FS4}^{t*}	P_{FS5}^{t*}	P_{FS6}^{t*}	P_{FS7}^{t*}	P_{FS8}^{t*}	P_{FS9}^{t*}	P_{FS10}^{t*}	P_{FS11}^{t*}	P_{FS12}^{t*}
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0660	1.0660	1.0154	1.0274	1.0519	1.0409	1.0266	1.0223	1.0253	1.0415	1.0529	1.0541
3	1.1206	1.1765	1.1206	1.1340	1.1637	1.0901	1.0730	1.0677	1.0742	1.0922	1.1058	1.1070
4	1.1370	1.1798	1.1235	1.1370	1.1589	1.1102	1.0919	1.0887	1.0728	1.0795	1.1144	1.1224
5	1.1624	1.1780	1.1192	1.1404	1.1624	1.2880	1.4462	1.7124	1.4438	1.2416	1.1292	1.1443
6	1.0919	1.1183	1.1224	1.1182	0.9854	1.0919	1.2534	1.5766	1.4340	1.2197	1.0899	1.0749
7	1.2646	1.3132	1.3207	1.3169	1.0164	1.1017	1.2646	1.6030	1.4584	1.2577	1.1568	1.2275
8	1.6720	1.7436	1.7548	1.7462	1.1350	1.1580	1.3191	1.6720	1.5190	1.3514	1.3164	1.6277
9	1.4099	1.4660	1.4707	1.4942	1.1351	1.0735	1.2226	1.5520	1.4099	1.2481	1.2167	1.3729
10	1.2621	1.2918	1.2949	1.3293	1.1815	1.1298	1.2691	1.5616	1.4256	1.2621	1.1919	1.2498
11	1.0411	1.0541	1.0550	1.0622	1.0717	1.0429	1.1381	1.3224	1.2064	1.1024	1.0411	1.0407
12	0.9759	0.9870	0.9879	0.9886	0.9913	0.9913	1.0055	1.0025	1.0022	0.9855	0.9762	0.9759
13	0.9981	1.0043	1.0042	0.9878	0.9722	0.9729	0.9861	0.9813	1.0010	1.0069	0.9988	0.9968
14	1.0359	1.1744	1.1230	1.1139	1.1368	0.9887	0.9953	0.9917	1.0069	1.0318	1.0297	1.0290
15	1.0776	1.0806	1.0301	1.0285	1.0453	1.0365	1.0383	1.0329	1.0375	1.0616	1.0678	1.0680
16	1.1480	1.1629	1.1105	1.1204	1.1404	1.1398	1.1497	1.1417	1.1352	1.1168	1.1343	1.1423

t	P_{FS1}^{t*}	P_{FS2}^{t*}	P_{FS3}^{t*}	P_{FS4}^{t*}	P_{FS5}^{t*}	P_{FS6}^{t*}	P_{FS7}^{t*}	P_{FS8}^{t*}	P_{FS9}^{t*}	P_{FS10}^{t*}	P_{FS11}^{t*}	P_{FS12}^{t*}
17	1.1920	1.2251	1.2303	1.2187	1.0856	1.1957	1.3729	1.6517	1.4872	1.2798	1.1686	1.1885
18	1.3415	1.3953	1.4093	1.3719	1.1463	1.4244	1.5595	1.7902	1.7022	1.4552	1.3326	1.3212
19	1.3659	1.4105	1.4011	1.4516	1.1545	1.4529	1.6050	1.8714	1.7432	1.5066	1.2772	1.3438
20	1.3894	1.4334	1.4240	1.4748	1.2245	1.3062	1.5219	1.9546	1.7169	1.4841	1.2628	1.3525
21	1.2859	1.3152	1.3177	1.3228	1.1738	1.1858	1.3915	1.7609	1.5828	1.3716	1.2407	1.2585
22	1.2120	1.2223	1.2224	1.2179	1.2045	1.2021	1.3679	1.6155	1.4592	1.3215	1.2318	1.1998
23	1.0702	1.0767	1.0788	1.0869	1.0750	1.0735	1.0837	1.0819	1.0939	1.0769	1.0691	1.0691
24	1.0486	1.0519	1.0534	1.0507	1.0280	1.0249	1.0310	1.0290	1.0469	1.0489	1.0440	1.0437
25	1.0258	1.0300	1.0316	1.0202	0.9954	0.9922	0.9993	0.9956	1.0103	1.0228	1.0195	1.0192
26	1.0709	1.2436	1.1861	1.1853	1.2134	1.0186	1.0115	1.0045	1.0261	1.0589	1.0608	1.0590
27	1.1314	1.1751	1.1204	1.1183	1.1453	1.0858	1.0722	1.0660	1.0739	1.1164	1.1143	1.1134
28	1.2477	1.2512	1.1939	1.2081	1.2228	1.2052	1.1918	1.1853	1.1929	1.1960	1.2223	1.2305
29	1.3413	1.3634	1.3680	1.3677	1.2133	1.4527	1.5736	1.7720	1.5298	1.3298	1.3015	1.3176
30	1.4282	1.4522	1.4617	1.4642	1.0565	1.2828	1.3943	1.6184	1.6835	1.4441	1.4074	1.3798
31	1.6860	1.7403	1.7475	1.7547	1.0734	1.2637	1.3995	1.6708	1.7864	1.5648	1.4925	1.5978
32	1.8005	1.8601	1.8686	1.8639	1.2578	1.3339	1.4890	1.8994	1.8232	1.6102	1.4988	1.7178
33	1.4029	1.4408	1.4436	1.4597	1.3212	1.3322	1.5096	1.8692	1.6963	1.4770	1.3189	1.3681
34	1.3373	1.3531	1.3537	1.3650	1.3675	1.3578	1.4647	1.6770	1.5104	1.3929	1.3107	1.3160
35	1.0826	1.0843	1.0855	1.0966	1.0790	1.0749	1.1187	1.2892	1.2093	1.1146	1.0854	1.0779
36	1.0179	1.0194	1.0210	1.0364	1.0174	1.0097	1.0072	1.0068	1.0166	1.0091	1.0126	1.0131
37	1.0108	1.0151	1.0171	1.0338	1.0331	1.0240	1.0187	1.0174	1.0140	1.0046	1.0063	1.0070
38	1.0182	1.1742	1.1205	1.1309	1.1747	1.0059	0.9998	0.9993	1.0039	1.0156	1.0156	1.0153
39	1.1031	1.1686	1.1161	1.1280	1.1728	1.1080	1.0993	1.0970	1.0849	1.0965	1.0970	1.0974
40	1.4040	1.3916	1.3323	1.3581	1.4302	1.4698	1.4547	1.4398	1.4032	1.3799	1.3884	1.3943
41	1.8975	1.9304	1.9283	1.9375	1.5129	1.7996	1.9161	2.0870	1.9017	1.6845	1.8214	1.8410
42	1.9937	2.0489	2.0532	2.0591	1.3405	1.5049	1.6691	1.9928	1.9347	1.6979	1.8456	1.9131
43	2.2578	2.3145	2.2994	2.3769	1.1826	1.2955	1.4167	1.7577	1.8205	1.6248	1.7234	2.0790
44	1.9183	1.9686	1.9552	2.0226	1.1290	1.1592	1.3084	1.6909	1.6882	1.5070	1.4868	1.8028
45	1.5254	1.5681	1.5690	1.5951	1.3664	1.3324	1.5316	1.9316	1.7427	1.5262	1.4159	1.4917
46	1.4305	1.4576	1.4601	1.4931	1.4021	1.3760	1.5702	1.9258	1.7498	1.5288	1.3927	1.4215
47	1.2525	1.2695	1.2696	1.2794	1.3005	1.2797	1.3878	1.5328	1.4330	1.3225	1.2683	1.2567
48	1.1275	1.1414	1.1427	1.1278	1.1386	1.1486	1.1684	1.1641	1.1529	1.1488	1.1345	1.1327
49	1.1255	1.1384	1.1393	1.1156	1.1301	1.1338	1.1462	1.1384	1.1317	1.1459	1.1345	1.1309
50	1.2165	1.3890	1.3279	1.3258	1.4179	1.2646	1.2728	1.2641	1.2310	1.2358	1.2200	1.2185
51	1.2425	1.2701	1.2106	1.2255	1.3003	1.3274	1.3389	1.3261	1.2796	1.2422	1.2470	1.2554
52	1.4108	1.4129	1.3594	1.3702	1.4568	1.5267	1.5331	1.5071	1.4519	1.4065	1.4122	1.4239
53	1.6510	1.6917	1.7000	1.6739	1.3661	1.6056	1.6885	1.8991	1.7467	1.5371	1.6035	1.6283
54	1.9296	1.9827	1.9910	1.9807	1.1841	1.3865	1.4792	1.7302	1.8456	1.6221	1.8150	1.8783
55	2.0418	2.0824	2.0649	2.1376	1.2521	1.4315	1.5784	1.9331	1.8834	1.6635	1.6895	1.9618
56	1.6455	1.6827	1.6689	1.7285	1.2958	1.3840	1.5991	2.0519	1.8024	1.5702	1.4725	1.6028
57	1.4359	1.4774	1.4810	1.4925	1.3644	1.3930	1.6116	2.0057	1.8083	1.5667	1.4016	1.4162
58	1.3727	1.3927	1.3962	1.4042	1.4258	1.4135	1.4202	1.5897	1.4549	1.3947	1.3568	1.3620
59	1.0984	1.1079	1.1102	1.1079	1.1039	1.0947	1.0910	1.0853	1.0793	1.0986	1.0924	1.0909
60	1.0215	1.0289	1.0311	1.0194	1.0112	1.0045	1.0031	0.9965	1.0066	1.0286	1.0188	1.0165
61	1.0698	1.0769	1.0784	1.0718	1.0788	1.0753	1.0741	1.0690	1.0602	1.0751	1.0680	1.0664
62	1.2029	1.4016	1.3388	1.3574	1.4746	1.2851	1.2714	1.2673	1.2116	1.2136	1.2049	1.2038

(*Continued*)

Table A.26 (Continued)

t	P_{FS1}^{t*}	P_{FS2}^{t*}	P_{FS3}^{t*}	P_{FS4}^{t*}	P_{FS5}^{t*}	P_{FS6}^{t*}	P_{FS7}^{t*}	P_{FS8}^{t*}	P_{FS9}^{t*}	P_{FS10}^{t*}	P_{FS11}^{t*}	P_{FS12}^{t*}
63	1.2938	1.3207	1.2591	1.2765	1.3642	1.3644	1.3397	1.3370	1.2725	1.2960	1.2935	1.2915
64	1.4493	1.3835	1.3187	1.3605	1.4623	1.5829	1.5447	1.5409	1.4425	1.4287	1.4463	1.4524
65	1.8187	1.8378	1.8364	1.8513	1.5348	1.6835	1.8128	2.1547	1.9214	1.6963	1.7452	1.7653
66	1.8654	1.9123	1.9208	1.9220	1.2138	1.3924	1.4919	1.7687	1.8118	1.5912	1.7562	1.7934
67	2.2309	2.2936	2.2795	2.3570	1.1303	1.2842	1.4109	1.7295	1.8490	1.6399	1.7690	2.0736
68	1.9754	2.0131	1.9958	2.0662	1.1684	1.2144	1.3826	1.7928	1.8026	1.5942	1.6048	1.8980
69	1.7245	1.7767	1.7783	1.8012	1.4971	1.4525	1.6367	2.0508	1.8413	1.6320	1.5844	1.7190
70	1.5060	1.5383	1.5420	1.5684	1.4961	1.4915	1.6850	2.0395	1.8434	1.6157	1.4853	1.5069
71	1.1912	1.2098	1.2135	1.2268	1.2440	1.2490	1.3792	1.5776	1.4281	1.2874	1.2135	1.1969
72	1.1733	1.1892	1.1936	1.1963	1.2131	1.2254	1.2386	1.2286	1.2096	1.1821	1.1806	1.1843
Mean	1.3552	1.3916	1.3774	1.3897	1.2052	1.2403	1.3196	1.4841	1.4165	1.3095	1.2848	1.3307

that use scanner data. Given the recent use of GK indices by several national statistical agencies, it seems to be useful to calculate these indices using our data set.

The GK system of equations for T time periods involves *T price levels* $p_{GK}^{1},\ldots,p_{GK}^{T}$ and *N quality adjustment factors* $\alpha_1,\ldots,\alpha_N$. Let p^t and q^t denote the N-dimensional price and quantity vectors for month t (with components $p_{t,n}$ and $q_{t,n}$ as usual). The total consumption (or sales) vector q over the entire window of observations is defined as the following simple sum of the period-by-period consumption vectors:

$$q \equiv \Sigma_{t=1}^{T} q^t, \qquad \text{(A1)}$$

where $q \equiv [q_1,q_2,\ldots,q_N]$. The equations which determine the *GK price levels* $p_{GK}^{1},\ldots,p_{GK}^{T}$ and *quality adjustment factors* $\alpha_1,\ldots,\alpha_N$ (up to a scalar multiple) are as follows:

$$\alpha_n = \Sigma_{t=1}^{T} [q_{t,n}/q_n][p_{t,n}/p_{GK}^{t}];\ n = 1,\ldots,N; \qquad \text{(A2)}$$

$$p_{GK}^{t} = p^t\cdot q^t/\alpha\cdot q^t = \Sigma_{n=1}^{N} [\alpha_n q_{t,n}/\alpha\cdot q^t][p_{t,n}/\alpha_n];\quad t = 1,\ldots,T, \qquad \text{(A3)}$$

where $\alpha \equiv [\alpha_1,\ldots,\alpha_N]$ is the vector of GK quality adjustment factors. The sample share of period t's purchases of product n in total sales of product n over all T periods can be defined as $S_{t,n} \equiv q_{t,n}/q_n$ for $n = 1,\ldots,N$ and $t = 1,\ldots,T$. Thus, $\alpha_n \equiv \Sigma_{t=1}^{T} S_{t,n}[p_{t,n}/p_{GK}^{t}]$ is a (real) share-weighted average of the period t inflation-adjusted prices $p_{t,n}/p_{GK}^{t}$ for product n over all T periods. The period t quality-adjusted sum of quantities sold is defined as the *period t GK quantity level*, $q_{GK}^{t} \equiv \alpha\cdot q^t = \Sigma_{n=1}^{N} \alpha_n q_{t,n}$.[81] Thus, the aggregate quantity or volume or utility function for period t is a *simple linear function* of the quantities $q_{t,n}$ consumed during period t. This period t quantity level is divided into the value of period t sales, $p^t\cdot q^t = \Sigma_{n=1}^{N} p_{tn}q_{tn}$, in order to obtain the period t GK price level, p_{GK}^{t}. Thus, the GK price level for period t can be interpreted as a *quality-adjusted unit value index*, where α_n act as the quality adjustment factors.

Note that the GK price level, p_{GK}^{t}, defined by (A3) *does not depend on the estimated reservation prices*; that is, the definition of p_{GK}^{t} zeros out any reservation prices that are applied to missing products, and thus $P_{GK}^{t} \equiv p_{GK}^{t}/p_{GK}^{1}$ also does not depend on reservation prices.[82]

It can be seen that if a solution to equations (A2) and (A3) exists, then if all of the period price levels p_{GK}^{t} are multiplied, say, by a positive scalar λ say and all of the quality adjustment factors α_n are divided by the same λ, then another solution to (A2) and (A3) is obtained. Hence, a_n and p_{GK}^{t} are only determined up to a scalar multiple, and an additional normalization such as $p_{GK}^{1} = 1$ or $\alpha_1 = 1$ is required to determine a unique solution to the system of equations defined by (A2) and (A3).

A traditional method for obtaining a solution to (A2) and (A3) is to iterate between these equations. Thus, set $\alpha = 1_N$, a vector of ones, and use equations (A3) to obtain an initial sequence for p_{GK}^{t}. Substitute these p_{GK}^{t} estimates into equations (A2) and obtain α_n estimates. Substitute these α_n estimates into equations (A3) and obtain a new sequence of p_{GK}^{t} estimates. Continue iterating between the two systems until convergence is achieved. This method was used to calculate the GK price levels p_{GK}^{t}. Using our data set, the iterative method took 20 iterations to converge to three decimal places.

However, there is a more efficient non-iterative method which can be used to compute the GK indices. Following Diewert (1999b; 26),[83] substitute equations (A3) into equations (A2) and after some simplification, obtain the following system of equations that will determine the components of the α vector:

$$[I_N - C]\alpha = 0_N, \qquad \text{(A4)}$$

where I_N is the N by N identity matrix, 0_N is a vector of zeros of dimension N, and the C matrix is defined as:

$$C \equiv \hat{q}^{-1}\, \Sigma_{t=1}^{T} s^t q^{tT}, \qquad \text{(A5)}$$

[81] Khamis (1972; 101) also derived this equation in the time series context.

[82] In equations (A2) and (A3), each price p_{tn} always appears with the multiplicative factor q_{tn}. Thus, if p_{tn} is an imputed price, it will always be multiplied by $q_{tn} = 0$, and thus any imputed price will have no impact on α_n and p_{GK}^{t}.

[83] See also Diewert and Fox (2020) for additional discussion on this solution method.

where $\hat{q}$ is an N by N diagonal matrix with the elements of the sample total purchase vector q running down the main diagonal and $\hat{q}^{-1}$ denotes the inverse of this matrix, s^t is the period t expenditure share column vector, q^t is the column vector of quantities purchased during period t, and q_n is the nth element of the sample total q defined by (A1).

The matrix $I_N - C$ is singular which implies that the N equations in (A4) are not all independent. In particular, if the first N-1 equations in (A4) are satisfied, then the last equation in (A4) will also be satisfied. It can also be seen that the N equations in (A4) are homogeneous of degree one in the components of the vector α. Thus, to obtain a unique α solution to (A4), set α_N equal to 1, drop the last equation in (A4) and solve the remaining N–1 equations for $\alpha_1,\alpha_2,\ldots,\alpha_{N-1}$. Once the α_n values are known, equations (A3) can be used to determine the GK price levels, $p_{GK}^{\ t} = p^t \cdot q^t / \alpha \cdot q^t$ for $t = 1,\ldots,T$. These price levels were then divided by the first price level $p_{GK}^{\ 1}$ in order to form the GK indices, $P_{GK}^{\ t} \equiv p_{GK}^{\ t}/p_{GK}^{\ 1}$, which are listed in Table A.27.[84] For comparison purposes, we also list the indices from Table 9.17 in the main text. The indices listed in Table A.27 are plotted in Figure A9.1.

The GK indices $P_{GK}^{\ t}$ end up capturing the trend as was the case for all of the other indices with the exception of the chained Paasche and Fisher indices, $P_{PCH}^{\ t*}$ and $P_{FCH}^{\ t*}$, which suffer from severe downward chain drift. The seasonal fluctuations in the GK indices are smaller than the fluctuations in the fixed-base Fisher and GEKS indices, $P_{FB}^{\ t*}$ and $P_{GEKS}^{\ t*}$ but the mean of $P_{GK}^{\ t}$ is 1.2982 compared to the mean of the similarity-linked indices $P_S^{\ t*}$ of 1.4749. The following chart shows that the seasonal fluctuations in the GK indices are substantial.

Table A.27 GK Price Levels and Indices and Alternative Month-to-Month Price Indices Using Maximum Overlap Bilateral Indices as Building Blocks

t	$P_{LFB}^{\ t*}$	$P_{LCH}^{\ t*}$	$P_{PFB}^{\ t*}$	$P_{PCH}^{\ t*}$	$P_{FCH}^{\ t*}$	$P_{FFB}^{\ t*}$	$P_{GEKS}^{\ t*}$	$P_S^{\ t*}$	$P_{GK}^{\ t}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.07104	1.07104	1.06104	1.06104	1.06603	1.06603	1.03802	1.06603	1.00629
3	1.12812	1.18503	1.11303	1.16798	1.17647	1.12055	1.10386	1.17647	1.10863
4	1.15044	1.19078	1.12373	1.16845	1.17956	1.13701	1.11167	1.17956	1.11325
5	1.18406	1.19694	1.14104	1.16942	1.18310	1.16235	1.28331	1.18310	1.34728
6	1.10502	1.03417	1.07887	0.97269	1.00296	1.09186	1.17550	1.00296	1.15685
7	1.24566	1.06832	1.28386	0.95860	1.01198	1.26462	1.28536	1.01198	1.25985
8	1.64472	1.13041	1.69981	0.98562	1.05554	1.67204	1.53539	1.05554	1.35030
9	1.33555	1.05897	1.48835	0.90641	0.97973	1.40988	1.34806	0.97973	1.20870
10	1.23076	1.08596	1.29420	0.90374	0.99067	1.26208	1.29133	0.99067	1.24074
11	1.03294	0.96785	1.04925	0.77360	0.86529	1.04107	1.08720	1.04107	1.05898
12	0.97081	0.90818	0.98105	0.72496	0.81141	0.97592	0.99061	0.97592	0.97584
13	0.99746	0.92454	0.99881	0.74299	0.82881	0.99813	0.99017	0.99684	0.99408
14	1.04362	0.96387	1.02824	0.76965	0.86130	1.03590	1.03346	1.17902	1.09779
15	1.08902	0.91833	1.06632	0.69057	0.79635	1.07761	1.04121	1.08056	1.01912
16	1.15867	0.99822	1.13743	0.75088	0.86576	1.14800	1.12905	1.17474	1.09612
17	1.23204	1.07256	1.15330	0.78529	0.91776	1.19202	1.30850	1.10498	1.37790
18	1.40150	1.12341	1.28409	0.82835	0.96466	1.34151	1.44174	1.30841	1.53226
19	1.29310	1.21533	1.44276	0.86212	1.02360	1.36588	1.49357	1.18142	1.54893
20	1.32299	1.27417	1.45909	0.88364	1.06109	1.38937	1.50199	1.23391	1.54799
21	1.27093	1.19560	1.30112	0.81994	0.99011	1.28593	1.35277	1.09986	1.36646
22	1.20769	1.24737	1.21638	0.79472	0.99564	1.21203	1.26983	1.23179	1.26559
23	1.07109	1.10288	1.06924	0.70461	0.88153	1.07017	1.07993	1.06906	1.06493
24	1.05248	1.07832	1.04479	0.68717	0.86081	1.04863	1.04214	1.04392	1.04178
25	1.03276	1.05627	1.01894	0.67328	0.84331	1.02583	1.01037	1.02270	1.01509
26	1.07388	1.09687	1.06790	0.69442	0.87275	1.07089	1.07080	1.22856	1.15710
27	1.14208	1.05809	1.12088	0.64996	0.82929	1.13143	1.09977	1.17215	1.11378
28	1.26148	1.14177	1.23415	0.70168	0.89507	1.24774	1.20934	1.25327	1.18622

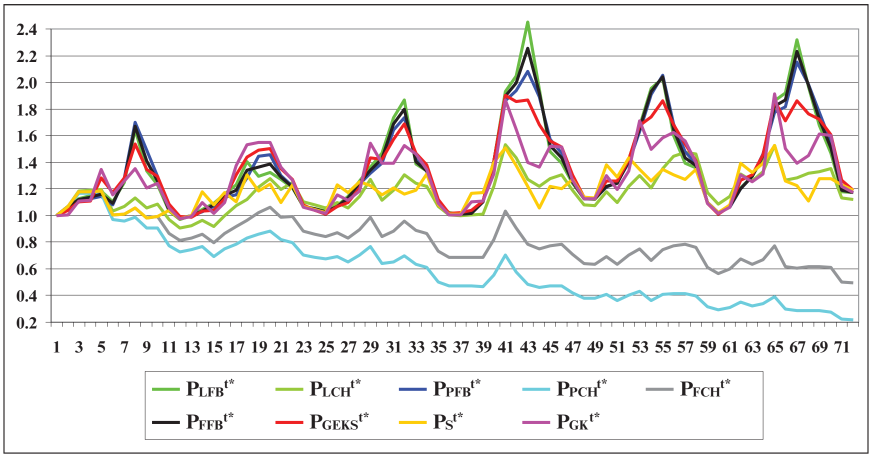
(Continued)

[84] Table A.27 lists the GK indices using the efficient method. The efficient method will always work if the elements in the C matrix are all positive. If the elements of C are only nonnegative, then in rare cases, the efficient method may not work; see Diewert (1999b; 26).

Table A.27 (Continued)

t	P_{LFB}^{t*}	P_{LCH}^{t*}	P_{PFB}^{t*}	P_{PCH}^{t*}	P_{FCH}^{t*}	P_{FFB}^{t*}	P_{GEKS}^{t*}	P_{S}^{t*}	P_{GK}^{t}
29	1.36612	1.26826	1.31684	0.76739	0.98654	1.34125	1.43467	1.22223	1.54415
30	1.46661	1.11351	1.39075	0.63881	0.84340	1.42818	1.42150	1.15449	1.39074
31	1.73296	1.18814	1.64024	0.65250	0.88049	1.68596	1.57446	1.20526	1.39273
32	1.86733	1.30512	1.73601	0.70065	0.95626	1.80047	1.68844	1.16278	1.52494
33	1.38675	1.24237	1.41926	0.63598	0.88889	1.40291	1.47063	1.18929	1.46106
34	1.33751	1.21976	1.33703	0.61354	0.86509	1.33727	1.38167	1.31066	1.34770
35	1.08703	1.06732	1.07810	0.50379	0.73329	1.08256	1.11771	1.07810	1.09437
36	1.02305	1.00314	1.01285	0.47308	0.68889	1.01793	1.01873	1.01195	1.00484
37	1.01159	0.99864	1.00992	0.47009	0.68516	1.01076	1.01965	1.01076	1.00324
38	1.02156	1.00610	1.01478	0.47200	0.68912	1.01816	1.03899	1.16812	1.10257
39	1.10562	1.01261	1.10047	0.46755	0.68807	1.10305	1.10616	1.17108	1.11092
40	1.40435	1.21657	1.40366	0.55437	0.82124	1.40401	1.40571	1.39663	1.31260
41	1.93372	1.52949	1.86189	0.70095	1.03542	1.89746	1.90004	1.50841	1.86742
42	2.04948	1.43492	1.93935	0.57426	0.90775	1.99365	1.85555	1.37756	1.64465
43	2.44964	1.26995	2.08106	0.48573	0.78540	2.25784	1.86789	1.22151	1.39593
44	1.94826	1.22130	1.88870	0.45816	0.74803	1.91825	1.68235	1.05506	1.36349
45	1.47755	1.27915	1.57473	0.46934	0.77483	1.52537	1.55868	1.22173	1.51780
46	1.39276	1.30456	1.46928	0.47089	0.78377	1.43051	1.51589	1.19999	1.51189
47	1.24213	1.18988	1.26300	0.42194	0.70856	1.25252	1.30463	1.26828	1.26399
48	1.12808	1.08051	1.12696	0.37820	0.63926	1.12752	1.13747	1.13921	1.12666
49	1.12212	1.07637	1.12896	0.37668	0.63675	1.12554	1.12370	1.13475	1.12367
50	1.21916	1.17596	1.21377	0.40957	0.69400	1.21646	1.25767	1.38339	1.30070
51	1.23969	1.09662	1.24530	0.36458	0.63230	1.24249	1.25967	1.29063	1.19632
52	1.42259	1.21667	1.39905	0.40512	0.70206	1.41077	1.42183	1.43303	1.33554
53	1.67018	1.29789	1.63196	0.43211	0.74889	1.65096	1.67706	1.34386	1.70876
54	1.95352	1.20862	1.90595	0.36369	0.66299	1.92959	1.74172	1.25757	1.49425
55	2.03052	1.35442	2.05315	0.40619	0.74173	2.04180	1.85986	1.34547	1.58188
56	1.60294	1.44271	1.68914	0.41245	0.77140	1.64547	1.68088	1.30412	1.62578
57	1.39502	1.47625	1.47791	0.41505	0.78276	1.43587	1.53684	1.26875	1.55911
58	1.35851	1.46359	1.38700	0.39760	0.76284	1.37268	1.41013	1.34737	1.39857
59	1.09904	1.17136	1.09780	0.31761	0.60994	1.09842	1.09276	1.09738	1.08979
60	1.02215	1.08744	1.02087	0.29512	0.56650	1.02151	1.01005	1.01922	1.01532
61	1.07410	1.14641	1.06543	0.31177	0.59784	1.06976	1.06802	1.07767	1.06331
62	1.20643	1.30964	1.19934	0.34893	0.67600	1.20288	1.26692	1.39115	1.30698
63	1.29331	1.25722	1.29424	0.32078	0.63505	1.29377	1.30348	1.32072	1.25348
64	1.43515	1.32893	1.46351	0.33813	0.67033	1.44926	1.45723	1.39001	1.31303
65	1.86123	1.52850	1.77723	0.38896	0.77106	1.81874	1.86177	1.52597	1.91247
66	1.91700	1.26434	1.81516	0.30051	0.61640	1.86539	1.70997	1.25740	1.50202
67	2.31683	1.28283	2.14817	0.28713	0.60691	2.23091	1.86260	1.22459	1.39528
68	1.96840	1.31863	1.98236	0.28708	0.61526	1.97537	1.76427	1.11160	1.44934
69	1.67463	1.32951	1.77584	0.28527	0.61585	1.72450	1.72233	1.27951	1.61471
70	1.46701	1.35499	1.54608	0.27515	0.61059	1.50602	1.60516	1.27885	1.60223
71	1.18124	1.13072	1.20127	0.22115	0.50006	1.19121	1.26341	1.23088	1.21718
72	1.17122	1.11995	1.17533	0.21988	0.49624	1.17327	1.18952	1.19115	1.17489
Mean	1.35950	1.17720	1.35160	0.59613	0.81450	1.35520	1.34680	1.18920	1.29820

Figure A9.1 GK, GEKS, and Similarity-Linked Indices with Laspeyres, Paasche, and Fisher Fixed-Base and Chained Maximum Overlap Indices



References

Allen, Robert C., and W. Erwin Diewert. 1981. "Direct versus Implicit Superlative Index Number Formulae." *Review of Economics and Statistics* 63: 430–35.

Alterman, William F., W. Erwin Diewert, and Robert C. Feenstra. 1999. *International Trade Price Indexes and Seasonal Products*. Washington DC: Bureau of Labor Statistics.

Anderson, Oskar. 1927. "On the Logic of the Decomposition of Statistical Series into Separate Components." *Journal of the Royal Statistical Society* 90: 548–69.

Armknecht, Paul A., and Fenella Maitland-Smith. 1999. "Price Imputation and Other Techniques for Dealing with Missing Observations, Seasonality and Quality Change in Price Indices." IMF Working Paper No. 99/78, Washington, DC, June.

Aten, Bettina, and Alan Heston. 2009. "Chaining Methods for International Real Product and Purchasing Power Comparisons: Issues and Alternatives." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao (pp. 245–73). Cheltenham, UK: Edward Elgar.

Australian Bureau of Statistics. 2016. "Making Greater Use of Transactions Data to Compile the Consumer Price Index." Information Paper 6401.0.60.003, November 29, Canberra: ABS.

Baldwin, Andrew. 1990. "Seasonal Baskets in Consumer Price Indexes." *Journal of Official Statistics* 6 (3): 251–73.

Balk, Bert M. 1980a. *Seasonal Products in Agriculture and Horticulture and Methods for Computing Price Indices*, Statistical Studies no. 24, The Hague: Netherlands Central Bureau of Statistics.

Balk, Bert M. 1980b. "Seasonal Products and the Construction of Annual and Monthly Price Indexes." *Statistische Hefte* 21(2): 110–16.

Balk, Bert M. 1980c. "A Method for Constructing Price Indices for Seasonal Products." *Journal of the Royal Statistical Society A* 143: 68–75.

Balk, Bert M. 1981. "A Simple Method for Constructing Price Indices for Seasonal Products." *Statistische Hefte* 22 (1): 1–8.

Balk, Bert M. 1996. "A Comparison of Ten Methods for Multilateral International Price and Volume Comparisons." *Journal of Official Statistics* 12: 199–222.

Balk, Bert M. 2008. *Price and Quantity Index Numbers*. New York: Cambridge University Press.

Bean, Louis H., and Oscar C. Stine. 1924. "Four Types of Index Numbers of Farm Prices." *Journal of the American Statistical Association* 19: 30–35.

Carli, Gian-Rinaldo. 1804. "Del valore e della proporzione de' metalli monetati." In *Scrittori classici italiani di economia politica*, Volume 13 (pp. 297-366). Milano: G.G. Destefanis (originally published in 1764).

Caves, Douglas W., Laurits R. Christensen, and W. Erwin Diewert. 1982. "Multilateral Comparisons of Output, Input and Productivity using Superlative Index Numbers." *Economic Journal* 92: 73–86.

Chessa, Antonio G. 2016. "A New Methodology for Processing Scanner Data in the Dutch CPI." *Eurona* 2016 (1): 49–69.

Chessa, Antonio G. 2016. "MARS: A Method for Defining Products and Linking Barcodes of Item Relaunches." Paper presented at the 16th Meeting of the Ottawa Group, Rio de Janeiro, Brazil, May 8–10. https://www.ottawagroup.org/Ottawa/ottawagroup.nsf/home/Meeting + 16/$FILE/A%20method%20for%20defining%20products%20paper.pdf

Court, Andrew T. 1939. "Hedonic Price Indexes with Automotive Examples." In *The Dynamics of Automobile Demand* (pp. 99–117). New York: General Motors Corporation.

Crump, Norman. 1924. "The Interrelation and Distribution of Prices and Their Incidence upon Price Stabilization." *Journal of the Royal Statistical Society* 87: 167–206.

Dalén, Jörgen. 1992. "Computing Elementary Aggregates in the Swedish Consumer Price Index." *Journal of Official Statistics* 8: 129–47.

de Haan, Jan. 2015. "Rolling Year Time Dummy Indexes and the Choice of Splicing Method." Room Document at the 14th meeting of the Ottawa Group, May 22, Tokyo. http://www.stat.go.jp/english/info/meetings/og2015/pdf/t1s3room

de Haan, Jan, and Heymerik van der Grient. 2011. "Eliminating Chain Drift in Price Indexes Based on Scanner Data." *Journal of Econometrics* 161: 36–46.

Diewert, W. Erwin. 1976. "Exact and Superlative Index Numbers." *Journal of Econometrics* 4: 114–45.

Diewert, W. Erwin. 1978. "Superlative Index Numbers and Consistency in Aggregation." *Econometrica* 46: 883–900.

Diewert, W. Erwin. 1983. "The Treatment of Seasonality in a Cost of Living Index." In *Price Level Measurement*, edited by W.E. Diewert and Claude Montmarquette (pp. 1019–1045). Ottawa: Statistics Canada.

Diewert, W. Erwin. 1988. "Test Approaches to International Comparisons." In *Measurement in Economics: Theory and Applications of Economic Indices*, edited by W. Eichhorn (pp. 67–86). Heidelberg: Physica-Verlag.

Diewert, W. Erwin. 1992. "Fisher Ideal Output, Input and Productivity Indexes Revisited." *Journal of Productivity Analysis* 3: 211–48.

Diewert, W. Erwin. 1995. "Axiomatic and Economic Approaches to Elementary Price Indexes." Discussion Paper No. 95–01, Department of Economics, University of British Columbia, Vancouver, Canada.

Diewert, W. Erwin. 1998. "High Inflation, Seasonal Products and Annual Index Numbers." *Macroeconomic Dynamics* 2: 456–71.

Diewert, W. Erwin. 1999a. "Index Number Approaches to Seasonal Adjustment." *Macroeconomic Dynamics* 3: 1–21.

Diewert, W. Erwin. 1999b. "Axiomatic and Economic Approaches to International Comparisons." In *International and Interarea Comparisons of Income, Output and Prices*, edited by Alan Heston and Robert E. Lipsey (pp. 13–87), Studies in Income and Wealth, Volume 61. Chicago: The University of Chicago Press.

Diewert, W. Erwin. 2009. "Similarity Indexes and Criteria for Spatial Linking." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao (pp. 183–216). Cheltenham, UK: Edward Elgar.

Diewert, W. Erwin. 2021a. "Elementary Indexes." Draft Chapter 6 in *Consumer Price Index Theory*, Washington DC: International Monetary Fund, published online at: https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W. Erwin. 2021b. "The Chain Drift Problem and Multilateral Indexes." Draft Chapter 7 in *Consumer Price Index Theory*, Washington DC: International Monetary Fund, published online at: https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W. Erwin. 2021c. "Quality Adjustment Methods." Draft Chapter 8 in *Consumer Price Index Theory*, Washington DC: International Monetary Fund, published online at: https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W. Erwin, William F. Alterman, and Robert C. Feenstra. 2012. "Time Series versus Index Number Methods of Seasonal Adjustment." In *Price and Productivity Measurement: Volume 2—Seasonality*, Chapter 3 in Diewert, edited by W. Erwin, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura (pp. 29–52). Bloomington, IN: Trafford Press.

Diewert, W. Erwin, Yoel Finkel, and Yevgeny Artsev. 2011. "Empirical Evidence on the Treatment of Seasonal Products: The Israeli CPI Experience." In *Price and Productivity Measurement: Volume 2—Seasonality*, Chapter 4 in Diewert, edited by W. Erwin, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura (pp. 53–78). Bloomington, IN: Trafford Press.

Diewert, W. Erwin, and Kevin J. Fox. 2017. "Output Growth and Inflation Across Space and Time." *EURONA* 2017 (1): 7–40.

Diewert, W. Erwin, and Kevin J. Fox. 2018. "Addendum to Output Growth and Inflation Across Space and Time." *EURONA* 2018(2): 1–10.

Diewert, W. Erwin and Kevin J. Fox. 2021. "Substitution Bias in Multilateral Methods for CPI Construction Using Scanner Data." *Journal of Business and Economic Statistics* 40 (1): 355–69.

Diewert, W. Erwin, Kevin J. Fox, and Paul Schreyer. 2018. "The Digital Economy, New Products and Household Welfare." Economic Statistics Center of Excellence (ESCoE), London, UK.

Dutot, Charles. 1738. *Réflexions politiques sur les finances et le commerce*, Volume 1, La Haye: Les frères Vaillant et N. Prevost.

Eichhorn, Wolfgang. 1978. *Functional Equations in Economics*. Reading, MA: Addison-Wesley Publishing Company.

Eltetö, Ödon, and Köves, Pal. 1964. "On a Problem of Index Number Computation Relating to International Comparisons." (in Hungarian), *Statisztikai Szemle* 42: 507–18.

European Economic Commission. 2020. "Commission Implementing Regulation (EUO 2020/1148 of 31 July, 2020." *Official Journal of the European Union* L 252: 12–23.

Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton Mifflin Co.

Flux, Alfred W. 1921. "The Measurement of Price Change." *Journal of the Royal Statistical Society* 84: 167–99.

Geary, Roy. C. 1958. "A Note on Comparisons of Exchange Rates and Purchasing Power between Countries." *Journal of the Royal Statistical Society Series A* 121: 97–99.

Gini, Corrado. 1924. "Quelques considérations au sujet de la construction des nombres indices des prix et des questions analogues." *Metron* 4 (1): 3–162.

Gini, Corrado. 1931. "On the Circular Test of Index Numbers." *Metron* 9 (9): 3–24.

Hill, Robert J. 1997. "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities." *Review of Income and Wealth* 43 (1): 49–69.

Hill, Robert J. 1999a. "Comparing Price Levels across Countries Using Minimum Spanning Trees." *The Review of Economics and Statistics* 81: 135–42.

Hill, Robert J. 1999b. "International Comparisons using Spanning Trees." in *International and Interarea Comparisons of Income, Output and Prices*, edited by Alan Heston and Robert E. Lipsey (pp. 109–120), Studies in Income and Wealth Volume 61, NBER. Chicago: The University of Chicago Press.

Hill, Robert J. 2001. "Measuring Inflation and Growth Using Spanning Trees." *International Economic Review* 42: 167–85.

Hill, Robert J. 2004. "Constructing Price Indexes Across Space and Time: The Case of the European Union." *American Economic Review* 94: 1379–410.

Hill, Robert J. 2009. "Comparing Per Capita Income Levels Across Countries Using Spanning Trees: Robustness, Prior Restrictions, Hybrids and Hierarchies." In *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, edited by D.S. Prasada Rao (pp. 217–244). Cheltenham: Edward Elgar.

Hill, Robert J., and Marcel P. Timmer. 2006. "Standard Errors as Weights in Multilateral Price Indexes." *Journal of Business and Economic Statistics* 24 (3): 366–77.

Inklaar, Robert, and W. Erwin Diewert. 2016. "Measuring Industry Productivity and Cross-Country Convergence." *Journal of Econometrics* 191: 426–33.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2009. "Scanner Data, Time Aggregation and the Construction of Price Indexes." Discussion Paper 09–09, Department of Economics, University of British Columbia, Vancouver, Canada.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2011. "Scanner Data, Time Aggregation and the Construction of Price Indexes." *Journal of Econometrics* 161: 24–35.

Jevons, William S. 1865. "The variation of Prices and the Value of the Currency since 1782." *Journal of the Statistical Society of London* 28: 294–320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119–50.

Keynes, John M. 1909. "The Method of Index Numbers with Special Reference to the Measurement of General Exchange Value." In *The Collected Writings of John Maynard Keynes* (1983), Volume 11, edited by Don Moggridge (reprinted as pp. 49–156). Cambridge: Cambridge University Press.

Keynes, John M. 1930. *Treatise on Money*, Volume 1. London: Macmillan.

Khamis, Salem H. 1970. "Properties and Conditions for the Existence of a New Type of Index Number." *Sankhya B* 32: 81–98.

Khamis, Salem H. 1972. "A New System of Index Numbers for National and International Purposes." *Journal of the Royal Statistical Society Series A* 135: 96–121.

Kravis, Irving B., Alan Heston, and Robert Summers. 1982. *World Product and Income: International Comparisons of Real Gross Product*. Statistical Office of the United Nations and the World Bank, Baltimore: The Johns Hopkins University Press.

Krsinich, Frances. 2016. "The FEWS Index: Fixed Effects with a Window Splice." *Journal of Official Statistics* 32: 375–404.

Laspeyres, Etienne. 1871. "Die Berechnung einer mittleren Waarenpreissteigerung." *Jahrbücher für Nationalökonomie und Statistik* 16: 296–314.

Lowe, Joseph. 1823. *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition. London: Longman, Hurst, Rees, Orme and Brown.

Marshall, Alfred. 1887. "Remedies for Fluctuations of General Prices." *Contemporary Review* 51: 355–75.

Mendershausen, Horst. 1937. "Annual Survey of Statistical Technique: Methods of Computing and Eliminating Changing Seasonal Fluctuations." *Econometrica* 5: 234–62.

Mitchell, Wesley C. 1927. *Business Cycles*. New York: National Bureau of Economic Research.

Mudgett, Bruce D. 1955. "The Measurement of Seasonal Movements in Price and Quantity Indexes." *Journal of the American Statistical Association* 50: 93–98.

O' Donnell, Gerry and Clement Yélou. 2021. *Adjusted Price Index and Monthly Adjusted Consumer Expenditure Basket Weights*. Catalogue no. 62F0014M, Release date: November 10, 2021, Ottawa: Statistics Canada.

Paasche, Hermann. 1874. "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen." *Jahrbücher für Nationalökonomie und Statistik* 12: 168–78.

Stone, Richard. 1956. *Quantity and Price Indexes in National Accounts*. Paris: OECD.

Summers, Robert. 1973. "International Comparisons with Incomplete Data." *Review of Income and Wealth* 29 (1): 1–16.

Szulc, Bohdan J. 1964. "Indices for Multiregional Comparisons", (in Polish), *Przeglad Statystyczny* 3: 239–54.

Szulc, Bohdan J. 1983. "Linking Price Index Numbers." In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette (pp. 537–566). Ottawa: Statistics Canada.

Szulc, Bohdan J. 1987. "Price Indices below the Basic Aggregation Level." *Bulletin of Labour Statistics* 2: 9–16.

Theil, Herni. 1967. *Economics and Information Theory*. Amsterdam: North-Holland Publishing.

Törnqvist, Leo. 1936. "The Bank of Finland's Consumption Price Index." *Bank of Finland Monthly Bulletin* 10: 1–8.

Törnqvist, Leo, and Egil Törnqvist. 1937. "Vilket är förhällandet mellan finska markens och svenska kronans köpkraft?" *Ekonomiska Samfundets Tidskrift* 39: 1–39 reprinted as pp. 121–60 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

Triplett, Jack. 2004. *Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes*. Directorate for Science, Technology and Industry, DSTI/DOC(2004)9, Paris: OECD.

Triplett, Jack E., and Richard J. McDonald. 1977. "Assessing the Quality Error in Output Measures: The Case of Refrigerators." *The Review of Income and Wealth* 23 (2): 137–56.

Turvey, Ralph. 1979. "The Treatment of Seasonal Items in Consumer Price Indices." *Bulletin of Labour Statistics*, Fourth Quarter, International Labour Office, Geneva, 13–33.

Walsh, C. Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

Walsh, C. Moylan. 1921. "Discussion." *Journal of the American Statistical Association* 17: 537–44.

Young, Arthur. 1812. *An Inquiry into the Progressive Value of Money in England as Marked by the Price of Agricultural Products*. London: B. McMillan.

Yule, G. Udny. 1921. "Discussion of Mr. Flux's Paper", *Journal of the Royal Statistical Society* 84: 199–202.

Zarnowitz, Victor. 1961. "Index Numbers and the Seasonality of Quantities and Prices." In *The Price Statistics of the Federal Government*, edited by G.J. Stigler (Chairman) (pp. 233–304). New York: National Bureau of Economic Research.

Zhang, Li-Chun, Ingvild. Johansen, and Ragnhild Nygaard. 2019. "Tests for Price Indices in a Dynamic Item Universe." *Journal of Official Statistics* 35 (3): 683–97.

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THE TREATMENT OF DURABLE GOODS AND HOUSING* 10

1. Introduction

When a durable good (other than housing) is purchased by a consumer, national CPIs typically attribute *all* of that expenditure to the period of purchase even though the use of the good extends beyond the period of purchase.[1] By definition, a durable good delivers services longer than the accounting period under consideration.[2] The *System of National Accounts, 1993* (SNA), defines a *durable good* as follows:

> In the case of goods, the distinction between acquisition and use is analytically important. It underlies the distinction between durable and non-durable goods extensively used in economic analysis. In fact, the distinction between durable and non-durable goods is not based on physical durability as such. Instead, the distinction is based on whether the goods can be used once only for purposes of production or consumption or whether they can be used repeatedly, or continuously. For example, coal is a highly durable good in a physical sense, but it can be burnt only once. A durable good is therefore defined as one which may be used repeatedly or continuously over a period of more than a year, assuming a normal or average rate of physical usage. A consumer durable is a good that may be used for purposes of consumption repeatedly or continuously over a period of a year or more.
>
> System of National Accounts (1993, 208)

This chapter will be concerned with the problems involved in pricing the services provided by durable goods according to the previous definition. Thus, durability is more than the fact that a good can physically persist for more than a year (this is true of most goods): A *durable good* is distinguished from a *nondurable good* due to its property that it can deliver useful services to a consumer through repeated use over an extended period of time. Examples of durable goods are automobiles and washing machines. A *storable good* is a good that can be stored over at least two periods of time but can only be used in a single period—for example, a can of beans. A *perishable good* is a good that can be stored for only a limited period of time—for example, a carton of milk. Thus, perishable goods are like services: depending on the length of the period, they must be consumed in their period of purchase. Most of this chapter will be concerned with the treatment of durable goods, but Section 10 will look at the treatment of storable goods.

Since the benefits of using the consumer durable extend over more than one period, *it is not appropriate to charge the entire purchase cost of the durable to the initial period of purchase*. If this point of view is taken, then the initial purchase cost must be distributed somehow over the useful life of the asset. This is the *fundamental problem of accounting*.[3] Hulten (1990) explained the consequences for accountants of the durability of a purchase as follows:

> Durability means that a capital good is productive for two or more time periods, and this, in turn, implies that a distinction must be made between the value of using or renting capital in any year and the value of owning the capital asset. This distinction would not necessarily lead to a measurement

* This chapter draws on Chapter 23 of the *Consumer Price Index Manual*; see ILO, IMF, OECD, UNECE, Eurostat, World Bank (2004, 419–441) and Chapter 6 of Diewert et al. (2020, 223–298). The authors thank Paul Armknecht, John Astin, Corinne Becker-Vermeulen, David Fenwick, Dennis Fixler, Elspeth Hazell, Michael Henderson, Brian Graf, Ronald Johnson, Shaima Kamleh, Jill Leyland, Jens Mehrhoff, Paul Schreyer, Nigel Stapledon, Valentina Stoevska, Randall Verbrugge, and Alice Xu for their helpful comments on earlier drafts.

[1] This treatment of the purchases of durable goods dates back to Alfred Marshall (1898, 594–595) at least: "We have noticed also that though the benefits which a man derives from living in his own house are commonly reckoned as part of his real income, and estimated at the net rental value of his house; the same plan is not followed with regard to the benefits which he derives from the use of his furniture and clothes. It is best here to follow the common practice, and not count as part of the national income or dividend anything that is not commonly counted as part of the income of the individual."

[2] An alternative definition of a durable good is that the good delivers services to its purchaser for a period exceeding three years: "The Bureau of Economic Analysis defines consumer durables as those durables that have an average life of at least 3 years" (Arnold J. Katz, 1983, 422).

[3] "The third convention is that of the annual accounting period. It is this convention which is responsible for most of the difficult accounting problems. Without this convention, accounting would be a simple matter of recording completed and fully realized transactions: an act of primitive simplicity" (Stephen Gilman, 1939, 26).

"*All* the problems of income measurement are the result of our desire to attribute income to arbitrarily determined short periods of time. Everything comes right in the end; but by then it is too late to matter" (David Solomons, 1961, 378). Note that these authors do not mention the additional complications that are due to the fact that future revenues and costs must be discounted to yield values that are equivalent to present dollars. For more recent papers on the fundamental problem of accounting, see Diewert (2005a, 480), Cairns (2013, 634), and Diewert and Fox (2016).

> problem if the capital services used in any given year were paid for in that year; that is, if all capital were rented. In this case, transactions in the rental market would fix the price and quantity of capital in each time period, much as data on the price and quantity of labor services are derived from labor market transactions. But, unfortunately, much capital is utilized by its owner and the transfer of capital services between owner and user results in an implicit rent typically not observed by the statistician. Market data are thus inadequate for the task of directly estimating the price and quantity of capital services, and this has led to the development of indirect procedures for inferring the quantity of capital, like the perpetual inventory method, or to the acceptance of flawed measures, like book value.
>
> Charles R. Hulten (1990, 120–121)

There are three main methods for dealing with the durability problem:

- Ignore the problem of distributing the initial cost of the durable over the useful life of the good and allocate the entire charge to the period of purchase. This is known as the *acquisitions approach*, and it is the present approach used by CPI statisticians for all durables with the exception of housing.
- The *rental equivalence* or *leasing equivalence approach*: In this approach, a price is imputed for the durable; this price is equal to the rental price or leasing price of an equivalent consumer durable for the same period of time.
- The *user cost approach*: In this approach, the initial purchase cost of the durable is decomposed into two parts: One part that reflects an estimated cost of using the services of the durable for the period and another part that is regarded as an investment that must earn some exogenous rate of return.

These three major approaches will be discussed more fully in Sections 2, 3, 4, and 9.[4] There is a fourth approach that has not been applied but seems conceptually attractive. It will be discussed in Section 5: the *opportunity cost approach*. This approach takes the *maximum* of the rental equivalence and user cost as the price for the use of the services of a consumer durable over a period of time. Finally, there is a fifth approach to the treatment of consumer durables that has only been used in the context of pricing OOH and that is the *payments approach*.[5] This is a kind of cash flow approach, which will be discussed in Section 18 after we have discussed the other approaches in more detail.

The three main approaches to the treatment of durable purchases can be applied to the purchase of any durable commodity. However, historically, it turns out that the rental equivalence and user cost approaches have only been applied to OOH. In other words, the acquisitions approach to the purchase of consumer durables has been universally used by statistical agencies, with the exception of OOH. A possible reason for this is tradition; that is, Marshall (1898) set the standard and statisticians have followed his example for the past century. However, another possible reason is that unless the durable good has a very long useful life, it usually will not make a great deal of difference in the long run whether the acquisitions approach or one of the two alternative approaches is used. This point is discussed in more detail in Section 10.

A major component of the user cost approach to valuing the services of OOH is the depreciation component. In Section 6, a general model of depreciation for a consumer durable is presented, and then it is specialized in Sections 7 and 8 to the three models of depreciation that are widely used.

The general model presented in Section 6 assumes that homogeneous units of the durable good are produced in each period, and it also assumes that used units of the durable trade on secondhand markets so that information on the prices of the various vintages of the durable at any point in time can be used to determine the pattern of depreciation. However, many durables (like housing) are custom produced (that is, they are unique goods), and thus the methods for determining the form of depreciation explained in Section 6 are not immediately applicable. The special problems associated with the measurement of housing services are considered in Sections 11–18.

Sections 11 and 12 show how information on the sales of dwelling units can be used to decompose the sales price into land and structure components. This information is required for the country's national balance sheet accounts. The decomposition into land and structure components is also required for the construction of rental prices and user costs and for measures of multifactor productivity for the rental housing sector of the economy.[6] Section 11 looks at land and structure decompositions for the sale of detached housing units, while Section 12 does the same for the sales of condominium units. The hedonic regression models that are explained in Sections 11 and 12 are basically supply-side models, while Section 13 looks at demand-side hedonic regression models for the sales of detached houses. Sections 14 and 15 look at the problems associated with the construction of rent indices. Section 14 shows how a very simple repeat rents model can be modified in order to deal with depreciation of the structure, which causes the quality of a rental unit to decline over time. However, there are some problems with the modified repeat rents model, so Section 15 considers more general hedonic regression models for rents. Sections 16 and 17 look at the problems associated with valuing the services of OOH in a CPI. Section 16 notes that, in principle, there are separate user costs for the owned structure and for the land that the structure sits on. Section 17 compares the rental equivalence and user cost approaches for the treatment of OOH. This section also explains why the

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[4] It should be noted that, in principle, the user cost and rental equivalence approaches *should* be much the same: The owner of a rental property needs to construct a user cost for the current period (using its opportunity cost of capital as the interest rate that appears in the user cost formula) so that the resulting user cost can be used as the rental price that will just allow the owner to make the target rate of return on the property investment. In practice, the exact equality does not hold due to various market imperfections which will be discussed later.

[5] This is the term used by Goodhart (2001, F350–F351).

[6] Depreciation applies to the structure part of property value but not to the land part.

amount that an owned dwelling unit could rent for is in general different from the user cost that could be used to price the services of the unit to an owner. Section 18 looks at some alternative approaches to measuring housing services in a CPI such as the payments approach and the household costs approach.

Section 19 applies the user cost approach to household holdings of monetary balances. The difficult issues associated with defining real monetary balances are also addressed.

Section 20 concludes.

2. The Acquisitions Approach

The net acquisitions approach to the treatment of OOH is described by Goodhart as follows:

> The first is the net acquisitions approach, which is the change in the price of newly purchased owner occupied dwellings, weighted by the net purchases of the reference population. This is an asset based measure, and therefore comes close to my preferred measure of inflation as a change in the value of money, though the change in the price of the stock of existing houses rather than just of net purchases would in some respects be even better. It is, moreover, consistent with the treatment of other durables. A few countries, e.g., Australia and New Zealand, have used it, and it is, I understand, the main contender for use in the Euro-area Harmonized Index of Consumer Prices (HICP), which currently excludes any measure of the purchase price of (new) housing, though it does include minor repairs and maintenance by home owners, as well as all expenditures by tenants.
>
> Charles Goodhart (2001, F350)

Thus, the weights for the net acquisitions approach are the net purchases of the household sector of houses from other institutional sectors in the base period. Note that, in principle, purchases of secondhand dwellings from other sectors are relevant here; for example, a local government may sell rental dwellings to owner occupiers. However, typically, newly built houses form a major part of these types of transactions. Thus, the long-term price relative to this category of expenditure will be primarily the price of (new) houses (quality adjusted) in the current period relative to the price of new houses in the base period.[7] If this approach is applied to other consumer durables, it is extremely easy to implement: The purchase of a durable is treated in the same way as a nondurable or service purchase.

One additional implication of the net acquisitions approach is that major renovations and additions to owner-occupied dwelling units could also be considered as being in scope for this approach. In practice, major renovations to a house are treated as investment expenditures and not covered as part of a CPI. Normal maintenance expenditures on a dwelling unit are usually treated in a separate category in the CPI.

Traditionally, the net acquisitions approach also includes transfer costs related to the buying and selling of secondhand houses as expenditures that are in scope for an acquisitions-type CPI. These costs are mainly the costs of using a real estate agent's services and asset transfer taxes. These costs can be measured, but the question arises as to what is the appropriate deflator for these costs. An overall property price index is probably a satisfactory deflator.[8]

The major advantage of the acquisitions approach is that it treats durable and nondurable purchases in a completely symmetric manner, and thus no special procedures have to be developed by a statistical agency to deal with durable goods.[9] As will be seen in Section 10, the major disadvantage of this approach is that the expenditures associated with this approach will tend to understate the corresponding expenditures on durables that are implied by the rental equivalence and user cost approaches.

Some differences between the acquisitions approach and the other approaches are as follows:

- If rental or leasing markets for the durable exist and the durable has a long useful life, then, as mentioned earlier, the expenditure weights implied by the rental equivalence or user cost approaches will typically be much larger than the corresponding expenditure weights implied by the acquisitions approach; see Section 17.
- If the base year corresponds to a boom year (or a slump year) for the durable, then the base period expenditure weights may be too large or too small. Put another way, the aggregate expenditures that correspond to the acquisitions approach are likely to be more volatile than the expenditures for the aggregate that are implied by the rental equivalence or user cost approaches.[10]
- In making comparisons of consumption across countries where the proportion of owning versus renting or leasing the durable varies greatly,[11] the use of the acquisitions approach may lead to misleading cross-country comparisons. The reason for this is that opportunity costs of capital are excluded

[7] This price index may or may not include the price of the land that the new dwelling unit sits on; for example, a new house price construction index would typically not include the land cost. The acquisitions approach concentrates on the purchases by households of goods and services that are provided by suppliers from outside the household sector. Thus, if the land on which a new house sits was previously owned by the household sector, then presumably, the cost of this land would be excluded from an acquisitions type new house price index. In this case, the price index that corresponds to the acquisitions approach is basically a new house price index (excluding land) or a modification of a construction cost index where the modification takes into account builder's margins.

[8] See the discussion in Section 17 on transfer costs.

[9] The acquisitions approach is straightforward and simple for most durable goods but not for housing if the land component of property value is regarded as out of scope. Properties are sold with a single price that includes both the land and structure components of housing and so if the land part of property value is regarded as out of scope for the index, then there is a problem in decomposing property value into land and structure components. This decomposition problem can be avoided if information on the construction costs for building a new housing unit is available. In this case, the construction cost index (including builder's markups) can serve as the price index for newly constructed dwelling units.

[10] Hill, Steurer, and Waltl (2020) make this point and list other problems with the acquisitions approach.

[11] From Hoffmann and Kurz (2002, 3–4), about 60 percent of German households lived in rented dwellings, whereas only about 11 percent of Spaniards rented their dwellings in 1999.

in the net acquisitions approach, whereas they are explicitly or implicitly included in the other two approaches.

More fundamentally, whether the acquisitions approach is the right choice or not depends on the overall purpose of the index number. If the purpose is to measure the price of current period consumption *services*, then the acquisitions approach can only be regarded as an approximation to a more appropriate approach (which would be either the rental equivalence or the user cost approach). If the purpose of the index is to measure monetary (or nonimputed) expenditures by the household sector during the period, then the acquisitions approach is preferable (provided the land component of property value is in scope), since the rental equivalence and user cost approaches necessarily involve imputations.[12]

The acquisitions approach (as applied to OOH) is discussed in detail in Eurostat (2017).[13] Eurostat is considering the use of the acquisitions approach for the treatment of OOH in its HICP, but at this date, no decision has been finalized. At present, OOH is simply omitted in the HICP. Eurostat considered the use of the acquisitions approach for OOH because at first sight it seems that no imputations have to be made in order to implement it. The HICP was created as an index of consumer prices that used actual transaction prices without the use of any imputations.[14] As such, it was thought to be particularly useful for monitoring inflation by central banks. However, the sale of a newly constructed dwelling unit typically includes a land component, which the Eurostat methodology excludes, but existing methods for excluding the land component involve imputations.[15]

3. The Rental Equivalence Approach

The rental equivalence approach simply values the services yielded by the use of a consumer durable good for a period by the corresponding market rental value for the same durable for the same period of time (if such a rental value exists).

The most important consumer durable in a CPI is housing that is owned (OOH). The international SNA 1993 recommended the use of the rental equivalence approach to measure the services of OOH:

> As well-organized markets for rented housing exist in most countries, the output of own-account housing services can be valued using the prices of the same kinds of services sold on the market with the general valuation rules adopted for goods and services produced on own account. In other words, the output of housing services produced by owner-occupiers is valued at the estimated rental that a tenant would pay for the same accommodation, taking into account factors such as location, neighbourhood amenities, etc. as well as the size and quality of the dwelling itself.
>
> Eurostat, IMF, OECD, UN and World Bank (1993, 134)

However, the SNA 1993 followed Marshall (1898, 595) and did not extend the rental equivalence approach to consumer durables other than housing. This seemingly inconsistent treatment of durables was explained in the SNA 1993 as follows:

> The production of housing services for their own final consumption by owner-occupiers has always been included within the production boundary in national accounts, although it constitutes an exception to the general exclusion of own-account service production. The ratio of owner-occupied to rented dwellings can vary significantly between countries and even over short periods of time within a single country, so that both international and intertemporal comparisons of the production and consumption of housing services could be distorted if no imputation were made for the value of own-account services.
>
> Eurostat, IMF, OECD, UN and World Bank (1993, 126)

Eurostat's (2001) *Handbook on Price and Volume Measures in National Accounts* also recommended the rental equivalence approach for the treatment of the dwelling services for OOH:

> The output of dwelling services of owner occupiers at current prices is in many countries estimated by linking the actual rents paid by those renting similar properties in the rented sector to those of owner occupiers. This allows the imputation of a notional rent for the service owner occupiers receive from their property.
>
> Eurostat (2001, 99)

To summarize the preceding material, it can be seen that the rental equivalence approach to the treatment of a durable good is conceptually simple: Use the current period rental or leasing price for a comparable unit of the consumer durable to measure its service flow. But where will the statistical agency find the relevant rental data to price the services of OOH? There are at least three possible methods:

- Ask homeowners what they think the market rent for their dwelling unit is.[16]
- Undertake a survey of owners of rental properties or managers of rental properties, and ask what rents they charge for their rental properties by type of property.

[12] Fenwick (2009, 2012) laid out the case for the use of the acquisitions approach as a useful measure of general inflation. He also argued for the construction of multiple CPIs to suit different purposes.

[13] This very useful publication also discusses the main methods for the treatment of OOH, and it also covers the methods used to construct residential property price indices. The latter topic is also covered in Eurostat (2013).

[14] However, with the passage of time, it became apparent that some imputations for changes in the quality of consumer goods and services had to be made. Thus the current HICP is not completely free from imputations. See Astin (1999) for the methodological foundations of the HICP.

[15] The use of a construction cost index to value the structure component of property value also involves an imputation, but it is a reasonably straightforward one.

[16] This approach is used by the Bureau of Labor Statistics (1983) in order to determine *expenditure weights* for OOH; that is, homeowners are asked to estimate what their house would rent for if it were rented to a third party.

- Use one of the preceding two methods to get a rent to value ratio for various types of property for a benchmark period, and then link these ratios to indices of purchase prices for the various types of property.[17]

There are some disadvantages associated with the use of the rental equivalence approach to the valuation of OOH services:

- Homeowners may not be able to provide very accurate estimates for the rental value of their dwelling unit.
- On the other hand, if the statistical agency tries to match the characteristics of an owned dwelling unit with a comparable unit that is rented in order to obtain the imputed rent for the owned unit, there may be difficulties in finding such comparable units. Furthermore, even if a comparable unit is found, the rent for the comparable unit may not be an appropriate opportunity cost for valuing the services of the owned unit.[18]
- The statistical agency should make an adjustment to these estimated rents over time in order to take into account the effects of depreciation, which causes the quality of the unit to slowly decline over time (unless this effect is completely offset by renovation and repair expenditures).[19]
- Care must be taken to determine exactly what extra services are included in the homeowner's estimated rent; that is, does the rent include insurance, electricity, and fuel or the use of various consumer durables in addition to the structure? If so, these extra services should be stripped out of the rent if they are covered elsewhere in the CPI.[20]

In order to overcome the first difficulty listed earlier, the Japanese government collects housing rent data from *property management companies* or owners who rent out their dwelling units; that is, Japan uses the second method to value the services of OOH. However, the characteristics of the owner-occupied population of dwelling units are generally quite different from the characteristics of the rental population.[21] Thus, typically, it is difficult to find rental units that are comparable to owned dwelling units. The use of hedonic regression techniques can mitigate this lack of matching problem. Moreover, the use of hedonic regressions can deal with the depreciation or quality decline problem mentioned earlier. We will discuss hedonic regression techniques later in this chapter in Sections 11–15.

In addition to these possible biases in using the rental equivalence approach to the valuation of the services of OOH, there are differences between *contract rent* and *market rent*. *Contract rent* or *roll-over rent* refers to the rent paid by a renter who has a long-term rental contract with the owner of the dwelling unit and (current) *market rent* is the rent paid by the renter in the first period after a rental contract has been negotiated. In a normal economy that is experiencing moderate or low general inflation, typically market rent will be higher than contract rent. However, if there are rent controls or a temporary glut of rental units, then market rent could be lower than contract rent. In any case, it can be seen that if we value the services of an owner-occupied dwelling at its current opportunity cost on the rental market, market rent should be used in the CPI to value the services of OOH rather than contract rent. However, contract rent or rollover rent (adjusted for depreciation and improvements) should be used to estimate the cost of rented dwellings in a CPI.

Finally, it is known that price adjustments are often not made for rollover contracts (that is, renewed leases). As a result, it is likely that new contract rents determined freely by the market will diverge considerably from rollover contract rents.[22] This is the *stickiness of rents problem*.

In the following section, we provide an introduction to user cost theory for a non-housing durable good. In subsequent sections, we will deal with the problems associated with measuring depreciation and the aggregation of user costs over different ages of the same good. In Sections 11–17, we will look at the additional difficulties that are associated with the formation of user costs for housing and the relationship between user costs and rental prices for housing services.

4. The User Cost Approach for Pricing the Services of a Durable Good

The user cost approach to the treatment of durable goods is in some ways very simple: It calculates the cost of purchasing the durable at the beginning of the period, using the services of the durable during the period, and then netting off from these costs the benefit that could be obtained by selling the durable good at the end of the period. However, there are several details of this procedure that are somewhat controversial. These details involve the use of opportunity costs (which are usually imputed costs), the treatment of interest, and the treatment of capital gains or holding gains.

Another complication with the user cost approach is that it involves making distinctions between current period

[17] Lebow and Rudd (2003, 169) note that the US Bureau of Economic Analysis applies a benchmark rent to value ratio for rented properties to the value of the owner-occupied stock of housing. It can be seen that this approach is essentially a simplified user cost method where all of the key variables in the user cost formula (to be discussed later) are held constant except the asset price of the property.

[18] We will return to this point after we have discussed the opportunity cost approach to the valuation of OOH services.

[19] This issue will be discussed in more detail in Section 17. Papers which discuss how to strip out utility and insurance costs out of rents include Verbrugge (2012), Coffey, McQuinn, and O'Toole (2020), and Adams and Verbrugge (2021). Also, in many countries, there are rent controls. A rent-controlled comparable property is not a correct opportunity cost to use to value the services of an owned dwelling unit.

[20] However, it could be argued that these extra services that might be included in the rent are mainly a weighting issue; that is, it could be argued that the trend in the homeowner's estimated rent would be a reasonably accurate estimate of the trend in the rents after adjusting for the extra services included in the rent.

[21] For example, according to the 2013 Japanese Housing and Land Survey, the average floor space (size) of OOH in Tokyo was 110.64 square meters for single family houses and 82.71 square meters for rental housing, a difference of over 30 square meters. For condominiums, an even greater discrepancy exists: the average floor space is 65.73 square meters for OOH and 37.64 square meters for rental housing. Moreover, in addition to the difference in floor space between rented and owned units, the quality of the owned units tends to be higher than the rented units and these quality differences need to be taken into account.

[22] On this point, see Genesove (2003), Shimizu, Nishimura, and Watanabe (2010b), Shimizu and Watanabe (2011), Lewis and Restieaux (2015), Gallin and Verbrugge (2019), and Suzuki, Asami, and Shimizu (2021).

(flow) purchases within the period under consideration and the holdings of physical stocks of the durable at the beginning and end of the accounting period. Typically, when constructing a CPI, we think of all quantity purchases as taking place at a single point in time, say the middle of the period under consideration, at the (unit value) average prices for the period. In constructing user costs, prices at the beginning and end of an accounting period play an important role.

To determine the net cost of using a durable good during period 0, it is assumed that one unit of the durable good is purchased at the beginning of period 0 at price P^0. The "used" or "second-hand" durable good can be sold at the end of period 0 at price $P_S^{\,1}$.[23] It might seem that a reasonable *net cost* for the use of one unit of the consumer durable during period 0 is its initial purchase price P^0 less its end of period 0 "scrap value," $P_S^{\,1}$. However, *money received at the end of the period is not as valuable as money that is received at the beginning of the period*. Thus, in order to convert the end-of-period value into its beginning of the period equivalent value, it is necessary to *discount* the term $P_S^{\,1}$ by the term $1 + r^0$, where r^0 is the beginning of period 0 nominal interest rate that the consumer faces. Hence, the *period 0 user cost* u^0 for the consumer durable[24] is defined as

$$u^0 \equiv P^0 - P_S^{\,1}/(1 + r^0). \tag{1}$$

There is another way to view the user cost formula (1): the consumer purchases the durable at the beginning of period 0 at price P^0 and charges himself or herself the rental price u^0. The remainder of the purchase price, I^0, defined as

$$I^0 \equiv P^0 - u^0 \tag{2}$$

can be regarded as an *investment*, which is to yield the appropriate opportunity cost of capital r^0 that the consumer faces. At the end of period 0, this rate of return could be realized provided that I^0, r^0, and the selling price of the durable at the end of the period $P_S^{\,1}$ satisfy the following equation:

$$I^0(1 + r^0) = P_S^{\,1}. \tag{3}$$

Given $P_S^{\,1}$ and r^0, (3) determines I^0, which in turn, given P^0, determines the user cost u^0 using (2).[25]

Thus, user costs are not like the prices of nondurables or services because the user cost concept involves pricing the durable at *two* points in time rather than at a single point in time. Because the user cost concept involves prices at two points in time, money received or paid out at the first point in time is more valuable than money paid out or received at the second point in time, and so *interest rates* creep into the user cost formula. Furthermore, because the user cost concept involves prices at two points in time, *expected prices* can be involved if the user cost is calculated at the beginning of the period under consideration instead of at the end. With all of these complications, it is no wonder that many price statisticians would like to avoid using user costs as a pricing concept. However, even for price statisticians who would prefer to use the rental equivalence approach to the treatment of durables over the user cost approach, there is some justification for considering the user cost approach in some detail, since this approach gives insights into the economic determinants of the *rental* or *leasing price* of a durable.

The user cost formula (1) can be put into a more familiar form if the period 0 *economic depreciation rate* 0 and the period 0 *ex-post asset inflation rate* i^0 are defined. Define δ by

$$(1 - \delta) \equiv P_S^{\,1}/P^1, \tag{4}$$

where $P_S^{\,1}$ is the price of a one-period old used asset at the end of period 0 and P^1 is the price of a new asset at the end of period 0. Typically, if a new asset and a one-period older asset are sold at the same time, then the new asset will be worth more than the used asset, and hence δ will be a positive number between 0 and 1. The *period 0 inflation rate* for the new asset, i^0, is defined by

$$1 + i^0 \equiv P^1/P^0. \tag{5}$$

Eliminating P^1 from equations (4) and (5) leads to the following formula for the *end of period 0 used asset price*:

$$P_S^{\,1} = (1 - \delta)(1 + i^0)P^0. \tag{6}$$

Substitution of (6) into (1) yields the following expression for the *period 0 user cost* u^0:

$$u^0 = [(1 + r^0) - (1 - \delta)(1 + i^0)]P^0/(1 + r^0). \tag{7}$$

Note that $r^0 - i^0$ can be interpreted as a period 0 *real interest rate*, and $\delta(1 + i^0)$ can be interpreted as an *inflation-adjusted depreciation rate*.

The user cost u^0 is expressed in terms of prices that are discounted to the *beginning* of period 0. However, it is also possible to express the user cost in terms of prices that are "anti-discounted" or *appreciated* to the *end* of period 0.[26] Thus, define the *end of period 0 user cost* p^0 as[27]

[23] Note that this approach to pricing the services of a durable good assumes the existence of secondhand markets for units of the durables that have aged. This assumption may not be satisfied for many consumer durables including unique assets such as dwelling units and works of art, which are not bought and sold every period.

[24] This approach to the derivation of a user cost formula was suggested by Diewert (1974), who in turn based it on an approach adopted by Hicks (1946, 326).

[25] This derivation for the user cost of a consumer durable was also made by Diewert (1974, 504).

[26] Thus, the beginning of the period user cost u^0 discounts all monetary costs and benefits into their dollar equivalent at the beginning of period 0, whereas p^0 discounts (or appreciates) all monetary costs and benefits into their dollar equivalent at the end of period 0. This leaves open how flow transactions that take place within the period should be treated. Following the conventions used in financial accounting suggests that *flow transactions* taking place within the accounting period be regarded as taking place at the end of the accounting period and hence, following this convention, end-of-period user costs should be used by the price statistician; see Peasnell (1981).

[27] Christensen and Jorgenson (1969) derived a user cost formula similar to (7) in a different way using a continuous time optimization model. If the inflation rate i equals 0, then the user cost formula (7) reduces to that derived by Walras (1954, 269, first edition 1874). This zero inflation rate user cost formula was also derived by the industrial engineer A. Hamilton Church (1901, 907–908), who perhaps drew on the work of Matheson: "In the case of a factory where the occupancy is assured for a term of years, and the rent is a first charge on profits, the rate of interest, to be an appropriate rate, should, so far as it applies to the buildings, be

$$p^0 \equiv (1 + r^0)u^0 = [(1 + r^0) - (1 - \delta)(1 + i^0)]P^0, \qquad (8)$$

where the last equation follows from (7). If the real interest rate r^{0*} is defined as the nominal interest rate r^0 less the asset inflation rate i^0 and the small term δi^0 is neglected, then the end-of-period user cost defined by (8) reduces to

$$p^0 = (r^{0*} + \delta)P^0. \qquad (9)$$

Abstracting from transaction costs and inflation, it can be seen that the end-of-period user cost defined by (9) is an *approximate rental cost*; that is, the rental cost for the use of a consumer (or producer) durable good should equal the (real) opportunity cost of the capital tied up, $r^{0*}P^0$, plus the decline in value of the asset over the period, δP^0. Formulae (8) and (9) thus cast some light on the economic determinants of rental or leasing prices for consumer durables.

If the simplified user cost formula defined by (9) is used, then, at first glance, forming a price index for the user cost of a durable good is not very much more difficult than forming a price index for the purchase price of the durable good, P^0. The price statistician needs only to

- make a reasonable assumption as to what an appropriate monthly or quarterly real interest rate r^{0*} should be;
- make an assumption as to what a reasonable monthly or quarterly depreciation rate δ should be;[28]
- collect purchase prices P^0 for the durable and use formula (9) to calculate the simplified user cost.[29]

If it is thought necessary to implement the more complicated user cost formula (8) in place of the simpler formula (9), then the situation is more complicated. As it stands, the end-of-period user cost formula (8) is an *ex-post* (or after the fact) *user cost*: The asset inflation rate i^0 cannot be calculated until the end of period 0 has been reached. Formula (8) can be converted into an *ex ante* (or before the fact) *user cost* formula if i^0 is interpreted as an *anticipated asset inflation rate*. The resulting formula should approximate a market rental rate for the durable good.[30]

Note that in the user cost approach to the treatment of consumer durables, the *entire* user cost formula (8) or (9) is the period 0 price. Thus, in the time series context, it is *not* necessary to deflate each component of the formula *separately*; the period 0 price $p^0 \equiv [r^0 - i^0 + \delta(1 + i^0)]P^0$ is compared to the corresponding period 1 price, $p^1 \equiv [r^1 - i^1 + \delta(1 + i^1)]P^1$, and so on.

In principle, depreciation rates can be estimated using information on the selling prices of used units of the durable good.[31] However, for housing, the situation is more complex, as will be explained later.

We conclude this introductory section by noting some practical problems that statistical agencies will face when calculating user costs for durable goods:[32]

- It is difficult to determine what the relevant nominal interest rate r^0 is for each household. If a consumer has to borrow to finance the cost of a durable good purchase, then this interest rate will typically be much higher than the safe rate of return that would be the appropriate opportunity cost rate of return for a consumer who had no need to borrow funds to finance the purchase.[33] It may be necessary to simply use a benchmark interest rate that would be determined by the government, a national statistical agency, or an accounting standards board.[34]
- It will generally be difficult to determine what the relevant depreciation rate is for the consumer durable.[35]

equal (including the depreciation rate) to the rental which a landlord who owned but did not occupy a factory would let it for." Ewing Matheson (1910, 169), first published in 1884. Additional derivations of user cost formulae in discrete time have been made by Katz (1983, 408–409) and Diewert (2005a). Hall and Jorgenson (1967) introduced tax considerations into user cost formulae.

28 The geometric model for depreciation to be explained in more detail in Section 6 requires only a single monthly or quarterly depreciation rate. Other models of depreciation may require the estimation of a sequence of vintage depreciation rates. If the estimated annual geometric depreciation rate is δ_a, then the corresponding monthly geometric depreciation rate δ can be obtained by solving the equation $(1 - \delta)^{12} = 1 - \delta_a$. Similarly, if the estimated annual real interest rate is r_a^*, then the corresponding monthly real interest rate r^* can be obtained by solving the equation $(1 + r^*)^{12} = 1 + r_a^*$.

29 Iceland uses a variant of the simplified user cost formula (9) to estimate the services of OOH with a real interest rate approximately equal to 4 percent and a depreciation rate of 1.25 percent. The depreciation rate is relatively low because it is applied to the entire property value and not to just the structure portion of property value; see Guðnason and Jónsdóttir (2011). Eurostat (2005) also uses a simplified user cost formula. Additional simplified user cost formulae have been developed by Verbrugge (2008), Hill, Steurer, and Waltl (2020) and many others; see Section 17.

30 Since landlords must set their rent at the beginning of the period (in actual practice, they usually set their rent for an extended period of time) and if the user cost approach is used to model the economic determinants of market rental rates, then the asset inflation rate i^0 should be interpreted as *an expected inflation rate* rather than an after the fact actual inflation rate. This use of ex ante prices in this price measurement context should be contrasted with the preference of national accountants to use actual or ex-post prices in the system of national accounts.

31 For housing, the situation is more complex because typically a dwelling unit is a *unique good*; its location is a price-determining characteristic and each housing unit has a unique location and thus is a unique good. It also changes its characteristic over time due to renovations and depreciation of the structure. Thus the treatment of housing is much more difficult than the treatment of other durable goods. Note that the definitions (4) and (5) of the depreciation rate δ and the asset inflation rate i^0 implicitly assumed that prices for a new asset and a one period old asset were available in both periods 0 and 1. This assumption is not satisfied for a unique asset.

32 For additional material on difficulties with the user cost approach, see Diewert (1980, 475–479) and Katz (1983, 415–422).

33 Katz (1983, 415–416) comments on the difficulties involved in determining the appropriate rate of interest to use: "There are numerous alternatives: a rate on financial borrowings, on savings, and a weighted average of the two; a rate on nonfinancial investments. e.g., residential housing, perhaps adjusted for capital gains; and the consumer's subjective rate of time preference. Furthermore, there is some controversy about whether it should be the maximum observed rate, the average observed rate, or the rate of return earned on investments that have the same degree of risk and liquidity as the durables whose services are being valued."

34 One way for choosing the nominal interest rate for period t, r^t, is to set it equal to $(1 + r^*)(1 + \rho^t) - 1$, where ρ^t is a consumer price inflation rate for period t and r^* is a reference real interest rate. The Australian Bureau of Statistics has used this method for determining r^t with $r^* \equiv 0.04$; that is, a 4 percent real interest rate was chosen. Other methods for determining the appropriate interest rate that should be inserted into user cost formula are discussed by Harper, Berndt, and Wood (1989), Schreyer (2001), and Hill, Steurer, and Waltl (2020).

35 We will discuss geometric or declining balance depreciation and one-hoss-shay depreciation below. For references to the depreciation literature and for empirical methods for estimating depreciation rates, see Hulten and Wykoff (1981a, 1981b, 1996), Beidelman (1973, 1976), Jorgenson (1996), and Diewert and Lawrence (2000).

- *Ex-post user costs* based on formula (8) may be too volatile to be acceptable to users[36] (due to the volatility of the ex-post asset inflation rate i^0), and hence an *ex ante user cost* concept may have to be used. For most durable goods, the asset inflation rates are smaller than the reference nominal interest rate so that subtracting an ex-post asset inflation rate from the sum of the nominal interest rate plus the asset depreciation rate will usually lead to reasonably stable positive user costs. However, for durable goods with very low depreciation rates, like a housing structure or like land (which has a zero depreciation rate), the resulting ex-post user costs may turn out to be negative for some periods. This means that the resulting negative user costs are not useful approximations to rental prices for these long-lived durable goods. This creates difficulties in that different national statistical agencies will generally make different assumptions and use different methods in order to construct anticipated inflation rates for structures and land, and hence the resulting ex ante user costs of the durable may not be comparable across countries.[37]
- The user cost formula (8) should be generalized to accommodate various taxes that may be associated with the purchase of a durable or with the continuing use of the durable.[38]

Some of the problems associated with estimating depreciation rates will be discussed in Section 6.

[36] Goodhart (2001, F351) commented on the practical difficulties of using ex-post user costs for housing as follows: "An even more theoretical user cost approach is to measure the cost foregone by living in an owner-occupied property as compared with selling it at the beginning of the period and repurchasing it at the end. . . . But this gives the absurd result that as house prices rise, so the opportunity cost falls; indeed the more virulent the inflation of housing asset prices, the more negative would this measure become. Although it has some academic aficionados, this flies in the face of common sense; I am glad to say that no country has adopted this method." As noted above, Iceland and Eurostat have in fact adopted a simplified user cost framework which seems to work well enough. Moreover, the user cost concept is used widely in production function and productivity studies and by national statisticians who construct multifactor productivity accounts for their countries.

[37] For additional material on the difficulties involved in constructing ex ante user costs, see Diewert (1980, 475–486) and Katz (1983, 419–420). For empirical comparisons of different user cost formulae, see Harper, Berndt and Wood (1989), Diewert and Lawrence (2000) and Hill, Steurer, and Waltl (2020). In Diewert and Fox (2018), the authors calculated Jorgensonian (ex-post) user costs for US land used in residential housing for the years 1960–2014 and found that negative user costs occurred. Diewert and Fox then replaced the ex-post capital gains term in the user cost for land with the long-term inflation rate for land over the previous rolling window of 25 years (as an approximation to the ex ante or expected asset inflation rate) and this substitution led to positive user costs for land that were relatively smooth. Hill, Steurer, and Waltl (2020) also recommend the use of long-run asset inflation rates to avoid chain drift in housing indices based on user costs.

[38] For example, property taxes are associated with the use of housing services and hence should be included in the user cost formula; see Section 16. As Katz (1983, 418) noted, taxation issues also impact the choice of the interest rate: "Should the rate of return be a before or after tax rate?" From the viewpoint of a household that is not borrowing to finance the purchase of the durable, an after tax rate of return seems appropriate but from the point of a leasing firm, a before tax rate of return seems appropriate. This difference helps to explain why rental equivalence prices for the durable might be higher than user cost prices; see also Section 16.

5. The Opportunity Cost Approach

The opportunity cost approach to the valuation of the services of a consumer durable during a time period is quite easy to describe: The opportunity cost valuation is simply the *maximum* of the foregone rental or leasing price for the services of the durable during a period of time and the corresponding user cost for the durable.

It is easy to see that when a household has a consumer durable in its possession, the household forgoes the money that one could earn by renting out the services of the durable good for the period of time under consideration. (Such rental markets may not exist, in which case, this opportunity cost is 0.) Thus, the rental equivalent (at current market rates) is one opportunity cost that the household incurs by continuing to own and use the services of the durable during the period.

However, there is another opportunity cost that is applicable to using the services of the durable good during the period under consideration. By using the services of the durable good, the household also forgoes a *financial opportunity cost*. Thus, the durable good could be sold on the secondhand market at the beginning of the period at the price P^0. This amount of money could be invested in some financial instrument that earns the one period rate of return of r^0. Thus, at the end of the period, the household would have accumulated $P^0(1 + r^0)$ dollars as a result of selling the consumer durable at the beginning of the period. Now suppose at the end of the period, the household buys back the consumer durable that it sold at the beginning of the period. The value of the durable good at the end of the period will be $(1 + i^0)(1 - \delta^0)P^0$, where i^0 is the asset appreciation rate over period 0 and d^0 is the depreciation rate for the durable good. Thus, the net opportunity cost of using the services of the durable for period 0 from the financial perspective is $P^0(1 + r^0) - (1 + i^0)(1 - \delta^0)P^0$, which is exactly the *end-of-period user cost* for the durable good that was derived earlier; see equation (8).

A true opportunity cost for using the services of a durable good should equal the *maximum* of the benefits that are foregone by not using these services. Thus the opportunity cost approach to pricing the services of a consumer durable is equivalent to taking the maximum of the rent and user cost that the durable could generate over the period under consideration.[39]

6. A General Model of Depreciation for Consumer Durables

In this section, a "general" model of depreciation for durable goods that appear on the market each period without undergoing quality change will be presented. In the following two sections, this general model will be specialized to the three most common models of depreciation that appear in the literature.

The main tool that can be used to identify depreciation rates for a durable good is the *cross-sectional sequence of*

[39] The opportunity cost approach to pricing the services of OOH was first proposed by Diewert (2008). It was further developed by Diewert and Nakamura (2011) and Diewert, Nakamura, and Nakamura (2011). There have been at least two studies that implemented the opportunity cost approach to the valuation of the services of OOH; see Shimizu et al. (2012) and Aten (2018).

asset prices classified by their age that units of the good sell for on the secondhand market at any point of time.[40] Thus, in order to apply this method for the measurement of depreciation, it is necessary that such secondhand markets exist.

Some notation is required. Let $P_0^{\,t}$ be the price of a newly produced unit of the durable good at the *beginning* of period t. Let $P_v^{\,t}$ be the secondhand market price at the beginning of period t of a unit of the durable good that is v periods old.[41] The *beginning of period t cross-sectional depreciation rate* for a brand new unit of the durable good, $\delta_0^{\,t}$, is defined as follows:

$$1 - \delta_0^{\,t} \equiv P_1^{\,t}/P_0^{\,t}. \tag{10}$$

Once $\delta_0^{\,t}$ has been defined by (10), the *period t cross-sectional depreciation rate* for a unit of the durable good that is one-period old at the beginning of period t, $\delta_1^{\,t}$, can be defined using the following equation:

$$(1 - \delta_1^{\,t})(1 - \delta_0^{\,t}) \equiv P_2^{\,t}/P_0^{\,t}. \tag{11}$$

Note that $P_2^{\,t}$ is the beginning of period t asset price of a unit of the durable good that is two periods old, and it is compared to the price of a brand new unit of the durable $P_0^{\,t}$.

Given that the period t cross-sectional depreciation rates for units of the durable that are 0, 1, 2, . . ., v - 1 periods old at the beginning of period 0 are defined (these are the depreciation rates $\delta_0^{\,t}, \delta_1^{\,t}, \delta_2^{\,t}, \ldots, \delta_{v-1}^{\,t}$), the *period t cross-sectional depreciation rate for units of the durable that are v periods old* at the beginning of period t, $\delta_v^{\,t}$, can be defined using the following equation:

$$(1 - \delta_v^{\,t})(1 - \delta_{v-1}^{\,t}) \ldots (1 - \delta_1^{\,t})(1 - \delta_0^{\,t}) \equiv P_{v+1}^{\,t}/P_0^{\,t}. \tag{12}$$

Thus, it is clear how the sequence of *period 0 vintage asset prices* $P_v^{\,t}$ can be converted into a sequence of *period t vintage depreciation rates*, $\delta_v^{\,t}$. In the depreciation literature, it is usually assumed that the sequence of vintage depreciation rates, $\delta_v^{\,t}$, is independent of the period t so that

$$\delta_v^{\,t} = \delta_v \text{ for all periods } t \text{ and all ages } v. \tag{13}$$

This material shows how the sequence of vintage or used durable goods prices at a point in time can be used in order to estimate depreciation rates. This method for estimating depreciation rates using data on secondhand assets, with a few extra modifications to account for differing ages of retirement, was pioneered by Beidelman (1973, 1976) and Hulten and Wykoff (1981a, 1981b, 1996).[42]

Recall the user cost formula for a new unit of the durable good under consideration, which was defined by (1). The same approach can be used in order to define a sequence of period 0 user costs for all vintages v of the durable. Thus, suppose that $P_{v+1}^{\,1a}$ is the *anticipated end of period 0 price* of a unit of the durable good that is v periods old at the beginning of period 0, and let r^0 be the consumer's opportunity cost of capital for period 0. Then the discounted to the beginning of period 0 *user cost* of a unit of the durable good that is v periods old at the beginning of period 0, $u_v^{\,0}$, is defined as follows:

$$u_v^{\,0} \equiv P_v^{\,0} - P_{v+1}^{\,1a}/(1 + r^0); \quad v = 0,1,2, \ldots \tag{14}$$

It is now necessary to specify how the *end* of period 0 anticipated vintage asset prices $P_v^{\,1a}$ are related to their counterpart *beginning* of period 0 vintage asset prices $P_v^{\,0}$. The assumption that is made now is that the entire sequence of vintage asset prices at the end of period 0 is equal to the corresponding sequence of asset prices at the beginning of period 0 times a general anticipated period 0 inflation rate factor, $(1 + i^0)$, where i^0 is the *anticipated period 0 (general) asset inflation rate*. Thus, it is assumed that[43]

$$P_v^{\,1a} = (1 + i^0)P_v^{\,0}; \quad v = 0,1,2, \ldots. \tag{15}$$

Substituting (15) and (10)–(13) into (14) leads to the following beginning of *period 0 sequence of vintage user costs*:[44]

$$\begin{aligned} u_v^{\,0} &= (1 - \delta_{v-1})(1 - \delta_{v-2}) \ldots (1 - d_0)[(1 + r^0) \\ &\quad - (1 - \delta_v)(1 + i^0)]P_0^{\,0}/(1 + r^0) \\ &= (1 - \delta_{v-1})(1 - \delta_{v-2}) \ldots (1 - \delta_0)[\,r^0 - i^0 + \delta_v(1 + i^0)]P_0^{\,0}/ \\ &\quad (1 + r^0); \qquad v = 1,2, \ldots. \end{aligned} \tag{16}$$

If $v = 0$, then $u_0^{\,0} \equiv [r^0 - i^0 + \delta_0(1 + i^0)]P_0^{\,0}/(1 + r^0)$, and this agrees with the user cost formula for a new purchase of the durable u^0 that was derived earlier in (7) (with our changes in notation; that is, P^0 is now called $P_0^{\,0}$).

The sequence of vintage user costs $u_v^{\,0}$ defined by (16) is expressed in terms of prices that are discounted to the *beginning* of period 0. However, as was done in Section 4, it is also possible to express the user costs in terms of prices that are "anti-discounted" to the *end* of period 0. Thus, define the sequence of vintage *end of period 0 user cost* $p_v^{\,0}$ as follows:

$$\begin{aligned} p_v^{\,0} &\equiv (1 + r^0)u_v^{\,0} = (1 - \delta_{v-1})(1 - \delta_{v-2}) \ldots (1 - \delta_0)[r^0 - i^0 \\ &\quad + \delta_v(1 + i^0)]P_0^{\,0}; \qquad v = 1,2, \ldots, \end{aligned} \tag{17}$$

[40] Another information source that could be used to identify depreciation rates for the durable good is the sequence of vintage rental or leasing prices that might exist for some consumer durables. In the closely related capital measurement literature, the general framework for an internally consistent treatment of capital services and capital stocks in a set of vintage accounts was set out by Jorgenson (1989) and Hulten (1990, 127–129; 1996, 152–160).

[41] If these secondhand vintage prices depend on how intensively the durable good has been used in previous periods, then it will be necessary to further classify the durable good not only by its vintage v but also according to the intensity of its use. In this case, think of the sequence of vintage asset prices $P_v^{\,t}$ as corresponding to the prevailing market prices of the various vintages of the good at the beginning of period t for assets that have been used at "average" intensities.

[42] See also Jorgenson (1996) for a review of the empirical literature on the estimation of depreciation rates.

[43] More generally, we assume that assumptions (15) hold for subsequent periods t as well; that is, it is assumed that $P_v^{\,t+1a} = (1 + i^t)P_v^{\,t}$ for $v = 0,1,2, \ldots$ and $t = 0,1,2, \ldots$, where $P_v^{\,t+1a}$ is the anticipated price of a unit of the durable good that is v periods old at the end of period t, i^t is a period t expected asset inflation rate for all ages of the durable and $P_v^{\,t}$ is the secondhand market price for a unit of the durable good that is v periods old at the beginning of period t.

[44] When $v = 0$, define $\delta_{-1} \equiv 1$; that is, the terms in front of the square brackets on the right-hand side of (16) are set equal to 1.

with $p_0^{\,0}$ defined as follows:

$$p_0^{\,0} \equiv (1 + r^0)u_0^{\,0} = [r^0 - i^0 + \delta_v(1 + i^0)]P_0^{\,0}. \qquad (18)$$

Thus, if the price statistician has estimates for the vintage depreciation rates δ_v, the nominal interest rate r^0, the expected asset inflation rate, and is also able to collect a sample of prices for new units of the durable good $P_0^{\,0}$, then the sequence of vintage user costs defined by (17) can be calculated. To complete the model, the price statistician should gather information on the stocks held by the household sector of each vintage of the durable good, and then normal index number theory can be applied to these p's and q's, with the p's being vintage user costs and the q's being the vintage stocks pertaining to each period. For some worked examples of this methodology under various assumptions about depreciation rates and the calculation of expected asset inflation rates, see Diewert and Lawrence (2000) and Diewert (2005a).[45]

In the following two sections, the general methodology described earlier is specialized by making additional assumptions about the form of the vintage depreciation rates δ_v.[46]

7. Geometric or Declining Balance Depreciation

The *declining balance method of depreciation* dates back to Matheson (1910, 55) at least.[47] In terms of the algebra presented in the previous section, the method is very simple: all of the cross-sectional vintage depreciation rates $\delta_v^{\,t}$ defined by (10)–(12) are assumed to be equal to the same rate d, where δ is a positive number less than one; that is, for all time periods t and all vintages v, it is assumed that

$$\delta_v^{\,t} = \delta; \quad v = 0,1,2,\ldots. \qquad (19)$$

Substitution of (19) into (17) leads to the following formula for the sequence of *end of period 0 vintage user costs*:

$$\begin{aligned} p_v^{\,0} &= (1-\delta)^v[\,r^0 - i^0 + \delta(1 + i^0)]P^0; \quad v = 0,1,2,\ldots \qquad (20)\\ &= (1-\delta)^v p_0^{\,0}, \end{aligned}$$

where the second equation follows from definition (18). The second set of equations in (20) says *that all of the vintage user costs are proportional to the user cost for a new asset*. This proportionality means that it is not necessary to use an index number formula to aggregate over vintages to form a durable services aggregate. To see this, it is useful to calculate the aggregate value of services yielded by all vintages of the consumer durable at the beginning of period 0. Let q^{-v} be the quantity of the new durable purchased by the household sector v periods ago for $v = 1,2,\ldots.$, and let q^0 be the new purchases of the durable during period 0. The beginning of period 0 user cost for the holdings of the durable of age v will be $p_v^{\,0}$ defined by (20). Thus, the aggregate value of services over all vintages of the good, including those purchased in period 0, will have the value v^0 defined as follows:

$$\begin{aligned} v^0 &= p_0^{\,0}q^0 + p_1^{\,0}q^{-1} + p_2^{\,0}q^{-2} + \ldots \qquad (21)\\ &= p_0^{\,0}q^0 + (1-\delta)\,p_0^{\,0}q^{-1} + (1-\delta)^2\,p_0^{\,0}q^{-2} + \ldots \quad \text{using (20)}\\ &= p_0^{\,0}\,[q^0 + (1 - d)q^{-1} + (1-\delta)^2q^{-2} + \ldots]\\ &= p_0^{\,0}Q^0, \end{aligned}$$

where the period 0 aggregate (quality-adjusted) quantity of durable services consumed in period 0, Q^0, is defined as

$$Q^0 \equiv q^0 + (1-\delta)q^{-1} + (1-\delta)^2q^{-2} + \ldots. \qquad (22)$$

Thus, the *period 0 services quantity aggregate* Q^0 is equal to new purchases of the durable in period 0, q^0, plus one minus the depreciation rate δ times the purchases of the durable in the previous period, q^{-1}, plus the square of one minus the depreciation rate times the purchases of the durable two periods ago, q^{-2}, and so on. The service price that can be applied to this quantity aggregate is $p_0^{\,0}$, the imputed rental price or user cost for a new unit of the durable purchased in period 0.

This result greatly simplifies the valuation of consumer durables. Normally, the price statistician would have to keep track of all new purchases of the durable good by the reference population by period, calculate the relevant user costs $p_v^{\,0}$ and $p_v^{\,t}$ for periods 0 and t, and apply the relevant index number formula (Laspeyres, Paasche, Fisher, or whatever formula is used in the CPI) to these age-specific prices and quantities for periods 0 and t. But because under assumptions (13), (15), and (19), *all vintage user costs vary in a proportional manner over time*,[48] and thus any reasonable index number formula will find that the price index going from period 0 to t is equal to $p_0^{\,t}/p_0^{\,0}$, the ratio of user costs for a new unit of the durable good. Moreover, the corresponding *aggregate quantity index* will be equal to Q^t/Q^0, where Q^0 is defined by (22) and Q^t is defined by

$$\begin{aligned} Q^t &\equiv q^t + (1-\delta)q^{t-1} + (1-\delta)^2q^{t-2} + \ldots. \qquad (23)\\ &= q^t + (1-\delta)Q^{t-1}. \end{aligned}$$

Note that the second equation simplifies the calculation of the period t aggregate service flow (in real terms) over all vintages of the consumer durable: The period t aggregate

45 Additional examples and discussion can be found in two OECD Manuals on productivity measurement and the measurement of capital; see Schreyer (2001, 2009).

46 In the case of one-hoss-shay depreciation, assumptions are made about the sequence of user costs, $u_v^{\,t}$, as the asset age v increases.

47 A case for attributing the method to Walras (1954, 268–269) could be made but he did not lay out all of the details. Matheson (1910, 91) used the term "diminishing value" to describe the method. Hotelling (1925, 350) used the term "the reducing balance method," while Canning (1929, 276) used the term the "declining balance formula." For a more recent exposition of the geometric model of depreciation, see Jorgenson (1989).

48 Equations (20) for period t are as follows: $p_v^{\,t} = (1-\delta)^v p_0^{\,t}$ for $v = 1,2,\ldots$ and so the entire sequence of user costs by age of asset vary in a proportional manner over time under our assumptions. Thus, an aggregate period t price for the entire group of assets of varying ages is $p_0^{\,t}$ and the corresponding aggregate quantity will be Q^t defined by (23). This is an application of Hicks' (1946, 312–313) aggregation theorem: "Thus we have demonstrated mathematically the very important principle, used extensively in the text, that if the prices of a group of goods change in the same proportion, that group of goods behaves just as if it were a single commodity."

flow, Q^t, is equal to period t new purchases of the durable, q^t, plus $(1 - \delta)$ times the aggregate flow of services in the previous period, Q^{t-1}.

If the depreciation rate δ and the purchases of the durable in prior periods are known, then the aggregate service quantity Q^0 can readily be calculated using (22). Then using (21), it can be seen that the period 0 value of the services of the durable (over all vintages), v^0, decomposes into the price term $p_0^{\,0}$ times the quantity term Q^0. Hence, it is not necessary to use an index number formula to aggregate over vintages using this depreciation model.

The stock of consumer durables held by the household sector of a country should appear in the balance sheets of the country.[49] Using the geometric model of depreciation, it is very easy to calculate the nominal and real value of the stock of consumer durables held by households. At time t, the stocks held by the household sector for the particular type of consumer durable under consideration are $q^t, q^{t-1}, q^{t-2}, \ldots$ and the corresponding asset prices by age of asset are $P_0^{\,t}$, $P_1^{\,t}, P_2^{\,t}, \ldots$. Assumptions (12), (13), and (19) imply that these period t asset prices satisfy the following equations:

$$P_v^{\,t} = (1 - \delta)^v\, P_0^{\,t}; \quad v = 1,2, \ldots . \tag{24}$$

Equation (24) can be used to define period t *aggregate asset value* for the stocks held by households for the durable good over all ages of the durable good V^t:

$$\begin{aligned} V^t &\equiv P_0^{\,t}q^t + P_1^{\,t}q^{t-1} + P_2^{\,t}q^{t-2} + P_3^{\,t}q^{t-3} + \ldots \quad (25)\\ &= P_0^{\,t}[q^t + (1 - \delta)^1 q^{t-1} + (1 - \delta)^2 q^{t-2} + \ldots] \quad \text{using (24)}\\ &= P_0^{\,t}\, Q^t, \end{aligned}$$

where Q^t is defined by (23). Thus, Q^t serves as a measure of the real capital stock of the consumer durable at the end of period t, *and* it also serves as a measure of the real consumption services provided by this capital stock during period t.

This algebra explains why the geometric model of depreciation is used so widely in production function studies and in the measurement of total factor productivity or multifactor productivity in the production accounts of countries: It is very simple to work with![50]

8. Alternative Depreciation Models

Another very common model of depreciation is the *straight line model*.[51] In this model, the most probable length of life for the durable is somehow determined, say L periods, so that after being used for L periods, the durable is scrapped. In the straight line depreciation model, it is assumed that the period 0 cross-sectional vintage asset prices $P_v^{\,0}$ decline in a linear fashion relative to the period 0 price of a new asset $P_0^{\,0}$:

$$P_v^{\,0}/P_0^{\,0} = [L - v]/L \qquad \text{for } v = 0, 1, 2, \ldots , L - 1. \tag{26}$$

For $v = L, L + 1, \ldots$, it is assumed that $P_v^{\,0} = 0$. Now use definitions (14) and (17) along with assumptions (15) in order to obtain the following sequence of *end of period 0 vintage user costs* for a unit of the durable good of age v at the beginning of period 0:

$$\begin{aligned} p_v^{\,0} &= P_v^{\,0}(1 + r^0) - (1 + i^0)P_{v+1}^{\,0} \quad \text{for } v = 0, 1, 2, \ldots , L - 1 \quad (27)\\ &= [1/L][(L - v)(1 + r^0) - (L - v - 1)(1 + i^0)]P_0^{\,0}\\ &\quad \text{using assumptions (26)}\\ &= [(r^0 - i^0)(L - v)L^{-1} + (1 + i^0)L^{-1}]P_0^{\,0}. \end{aligned}$$

The user costs for units of the durable good that are older than L periods are zero; that is, $p_v^{\,0} \equiv 0$ for $v \geq L$. Looking at the terms in square brackets on the right-hand side of (27), it can be seen that the first term $(r^0 - i^0)(L - v)L^{-1}P_0^{\,0}$ is a real interest opportunity cost for holding and using the unit of the durable that is v periods old (and this imputed real interest cost declines as the durable good ages; that is, as the age v increases), and the second term $(1 + i^0)(1/L)P_0^{\,0}$ is an inflation-adjusted depreciation term that is equal to the constant straight line depreciation rate $1/L$ times the adjustment factor for asset inflation over the period, $(1 + i^0)$, times the price of a new unit of the durable good $P_0^{\,0}$. In period t, the corresponding end-of-period user cost for a unit of the durable good that is v periods old is defined as $p_v^{\,t} \equiv [(r^t - i^t)(L - v)L^{-1} + (1 + i^t)L^{-1}]P_0^{\,t}$, for $v = 0,1,2, \ldots, L - 1$. Thus, in both periods 0 and t, the sequences of end-of-period user costs by age, $\{p_v^{\,0}\}$ and $\{p_v^{\,t}\}$ for $v = 0,1,2, \ldots, L - 1$, are proportional to the price of a new unit of the durable for periods 0 and t, $P_0^{\,0}$ and $P_0^{\,t}$, respectively,[52] but if r^0 and/or i^0 change to a different r^t or i^t, then the factors of proportionality will change as we go from period 0 to t, and so we cannot apply Hicks' aggregation theorem in this case.

In the case of changing nominal interest rates r and/or changing expected or actual asset price inflation rates, i^t, we cannot assume that the overall inflation rate between periods 0 and t for all ages of the durable good is equal to $P_0^{\,t}/P_0^{\,0}$ as was the case with the geometric model of depreciation. Thus, for the straight line model of depreciation, it is necessary to keep track of household purchases of the durable for L periods and weight up each vintage quantity q^{-v} of these purchases by the corresponding end-of-period user costs vintage user cost $p_v^{\,0}$ defined by (27) for period 0, and a similar calculation will have to be made for period t. Once these vectors of prices and quantities have been calculated for both periods, then normal index number theory can

[49] However, for many countries, stocks of consumer durables will not be present in the country's balance sheets and so it will be necessary to use historical data on the purchases of durables along with estimated depreciation rates in order to form estimated stocks for consumer durables.

[50] See Jorgenson (1989) who popularized the use of the geometric model of depreciation in production function and total factor productivity studies. For an application of his methodology to valuing the services of consumer durables in the United States, see Christensen and Jorgenson (1969).

[51] This model of depreciation dates back to the late 1800s; see Matheson (1910, 55), Garcke and Fells (1893, 98), or Canning (1929, 265–266).

[52] Thus as the price of a new unit of the durable good changes over time, the value of depreciation will also change in line with the change in the price of the new unit. Thus economic depreciation as we have defined it is different from historical cost accounting depreciation which does not adjust depreciation allowances for changes in the levels of asset prices over time. Put another way, historical cost depreciation does not reflect current opportunity costs of using the services of consumer durable.

be applied to get the overall price index for the household holdings of the durable good, and this index can be used to deflate the user cost aggregate values to get an appropriate volume index.[53] It can be seen that the straight line model of depreciation is considerably more complicated to implement than the geometric model of depreciation explained in the previous section.[54]

The final model of depreciation that is in common use is the "light bulb" or *one-hoss-shay model of depreciation*.[55] In this model, the durable delivers the *same* services for each vintage: a chair is a chair, no matter what its age is (until it falls to pieces and is scrapped). Thus, this model also requires an estimate of the most probable life L of the consumer durable.[56] In this model, it is assumed that the sequence of vintage beginning of the period user costs u_v^0 defined by (14) and (15) is *constant* for all vintages younger than the asset lifetime L; that is, it is assumed that

$$u_v^0 \equiv P_v^0 - (1 + i^0)P_{v+1}^0/(1 + r^0) = u^0; \quad v = 0, 1, 2, \ldots, L-1, \tag{28}$$

where $u^0 > 0$ is a constant. Equation (28) can be rewritten in the following form:

$$u^0 = P_v^0 - \gamma P_{v+1}^0; \quad v = 0, 1, 2, \ldots, L-1, \tag{29}$$

where the *discount factor* γ is defined as

$$\gamma \equiv (1 + i^0)/(1 + r^0) \equiv 1/(1 + r^{0*}). \tag{30}$$

The interest rate r^{0*} can be regarded as an *asset-specific real interest rate*; that is, $1 + r^{0*} \equiv (1 + r^0)/(1 + i^0)$ so that one plus the nominal interest rate r^0 is deflated by one plus the expected asset price inflation rate, i^0. Note that equations (29) can be rewritten as follows:

$$P_v^0 = u^0 + \gamma P_{v+1}^0; \quad v = 0, 1, 2, \ldots, L-1. \tag{31}$$

Use equation (31) with $v = 0$ to express P_0^0 in terms of u^0 and P_1^0. Now use (31) with $v = 1$ to express P_2^0 in terms of u^0 and P_1^0 and then substitute P_1^0 using the previous expression that expressed P_1^0 in terms of P_0^0 and u^0. Continue this substitution process until finally it ends after L such substitutions when P_L^0 is reached and, of course, P_L^0 equals zero. The following equation is obtained:

$$\begin{aligned} P_0^0 &= u^0 + \gamma u^0 + \gamma^2 u^0 + \ldots + \gamma^{L-1}u^0 \\ &= u^0 [1 + \gamma + \gamma^2 + \ldots + \gamma^{L-1}] \\ &= \{u^0/(1-\gamma)\} - \{u^0 \gamma^L/(1-\gamma)\} \quad \text{provided that } \gamma < 1\,[57] \\ &= u^0 (1-\gamma^L)/(1-\gamma)\,. \end{aligned} \tag{32}$$

Now use the last equation in (32) in order to solve for the constant over vintages (beginning of the period) *user cost* for this model, u^0, in terms of the period 0 price for a new unit of the durable, P_0^0, and the discount factor γ defined by (31):

$$u^0 = (1-\gamma)P_0^0/(1-\gamma^L) = u_v^0; \quad v = 0,1,2, \ldots, L-1. \tag{33}$$

The sequence of *end of period 0 user cost*, p_v^0, is as usual equal to the corresponding beginning of the period 0 user cost, u_v^0, times the period 0 nominal interest rate factor, $1 + r^0$:

$$p_v^0 \equiv (1 + r^0)u_v^0 = [1 + r^0][1 - \gamma^0][1 - (\gamma^0)^L]^{-1}P_0^0 = p_0^0; \quad v = 0,1,2, \ldots, L-1, \tag{34}$$

and $p_v^0 = 0$ for $v = L, L + 1, \ldots$ and $\gamma^0 \equiv (1 + i^0)/(1 + r^0)$.

The *aggregate services of all vintages* of the good for period 0, including those purchased in period 0, will have the value v^0, which is given by:

$$\begin{aligned} v^0 &= p_0^0 q^0 + p_1^0 q^{-1} + p_2^0 q^{-2} + \ldots + p_{L-1}^0 q^{-(L-1)} \\ &= p_0^0 [q^0 + q^{-1} + q^{-2} + \ldots + q^{-(L-1)}] \\ &= p_0^0 Q^0, \end{aligned} \tag{35}$$

where the *period 0 aggregate (quality-adjusted) quantity of durable services* consumed in period 0, Q^0, is defined as follows for this depreciation model:

$$Q^0 \equiv q^0 + q^{-1} + q^{-2} + \ldots + q^{-(L-1)}\,. \tag{36}$$

Thus, in this model of depreciation, the service quantity aggregate is the simple sum of household purchases over the

[53] Diewert and Lawrence (2000) noted this problem with the straight line model of depreciation; that is, that in general, *an index number formula* should be used to aggregate over the different ages of the asset in order to obtain an aggregate of the capital services of the different vintages of the asset.

[54] However, if one is willing to assume that the reference interest rate for period t, r^t, and the expected asset inflation rate over all ages of the asset, i^t, both remain constant, then all reasonable index number formulae will estimate the overall rate of user cost inflation between periods 0 and t as the new consumer good purchase price ratio, P_0^t/P_0^0. However, the assumption that r^t and i^t remain constant over time is only a rough approximation to reality. Note that in order to calculate real and nominal consumption of the durable (over all ages of the durable), it will be necessary to use the vintage user costs defined by (27) for a constant r and i to weight up past purchases of the durable good. Thus, define the constants $\alpha_v \equiv [(r-i)(L-v)L^{-1} + (1+i)L^{-1}]$ for $v = 0,1,2, \ldots, L-1$ and $\alpha_v \equiv 0$ for $v \geq L$. Then the period *t nominal value of durable services* is defined as $v^t \equiv p_0^t q^t + p_1^t q^{t-1} + p_2^t q^{t-2} + \ldots + p_{L-1}^t q^{t-L+1} = \alpha_0 P_0^t q^t + \alpha_1 P_0^t q^{t-1} + \alpha_2 P_0^t q^{t-2} + \ldots + a_{L-1} P_0^t q^{t-L+1} = P_0^t Q^t$, where Q^t is the real value or volume of durable services defined as $Q^t \equiv \alpha_0 q^t + \alpha_1 q^{t-1} + \alpha_2 q^{t-2} + \ldots + \alpha_{L-1} q^{t-L+1}$. Define $\beta_v \equiv (L-v)/L$ for $v = 0,1,2, \ldots, L-1$. The period *t asset value* of consumer holdings of the durable good is defined as $V^t \equiv P_0^t q^t + P_1^t q^{t-1} + P_2^t q^{t-2} + \ldots + P_{L-1}^t q^{t-L+1} = P_0^t[\beta_0 q^t + \beta_1 q^{t-1} + \beta_2 q^{t-2} + \ldots + \beta_{L-1} q^{t-L+1}] = P_0^t Q^{t*}$, where we have used assumptions (26) applied to period t and the real value of durable stocks held by households at the end of period t is defined as $Q^{t*} \equiv \beta_0 q^t + \beta_1 q^{t-1} + \beta_2 q^{t-2} + \ldots + \beta_{L-1} q^{t-L+1}$. The decomposition of V^t into $P_0^t Q^{t*}$ does not require the assumption of constant r^t and i^t.

[55] This model can be traced back to Böhm-Bawerk (1891, 342). For a more comprehensive exposition, see Hulten (1990, 124) or Diewert (2005a).

[56] The assumption of a single life L for a durable can be relaxed using a methodology developed by Hulten: "We have thus far taken the date of retirement T to be the same for all assets in a given cohort (all assets put in place in a given year). However, there is no reason for this to be true, and the theory is readily extended to allow for different retirement dates. A given cohort can be broken into components, or subcohorts, according to date of retirement and a separate T assigned to each. Each subcohort can then be characterized by its own efficiency sequence, which depends among other things on the subcohort's useful life T_i." (Charles R. Hulten (1990, 125)). For more details on how this methodology works, see Schreyer (2009).

[57] If $\gamma \geq 1$, then use the second equation in (32) to express u^0 in terms of P_0^0 and the various powers of γ.

last L periods.[58] As was the case with the geometric depreciation model, the one-hoss-shay model does not require index number aggregation over vintages when calculating aggregate services from all vintages of the durable: There is a constant service price p_0^0 for all assets that are less than L periods old and the associated period 0 quantity Q^0 is the simple sum defined by (36) over the purchases of the last L periods.[59]

The first two models of depreciation considered earlier (the geometric and straight line models) made assumptions about the pattern of depreciation rates for durables of different ages. The light bulb model made assumptions about the pattern of user costs for a durable good by its age. For a more general model of depreciation that allows for an arbitrary pattern of user costs by age of asset, see Diewert and Wei (2017).

How can the different models of depreciation be distinguished empirically? For durable goods that do not change in quality over time, there are *three possible methods* for determining the sequence of vintage depreciation rates:[60]

- By making a rough estimate of the average length of life L for the durable good and then by *assuming* a depreciation model that seems most appropriate.[61]
- By using cross-sectional information on the sales of used durable prices at a single point in time and then using equations (10)–(12) to determine the corresponding sequence of vintage depreciation rates.[62]
- By using cross-sectional information on the rental or leasing prices of the durable as a function of the age of the durable and then equations (17) and (18), along with information on the appropriate nominal interest rate r^0 and expected durables inflation rate i^0 in addition to information on the price of a new unit of the durable good P^0, the corresponding sequence of vintage depreciation rates can be determined.

Which one of the three models of depreciation presented in this chapter should be used in empirical applications? It is not possible to give a universally valid answer to this question, but it is worth mentioning that the geometric model of depreciation is probably the most useful at the macro level. A problem with the models of depreciation considered in this section is that they assume that all assets in the asset class under consideration are retired at the same age. In real life, this is not the case. Thus, Hulten and Wykoff (1981a) and Schreyer (2009) generalized these models to allow for the assets to be retired at different ages, and they showed that under these conditions aggregate depreciation followed the geometric model to a reasonably high degree of approximation. The resulting geometric depreciation rates reflect the sum of wear and tear depreciation of unretired assets plus the average amount of additional depreciation that is due to premature retirement of the assets.

9. The Relationship between User Costs and Acquisition Costs

In this section, the user cost approach to the treatment of consumer durables will be compared to the acquisitions approach. Obviously, in the short run, the value flows associated with each approach could be very different. For example, if real interest rates, $r^0 - i^0$, are very high and the economy is in a severe recession or depression, then purchases of new consumer durables, say q^0, could be very low and even approach 0 for very long-lived assets like houses. On the other hand, using the user cost approach, existing stocks of consumer durables would be carried over from previous periods and priced out at the appropriate user costs, and the resulting consumption value flow could be quite large. Thus, in the short run, the monetary values of consumption under the two approaches could be vastly different. Hence, in what follows, a (hypothetical) longer-run comparison is considered where real interest rates are held constant.[63]

Suppose that in period 0, the reference population of households purchased q^0 units of a consumer durable at the purchase price P^0. Then *the period 0 value of consumption from the viewpoint of the acquisitions approach* is

$$V_A^0 \equiv P^0 q^0 . \tag{37}$$

Recall that the end-of-period user cost for one new unit of the asset purchased at the beginning of period 0 was p^0 defined by (8). In order to simplify the analysis, the geometric *model of depreciation* is assumed; that is, at the beginning of period 0, a one-period old asset is worth $(1 - \delta)P^0$, a two-period old asset is worth $(1\text{-d})^2P^0$, . . . , a t-period old asset is worth $(1 - \delta)^tP^0$, and so on. Under these hypotheses, the corresponding end of period 0 user cost for a new asset purchased at the beginning of period 0 is p^0; the end of period 0 user cost for a one-period old asset at the beginning of period 0 is $(1 - \delta)p^0$; the corresponding user cost for a two-period old asset at the beginning of period 0 is $(1 - \delta)^2p^0$; . . .; the corresponding user cost for a t-period old asset at the beginning of period 0 is $(1 - \delta)^tp^0$; and so on The final simplifying assumption is that household purchases of the consumer durable have been growing at the geometric rate g into the indefinite past. This means that if household purchases of the durable were q^0 in period 0, then in the previous period they purchased $q^0/(1 + g)$ new units, two periods ago, they purchased $q^0/(1 + g)^2$ new units, . . . , t periods ago, they purchased $q^0/(1 + g)^t$ new units, and so on. Putting all of these assumptions together, it can be seen that *the period 0 value of consumption services from the viewpoint of the user cost approach* is

$$V_U^0 \equiv p^0q^0 + [(1 - \delta)p^0q^0/(1 + g)] + [(1 - \delta)^2 p^0q^0/(1 + g)^2] + \ldots \tag{38}$$

[58] In the national income accounting literature, this measure is sometimes called the gross capital stock.

[59] Using equations (31), it can be shown that $P_v^0 = u^0[1 + (\gamma^0) + (\gamma^0)^2 + \ldots + (\gamma^0)^{L-1-v}]$ for $v = 0,1,2, \ldots, L - 1$, where $\gamma^0 \equiv (1 + i^0)/(1 + r^0)$ and $P_v^0 = 0$ for $v \geq L$. Thus, the period 0 value of the stock of consumer durables is $\Sigma_{v=0}^{L-1} P_v^0 q^{-v}$. The corresponding asset prices for period t are equal to $P_v^t = u^t[1 + (\gamma^t) + (\gamma^t)^2 + \ldots + (\gamma^t)^{L-1-v}]$ for $v = 0,1,2, \ldots, L - 1$, where $u^t \equiv [1 - (\gamma^t)]P_0^t/[1 - (\gamma^t)^L]$, $g^t \equiv (1 + i^t)/(1 + r^t)$, and $P_v^t = 0$ for $v \geq L$. The period t value of the stock of consumer durables is $\Sigma_{v=0}^{L-1} P_v^t q^{t-v}$. An index number formula will have to be used to form aggregate price and quantity indices for the *stocks* of consumer durables using the one-hoss-shay model of depreciation.

[60] These three classes of methods were noted in Malpezzi, Ozanne, and Thibodeau (1987, 373–375) in the housing context.

[61] A length of life L can be converted into an equivalent geometric depreciation rate δ by setting δ equal to a number between $1/L$ and $2/L$.

[62] This method will be pursued in Sections 11–15 for housing depreciation rates.

[63] The following material is based on Diewert (2002).

$= (1 + g)p^0q^0/(g + \delta)$ *summing the infinite series*

$= (1 + g)[(1 + r^0) – (1 – \delta)(1 + i^0)]P^0q^0/(g + \delta)$ using (8).

Equation (38) can be simplified by letting the asset inflation rate i^0 be 0 (or by replacing $r^0 – i^0$ by the real interest rate r^{0*} and by ignoring the small term δi^0), and under these conditions, the ratio of the user cost flow of consumption (38) to the acquisitions measure of consumption in period 0, (37) becomes

$$V_U{}^0/V_A{}^0 = (1 + g)(r^{0*} + \delta)/(g + \delta). \quad (39)$$

Using formula (39), it can be seen that if $1 + g > 0$ and $d + g > 0$, then $V_U{}^0/V_A{}^0$ will be greater than unity if $r^{0*} > g(1 – \delta)/(1 + g)$, a condition that will usually be satisfied. Thus, under normal conditions and over a longer time horizon, *household expenditures on consumer durables using the user cost approach will tend to exceed the corresponding expenditures on new purchases of the consumer durable*. Since the value of consumption services using the rental equivalence approach will tend to approximate the value of consumption services using the user cost approach, it can be seen that the acquisitions approach to household expenditures will tend to understate the value of consumption services estimated by the user cost and rental equivalence approaches. The difference between the user cost and acquisitions approach will tend to grow as the depreciation rate *d* decreases.

To get a rough idea of the possible magnitude of the value ratio for the two approaches, $V_U{}^0/V_A{}^0$, equation (39) is evaluated for a "housing" example using annual data, where the depreciation rate is 2 percent (that is, $\delta = .02$), the real interest rate is 3 percent (that is, $r^{0*} = .03$), and the growth rate for the production of new houses is 1 percent (that is, $g = .01$). In this base case, the ratio of user cost expenditures on housing to the purchases of new housing in the same period, $V_U{}^0/V_A{}^0$, is 1.68. If the depreciation rate is *decreased* to 1 percent, then $V_U{}^0/V_A{}^0$ *increases* to 2.02. If the real interest rate is *decreased* to 2 percent (with $\delta = .02$ and $g = .01$), then $V_U{}^0/V_A{}^0$ *decreases* to 1.35, and if the real interest rate is *increased* to 4 percent, then $V_U{}^0/V_A{}^0$ *increases* to 2.02. Thus, an acquisitions approach to housing in the CPI is likely to give a substantially smaller weight to housing services than a user cost approach would give.

However, for shorter-lived consumer durables like clothing, the difference between the acquisitions approach and the user cost approach will not be so large, and hence, the acquisitions approach can be justified as being approximately "correct" as a measure of consumption services for these high-depreciation-rate durable goods.[64]

For longer-lived durables such as houses, automobiles, and household furnishings, it would be useful for a national statistical agency to produce user costs for these goods and for the national accounts division to produce the corresponding consumption flows as "analytic series." This would extend the present national accounts treatment of housing to other long-lived consumer durables. Note also that this revised treatment of consumption in the national accounts would tend to make rich countries richer, since poorer countries hold fewer long-lived consumer durables on a per capita basis.

10. User Costs for Storable Goods

A *storable good* is similar to a durable good in that it can be purchased in one period and then consumed in a subsequent period. However, the services of a durable good can be utilized in multiple periods, whereas a storable good (such as a can of beans) can only be consumed in a single period. Stocks of storable goods that are held at the beginning of an accounting period tie up financial capital in a manner that is similar to the holdings of durable goods at the beginning of the period. Thus, the implicit (or explicit) interest cost of inventories of storable goods should be recognized in the household accounts. Furthermore, stocks of storable goods should be included in the balance sheets or wealth accounts of households.[65]

The *user cost* for a unit of a storable good held at the beginning of an accounting period can be formed using the same methodology that was used in Section 4 where the user cost of a durable good was set equal to its purchase cost less the discounted value of its price at the end of the accounting period.

Suppose that there are *N* storable goods that a household (or a group of households) can purchase during an accounting period *t*. Denote the vector of period *t* purchases of storable goods by the household group in scope as $q^t \equiv [q_{t1}, \ldots, q_{tN}] > 0_N$ and denote the corresponding period *t* (unit value) price vector by $p^t \equiv [p_{t1}, \ldots, p_{tN}] >> 0_N$ with $p^t \cdot q^t > 0$. However, the household group also holds some inventories of the *N* storable goods. Denote the vector of household inventory holdings of the *N* storable goods at the beginning of period *t* by $Q^t \equiv [Q_{t1}, \ldots, Q_{tN}] \geq 0_N$.[66] Denote the corresponding vector of storable goods prices at the beginning of period *t* by $P^t \equiv [P_{t1}, \ldots, P_{tN}] >> 0_N$. Typically, these prices would be the market prices for the storable goods that prevail at the beginning of period *t*.[67] In any case, the beginning of period *t* value of inventories of storable goods is equal to $P^t \cdot Q^t = \sum_{n=1}^N P_{tn} Q_{tn}$.

The *period t user cost for storable good n*, $U_{tn}{}^*$, is defined as the cost of purchase of a unit of the good at the beginning of the accounting period less the discounted price of a similar unit sold at the end of the accounting period; that is, $U_{tn}{}^*$, is defined as follows:

$$U_{tn}{}^* \equiv P_{tn} – P_{t+1,}n/(1 + r_t);\ n = 1, \ldots, N, \quad (40)$$

[64] Let $r^{0*} = .03$, $g = .01$, and $\delta = .2$. Under these assumptions, using (39), we find that $V_U{}^0/V_A{}^0 = 1.11$; that is, using a geometric depreciation rate of 20 percent, the user cost approach leads to an estimated value of consumption that is 11 percent higher than that obtained using the acquisitions approach under the conditions specified. Thus the acquisitions approach for consumer durables with high depreciation rates is probably satisfactory.

[65] The response of households to the lockdown restrictions prevailing at the time of writing has been to dramatically increase inventories of storable goods. Health authorities have encouraged households to make fewer trips to retail outlets and this advice has led to increased inventories of storables.

[66] It should be noted that most countries do not have estimates for inventories of household storable items. For Japan, some information on food inventories is collected by the Lifescape Marketing Company. This information was used in a study on storable goods by Ueda, Watanabe, and Watanabe (2020). This study has many references to the literature on the treatment of storable commodities in a CPI.

[67] If beginning of the period *t* prices for storable goods are not available, P^t could be approximated by $(½)p^{t–1} + (½)p^t$ or by $p^{t–1}$.

where r_t is the beginning of period t household cost of financial capital for the group of households under consideration; that is, r_t is an appropriate nominal interest rate.[68]

The user cost defined by (40) is a beginning of the period t user cost; that is, costs and benefits are discounted to the *beginning of period t*. If we anti-discount prices to the *end of period t*, the resulting user cost, U_{tn}, is defined as follows:

$$U_{tn} \equiv (1 + r_t)U_{tn}^* = (1 + r_t)P_{tn} - P_{t+1,}n = r_tP_{tn} - (P_{t+1,}n - P_{tn}); \quad n = 1, \ldots, N. \tag{41}$$

Thus, the end-of-period user cost for holding a unit of the nth storable good during period t, U_{tn}, is the imputed or actual interest cost of tying up financial capital during the period, r_tP_{tn}, less the actual or imputed capital gain the household would make on selling the unit of the storable good at the end of the period.[69]

Define the period t vector of user costs of storable products, U^t, as $[U_{t1}, \ldots U_{tN}]$. Using definitions (41), U^t is equal to the following vector:

$$U^t = r_tP^t - (P^{t+1} - P^t);\ t = 1,2, \ldots \tag{42}$$

Thus far, the treatment of inventories of storable products in the consumer context seems to be a straightforward extension of the earlier treatment of durable products in Section 4. But the situation is a bit more complicated than the preceding algebra would indicate. When dealing with storable products in the consumer context, there is an extra set of equations that does not occur when dealing with inventory items in the producer context. The extra equations are the following ones:

$$q^t = c^t + [Q^{t+1} - Q^t] = c^t + \Delta Q^t\ t = 1,2, \ldots, \tag{43}$$

where $c^t \equiv [c_{t1}, \ldots, c_{tN}] > 0_N$ is the period t vector of actual consumption of the N storable commodities and $\Delta Q^t \equiv Q^{t+1} - Q^t$ is the period t vector of change in inventories of storable goods. Equation (43) says that household period t purchases of storable commodities, q^t, equals household period t actual consumption of the commodities, c^t, plus the net change in inventories of the storable products, $Q^{t+1} - Q^t$, which in turn is equal to the end of period t stock of inventories, Q^{t+1}, less the beginning of period t stock of inventories, Q^t. Of course, equation t in equations (43) can be rearranged to give us the following supply equals demand equations:

$$Q^t + q^t = c^t + Q^{t+1};\ t = 1,2, \ldots \tag{44}$$

[68] As usual, it is difficult to determine this reference interest rate. If the household is borrowing money, then r_t is the appropriate borrowing rate or mortgage interest rate; if the household is loaning financial capital to others, then the appropriate interest rate is the expected rate of return on investments.

[69] The user costs U_{tn} are the counterparts to the end-of-period user cost for a durable good defined by (8) in Section 4. If the depreciation rate δ in equation (8) is equal to 0, then the user costs defined by (41) are exactly the same as the user costs defined by (8) using different notation. The user costs of inventories defined by definitions (41) are frequently used to value the services of business inventories; for example, see Christensen and Jorgenson (1969) and Diewert and Fox (2018). However, the valuation of the services of storable inventories in the household context has not been widespread.

Thus, the beginning of period t stock of storable goods, Q^t, plus new purchases of storable goods, q^t, equals consumption of the storable goods in period t, c^t, plus the end of period t stocks of storable goods, Q^{t+1}.

Recall that period t price and quantity vectors for household purchases of storable goods are p^t and q^t. Thus, the value v^t of household purchases of storable goods during period t is defined as follows:

$$\begin{aligned} v^t &\equiv p^t \cdot q^t\ t = 1,2, \ldots \\ &= p^t \cdot c^t + p^t \times [Q^{t+1} - Q^t] \quad \text{using (43)} \\ &= p^t \cdot c^t + p^t \times \text{D}Q^t. \end{aligned} \tag{45}$$

Thus, the period t value of household *purchases* of storables, $p^t \cdot q^t$, is equal to the period t value of household *consumption* of storable goods, $p^t \cdot c^t$, plus period t *net investment* in storables, $p^t \cdot \Delta Q^t$. All of these value aggregates use the vector of *average period t purchase prices* p^t to value q^t, c^t, and ΔQ^t.

When a product goes on sale, typically households will dramatically increase their purchases of it. However, not all of the purchased storable good will be consumed in the period of purchase, so inventories of the storable product will greatly increase. Basically, changes in inventory will tend to smooth purchases of storable goods so that consumption is relatively stable over time. Thus adjusting purchases of storable goods for changes in inventory will lead to estimates of household actual consumption of storables that are much smoother than household purchases of storables. Constructing CPIs using p^t and c^t as the basic price and quantity data to be used in an index number formula (rather than using p^t and q^t) will greatly mitigate the chain drift problem that will arise if household purchase data are used in place of household consumption data.

The length of the accounting period will affect the severity of the chain drift problem. If the accounting period length is a day, inventory changes may be large relative to daily consumption, leading to a big chain drift problem if daily purchase price and quantity data are used in an index number formula with variable weights. Furthermore, if the household data pertain to a single household or a small number of households, the vector of daily purchases of storable goods may have many zero components, leading to a lack of matching problem which affects the reliability of the resulting daily price index. On the other hand, the changes in storable inventories for an annual CPI will be small relative to the annual consumption of storables, and thus the difference between q^t and c^t in the case of an annual index will be small. Thus, for annual CPIs, it is probably not necessary to collect data on inventories of storables.[70] However, if a daily or weekly CPI is to be produced, it will be important to collect inventory data on storable goods and to adjust purchase data for changes in inventories.

This accounting treatment for storable goods does not give any insight into why large changes in storable inventories might occur. In order to provide an analytic framework for the treatment of storable goods in a cost of living index,

[70] However, if the national statistical agency also constructs measures of household wealth, it will be necessary to conduct periodic surveys of household inventories of storables.

it is necessary to introduce the concept of intertemporal cost minimization. The basic idea is that the consumer or household tries to minimize the discounted cost of consumption over a number of discrete time periods subject to attaining a certain level of (intertemporal) utility.[71] In order to minimize conceptual and notational complexity, we will look at the household's intertemporal cost minimization problem over a horizon that consists of just two periods.[72]

Suppose that the household's opportunity cost of capital at the beginning of period t is the interest rate r_t. As usual, let p^t and q^t be the price and quantity vectors for household *purchases* of storable goods for period t for t = 1,2,3. Define the household's (expected) discounted value of *household purchases* of storable goods $W > 0$ over the two-period horizon (discounted to the end of period 1) as follows:[73]

$$
\begin{aligned}
W &\equiv p^1\cdot q^1 + (1 + r_2)^{-1}p^2\cdot q^2 \\
&= p^1\cdot q^1 + p^{2*}\cdot q^2 \quad \text{defining } p^{2*} \equiv (1 + r_2)^{-1}p^2 \\
&= p^1\cdot[c^1 + Q^2] + p^{2*}\cdot[c^2 - Q^2] \quad \text{using } Q^1 \equiv 0_N,\ q^1 = c^1 + Q^2,\ q^2 = c^2 - Q^2 \ \textit{and}\ Q^3 \equiv 0_N \\
&= p^1\cdot c^1 + p^{2*}\cdot c^2 + [p^1 - p^{2*}]\cdot Q^2 \\
&= p^1\cdot c^1 + p^{2*}\cdot c^2 + u^2\cdot Q^2, \qquad (46)
\end{aligned}
$$

where

$$u^2 \equiv p^1 - p^{2*} = p^1 - (1 + r_2)^{-1}p^2 \qquad (47)$$

is the vector of *user costs of storable products* for the beginning of period 2 stocks of inventories.[74] In the previous model of consumer expenditures on storable goods, we are assuming that the household has no inventories of storables at the beginning of period 1 and at the end of period 2. Thus, inventories Q^2 are only held at the beginning of period 2.

In order to apply classical economic theory to the problem on deciding the level of inventories for storables, it is useful to regard W on the left-hand side of definition (46) as an exogenous amount of money that the household plans to spend on purchases of storable goods over the two-period horizon. Thus, we assume that the household is subject to a partial "wealth" constraint of the form $W \geq p^1\cdot c^1 + p^{2*}\cdot c^2 + u^2\cdot Q^2 = p^1\cdot q^1 + p^{2*}\cdot q^2$, where the household's decision variables are purchases of storables over the two periods, q^1, q^2, consumption of storables over the two periods, c^1, c^2, and holdings of inventories of storables at the beginning of period 2, Q^2.

If holdings of storables are not valued, except that they allow consumers to transfer purchases of consumption goods from one period where they are relatively cheap to another period where they are relatively expensive,[75] then if any component of the user cost vector u^2 is positive, say $u_{2n} = p_{1n} - (1 + r_2)^{-1}p_{2n} > 0$, then it does not make sense to purchase storable good n in period 1 in order to consume it in period 2 because it will be cheaper to purchase the good in period 2, taking into account the fact that the household will tie up financial capital if it holds the good as an inventory item. Thus good n will be held as an inventory item (so that $Q_{2n} > 0$) only if $u_{2n} \leq 0$. If $u_{2n} < 0$, then it definitely will be worthwhile to hold some inventories of storable good n. But how much inventory will be held? Furthermore, can we apply the usual exact index number theory that relies on static utility-maximizing behavior to the household's purchases of storable goods? In order to provide answers to these questions, we will look at an economic model of consumer behavior.

When modeling consumer behavior over a time horizon, economists assume that households have intertemporal utility functions to measure the relative worth of consuming the services of various commodities. A general utility function to model the relative value of storables over a two-period horizon is a function of the form $U(c^1,c^2)$. However, for producers of CPIs who might want to apply the simple exact index number theory explained in Chapter 5 to produce a price index for each separate period, it is necessary to assume a more restrictive functional form for the intertemporal utility function. Thus, we assume that $U(c^1,c^2) = F(f(c^1),f(c^2))$, where $F(C^1,C^2)$ is a "macro" utility function that describes the tradeoffs in consuming aggregate consumption in period 1, $C^1 \equiv f(c^1)$, against consuming aggregate consumption in period 2, $C^2 = f(c^2)$, where $f(c)$ is a within-the-period static utility function of the type studied in Chapter 5. As usual, we assume that the one-period utility function $f(c)$ is a differentiable, concave, linearly homogeneous, and increasing function of the nonnegative consumption vector c. We assume that the macro utility function, $F(C^1,C^2)$ is a differentiable,[76] increasing, and concave function of C^1 and C^2.

The household's intertemporal utility maximization problem is the problem of maximizing $F(f(c^1),f(c^2))$ subject to (i) the intertemporal budget constraint $W - [p^1\cdot q^1 + p^{2*}\cdot q^2] \geq 0$; (ii) the household material balance equations that relate purchases to consumption and inventory change for both periods, $q^1 = c^1 + Q^2$ and $q^2 + Q^2 = c^2$; and (iii) the nonnegativity constraints $q^1 \geq 0_N$, $q^2 \geq 0_N$, $Q^2 \geq 0_N$, $c^1 \geq 0_N$, and $c^2 \geq 0_N$. The decision variables for this constrained utility maximization problem are q^1, q^2, Q^2, c^1, and c^2. Use the

[71] The framework for intertemporal consumer theory is basically the consumer theory counterpart to Hicks' (1946, 325–328) intertemporal producer theory; see Diewert (1974, 1977).

[72] It is straightforward to extend the number of time periods under consideration to an arbitrary finite number.

[73] As was the case for our analysis of user costs in Section 4, we are following the conventions used in financial accounting that suggest that *flow transactions* taking place within the accounting period be regarded as taking place at the *end* of the accounting period and hence the period t cost of household purchases of storables, $p^t\cdot q^t$, is regarded as taking place at the end of period t; see Peasnell (1981). Thus the period 2 and 3 purchase costs, $p^2\cdot q^2$ and $p^3\cdot q^3$, in definition (46) are discounted to the end of period 1 which is the beginning of period 2.

[74] Compare these new user costs to our previous definition for the vector beginning of period 2 user costs given by equations (40) for t = 2. These definitions imply that $U^{2*} \equiv P^2 - (1 + r_2)^{-1}P^3$ where P^t is the vector of purchase prices for storable goods at the beginning of period t. It can be seen that $u^2 \equiv p^1 - (1 + r_2)^{-1}p^2$ has a similar form except in place of P^2 and P^3, the new definition uses the *average purchase prices* for storable goods for periods 1 and 2, p^1 and p^2.

[75] The user cost theory developed at the beginning of this section essentially assumed that holdings of storable goods increased household utility; that is, holding inventories of storable goods was assumed to be desirable even if the goods were never consumed. Our present perspective assumes that inventories are only valuable when they are consumed.

[76] We assume that the first-order partial derivatives of $F(C^1,C^2)$ are positive.

material balance equations to eliminate q^1 and q^2 from the intertemporal budget constraint. After eliminating q^1 and q^2 from the constraints, the household's utility maximization problem becomes the problem of maximizing $F(f(c^1),f(c^2))$ subject to (i) the intertemporal budget constraint $W - [p^1{\cdot}c^1 + p^{2*}{\cdot}c^2 + u^2{\cdot}Q^2] \geq 0$; (ii) $c^2 - Q^2 \geq 0_N$; and (iii) the nonnegativity constraints $Q^2 \geq 0_N$, $c^1 \geq 0_N$ and $c^2 \geq 0_N$. The decision variables for this constrained utility maximization problem are Q^2, c^1, and c^2. The *Lagrangian* function for this constrained maximization problem is defined as follows:

$$L(c^1,c^2,Q^2,\lambda,\kappa,\mu) \equiv F(f(c^1),f(c^2)) + \lambda\{W - [p^1{\cdot}c^1 + p^{2*}{\cdot}c^2 + u^2{\cdot}Q^2]\} + \mu{\cdot}[c^2 - Q^2] + \kappa{\cdot}Q^2, \quad (48)$$

where λ is a nonnegative scalar Lagrange or Kuhn and Tucker (1951) multiplier and μ and κ are nonnegative vectors of Lagrange multipliers.

Suppose $c^{1*} >> 0_N$, $c^{2*} >> 0_N$, and $Q^{2*} \geq 0_N$ is a solution to the household's intertemporal constrained maximization problem. Then there exist $\lambda^* > 0$,[77] $\mu^* \geq 0_N$, and $\kappa^* \geq 0_N$ such that the following *Kuhn–Tucker conditions* are satisfied:[78]

$$\begin{array}{ll}
(i)\ F_1^*\ \nabla f(c^{1*}) = \lambda^* p^1; & c^{1*} >> 0_N; \\
(ii)\ F_2^*\ \nabla f(c^{2*}) = \lambda^* p^{2*} - \mu^*; & c^{2*} >> 0_N; \\
(iii)\ -\lambda^* u^2 - \mu^* + \kappa^* \leq 0_N; & Q^{2*} \geq 0_N; [-\lambda^* u^2 - \mu^* + \kappa^*]Q^{2*} = 0; \\
(iv)\ W = p^1{\cdot}c^{1*} + p^{2*}{\cdot}c^{2*} + u^2{\cdot}Q^{2*}; & \lambda^* > 0; \\
(v)\ c^{2*} - Q^{2*} \geq 0_N; & \mu^* \geq 0_N; \mu^*{\cdot}[c^{2*} - Q^{2*}] = 0; \\
(vi)\ Q^{2*} \geq 0_N; & \kappa^* \geq 0_N; \kappa^*{\cdot}Q^{2*} = 0,
\end{array} \quad (49)$$

where $F_1^* \equiv \partial F(f(c^{1*}), f(c^{2*}))/\partial C^1$, $F_2^* \equiv \partial F(f(c^{1*}), f(c^{2*}))/\partial C^2$ are the first-order partial derivatives of the macro utility function with respect to aggregate consumption C^t in each period t and $\nabla f(c^{t*})$ is the vector of first-order partial derivatives of the micro utility function $f(c^t)$ with respect to the components of the period t consumption of storables vector c^t for $t = 1,2$. Conditions (49) are more complicated than the usual first-order necessary conditions that economists use when solving constrained optimization problems because, usually, we can assume that an interior solution to the optimization problem occurs and hence we can ignore nonnegativity constraints. But for this particular intertemporal utility maximization problem, nonnegativity constraints cannot be ignored; that is, usually, the solution to the problem will require that some decision variables be equal to zero. The conditions defined by (49) allow for zero decision variables.

Suppose the preceding solution to the household's intertemporal utility maximization problem satisfies conditions (49) and in addition, $\mu^* = 0_N$. Then conditions (49) (ii) become $F_2^*\ \nabla f(c^{2*}) = \lambda^* p^{2*}$, which are the period 2 counterparts to conditions (49) (i): $F_1^*\ \nabla f(c^{1*}) = \lambda^* p^1$. Using these two sets of equations and the linear homogeneity of $f(c)$, we can establish the following equations:[79]

$$p^t/p^t{\cdot}c^{t*} = \nabla f(c^{t*})/f(c^{t*});\ t = 1,2. \quad (50)$$

But equation (50) is the equation for *Wold's Identity* (1944, 69–71); see equation (15) in Chapter 5. Thus, if the vector of Kuhn–Tucker multipliers μ^* turns out to be a vector of zeros, then we can apply the exact index number theory that was explained in Chapter 5 to the consumer's demand for storable goods in our highly simplified model of inventory behavior.

The question that now needs to be addressed is "Under what conditions will $\mu^* = 0_N$?" An answer is provided subsequently. We will look at each component μ_n^* of μ^* in turn.

Case (i): Suppose that the user cost for storable good n for beginning of period 2 inventories is positive; that is, suppose that $u_n^{\,2} \equiv p_n^{\,1} - p_n^{\,2*} = p_n^{\,1} - (1 + r_2)^{-1}p_n^{\,2} > 0$. Thus, we have $p_n^{\,1} > (1 + r_2)^{-1}p_n^{\,2}$ so that the price of storable good n in period 1 is greater than its discounted period 2 expected price. Under these conditions, it makes no sense to purchase storable good n in period 1 to use in period 2 so that under these conditions, there will be no accumulation of inventories so that $Q_n^{\,2*}$ will equal 0. To see that $Q_n^{\,2*} = 0$ follows from conditions (49), *suppose* that $Q_n^{\,2*} > 0$. Using conditions (49) (vi), it can be seen that our supposition implies that $\kappa_n^* = 0$. Using $\kappa_n^* = 0$ and $Q_n^{\,2*} > 0$, condition (49) (iii) implies that $-\lambda^* u_n^{\,2} - \mu_n^* = 0$ or $\mu_n^* = -\lambda^* u_n^{\,2} < 0$ using $\lambda^* > 0$ and $u_n^{\,2} > 0$. But $\mu_n^* < 0$ contradicts conditions (49) (v) which implies $\mu_n^* \geq 0$. This contradiction means that our *supposition* that $Q_n^{\,2*} > 0$ is false, and hence $Q_n^{\,2*} = 0$. Using $Q_n^{\,2*} = 0$ along with (49) (ii) which implies $c_n^{\,2*} > 0$ means that the equations $0 = \mu_n^*[c_n^{\,2*} - Q_n^{\,2*}] = \mu_n^* c_n^{\,2*}$ will hold using (49) (v), which in turn implies that $\mu_n^* = 0$. This algebra can be summarized as follows: If the user cost of storable good n at the beginning of period 2, $u_n^{\,2}$, is positive, then no inventories of good n will be accumulated (so that $Q_n^{\,2*} = 0$) and the Lagrange multiplier for the nonnegativity constraints pertaining to purchases of good n will also be equal to zero (so that $\kappa_n^* = \mu_n^* = 0$). Thus, if all N user costs of storables, $u_n^{\,2}$, are positive, then there will be no purchases of inventories so that actual consumption in period t, c^{t*}, will equal market purchases for period t, q^{t*}, for periods $t = 1,2$.[80]

Case (ii): Suppose that the user cost for storable good n for beginning of period 2 inventories is negative; that is, suppose that $u_n^{\,2} \equiv p_n^{\,1} - (1 + r_2)^{-1}p_n^{\,2} < 0$. In this case, it makes sense to accumulate inventories of good n in period 1 because the period 1 price is less than the

[77] The Kuhn–Tucker conditions imply the existence of $\lambda \geq 0$, and we assumed the existence of only $\lambda > 0$. Our stronger assumption is justified if the first-order partial derivatives of the utility function are positive.

[78] See Kuhn and Tucker (1951) or Karlin (1959, 204). In the original constrained utility maximization problem that involved q^1, q^2, c^1, c^2, and Q^2, all of these decision variables were restricted to be nonnegative. Recall that $q^1 = c^1 + Q^2$. Thus, if $c^1 \geq 0_N$ and $Q^2 \geq 0_N$, then we also have $q^1 \geq 0_N$. However, $q^2 = c^2 - Q^2$ and so even though the simplified constrained utility maximization problem involved only the decision variables c^1, c^2, and Q^2, we still need to impose the restriction $q^2 \geq 0_N$, which implies the restriction (49) (v).

[79] Premultiply both sides of $\nabla f(c^{t*}) = [\lambda^*/F_t^*]\ p^t$ by c^{t*} for $t = 1,2$. Using Euler's theorem on linearly homogeneous functions, $f(c^{t*}) = c^{t*}{\cdot}\nabla f(c^{t*})$ for $t = 1,2$. Use the equations $c^{t*}{\cdot}\nabla f(c^{t*}) = f(c^{t*}) = [\lambda^*/F_t^*]\ p^t{\cdot}c^{t*}$ to solve for $[\lambda^*/F_t^*] = f(c^{t*})/p^t{\cdot}c^{t*}$ for $t = 1,2$. Thus we obtain the equations $\nabla f(c^{t*}) = [\lambda^*/F_t^*]\ p^t = p^t f(c^{t*})/p^t{\cdot}c^{t*}$ for $t = 1,2$, which are equivalent to equations (50).

[80] This very simple economic approach to the accumulation of inventories of storable goods *neglects the costs of shopping* which will imply some short-term inventory accumulation even if all user costs are positive.

discounted period 2 price for storable good *n*. It turns out that our simple model will imply that all of the purchases of good *n* are made in period 1; that is, we will have $c_n^{2*} = Q_n^{2*}$. Thus, there is a maximal amount of inventory accumulation that takes place in period 1.[81] We explain how conditions (49) can be used to explain this result. *Suppose* $0 \le Q_n^{2*} < c_n^{2*}$. Conditions (49) (v) and our supposition imply that $\mu_n^* = 0$. Conditions (49) (iii) and $\mu_n^* = 0$ imply that $-\lambda^* u_n^2 + \kappa_n^* \le 0$, and this condition along with $u_n^2 < 0$ and $\lambda^* > 0$ imply that $0 < -\lambda^* u_n^2 \le -\kappa_n^*$, which in turn implies that $\kappa_n^* < 0$. This contradicts part of conditions (49) (vi). Thus, our *supposition* is false. Since we also have the constraint $q_n^{2*} \equiv c_n^{2*} - Q_n^{2*} \ge 0$, we see that we must have $q_n^{2*} \equiv c_n^{2*} - Q_n^{2*} = 0$. In this case, we also have $k_n^* = 0$ and $\mu_n^* \ge 0$.

It can be seen that using an economic approach to model household purchases of storable goods is a difficult task. More realistic models of inventory accumulation need to take into account the costs of storing the inventories and they need to take into account the costs of shopping, which would include not only the transportation costs to the retail outlets but also the expenditure of *time* during the shopping process. A more realistic model of inventory accumulation would require a great deal of household information—information that is unlikely to be available to national statistical agencies in the near future.

What practical implications for statistical agencies can be drawn from the preceding analysis?

- The simplest strategy would be to just apply the acquisitions approach to purchases of storable goods; that is, simply assume that purchases of storables over a month are equal to the actual consumption of the goods over the month. Over the course of a year, the value of average inventory holdings of storable goods to total household consumption of storables will typically be a small stable fraction,[82] and thus the overall accuracy of the CPI will not be greatly affected.
- If periodic surveys of household inventories of storable goods are made and if the statistical agency target index is a cost of living index, then it would be useful to treat holdings of storable goods in the same manner as holdings of durable goods are treated; that is, a user cost approach should be applied to storable goods.[83] If monthly surveys for household inventories of storable goods could be conducted, then estimates for the actual consumption of storables could be made along with estimates for the user cost value for the household holdings of storable inventories. Users could decide to use the estimates for actual consumption or for actual consumption plus the services of household inventories of storables, depending on their needs.
- For the country's balance sheet accounts, household inventories of storable goods are part of household wealth. Thus, for the construction of the balance sheet accounts, it is necessary for the national statistical agency to provide quarterly or annual estimates of household holdings of storable goods.[84]

In the following eight sections of this chapter, the focus will be on the special problems that are associated with both measuring the value of the housing stock and valuing the services of OOH.

11. Decomposing Residential Property Prices into Land and Structure Components

In this section, the problems associated with the construction of constant quality residential property price indices will be studied. The user cost approach to valuing the services of a durable good discussed in Section 4 cannot be applied directly to the construction of user costs for OOH because a residential property has two main components: a *structure* (which depreciates) and a *land plot* (which does not depreciate).[85] In this section, we will look at the resulting problems associated with the construction of *constant quality indices* for the *stock of residential housing units*; in subsequent sections, we will look at the problems associated with pricing the services of a residential dwelling unit.

There are two difficult measurement problems associated with the construction of a constant quality house price index:

- A dwelling unit is a *unique consumer durable good*; that is, the location of a housing unit is a price-determining characteristic of the unit and each house or apartment has a unique location.
- As mentioned earlier, there are two main components of a dwelling unit: (i) *the size of the structure* (measured in square meters of floor space) and (ii) *the size of the land plot* that the structure sits on (also measured in square meters). However, the purchase or selling price of a dwelling unit is for the entire property and thus *the decomposition of property price into its two main components will involve imputations.*

The first problem area listed here might not be a problem if the same dwelling unit sold at market prices at a frequent rate so that the location would be held constant and it would seem that the usual matched model methodology that is used in constructing price indices could be applied. But houses do not transact all that frequently; typically, a house is held for 10–20 years by the same owner before it is resold. Moreover, the structure is not constant over time; depreciation of the structure occurs over time and owners renovate and replace aging components of the structure. For example, the roofing materials for many dwellings are replaced every 20

[81] Our simple model of inventory accumulation *neglects any costs of inventory storage*, which helps explain our all or none results.

[82] This assumption will not be satisfied if the country is under a COVID-19 lockdown. Inventories of storable goods will be much larger than usual and may be quite variable.

[83] See (47) which defines the vector of user costs for storable goods.

[84] If balance sheet estimates are made at a quarterly frequency, approximate monthly estimates for holdings of storable goods could be constructed using various interpolation methods.

[85] It is important to recognize that a residential property is a bundle of two important components: a land component and a structure component. Knoll, Schularick, and Steger (2017, 331) summarize their study of house prices in 14 countries over the period 1870–2012 as follows: "Land prices, not replacement costs, are the key to understanding the trajectory of house prices. Rising land prices explain about 80 percent of the global house price boom that has taken place since World War II."

or 30 years. Thus depreciation and renovation constantly change the quality of the structure.

The second problem area is associated with the difficulty of decomposing the transaction price for a housing unit into *separate components* representing the structure value and the land value; that is, the single property price is for both components of the housing unit but for many purposes, we require separate valuations for the two components. The international SNA requires separate valuations for the land and structure components of residential housing in the national balance sheets of the country. Many countries construct estimates for the total factor productivity or multifactor productivity of the various sectors in the economy and the methodology used to construct these estimates requires separate price and quantity information on both structures and the land that the structures sit on. In this section, we indicate a possible method that can be used to accomplish this decomposition of property value into constant quality land and structure components.

The *builder's model* for valuing a detached dwelling unit postulates that the value of the property is the sum of two components: the value of the land that the structure sits on plus the value of the structure. This model can be justified in two situations:

- A household purchases a residential land plot with no structure on it (or if there are structures on the land plot, they are immediately demolished).[86]
- A household purchases a land plot and immediately builds a new dwelling unit on the property.

In the first case, it is clear that the property value is equal to the land value. In the second case, the total cost of the property after the structure is completed will be equal to the floor space area of the structure, say S square meters, times the building cost per square meter β_t during period t, plus the cost of the land, which will be equal to the cost per square meter α_t times the area of the land site, say L square meters. Now think of a sample of properties of the same general type in the same general location, which have prices or values V_{tn} in period t (where $t = 1, \ldots, T$) and structure floor space areas S_{tn} and land areas L_{tn} for $n = 1, \ldots, N(t)$, where $N(t)$ is the number of observations in period t. Assume that these prices are equal to the sum of the land and structure costs plus error terms ε_{tn}, which we assume are independently normally distributed with zero means and constant variances. This leads to the following *hedonic regression model* for period t, where α_t and β_t are the parameters to be estimated in the regression:[87]

$$V_{tn} = \alpha_t L_{tn} + \beta_t S_{tn} + \varepsilon_{tn};\ t = 1, \ldots, T;\ n = 1, \ldots, N(t). \quad (51)$$

The hedonic regression model defined by (51) applies to new structures and to purchases of vacant residential lots in the neighborhood under consideration, where $S_{tn} = 0$. Note that there are some strong simplifying assumptions built into the model defined by (51): (i) the period t land price α_t (per *square meter*) is assumed to be constant across all properties in the neighborhood under consideration and (ii) the construction cost (per *square meter*) is also assumed to be constant across all housing units built in the neighborhood during period t. The model here applies to raw land purchases and the purchases of new dwelling units during period t in the neighborhood under consideration. It is likely that a model that is similar to (51) applies to sales of older structures as well. Older structures will be worth less than newer structures due to the *depreciation* of the structure. Assuming that we have information on the age of the structure n at time t, say $A(t,n)$, and assuming a geometric (or declining balance) depreciation model, a more realistic hedonic regression model than that defined by (51) is the following *basic builder's model*:

$$V_{tn} = \alpha_t L_{tn} + \beta_t (1 - \delta)^{A(t,n)} S_{tn} + \varepsilon_{tn};\quad t = 1, \ldots, T;\ n = 1, \ldots, N(t), \quad (52)$$

where the parameter δ reflects the *net geometric depreciation rate* as the structure ages one additional period. Thus, if the age of the structure is measured in years, we would expect an annual *net* depreciation rate to be around 1 to 3 percent per year.[88] Note that (52) is now a nonlinear regression model, whereas (51) was a simple linear regression model. The period t constant quality price of land will be the estimated coefficient for the parameter α_t and the price of a unit of a newly built structure for period t will be the estimate for the parameter β_t. The period t quantity of land for property n is L_{tn} and the period t quantity of structure for property n, expressed in equivalent units of a new structure, is $(1 - \delta)^{A(t,n)} S_{tn}$, where S_{tn} is the floor space area of the structure for property n in period t.

Note that the preceding model can be viewed as a *supply-side model* as opposed to a *demand-side model*.[89] Basically, we are assuming a valuation of a housing structure that is equal to the cost per unit floor space area of a new unit times the floor space area times an adjustment for structure depreciation. The corresponding land value of the property is determined residually as total property value minus the imputed value of structures quality adjusted for the age of the structure. This assumption is justified for the case of newly built houses and sales of vacant lots, but it is less well justified for sales of properties with older structures, where a demand-side model may be more relevant.

There is a major practical problem with the hedonic regression model defined by (52): the *multicollinearity problem*. Experience has shown that it is usually not possible to estimate sensible land and structure prices in

[86] The cost of the demolition should be added to the purchase price for the land to get the overall land price for the land plot.

[87] Other papers that have suggested hedonic regression models that lead to additive decompositions of property values into land and structure components include Clapp (1980, 257–258), Davis and Heathcote (2007), Bostic, Longhofer, and Readfearn (2007, 184), Francke and Vos (2004), Diewert (2008, 19–22; 2010), Francke (2008, 167), Koev and Santos Silva (2008), Rambaldi et al. (2010), Diewert, Haan, and Hendriks (2011, 2015), Eurostat (2013), Diewert and Shimizu (2015, 2016, 2020), Diewert, Huang, and Burnett-Issacs (2017) and Burnett-Issacs, Huang, and Diewert (2021).

[88] This estimate of depreciation is regarded as a *net depreciation rate* because it is equal to a "true" gross structure depreciation rate less an average renovations appreciation rate. Since typically information on renovations and major repairs to a structure is not available, the age variable will only pick up average gross depreciation less average real renovation expenditures.

[89] We will pursue a demand-side model in Section 13.

a hedonic regression like that defined by (52) due to the multicollinearity between lot size and structure size.[90] Thus, in order to deal with the multicollinearity problem, the parameter β_t in (52) is replaced by p_{St}, an *exogenous period t construction cost price* for houses in the area under consideration.[91] The exogenous construction price index may be an official construction price index estimated by the national statistical agency or a relevant commercially available residential construction price index. Thus, the new model that replaces (52) is the following nonlinear hedonic regression model:

$$V_{tn} = \alpha_t L_{tn} + p_{St}(1 - d)^{A(t,n)}S_{tn} + \varepsilon_{tn};$$
$$t = 1, \ldots, T;\ n = 1, \ldots, N(t). \qquad (53)$$

This model has T land price parameters (the α_t) and one (net) geometric depreciation rate δ. Note that the replacement of β_t by the exogenous construction price level, p_{St}, means that we have saved T degrees of freedom as well as eliminated the multicollinearity problem.

In order to allow for a finer structure of local land prices, the sales data may be further classified into a finer classification of locations. For example, the initial regression (53) may be applied to say city-wide sales of residential properties. Suppose that the postal code of each sale is also available, and there are J postal codes. Then one can introduce the following *postal code dummy variables*, $D_{PC,tn,j}$, into the hedonic regression (53). These J dummy variables are defined as follows: for $t = 1, \ldots, T$; $n = 1, \ldots, N(t)$; $j = 1, \ldots, J$:

$$D_{PC,tn,j} \equiv 1 \text{ } \textit{if} \text{ observation } n \text{ in period } t \text{ is in Postal Code } j;$$
$$\equiv 0 \text{ if observation } n \text{ in period } t \text{ is not in Postal Code } j. \qquad (54)$$

We now modify the model defined by (53) to allow the *level* of land prices to differ across the J postal codes. The new nonlinear regression model is as follows:

$$V_{tn} = \alpha_t(\textstyle\sum_{j=1}^{J} \omega_j D_{PC,tn,j})L_{tn} + p_{St}(1 - \delta)^{A(t,n)}S_{tn} + \varepsilon_{tn};$$
$$t = 1, \ldots, T;\ n = 1, \ldots, N(t). \qquad (55)$$

Comparing the models defined by equations (53) and (55), it can be seen that we have added additional *J neighborhood relative land value parameters*, $\omega_1, \ldots, \omega_J$, to the model defined by (53). However, looking at (55), it can be seen that the T land time parameters (a_t) and the J location parameters (ω_j) cannot all be identified. Thus, it is necessary to impose at least one identifying normalization on these parameters. The following normalization is a convenient one:[92]

$$\omega_1 \equiv 1. \qquad (56)$$

Thus, Model 2 defined by equations (55) and (56) has J-1 additional parameters compared to Model 1 defined by (53). Note that if we initially set *all* of the ω_j values equal to unity, Model 2 collapses to Model 1. It is useful to make use of this fact in running a sequence of nonlinear hedonic regressions. The models that are proposed in this section are *nested* so that the final parameter estimates from a previous model can be used as starting parameter values in the next model's nonlinear regression.[93]

Model 2 makes the price of a residential land a nonsmooth function of the postal code or local neighborhood area; that is, the estimated price of land will exhibit discrete jumps as we move from one local area to an adjacent local area that has a different ω_j. If it is possible to collect spatial coordinate information for the properties in the sample, then it is possible to estimate a continuous land price surface for the hedonic regression model in place of the discrete plateau model that is defined by (55). These continuous surface models are very complex and not easy to estimate. However, Hill and Scholz (2018) and Diewert and Shimizu (2019) showed that for their particular samples of Australian and Japanese properties, the continuous surface models generated very similar price indices to their counterpart discrete models. Thus, if the purpose of the hedonic regressions is to generate residential land or property price indices, it is not necessary to estimate complex continuous surface models.

In the next model, some nonlinearities in the pricing of the land area for each property are introduced. The land plot areas in a typical sample of properties can vary five- or tenfold.[94] Up to this point, we have assumed that land plots in the same neighborhood sell at a constant price per square meter of lot area. However, it is likely that there is some nonlinearity in this pricing schedule; for example, it is likely that large lots sell at a per *square meter* price that is well below the per *square meter* price of medium-sized lots. In order

[90] See Schwann (1998) and Diewert, de Haan, and Hendriks (2011, 2015) for details on the multicollinearity problem.

[91] This formulation follows that of Diewert (2010), Diewert, de Haan, and Hendriks (2011, 2015), Eurostat (2013), Diewert and Shimizu (2015, 2016, 2020), Diewert, Huang, and Burnett-Issacs (2017) and Burnett-Issacs, Huang, and Diewert (2021). These authors assume that property value is the sum of land and structure components but movements in the price of structures are proportional to an exogenous structure price index. Note that the index p_{St} should be a levels price that gives the period t cost of building one square meter of structure.

[92] Equivalently, one could make the normalization $\alpha_1 = 1$ and not normalize ω_j. The resulting estimated a_t for $t = 2,3, \ldots, T$ can then be interpreted as a constant quality land price index for the entire region relative to period 1 where $\alpha_1 \equiv 1$. In this section, we are drawing on the formulation of Diewert, Huang, and Burnett-Issacs (2017) and using the normalization used in that paper.

[93] In order to obtain sensible parameter estimates in our final (quite complex) nonlinear regression model, it is absolutely necessary to follow our procedure of sequentially estimating gradually more complex models, using the final coefficients from the previous model as starting values for the next model. The models that are being described in this section were implemented in Diewert, Huang, and Burnett-Issacs (2017) where the econometric software Shazam was used to perform the nonlinear regressions; see White (2004).

[94] This brings up an important point that has not been mentioned until now. Panel data on the selling prices of properties and on the characteristics of the properties are subject to tremendous variations in the ratio of the highest price property to the lowest price property, to the largest lot size to the smallest lot size, to the largest floor space area to the smallest floor space area, and so on. The observations that appear in the tails of the distribution of prices and in the distributions of property characteristics are inevitably sparse and subject to measurement error. Thus in order to obtain sensible estimates in running these hedonic regressions, it is typically necessary to delete the observations that are in the tails of these distributions.

to capture this nonlinearity, divide up the total number of observations into K groups of observations based on their lot size. The Group 1 properties have lot size less than L_1 *square meter*; the Group 2 properties L_{tn} have lot sizes that satisfy the inequalities $L_1 \leq L_{tn} < L_2$; the Group 3 properties L_{tn} have lot sizes that satisfy the inequalities $L_2 \leq L_{tn} < L_3$; . . . ; the Group K properties L_{tn} have lot sizes that satisfy the inequalities $L_{K-1} \leq L_{tn}$. The break points $L_1 < L_2 < \ldots < L_{K-1}$ should be chosen so that the sample probability that any property in the sample will fall into any one of the groups is approximately equal. For each observation n in period t, the K *land dummy variables*, $D_{L,tn,k}$, for $k = 1, \ldots, K$ are defined as follows:

$$D_{L,tn,}k \equiv 1 \textit{ if} \text{ observation } tn \text{ has land area that belongs to group } k;$$
$$\equiv 0 \text{ if observation } tn \text{ has land area that does not belong to group } k. \quad (57)$$

These dummy variables are used in the definition of the following piecewise linear function of L_{tn}, $f_L(L_{tn})$:

$$f_L(L_{tn}) \equiv D_{L,tn,1}\lambda_1 L_{tn} + D_{L,tn,2}[\lambda_1 L_1 + \lambda_2(L_{tn} - L_1)] + D_{L,tn,3}[\lambda_1 L_1 + \lambda_2(L_2 - L_1) + \lambda_3(L_{tn} - L_2)] + \ldots + D_{L,tn,}K[\lambda_1 L_1 + \lambda_2(L_2 - L_1) + \ldots + \lambda_K(L_{tn} - L_{K-1})], \quad (58)$$

where λ_k are unknown parameters. The function $f_L(L_{tn})$ defines a *relative valuation function for the land area of a house* as a function of the plot area, L_{tn}. The new nonlinear regression model is the following one:

$$V_{tn} = \alpha_t(\textstyle\sum_{j=1}^{J} \omega_j D_{PC,tn,j}) f_L(L_{tn}) + p_{St}(1 - \delta)^{A(t,n)} S_{tn} + \varepsilon_{tn}; \quad t = 1, \ldots, T;\ n = 1, \ldots, N(t). \quad (59)$$

Comparing the models defined by equations (55) and (59), it can be seen that we have added an additional K *land plot size parameters*, $\lambda_1, \ldots, \lambda_K$, to the model defined by (55). However, looking at (59), it can be seen that the T land time parameters (α_t), the J postal code parameters (ω_j), and the K land plot size parameters (λ_k) all cannot be identified. Thus, the following identification normalizations on the parameters for Model 3 defined by (59) and (60) are imposed:

$$\omega_1 \equiv 1;\ \lambda_1 \equiv 1. \quad (60)$$

Note that if all of the λ_k parameters are set equal to unity, Model 3 collapses to Model 2. Typically, the log likelihood for Model 3 will be considerably higher than that for Model 2.[95] Land prices as functions of lot size do not always decline monotonically, but for very large land plots, the marginal price of an extra square foot of land is typically quite low.

The next model is similar to Model 3 except that now the marginal price of adding an extra amount of structure is allowed to vary as the size of the structure increases. It is likely that the quality of the structure increases as the size of the structure increases. In order to capture this nonlinearity, divide up the sample observations into M groups of observations based on their structure size. The Group 1 properties have structures with floor space area S_{tn} less than S_1 *square meter*, the Group 2 properties have structure areas S_{tn} satisfying the inequalities $S_1 \leq S_{tn} < S_2, \ldots$, the Group $M - 1$ properties have structure areas S_{tn} satisfying the inequalities $S_{M-2} \leq S_{tn} < S_{M-1}$, and the Group M properties have structure areas S_{tn} satisfying the inequalities $S_{M-1} \leq S_{tn}$, where the $M - 1$ break points satisfy the inequalities $S_1 < S_2 < \ldots < S_{M-1}$. Again, the break points should be chosen so that the sample probability that any property in the sample will fall into any one of the groups is approximately equal. For each observation n in period t, we define the M *structure dummy variables*, $D_{S,tn,m}$, for $m = 1, \ldots, M$ as follows:

$$D_{S,tn,}m \equiv 1 \text{ if observation } tn \text{ has structure area that belongs to structure group } m;$$
$$\equiv 0 \text{ if observation } tn \text{ has structure area that does not belong to group } m. \quad (61)$$

These dummy variables are used in the definition of the following piecewise linear function of S_{tn}, $g_S(S_{tn})$:

$$g_S(S_{tn}) \equiv D_{S,tn,1}\mu_1 S_{tn} + D_{S,tn,2}[\mu_1 S_1 + \mu_2(S_{tn} - S_1)] + D_{S,tn,3}[\mu_1 S_1 + \mu_2(S_2 - S_1) + \mu_3(S_{tn} - S_2)] + D_{S,tn,4}[\mu_1 S_1 + \mu_2(S_2 - S_1) + \mu_3(S_3 - S_2) + \mu_4(S_{tn} - S_3)] + \ldots + D_{S,tn,}M[\mu_1 S_1 + \mu_2(S_2 - S_1) + \mu_3(S_3 - S_2) + \ldots + \mu_M(S_{tn} - S_{M-1})], \quad (62)$$

where the μ_m are unknown parameters. The function $g_S(S_{tn})$ defines a *relative valuation function for the structure area of a house* as a function of the structure area.

The new nonlinear regression model is the following Model 4:

$$V_{tn} = \alpha_t(\textstyle\sum_{j=1}^{J} \omega_j D_{PC,tn,j}) f_L(L_{tn}) + p_{St}(1 - \delta)^{A(t,n)} g_S(S_{tn}) + \varepsilon_{tn}; \quad t = 1, \ldots, T;\ n = 1, \ldots, N(t). \quad (63)$$

Comparing the models defined by equations (59) and (63), it can be seen that additional M *structure floor space parameters*, $\mu_1, \ldots, \mu_M$, have been added to the model defined by (59).[96] Again, we add the normalizations (60) in order to identify all of the parameters in the model. Note that if all of the μ_m parameters are set equal to unity, Model 4 collapses down to Model 3. Typically, the log likelihood for Model 4 will be considerably higher than that for Model 3.[97]

[95] For the example in Diewert, Huang, and Burnett-Issacs (2017) where the models described in this section were estimated, the log likelihood increased by 1762 log likelihood points and the R^2 value jumped from 0.7662 for Model 2 to 0.8283 for Model 3 for the addition of six new λ_k parameters.

[96] At this stage of the sequential estimation procedure, it is usually not necessary to impose a normalization on the parameters μ_1-μ_M. This lack of a normalization means that the scale of the exogenous structure price levels p_{St} is allowed to change; that is, essentially, allowance is now made to quality adjust the exogenous index to a certain extent. However, if the resulting estimated structure values turn out to be unreasonably large or small, then it will be necessary to set one of the μ_m equal to 1.

[97] For the example in Diewert, Huang, and Burnett-Issacs (2017), the log likelihood increased by 935 log likelihood points and the R^2 value jumped from 0.8283 for Model 3 to 0.8520 for Model 4 for the addition of five new μ_m parameters.

At this stage, it is often the case that an acceptable model has been estimated. How can the estimated parameters from the final model be used in order to form price and quantity indices?

The sequence of price levels for the land component of residential property sales is defined as $\alpha_1, \alpha_2, \ldots, \alpha_T$, and the corresponding sequence of price levels for the structure component of residential property sales in the T periods is defined as the exogenous sequence of indices, $p_{S1}, p_{S2}, \ldots, p_{ST}$. The land and structure values of properties transacted in period t, V_{Lt} and V_{St}, are defined by using the estimated land and structure-additive components of transacted properties in period t, $\alpha_t(\Sigma_{j=1}{}^J \omega_j D_{PC,tn,j}) f_L(L_{tn})$ and $p_{St}(1 - \delta)^{A(t,n)}\, g_S(S_{tn})$, respectively, and summing over properties that were sold in period t, we have

$$V_{Lt} \equiv \Sigma_{n\in} N_{(t)}\ \alpha_t(\Sigma_{j=1}{}^J \omega_j D_{PC,tn,j}) f_L(L_{tn});\ t = 1, \ldots, T; \quad (64)$$

$$V_{St} \equiv \Sigma_{n\in} N_{(t)}\ p_{St}(1 - \delta)^{A(t,n)}\, g_S(S_{tn});\ t = 1, \ldots, T. \quad (65)$$

Using the prices $\alpha_1, \alpha_2, \ldots, \alpha_T$, the corresponding estimated land values $V_{L1}, \ldots, V_{LT}$, the prices $p_{S1}, p_{S2}, \ldots, p_{ST}$, and the corresponding estimated structure values $V_{S1}, \ldots, V_{ST}$, one can just apply normal index number theory using these data to construct Laspeyres, Paasche, Fisher, or whatever index formula is being used by the statistical agency in order to construct constant quality price and quantity overall property indices for the sales of residential properties in the area under consideration for the T periods.

However, constant quality land and structure price indices for *sales* of owner-occupied residential houses are not what are needed for most purposes; what is required are constant quality price and quantity indices for the *stock* of residential houses. In order to accomplish this task, it is necessary to have a census of the housing stock in the country, which would include information on the characteristics that are used in the hedonic regression model that is defined by (63). The information that is required in order to estimate (63) is as follows:

- The selling price of the residential properties (P_{tn})
- The age of the structure on the property (A_{tn})
- The area of the land plot (L_{tn})
- The floor space area of the structure (S_{tn})
- The neighborhood of the property (or the postal code)
- The exogenous structure price index that provides the construction cost of a new structure per meter square or per square foot (p_{St})

If a national housing census has information on these property characteristics (excluding the information on selling prices P_{tn} and on the exogenous structure price index p_{St}),[98] then it will be possible to insert the characteristics of each residential dwelling unit into the right-hand side of (63), and then using appropriate modifications of definitions (64) and (65), it will be possible to obtain estimates for the land and structure value for each dwelling unit in the area covered by the regression. If there is no national housing census information or the required characteristics are not included in the census, then it will be very difficult to form estimates for the value of residential land.

Additional information on house and property characteristics will lead to more accurate land and structure decompositions of property value. Examples of useful additional structure price determination characteristics are (i) the number of bathrooms, (ii) the number of bedrooms, (iii) the type of construction material, (iv) the number of stories, and so on. Examples of useful additional land price determination characteristics are (i) the distance to the nearest subway station, (ii) the distance to the city core, (iii) the quality of neighborhood schools, (iv) the existence of various neighborhood amenities, and so on. For examples of how these characteristics can be integrated into the builder's model, see Diewert, de Haan, and Hendriks (2011, 2015), Eurostat (2013, 2017), Diewert and Shimizu (2015), and Diewert, Huang, and Burnett-Issacs (2017).[99]

The estimates for the geometric depreciation rate generated by the application of the builder's model are useful for national income accountants because they facilitate the accurate estimation of structure depreciation, which is required for the national accounts. However, the depreciation estimates that are generated by the builder's model are *wear and tear depreciation* estimates that apply to structures that continue in existence over the sample period. The estimated depreciation rate measures (net) depreciation[100] of a structure that has survived from its birth to the period of its sale. However, there is another form of structure depreciation that the estimated depreciation rate misses, namely the loss of residual structure value that results from the *early demolition* of the structure. This problem was noticed and addressed by Hulten and Wykoff (1981a, 377–379; 1981b; 1996). Wear and tear depreciation is often called *deterioration* depreciation, and *demolition* or *early retirement depreciation* is sometimes called *obsolescence* depreciation.[101] Methods for estimating this form of depreciation have been proposed by Hulten and Wykoff as mentioned earlier and by Diewert and Shimizu (2017, 512–516). Both methods require information on the distribution of the ages of retirement for the asset class. The Hulten and Wykoff method absorbs demolition depreciation into the wear and tear depreciation rate, whereas the Diewert and Shimizu method uses the wear and tear depreciation rate that is generated by sales of surviving buildings but adds a separate depreciation rate that is due to early demolition of the structures in the asset class. Both methods

[98] Every country will have a national residential construction deflator because this deflator is required to form estimates of real investment in residential structures. However, this national deflator may not be entirely appropriate for the type of buildings in a particular neighbourhood.

[99] It is also possible to estimate more general models of depreciation using the builder's model; see Diewert and Shimizu (2017) and Diewert, Huang, and Burnett-Issacs (2017).

[100] It is a net estimate since renovation and replacement investments in the building tend to extend the life of the building or augment its value. Thus, the gross wear and tear depreciation rate for the structure will tend to be larger than the estimated net depreciation rate.

[101] Crosby, Devaney, and Law (2012, 230) distinguish the two types of depreciation and in addition, they provide a comprehensive survey of the depreciation literature as it applies to commercial properties.

require information on the age of structures when they are demolished.[102]

The previous paragraph simply warns the reader that wear and tear depreciation[103] for surviving buildings is not the entire depreciation story: there is also a loss of asset value that results from the early retirement of a building that needs to be taken into account when constructing national income accounting estimates of depreciation.

There is one additional complication that needs to be taken into account when running a hedonic regression on the sales of houses; that is, what happens when the sales information for an additional period becomes available? The simplest way of dealing with this problem dates back to Court (1939). His method works as follows: set $T = 2$ and run a hedonic regression that has a time dummy variable in it. In the context of the hedonic regression model defined by (63), estimates for the price of land for periods 1 and 2 would be obtained, say α_1^1 and α_2^1. The price index for land for periods 1 and 2 is defined as $P_L^1 = 1$ and $P_L^2 = \alpha_2^1/\alpha_1^1$. Now run a new hedonic regression using (63) for $t = 2,3$ and obtain new estimates for the price of land in periods 2 and 3, say α_2^2 and α_3^2. The price index for land in period 3 is defined as $P_L^3 = P_L^2(a_3^2/\alpha_2^2)$; that is, we update the price index value for period 2, P_L^2, by the rate of change in land prices going from period 2 to 3, (α_3^2/α_2^2). Thus, the previously estimated index is updated each period as new information becomes available. This *adjacent period time dummy model* has the advantage that it does not revise the previously estimated indices as the new information becomes available.[104]

The preceding method does not always work well in the context of estimating property price indices due to the sparseness of sales in a neighborhood and the multiplicity of parameters that are required to adequately control for differences in housing characteristics. Thus, Shimizu, Nishimura, and Watanabe (2010a, 797) suggested extending the number of periods from two to a longer window of T consecutive periods, leading to the *rolling window time dummy hedonic regression model*. Thus, for the model defined by (63), the land price parameters that are estimated by the first regression using the data for periods 1 to T are $\alpha_1^1, \alpha_2^1, \ldots, \alpha_T^1$, and the corresponding land price indices for periods 1 to t are $P_L^t \equiv a_t^1/\alpha_1^1$ for $t = 1, \ldots, T$. The second hedonic regression uses the data for periods 2, 3, . . . , T, $T + 1$ and the estimated land price parameters are $\alpha_2^2, \alpha_3^2, \ldots, \alpha_T^2, \alpha_{T+1}^2$. The price index for land in period $T + 1$ is defined as $P_L^{T+1} = P_L^T(\alpha_{T+1}^2/\alpha_T^2)$; that is, the price index for period T, P_L^t, is updated by the rate of change in land prices going from period T to $T + 1$, $\alpha_{T+1}^2/\alpha_T^2$.

There are two additional issues that need to be addressed when using a rolling window time dummy hedonic regression model:

- How long should the window length be? A longer window length will usually lead to more stable estimates for the unknown parameters in the hedonic regression. A shorter window length will allow for taste changes to take place more quickly. A window length of one year plus one period will allow for seasonal effects. At this stage of our knowledge, it is difficult to give definitive advice on the length of the window.
- When a new window is computed, how should the index results from the new window be linked to the previous index values? The same issue applies when a multilateral method is used in the time series context. Ivancic, Diewert, and Fox (2011) along with Shimizu, Nishimura, and Watanabe (2010a) and Shimizu et al. (2010) suggested that the movement of the indices for the last two periods in the new window be linked to the last index value generated by the previous window. However, Krsinich (2016) suggested that the movement of the indices generated by the new window over the entire new window period be linked to the window index value for the second period in the previous window. Krsinich called this a *window splice* as opposed to the *movement splice* explained earlier. De Haan (2015, 27) suggested that perhaps the linking period should be in the middle of the old window, which the Australian Bureau of Statistics (2016, 12) termed a *half splice*. Ivancic, Diewert, and Fox (2011, 33) suggested that the average of all possible links of the new window to the old window be used, and they called this a *mean splice* method for linking the results of the new window to the previous window.[105] Again, there is no consensus at this time on which linking method is "best." However, it is likely that all of these linking methods will generate much the same results.

It can be seen that estimating price indices for houses (or detached dwelling units) is not a straightforward task, particularly if one wants separate constant quality indices for the land and structure components of property value.[106] In the following section, it will be seen that it is even more

[102] The Hulten and Wykoff method estimates the age of retirement in a somewhat arbitrary fashion, whereas the Diewert and Shimizu method relies on mortality distributions on the age of buildings at the time they are demolished. Over long periods of time and using country wide data, the two methods should be equivalent. However, the Diewert and Shimizu method should give more accurate results at the firm and regional levels since their method is consistent with the hedonic estimation of structure depreciation rates as explained in this section.

[103] What has been labeled as wear and tear depreciation could be better described as *anticipated amortization of the structure* rather than wear and tear depreciation. Once a structure is built, it becomes a fixed asset which cannot be transferred to alternative uses (like a truck or machine). Thus amortization of the cost of the structure should be proportional to the cash flows or to the service flows of utility that the building generates over its expected lifetime. However, technical progress, obsolescence, or unanticipated market developments can cause the building to be demolished before it is fully amortized. See Diewert and Fox (2016) for a more complete discussion of the fixity problem.

[104] The two-period time dummy variable hedonic regression (and its extension to many periods) was first considered explicitly by Court (1939, 109–111) as his hedonic suggestion number two. Court used adjacent period time dummy hedonic regressions as links in a longer chain of comparisons extending from 1920 to 1939 for US automobiles: "The net regressions on time shown above are in effect price link relatives for cars of constant specifications. By joining these together, a continuous index is secured." If the two periods being compared are consecutive years, Griliches (1971, 7) coined the term "adjacent year regression" to describe this method for updating the index as new information becomes available. Diewert (2005b) looked at the axiomatic properties of adjacent year time dummy hedonic regressions.

[105] For the details on how the mean splice method works, see Diewert and Fox (2020).

[106] For additional hedonic regression models for detached houses, see Verbrugge (2008), Garner and Verbrugge (2011), Eurostat (2013, 2017), Hill (2013), Hill et al. (2018), Rambaldi and Fletcher (2014), and Silver (2018).

complicated to obtain separate indices for the land and structure components for condominium sales.

12. Decomposing Condominium Sales Prices into Land and Structure Components

A starting point for applying the builder's model to condominium sales is the hedonic regression model defined by equation (53) in the previous section.[107] For convenience, equation (53) is repeated as equation (66):

$$V_{tn} = \alpha_t L_{tn} + p_{St}(1 - d)^{A(t,n)}S_{tn} + \varepsilon_{tn};\quad t = 1,\ldots,T;\ n = 1,\ldots,N(t), \qquad (66)$$

where V_{tn} is the selling price of a condominium property in a neighborhood in period t, α_t is the price of the land that the structure sits on (per square meter), L_{tn} is the land area that can be attributed to the condo unit, p_{St} is the exogenous period t construction cost for the type of condo under consideration (per square meter), δ is the one-period wear and tear geometric depreciation rate for the structure, $A_{tn} = A(t,n)$ is the age of the structure in periods, S_{tn} is the floor space of unit n that is sold in period t (in square meters), and ε_{tn} is an error term.

A problem with the preceding model is that it is not appropriate to allocate the entire land value of the condominium property to any particular unit that is sold in period t. Thus, each condo unit in the building should be allocated a *share* of the total land value of the property. The problem is "How exactly should this imputed land share be calculated?" There are two simple methods for constructing an appropriate land share: (i) Use the unit's share of floor space to total structure floor space or (ii) simply use $1/N$ as the share, where N is the total number of units in the building. Thus, define the following two land share imputations for unit n in period t:

$$L_{Stn} \equiv (S_{tn}/TS_{tn})TL_{tn};\ L_{Ntn} \equiv (1/N_{tn})TL_{tn};\quad t = 1,\ldots,T;\ n = 1,\ldots,N(t), \qquad (67)$$

where S_{tn} is the floor space area of unit n that is sold in period t, TS_{tn} is the total building floor space area, TL_{tn} is the total land area of the building, and N_{tn} is the total number of units in the building for unit n sold in period t. The first method of land share imputation is used by the Japanese land tax authorities. The second method of imputation implicitly assumes that each unit can enjoy the use of the entire land area, and so an equal share of land for each unit seems "fair."

There is a problem with the definition of L_{Stn} in (67): The floor space "share" of unit n, S_{tn}/TS_{tn}, if summed over all units in the building would be less than 1 because the privately held floor space of each unit in the building does not account for shared building floor spaces such as halls, elevators, storage spaces, furnace rooms, and other "public" floor spaces, which are included in total building floor space, TS_{tn}. Thus, the "share" S_{tn}/TS_{tn} must be adjusted upward by some percentage to account for these shared building facilities.[108] In what follows, it is assumed that this adjustment has been made to S_{tn} (so that S_{tn} is now interpreted as adjusted condo floor space area).

In order to obtain sensible decompositions of the condominium selling price into land and structure components, it may be necessary to assume a structure value and focus on the determinants of land value at the initial stages of the sequential estimation procedure. Thus, following Diewert and Shimizu (2016), assume that the *imputed structure value* for unit n in period t, V_{Stn}, is defined as follows:

$$V_{Stn} \equiv p_{St}(1 - \delta)^{A(t,n)}S_{tn};\ t = 1,\ldots,T;\ n = 1,\ldots,N(t), \qquad (68)$$

where δ is an assumed geometric depreciation rate.[109] Once the imputed value of the structure has been defined by (68), the *imputed land value* for condo n in period t, V_{Ltn}, is defined by subtracting the imputed structure value from the total value of the condo unit, which is V_{tn}:

$$V_{Ltn} \equiv V_{tn} - V_{Stn};\ t = 1,\ldots,T;\ n = 1,\ldots,N(t). \qquad (69)$$

In the hedonic regressions that follow immediately, the imputed value of land for the condominium unit, V_{Ltn}, is used as the dependent variable in a hedonic regression. The following regressions explain variations in these imputed land values in terms of the property characteristics.

Suppose that the postal code of each sale is also available, and there are J postal codes. Then one can introduce the following *postal code dummy variables*, $D_{PC,tn,j}$, as explanatory variables into a hedonic regression. Define these J dummy variables using definitions (54) in the previous section, and estimate the following hedonic regression, which is a *land counterpart* to the hedonic regression defined by (55) in the previous section:

$$V_{Ltn} = \alpha_t(\textstyle\sum_{j=1}^{J} \omega_j D_{PC,tn,j})L_{Stn} + \varepsilon_{tn};\quad t = 1,\ldots,T;\ n = 1,\ldots,N(t). \qquad (70)$$

Note that the imputed value of land, V_{Ltn} defined by (69), replaces the total property value V_{tn}, which was the dependent variable in (55).[110]

[107] The analysis in this section follows that of Diewert and Shimizu (2016) and Burnett-Issacs, Huang, and Diewert (2021).

[108] Diewert and Shimizu (2016, 303) constructed estimates of Tokyo total building private floor space to total building floor space for each observation nt as $N_{tn}S_{tn}/TS_{tn}$, where N_{tn} is the number of units in the building which contained condo sale n in period t, S_{tn} is the private floor space of the sold unit, and TS_{tn} is the total floor space of the building. The sample wide average of these ratios was 0.899. Thus the first imputation method in definitions (67) was changed from $L_{Stn} \equiv (S_{tn}/TS_{tn})TL_{tn}$ to $L_{Stn} \equiv (1/0.899)(S_{tn}/TS_{tn})TL_{tn} = (1.1)(S_{tn}/TS_{tn})TL_{tn}$. Burnett-Issacs, Huang, and Diewert (2021) estimated a similar condo model and consulted with construction experts and determined that on average, the ratio of total space to private space for Ottawa condominium apartments was approximately 1.33. Thus they changed $L_{Stn} \equiv (S_{tn}/TS_{tn})TL_{tn}$ to $L_{Stn} \equiv (1.33)(S_{tn}/TS_{tn})TL_{tn}$.

[109] Diewert and Shimizu (2016) assumed δ = 0.03 and Burnett-Issacs, Huang, and Diewert (2021) assumed δ = 0.02 where the age variable A_{tn} is measured in years. Later, δ will be estimated.

[110] As usual, we need a normalization on the parameters such as $\alpha_1 = 1$ in order to identify all of the remaining parameters, $\alpha_2,\ldots,\alpha_T, \omega_1,\ldots,\omega_J$. Note that this regression uses the first method of land imputation defined by (67). Later, the second method will also be considered.

It is likely that the height of the building (number of stories) increases the value of the land plot supporting the building, all else equal. Thus, define the number of stories dummy variables, $D_{NS,tn,}s$, as follows: $t = 1, \ldots, T$; $n = 1, \ldots, N(t)$; $s = 1, \ldots, NS$:

$$D_{NS,tn,}s \equiv 1 \text{ if observation } n \text{ in period } t \text{ is in a building with } s \text{ stories;} \\ \equiv 0 \text{ if observation } n \text{ in period } t \text{ is not in building with } s \text{ stories.} \quad (71)$$

The new nonlinear regression model is as follows:

$$V_{Ltn} = \alpha_t(\Sigma_{j=1}^{J} \omega_j D_{PC,tn,j})(\Sigma_{s=1}^{NS} \chi_s D_{NS,tn,}s)L_{Stn} + \varepsilon_{tn}; \\ t = 1, \ldots, T; n = 1, \ldots, N(t). \quad (72)$$

Comparing the models defined by equations (70) and (72), it can be seen that additional NS *building height parameters*, $\chi_1, \ldots, \chi_{NS}$, have been added to the model defined by (70).[111] As usual, the models defined by (70) and (72) are nested so that the finishing parameter values from the nonlinear regression (70) can be used as starting values for (72) along with the starting values $\chi_1 = \chi_2 = \ldots = \chi_{NS} = 1$.

The higher up a unit is, the better is the view on average, and so it could be expected that the price of the unit increases as its height increases. The quality of the structure probably does not increase as the height of the unit increases, so it seems reasonable to impute the height premium as an adjustment to the land price component of the unit.

It is possible to introduce the height of the unit (the H variable) as a categorical variable (like the number of stories NS in the last hedonic regression model). However, both Diewert and Shimizu (2016) (hereafter DS) and Burnett-Issacs, Huang, and Diewert (2021) (hereafter BHD) found that this dummy variable approach could be replaced by using H as a continuous variable with little change in the fit of the model. Thus, the new nonlinear regression model is the following one, where $t = 1, \ldots, T$ and $n = 1, \ldots, N(t)$:

$$V_{Ltn} = \alpha_t(\Sigma_{j=1}^{J} \omega_j D_{PC,tn,j})(\Sigma_{s=1}^{NS} \chi_s D_{NS,tn,s}) \\ (1 + \gamma(H_{tn} - 3))L_{Stn} + \varepsilon_{tn}, \quad (73)$$

where H_{tn} is the height of the sold unit n in period t (measured as the number of stories from the ground level) and γ is a height of the unit parameter to be estimated.[112] The preceding model assumes that the lowest height for the units sold in the sample was $H_{tn} = 3$. Thus, for all the observations that correspond to the sold unit being located on the third floor of the building, the new parameter g in (73) will not affect the predicted value in the regression. However, for heights of the sold units that were greater than 3, the regression implies that the land value will increase by γ for each story that is above 3.[113]

As was mentioned earlier, there are two simple methods for imputing the share of the building's total land area to the sold unit. Up until now, we have used the first method of imputation defined by (67), which set the share of total land imputed to unit n in period t, L_{Stn}, equal to $(S_{tn}/TS_{tn})TL_{tn}$, whereas the second method set L_{Ntn} equal to $(1/N_{tn})TL_{tn}$. In the next model, the land imputation for unit n in period t is set equal to a *weighted average* of the two imputation methods, and the best-fitting weight, λ, is estimated. Thus, define

$$L_{tn}(\lambda) = [\lambda(S_{tn}/TS_{tn}) + (1 - \lambda)(1/N_{tn})]TL_{tn}; \\ t = 1, \ldots, T; n = 1, \ldots, N(t). \quad (74)$$

The new nonlinear regression model is the following one, where $t = 1, \ldots, T$ and $n = 1, \ldots, N(t)$, and $L_{tn}(\lambda)$ is defined by (74):[114]

$$V_{Ltn} = \alpha_t(\Sigma_{j=1}^{J} \omega_j D_{PC,tn,j})(\Sigma_{s=1}^{NS} \chi_s D_{NS,tn,s}) \\ (1 + \gamma(H_{tn} - 3))L_{tn}(\lambda) + \varepsilon_{tn}. \quad (75)$$

Conditional on the land area of the building, one would expect the sold unit's land imputation value to increase as the number of units in the building increases. Thus, one could use the total number of units in the building, N_{tn}, as a quality adjustment variable for the imputed land value of a condo unit. DS introduced this variable as a continuous variable. The smallest number of units in the buildings in their sample was 11. Thus, they introduced the term $1 + \kappa(N_{tn} - 11)$ as an explanatory term in the nonlinear regression. The new parameter κ is the percentage increase in the unit's imputed value of land as the number of units in the building grows by one unit. The new nonlinear regression model is the following one, where $t = 1, \ldots, T$ and $n = 1, \ldots, N(t)$, and $L_{tn}(\lambda)$ is defined by (74):

$$V_{Ltn} = \alpha_t(\Sigma_{j=1}^{J} \omega_j D_{PC,tn,j})(\Sigma_{s=1}^{NS} \chi_s D_{NS,tn,s}) \\ (1 + \gamma(H_{tn} - 3))(1 + \kappa(N_{tn} - 11))L_{tn}(\lambda) + \varepsilon_{tn}, \quad (76)$$

where $L_{tn}(\lambda)$ is defined by (74).

The next explanatory variable to be introduced into the hedonic regression model is one that is not obvious but turned out to be very significant in the regressions run by DS and BHD. The *footprint* of a building is the area of the land that directly supports the structure. An approximation to the footprint land for unit n in period t is the total structure area TS_{tn} divided by the total number of stories in the structure TH_{tn}. If footprint land is subtracted from the total

[111] Again normalizations like $\alpha_1 \equiv 1$; $\chi_1 \equiv 1$ are required in order to identify the remaining parameters. If all $\chi_s = 1$, then the model defined by (72) collapses to the model defined by (70).

[112] Normalizations like $\alpha_1 \equiv 1$; $\chi_1 \equiv 1$ need to be imposed in order to identify the remaining parameters.

[113] The studies that have implemented this model found that the estimated γ was in the 2–4 percent range. Thus the imputed land value of a unit increases by 2 to 4 percent for each story above the threshold level of 3.

[114] For the DS Tokyo condo data, the estimated λ turned out to be $\lambda^* = 0.3636$ ($t = 9.84$) so that the very simple land imputation method that just divided the total land plot size by the number of units in the building got a higher weight (0.6364) than the weight for the floor space allocation method (0.3636). For the Ottawa condo data, the estimated λ turned out to be $\lambda^* = 0.2525$ ($t = 12.10$).

land area, TL_{tn}, the resulting variable is *excess land*,[115] EL_{tn}, defined as follows:

$$EL_{tn} \equiv TL_{tn} - (TS_{tn}/TH_{tn});\ t = 1,\ldots,T;\ n = 1,\ldots,N(t). \quad (77)$$

In the Tokyo data used by DS, excess land ranged from 47 square meters to 2912 square meters. Now group the sample observations into M categories, depending on the amount of excess land that pertained to each observation. Group 1 consists of observations tn, where EL_{tn} is less than some number EL_1; Group 2 consists of observations such that $EL_1 \leq EL_{tn} < EL_2$; . . . ; Group M consists of observations such that $EL_{M-1} \leq EL_{tn}$. The break points $EL_1, EL_2, \ldots, EL_{M-1}$ should be chosen so that the number of observations in each group is approximately equal. Define the *excess land dummy variables*, $D_{EL,tn,m}$, as follows for $t = 1,\ldots,T$; $n = 1,\ldots,N(t)$; $m = 1,\ldots,M$:

$$\begin{aligned} D_{EL,tn,}m &\equiv 1 \text{ if observation } n \text{ in period } t \text{ is in excess land group } m; \\ &\equiv 0 \text{ if observation } n \text{ in period } t \text{ is not in excess land group } m. \end{aligned} \quad (78)$$

The new regression model is as follows:

$$\begin{aligned} V_{Ltn} = {} & \alpha_t(\Sigma_{j=1}{}^{J}\,\omega_j D_{PC,tn,j})(\Sigma_{s=1}{}^{NS}\,\chi_s D_{NS,tn,s})(\Sigma_{m=1}{}^{M}\,\mu_m D_{EL,tn,m})\times \\ & (1 + \gamma(H_{tn} - 3))(1 + \kappa(N_{tn} - 11))L_{tn}(\lambda) + \varepsilon_{tn}; \\ & t = 1,\ldots,T;\ n = 1,\ldots,N(t). \end{aligned} \quad (79)$$

Not all of the parameters in (79) can be identified, so the following normalizations on the parameters in (79) are imposed:

$$\alpha_1 \equiv 1;\ \chi_1 \equiv 1;\ \mu_1 \equiv 1. \quad (80)$$

Introducing the excess land dummy variables led to huge jumps in the log likelihoods for the hedonic regressions run by DS and BHS: 1020 for DS and 2652 for BHS.[116] Both studies found that the estimated μ_m were positive but their magnitudes decreased monotonically as the excess land variable increased.

There are three additional explanatory variables that were used by DS that may affect the price of land. Define TW as the walking time in minutes to the nearest subway station, TT as the subway running time in minutes to the Central Tokyo station from the nearest station, and the SOUTH dummy variable is set equal to 1 if the sold condo unit faces south and 0 otherwise. Let $D_{S,tn,2}$ equal the SOUTH dummy variable for sale n in period t. Define $D_{S,tn,2} = 1 - D_{S,tn,1}$. In the Tokyo data set used by DS, TW ranged from 1 to 19 minutes and TT ranged from 12 to 48 minutes. These new variables are inserted into the previous nonlinear regression model (79) in the following manner for $t = 1,\ldots,T$ and $n = 1,\ldots,N(t)$:

$$\begin{aligned} V_{Ltn} = {} & \alpha_t(\Sigma_{j=1}{}^{J}\,\omega_j D_{PC,tn,j})(\Sigma_{s=1}{}^{NS}\,\chi_s D_{NS,tn,s}) \\ & (\Sigma_{m=1}{}^{M}\,\mu_m D_{EL,tn,}m)(\varphi_1 D_{S,tn,1} + \varphi_2 D_{S,tn,2}) \times \\ & (1 + \gamma(H_{tn} - 3))(1 + \kappa(N_{tn} - 11))(1 + \eta(TW_{tn} - 1)) \\ & (1 + \theta(TT_{tn} - 12))L_{tn}(\lambda) + \varepsilon_{tn}, \end{aligned} \quad (81)$$

where $L_{tn}(\lambda)$ is defined by (74). Not all of the parameters in (81) can be identified, so the following normalizations (82) are imposed on the parameters in (81):

$$\alpha_1 \equiv 1;\ \chi_1 \equiv 1;\ \mu_1 \equiv 1;\ \varphi_1 \equiv 1. \quad (82)$$

Using the DS Tokyo data, the R^2 value for this model was found to be 0.6308 and the log likelihood increased by 406 points over the log likelihood of the previous model defined by (79) for the addition of three new parameters. The estimated parameters had the expected signs and had reasonable magnitudes.

At this point, DS concluded that the imputed land value for each condominium in their sample was predicted reasonably well by the hedonic regression model defined by (81) and (82). Thus, in the following regression, they switched from using the imputed land value V_{Ltn} defined by (69) as the dependent variable in the regressions to using the actual selling price of the property, V_{tn}. They used the specification for the land component of the property that is defined by (81) and (82), but they also added the structure term $p_{St}(1 - \delta)^{A(t,n)}S_{tn}$ to account for the structure component of the value of the condo unit. Note that the annual depreciation rate d is now estimated by the new hedonic regression model rather than assuming that it was equal to 3 percent. Thus, the number of unknown parameters in the new model increased by 1. They used the estimated values for the coefficients in (81) as starting values in this new nonlinear regression.[117]

Using their Tokyo data, DS found that R^2 for this new model was 0.8190 and the estimated depreciation rate was $\delta^* = 0.0367$ ($t = 27.1$). Note that R^2 is satisfactory; that is, the new model explains a substantial fraction of the variation in condo prices.

DS and BHD introduced some additional explanatory variables as quality-adjusting variables for the imputed value of structures. DS introduced the number of bedrooms and the type of building as quality adjusters for the value of the structure. BHD introduced the number of bedrooms, the number of bathrooms, the presence of balconies, the use of natural gas as the heating fuel, and whether there was commercial space in the building as additional variables that could determine the value of the structure. These variables were significant explanatory variables, but the overall R^2 for the final hedonic regression did not increase by

[115] This is land that is usable for purposes *other* than the direct support of the structure on the land plot.

[116] Recall the hedonic regression model defined by (59) in the previous section which introduced linear splines on the valuation of the land area of a stand-alone housing unit. This introduction also greatly increased the log likelihood of the regression. In the present context, the excess land dummy variables take the place of the linear spline functions in (59).

[117] Attempting to estimate the parameters in (83) without good starting values for the nonlinear regression will not lead to sensible parameter estimates. Thus it is necessary to obtain good starting values for (83) by estimating the rather long sequence of regressions explained above, starting with a very simple model and gradually introducing additional explanatory variables. Each regression in the sequence contains the previous one as a special case so that the final estimates of one regression can be used as starting values for the subsequent one.

a large amount with the addition of these variables to the regression. The details may be found in the work of Diewert and Shimizu (2016) and Burnett-Issacs, Huang, and Diewert (2021).

Once the final hedonic regression has been run, the sequence of land prices is given by $\alpha_1, \alpha_2, \ldots, \alpha_T$ and the sequence of condo structure prices is given by the exogenous structure price indices $p_{S1}, p_{S2}, \ldots, p_{ST}$. To obtain the overall property price indices for sales of condos, form the following counterparts to equations (64) and (65) in the previous section to obtain an estimate of period t condo land value, V_{Lt}, and estimated period t structure value, V_{St}, for $t = 1, \ldots, T$:

$$V_{Lt} \equiv \Sigma_{n\in} N_{(t)}\ \alpha_t(\Sigma_{j=1}{}^{J}\ \omega_j D_{PC,tn,j})(\Sigma_{s=1}{}^{NS}\ \chi_s D_{NS,tn,s})(\Sigma_{m=1}{}^{M}\ \mu_m D_{EL,tn,}m) \quad (83)$$

$$\times(\varphi_1 D_{S,tn,1} + \varphi_2 D_{S,tn,2})\ (1 + \gamma(H_{tn} - 3))(1 + \kappa(N_{tn} - 11))(1 + \eta(TW_{tn} - 1))(1 + \theta(TT_{tn} - 12))L_{tn}(\lambda);$$

$$V_{St} \equiv \Sigma_{n\in} N_{(t)}\ p_{St}(1 - \delta)^{A(t,n)} S_{tn}. \quad (84)$$

Using the prices $\alpha_1, \alpha_2, \ldots, \alpha_T$, the corresponding estimated land values $V_{L1}, \ldots, V_{LT}$, the prices $p_{S1}, p_{S2}, \ldots, p_{ST}$, and the corresponding estimated structure values $V_{S1}, \ldots, V_{ST}$, one can again apply normal index number theory using these data to construct Laspeyres, Paasche, Fisher, or whatever index formula is being used by the statistical agency in order to construct constant quality price and quantity overall property indices for the sales of condominium units in the area under consideration for T periods.

In summary, the builder's model can be modified to apply to the sales of condominium units, and reasonable decompositions of property value into land and structure components can be obtained. However, the nonlinear regressions that are required in order to implement the model end up being rather complex. In addition, information on more characteristics of the condominium properties needs to be collected in order to implement the models. The information that is required in order to estimate the final model and calculate (83) and (84) is as follows:

- The selling prices of the condominium properties in the sample (P_{tn})
- The age of the structure on the property (A_{tn})
- The total area of the land plot (TL_{tn})
- The floor space area of the condo unit (S_{tn})
- The total floor space area of the entire building (TS_{tn})
- The neighborhood of the property (or the postal code)
- The exogenous structure price index which provides the construction cost of a new structure per meter square or per square foot (p_{St})
- The number of stories of the building (NS_{tn})
- The height of the sold unit (the number of stories from the ground level) (H_{tn})
- The number of units in the building (N_{tn})
- The walking time in minutes to the nearest subway station (TW_{tn})
- The subway running time in minutes to the city center from the nearest station (TT_{tn})

The last two variables are not essential (and are not relevant in small towns and cities). Other non-essential variables that could be useful are the number of bedrooms, the number of bathrooms, the existence of balconies, the type of construction, the number of parking spaces, and so on.

The hedonic regression models that were considered in the last two sections are essentially modified supply-side models. In the following section, demand-side hedonic regressions are considered.

13. Demand-Side Property Price Hedonic Regressions

A way of rationalizing the traditional log price time dummy hedonic regression model for properties with varying amounts of land area L and constant quality structure area S^* is that the utility that these properties yield to consumers is proportional to the Cobb–Douglas utility function $L^{\alpha}S^{*\beta}$, where α and β are positive parameters (which do not necessarily sum to one).[118] Initially, assume that the constant quality structure area S^* is equal to the floor space area of the structure, S, times an age adjustment, $(1 - \delta)^A$, where A is the age of the structure in years and δ is a positive depreciation rate that is less than 1. Thus, S^* is related to S as follows:

$$S^* \equiv S(1 - \delta)^A. \quad (85)$$

In any given time period t, assume that the sale price of transacted property n, V_{tn}, with the amount of land L_{tn} and the amount of quality-adjusted structure $S_{tn}{}^*$, is given by the following expression:

$$\begin{aligned} V_{tn} &= p_t L_{tn}{}^{\alpha}[S_{tn}{}^*]^{\beta} \\ &= p_t L_{tn}{}^{\alpha}\ [S_{tn}(1 - d)^{A(t,n)}]^{\beta} \quad \text{using (85)} \\ &= p_t L_{tn}{}^{\alpha}\ S_{tn}{}^{b}(1 - \delta)^{\beta} A^{(t \cdot n)} \\ &= p_t L_{tn}{}^{\alpha}\ S_{tn}{}^{b} \varphi^{A(t,n)}, \end{aligned} \quad (86)$$

where $A(t,n) = A_{tn}$ is the age of house n sold in period t, p_t can be interpreted as the *period t property price index*, and the constant φ is defined as

$$\varphi \equiv (1 - \delta)^{\beta}. \quad (87)$$

Thus, if V_{tn} is deflated by the period t property price index p_t, the real value or utility u_{tn} of the property with characteristics L_{tn} and $S_{tn}{}^*$ is obtained:

$$V_{tn}/p_t = L_{tn}{}^{\alpha} S_{tn}{}^{*\beta} \equiv u_{tn}. \quad (88)$$

[118] The early analysis in this section follows that of McMillen (2003, 289 – 290), Shimizu, Nishimura, and Watanabe (2010a, 795), and Diewert, Huang, and Burnett-Issacs (2017). McMillen assumed that $\alpha + \beta = 1$. We follow Shimizu, Nishimura, and Watanabe in allowing α and β to be unrestricted. Knoll, Schularick, and Steger (2017, 344–345) assumed a Cobb–Douglas production function in order to decompose house prices into land and structure components; that is, they applied a production-side model in their decomposition instead of a demand-side decomposition as will be done in this section.

Thus, $u_{tn} \equiv q_t$ is the *aggregate real value of the property* with characteristics L_{tn} and S_{tn}^{*}.[119]

Define ρ_t as the logarithm of p_t and γ as the logarithm of φ; that is,

$$\rho_t \equiv lnp_t; \gamma \equiv ln\varphi. \tag{89}$$

After taking logarithms of both sides of the first equation in (88), using definitions (85) and (89) and adding the error terms, the following system of estimating equations is obtained:[120]

$$lnV_{tn} = \rho_t + \alpha lnL_{tn} + \beta lnS_{tn} + \gamma A_{tn} + \varepsilon_{tn}; \quad t = 1,\ldots,T; n = 1,\ldots,N(t), \tag{90}$$

where ε_{tn} are the independently distributed error terms with 0 means and constant variances. It can be seen that (90) is a traditional log price time dummy hedonic regression model with a minimal number of characteristics. The unknown parameters in (90) are the constant quality log property prices, $\rho_1, \ldots, r_T$, the taste parameters α, β, and the transformed depreciation rate γ. Once these parameters have been determined, the geometric depreciation rate δ that appears in equations (86) can be recovered from the regression parameter estimates as follows:

$$\delta \equiv 1 - e^{\gamma/\beta}. \tag{91}$$

We now explain how the hedonic pricing model defined by (86) can be manipulated to provide a decomposition of property value in period t into land- and quality-adjusted structure components.

Once estimates for α, β, and δ have been obtained, the defined period t value of a property with characteristics L_{tn} and S_{tn}^{*} is given by the following *period t property valuation function* by the right-hand side of (86); that is, define $V(p_t,L_{tn},S_{tn}^{*}) \equiv p_t L_{tn}^{\alpha} S_{tn}^{*\beta}$. In empirical applications of the hedonic regression model defined by (90), it will often happen that estimates for α and β are such that $\alpha + \beta$ is less than 1.[121] This means that a property in a given period that has double the land- and quality-adjusted structure than another property will sell for less than double the price of the smaller property. This follows from the fact that the Cobb–Douglas hedonic utility function, $u(L,S^{*}) \equiv L^{\alpha}S^{*\beta}$, exhibits diminishing returns to scale when $\alpha + \beta < 1$; that is, we have

$$u(\lambda L,\lambda S^{*}) = \lambda^{\alpha+\beta}u(L,S^{*}) \tag{92}$$

for all $\lambda > 0$. This behavior is roughly consistent with our builder's Models 5–7, where there was a tendency for property prices to increase less than the proportional increase of L and S^{*}.

The *marginal prices of land and constant quality structure* in period t for a property with characteristics L and S^{*}, $\pi_L(p_t,L,S^{*})$ and $\pi_{S*}(p_t,L,S^{*})$, are defined by partially differentiating the property valuation function with respect to L and S^{*}, respectively:

$$\pi_L(p_t,L_{tn},S_{tn}^{*}) \equiv \partial V(p_t,L_{tn},S_{tn}^{*})/\partial L \equiv p_t\, \alpha L_{tn}^{\alpha}S_{tn}^{*\beta}/L_{tn} = aV(p_t,L_{tn},S_{tn}^{*})/L_{tn}; \tag{93}$$

$$\pi_S^{*}(p_t,L_{tn},S_{tn}^{*}) \equiv \partial V(p_t,L_{tn},S_{tn}^{*})/\partial S^{*} \equiv p_t\, \beta L_{tn}^{\alpha}S_{tn}^{*\beta}/S_{tn}^{*} = \beta V(p_t,L_{tn},S_{tn}^{*})/S_{tn}^{*}. \tag{94}$$

Multiply the marginal price of land by the amount of land in the property, and add to this value of land the product of the marginal price of constant quality structure and the amount of constant quality structure on the property in order to obtain the following identity:

$$(\alpha+\beta)V(p_t,L_{tn},S_{tn}^{*}) = \pi_L(p_t,L_{tn},S_{tn}^{*})L_{tn} + \pi_S^{*}(p_t,L_{tn},S_{tn}^{*})S_{tn}^{*}. \tag{95}$$

If $\alpha + \beta$ is less than one, then using marginal prices to value the land and constant quality structure in a property will lead to a property valuation that is less than its selling price. Thus, to make the land and structure components of property value add up to property value, divide the marginal prices defined by (93) and (94) by $\alpha + \beta$ in order to obtain the following *adjusted prices of land and structures for property n sold in period t*, $p_{tL}(p_t,L_{tn},S_{tn}^{*})$ and $p_{tS*}(p_t,L_{tn},S_{tn}^{*})$:

$$p_{tL}(p_t,L_{tn},S_{tn}^{*}) \equiv \pi_L(p_t,L_{tn},S_{tn}^{*})/(\alpha+\beta) = \alpha(\alpha+\beta)^{-1}V(p_t,L_{tn},S_{tn}^{*})/L_{tn}; \tag{96}$$

$$p_{tS*}(p_t,L_{tn},S_{tn}^{*}) \equiv \pi_{S*}(p_t,L_{tn},S_{tn}^{*})/(\alpha+\beta) = \beta(\alpha+\beta)^{-1}V(p_t,L_{tn},S_{tn}^{*})/S_{tn}^{*}. \tag{97}$$

The preceding material outlines a theoretical framework that can generate a decomposition of property value into land and structure components using the results of a traditional log price time dummy hedonic regression model. To complete the analysis, it is necessary to fill in the details of how the individual property land and structure prices that are generated by the model can be aggregated into period t overall land and structure price indices.

Run the hedonic regression model defined by (90). Define the *constant quality property price index* p_t for period t as follows:

$$p_t \equiv \exp(\rho_t); t = 1,\ldots,T. \tag{98}$$

Define the geometric depreciation rate δ by (91). Once δ has been defined, the amount of quality-adjusted structure for property n in period t, S_{tn}^{*}, is defined as follows:

$$ln(S_{tn}^{*}) \equiv ln(S_{tn}) + A_{tn}\, ln(1-\delta); \quad t = 1,\ldots,T; n = 1,\ldots,N(t). \tag{99}$$

[119] For each property n in scope for period t, equations (88) can be rearranged to read as follows: $V_{tn}/u_{tn} = p_t$. Thus, the model assumes that purchasers of the type of property in scope for the sales index have the same property preferences over alternative properties n in period t (with land- and quality-adjusted structure quantities defined by L_{tn} and S_{tn}^{*}) given by the utility function $L_{tn}^{\alpha}S_{tn}^{*\beta}$. Competition between purchasers forces the price of the properties in scope per unit utility to equalize in period t; that is, we obtain the equations $V_{tn}/L_{tn}^{\alpha}S_{tn}^{*\beta} = p_t$. Of course, these assumptions will only be approximately correct so equations (88) will only hold approximately. If the R^2 value obtained for the hedonic regression (90) is low, then the underlying economic model will provide only a poor approximation to reality.

[120] Log price hedonic regressions for property prices date back to Bailey, Muth, and Nourse (1963).

[121] See, for example, the estimated model in Diewert, Huang, and Burnett-Issacs (2017).

Now that p_t, L_{tn}, S_{tn}^*, a and b have all been defined, we use these data in order to define the predicted prices for property n sold in period t, V_{tn}^*:

$$V_{tn}^* \equiv p_t (L_{tn})^{\alpha} (S_{tn}^*)^{\beta}; t = 1, \ldots, T; n = 1, \ldots, N(t). \quad (100)$$

Use equations (96) and (97) in order to define *constant quality land and structure prices* for sold property n in period t, p_{tnL} and p_{tnS*}, as follows:

$$p_{tnL} \equiv \alpha(\alpha + \beta)^{-1} V_{tn}^*/L_{tn}; t = 1, \ldots, T; n = 1, \ldots, N(t); \quad (101)$$
$$p_{tnS*} \equiv \beta(\alpha + \beta)^{-1} V_{tn}^*/S_{tn}^*; t = 1, \ldots, T; n = 1, \ldots, N(t). \quad (102)$$

Finally, *unit value constant quality land and structure prices* for all properties sold in period t, p_{tL} and p_{tS*}, are defined as follows:

$$p_{tL} \equiv \Sigma_{n=1}^{N(t)} p_{tnL} L_{tn}/\Sigma_{n=1}^{N(t)} L_{tn}; t = 1, \ldots, T; \quad (103)$$
$$p_{tS*} \equiv \Sigma_{n=1}^{N(t)} p_{tnS*} S_{tn}^*/\Sigma_{n=1}^{N(t)} S_{tn}^*; t = 1, \ldots, T. \quad (104)$$

The period t land and structure prices that are defined by (103) and (104) are reasonable summary statistic prices for land and structures sold in period t that are generated by the log price time dummy hedonic regression model defined by (90).

If the price of land grows at a different rate than the price of a constant quality structure, then the time dummy log price hedonic regression model defined by (90) will generate very different constant quality land and structure subindices when compared to the corresponding indices estimated by the builder's model. To see this, suppose that the same house n sold in period t is sold again in the following period $t + 1$. The period t data for this house are V_{tn}^*, L_{tn}, and S_{tn}^*, and the period $t + 1$ data are $V_{t+1}n^*$, $L_{t+1}n = L_{tn}$, and $S_{t+1}n^* = (1 - \delta)S_{tn}^*$. Use definitions (101) and (102) for this house for periods t and $t + 1$ and calculate the following land and structure inflation rates for this house going from period t to period $t + 1$:

$$p_{t+1}nL/p_{tnL} = [\alpha(\alpha + \beta)^{-1} V_{t+1}n^*/L_{tn}]/[\alpha(\alpha + \beta)^{-1} V_{tn}^*/L_{tn}]$$
$$= V_{t+1}n^*/V_{tn}^*; \quad (105)$$
$$p_{t+1}nS_*/p_{tnS*} = [\beta(\alpha + \beta)^{-1} V_{t+1}n^*/(1 - \delta)S_{tn}^*]/[\beta(\alpha + \beta)^{-1} V_{tn}^*/S_{tn}^*]$$
$$= (1 - \delta)^{-1}(V_{t+1}n^*/V_{tn}^*). \quad (106)$$

Thus, (one plus) the imputed land inflation rate, $p_{t+1}nL/p_{tnL}$, will equal (one plus) the growth in property value, $V_{t+1}n^*/V_{tn}^*$, and (one plus) the imputed constant quality structure inflation rate, $p_{t+1}nS_*/p_{tnS*}$, will equal $(1 - \delta)^{-1}(V_{t+1}n^*/V_{tn}^*)$. Hence, if δ is small, then the land and structure inflation rates will be almost identical and approximately equal to (one plus) the growth rate for the overall property value. Thus, the constant quality price indices for land and structures will move in an almost proportional manner. In most countries, the price of land will grow much more rapidly than the price of structures, so the hedonic regression model defined by (90) is not suitable for finding usable land price indices for residential housing.

However, the hedonic regression model defined by (90) (and its generalizations) can generate very reasonable overall constant quality property price indices, provided that the model generates a plausible estimate for the structure depreciation rate. To see why this result might occur, a highly simplified comparison of a builder's model and the log price traditional hedonic regression model studied in this section will be performed.

Consider the valuation of a representative property in periods 1 and 2 using both the builder's model and the traditional hedonic regression model explained in this section. In period 1, the quantity of land and constant quality structure are L_1 and S_1^*, with the total property value equal to V_1. In period 2, the quantity of land and constant quality structure are $L_2 = (1 + g_L)L_1$ and $S_2^* = (1 + g_S)S_1^*$, with the total property value equal to V_2. The L_t and S_t^* are known, and hence the growth rates g_L and g_S are also known. Using the property valuation function defined by (100), the two properties have the following value decompositions, where p_1 and p_2 are the constant quality property price levels for periods 1 and 2:

$$V_1 = p_1 L_1^{\alpha} S_1^{*\beta}; \quad (107)$$
$$V_2 = p_2 L_2^{\alpha} S_2^{*\beta}$$
$$= p_1(1 + \rho)[L_1(1 + g_L)]^{\alpha} [S_1^*(1 + g_S)]^{\beta}$$
$$\text{where } 1 + \rho = p_2/p_1$$
$$= V_1(1 + \rho)(1 + g_L)^{\alpha}(1 + g_S)^{\beta}$$
$$\approx V_1(1 + \rho)[\alpha(1 + g_L) + \beta(1 + g_S)], \quad (108)$$

where the last approximate equality follows if $\alpha + \beta = 1$ and the geometric mean $(1 + g_L)^{\alpha}(1 + g_S)^{\beta}$ is approximated by the corresponding arithmetic mean $\alpha(1 + g_L) + \beta(1 + g_S)$.

Now use the builder's model to value the same properties. Let p_{L1} and p_{L2} be the price levels for land in periods 1 and 2 and let p_{S1} and p_{S2} be the constant quality price levels for structures in periods 1 and 2. The builder's model imputes the following values for the properties in the two periods:

$$V_1 = p_{L1} L_1 + p_{S1} S_1^*; \quad (109)$$
$$V_2 = p_{L2} L_2 + p_{S2} S_2^* \quad (110)$$
$$= p_{L1}(1 + \rho_L)(1 + g_L)L_1 + p_{S1}(1 + \rho_S)(1 + g_S)S_1^*,$$

where the land and structure constant quality price indices are defined as $1 + \rho_L = p_{L2}/p_{L1}$ and $1 + \rho_S = p_{S2}/p_{S1}$. Define the land and structure share of property value in period 1 as $s_{L1} \equiv p_{L1}L_1/V_1$ and $s_{S1} \equiv p_{S1}S_1^*/V_1$, respectively. The *Laspeyres quantity* and *Paasche price indices* for properties, Q_L and P_P, are defined as follows:

$$Q_L \equiv s_{L1}(\backslash L_2/L_1) + s_{S1}(S_2^*/S_1^*) \quad (111)$$
$$= s_{L1}(1 + g_L) + s_{S1}(1 + g_S);$$
$$P_P \equiv [V_2/V_1]/Q_L \quad (112)$$
$$= [V_2/V_1]/[s_{L1}(1 + g_L) + s_{S1}(1 + g_S)],$$

where the last equality follows from (111). Using (108), we have the following approximate expression for 1 + ρ, which is the property price index generated by the traditional hedonic regression model:

$$1 + \rho \approx [V_2/V_1]/[\alpha(1 + g_L) + \beta(1 + g_S)]. \quad (113)$$

Comparing (112) to (113), it can be seen that the Paasche property price index that is generated by the builder's model, P_P, will be approximately equal to the property price index 1 + ρ that is generated by a traditional log price time dummy hedonic regression model provided that α is approximately equal to the land share s_{L1} and β is approximately equal to structure share s_{S1}.[122] Since the hedonic utility function for the traditional model is Cobb–Douglas, this approximate equality is likely to hold. Thus, the traditional model is likely to generate approximately the same overall property price indices as would be generated by the builder's model.[123]

The approximation result in the previous paragraph opens up another possible method for obtaining aggregate land values for residential housing. There are residential property price indices for many countries that are based on traditional hedonic regression models. Consider such a country that also conducts periodic censuses of housing where owners of residential dwelling units are asked to value their properties. Let the estimated value of housing in periods 1 and t be V_1 and V_t. Suppose the aggregate housing price index levels for these two periods are p_1 and p_t. Using these data, one can form aggregate volume estimates for residential housing as $q_1 \equiv V_1/p_1$ and $q_t \equiv V_t/p_t$. From the country's system of national accounts, it should be possible to obtain estimates for the aggregate price and quantity or volume of residential structures, which we denote by p_{S1} and q_{S1} for period 1 and p_{St} and q_{St} for period t. With these data in hand, aggregate Laspeyres, Paasche, and Fisher (1922) price and quantity indices for residential land can be obtained using (p_1,p_{S1}) and (p_t,p_{St}) as period 1 and t price vectors and using $(q_1,-q_{S1})$ and $(q_t,-q_{St})$ as period 1 and t quantity vectors. The resulting land prices (p_{L1},p_{Lt}) and volumes (q_{L1},q_{Lt}) would fill a gap in the SNA for the country. Real household wealth accounts could be constructed that had household land and household structures as separate assets.

For data series on residential property prices for either the sales of properties or the stock of properties, see the European Central Bank (2018) (which lists 228 series for European countries) and the Bank for International Settlements (2018), which lists long series for 18 advanced economies. For additional information on alternative approaches for the measurement of residential property price indices for sales of properties and for making estimates for the stock of residential properties, see Statistics Portugal (2009), Eurostat (2013, 2017), Hill (2013), Silver (2018), and Hill et al. (2020).

[122] To obtain this approximation result, it is also necessary that the depreciation rate that is estimated by the log price time dummy model be reasonable.

[123] For examples of studies where it was found that this approximate equality held, see Diewert (2010, 21), Diewert and Shimizu (2015, 1692), and Diewert, Huang, and Burnett-Issacs (2017, 32).

14. Price Indices for Rental Housing: The Modified Repeat Rents Approach

At first sight, it would seem that the construction of price indices for rental housing should be fairly straightforward, since typically, rents are paid to owners every month. Thus, all that seems to be necessary is to collect information on rents paid (from either the tenants or the owners), say R_{tn} and $R_{t+1}n$ for rental unit n in periods t and $t + 1$, form the price ratios, $R_{t+1}n/R_{tn}$, and take a suitable average of these ratios to form a rent index.

Specifically, suppose we have data on rents R_{tn} for a group of "somewhat homogeneous" rental dwelling units for $N(t)$ properties in period t for consecutive months t = 0, 1. Denote the set of available properties in period t by $S(t)$ for t = 0,1. Assume that there is a large overlap of properties between the two periods; that is, assume that the intersection set of properties $S(0)\cap S(1)$ consists of many properties. By "somewhat homogeneous" properties, we mean that the properties are similar in type (either detached, semi-detached, or high- or medium-rise apartments), located in a local area where a separate rent index could be produced (a postal code area or a neighborhood), either furnished or unfurnished, and the rental properties in scope have roughly similar ratios of land value to structure value. Later in this section, it will be assumed that estimates for the floor space of the structure of property n in period t, S_{tn}, and for the corresponding land area of the property, L_{tn}, are known. Typically, the floor space area and the land area of a specific rental property n will remain constant from period to period so that $S_{tn} = S_n$ and $L_{tn} = L_n$ for all time periods t that property n is in scope for the index. We also assume that the age of property n in period t, $A_{tn} = A(t,n)$ (in months if the index is a monthly index), is known (at least approximately).

Each rental property provides a unique service since the location of each rental property will in general be different and the location of the property is an important price-determining characteristic of each rental property. The quantity associated with each rent observation could be considered to be unity. Since periods 0 and 1 are close to each other, the characteristics that describe each rental property will not change much. Thus, a useful preliminary rent index going from period 0 to 1 is the following *repeat rents index*, P_{RR}, defined as the sum of rents paid in period 1 divided by the sum of rents paid in period 0 for all properties that are common to the two periods:

$$P_{RR} \equiv \Sigma_{n\in} S_{(0)}\cap S_{(1)}\, R_1 n/\Sigma_{n\in} S_{(0)}\cap S_{(1)}\, R_0 n. \quad (114)$$

Thus, this preliminary index P_{RR} is equal to the *maximum overlap rent value ratio*. This index can be interpreted as a Dutot index, but it can also be interpreted as a Laspeyres, Paasche, Lowe, or Fisher index since the quantity associated with rental property n in period t is 1 and the corresponding price is the rent R_{tn}.

Since rents usually do not change much from month to month, P_{RR} will be close to unity if months 0 and 1 are close to each other. Thus, the construction of rental property indices seems to be very straightforward!

However, there are *three problems* with the preceding maximum overlap rental index:

- The quantity (or utility) associated with each property does not remain constant from period to period *due to depreciation of the structure*. This depreciation can be offset by increased maintenance and renovation of the structure. But in general, there will be (on average) a *net depreciation rate* associated with the structure on the rental property.
- *New rental properties* may come into the location in scope during period 1. These properties are excluded from the continuing unit rent index defined by (114). Newly renovated properties also have the character of a new commodity that is not directly comparable to the corresponding rental property in period 0. If these properties can be identified, they should be excluded from the matched model index defined by (114) and they should be treated as a "new" property.
- Some rental properties that were rented out in period 0 may become *vacant* in period 1, and thus no household is getting utility from the vacant rental property in period 1, and hence these vacant properties should not be included in the CPI. Similarly, rental properties that were *demolished* in period 1 should be excluded from the matched model index.

Some solutions to the previous problems can be implemented at the cost of making additional assumptions.

To deal with the depreciation problem, assume that the statistical agency has an estimate for the annual structure geometric depreciation rate for the type of structure in scope for the local area rent index. This annual structure depreciation rate should be converted into a monthly rate. Thus, suppose the annual structure geometric depreciation rate is 1 percent or 2 percent. The corresponding monthly rate is 0.083 percent or 0.165 percent, respectively. But this monthly *structure depreciation rate*, δ, needs to be converted into a *property depreciation rate*; that is, it needs to be further reduced by the ratio of structure value to total property value (which includes land value). Suppose that the reduced value depreciation rate is known and is equal to the small fraction $\Delta > 0$.[124]

The estimated depreciation rate δ could equal 0. In this case, renters do not experience any reduction in the quality of the rented structure as the structure ages. This corresponds to one-hoss-shay or light bulb depreciation. If this case were to occur, it would imply that the aging bias adjustments made in these two models are not warranted and the estimating equations for those two models would need to be changed to reflect the one-hoss-shay depreciation of the structures. However, the available empirical evidence indicates that depreciation rates are positive.[125]

The next assumption that we make is that the *utility or real value of a rental property* declines at a geometric rate as the structure on the property ages. Thus, the utility of a rental property with a new structure on it in period t is set equal to one, and then its utility declines at a geometric rate as it ages. Thus, for rental property n in period t that has a structure on it of age $A_{tn} = A(t,n)$, its utility or *real quantity* q_{tn} as a function of the structure age is defined as follows:

$$q_{tn} \equiv (1 - \Delta)^{A(t,n)};\ t = 0,1;\ n \in S(t), \tag{115}$$

where Δ is the assumed geometric property depreciation rate that is due to structure depreciation. Thus, in order to measure the rental property quantity and adjust it for the change in the quality of the structure over time, it is necessary to have an estimate for Δ. We will address this problem later in this section.[126]

The rent for property $n \in S(t)$ in period t is R_{tn} and the corresponding quantity q_{tn} is defined by (115), so the *constant quality price* for property $n \in S(t)$ in period t is p_{tn} defined as the following value to quantity ratio:

$$p_{tn} \equiv R_{tn}/q_{tn} = R_{tn}/(1 - \Delta)^{A(t,n)};\ t = 0,1;\ n \in S(t). \tag{116}$$

Assuming that an estimate for the property depreciation rate Δ is available and the ages of the structures on the rental properties in scope are available, the prices and quantities defined by (116) and (115) can be used to form many indices, depending on statistical agency preferences. Thus, the *maximum overlap Laspeyres price index* is defined as follows:

$$\begin{aligned} P_{MOL} &\equiv \Sigma_{n\in S(0)\cap S(1)} p_{1n}q_{0n} / \Sigma_{n\in S(0)\cap S(1)} p_{0n}q_{0n} \\ &= \Sigma_{n\in S(0)\cap S(1)} p_{1n}q_{0n} / \Sigma_{n\in S(0)\cap S(1)} R_{0n} \\ &\quad \text{using (115) and (116) for } t = 0 \\ &= \Sigma_{n\in S(0)\cap S(1)} [R_{1n}/(1 - \Delta)] / \Sigma_{n\in S(0)\cap S(1)} R_{0n} \\ &\quad \text{using (116) for } t = 1 \text{ and (115) for } t = 0 \\ &= P_{RR}/(1 - \Delta) \text{ using definition (114)} \\ &> P_{RR}, \end{aligned} \tag{117}$$

where the inequality follows since $0 < 1 - \Delta < 1$. Thus, the simple repeat rents index P_{RR} defined by (114) will *understate* constant quality maximum overlap Laspeyres rental price inflation, P_{MOL}, defined by the first line in (117) by the factor $1/(1 - D)$, where Δ is the one-period geometric property depreciation rate.

The *maximum overlap Paasche price index* is defined as follows:

$$\begin{aligned} P_{MOP} &\equiv \Sigma_{n\in} S_{(0)\cap} S_{(1)} p_1nq_1n / \Sigma_{n\in} S_{(0)\cap} S_{(1)} p_0nq_1n \\ &= \Sigma_{n\in} S_{(0)\cap} S_{(1)} R_1n / \Sigma_{n\in} S_{(0)\cap} S_{(1)} [R_0n(1 - \Delta)] \\ &\quad \text{using (115) and (116)} \\ &= P_{MO}/(1 - \Delta) \text{ using definition (114).} \\ &> P_{MO}. \end{aligned} \tag{118}$$

[124] Later in this section, we will indicate how this property depreciation rate could be estimated using a hedonic regression. Malpezzi, Ozanne, and Thibodeau (1987, 382) found that for their US sample of rental properties, annual rent declined about 0.6 percent per year. This corresponds to a monthly Δ equal to 0.050 percent per month.

[125] See Malpezzi, Ozanne, and Thibodeau (1987) and the literature cited in thier paper.

[126] As indicated earlier, it may be possible to form an estimate for the property depreciation rate from a knowledge of the structure depreciation rate (obtained from national accounts information) and estimates of the relative value of the land and structure components of the rental properties in scope.

Using (117) and (118), we see that $P_{MOP} = P_{MOL}$. Define the *maximum overlap Fisher index* P_{MOF} as the geometric mean of the maximum overlap Laspeyres and Paasche indices:

$$P_{MOF} \equiv [P_{MOP}P_{MOL}]^{1/2} = P_{MOL} = P_{MOP} = P_{MO}/(1 - \Delta), \quad (119)$$

where P_{MO} is the rent-to-value ratio for the properties that are in the sample for periods 0 and 1, $\Sigma_{n\in S(0)\cap S(1)} R_1 n / \Sigma_{n\in S(0)\cap S(1)} R_0 n$.

As a point of interest, define the *maximum overlap unit value price index*, P_{MOUV}, as follows:

$$\begin{aligned} P_{MOUV} &\equiv [\Sigma_{n\in S(0)\cap S(1)} R_{1n} / \Sigma_{n\in S(0)\cap S(1)} q_{1n}] / \\ &\quad [\Sigma_{n\in S(0)\cap S(1)} R_{0n} / \Sigma_{n\in S(0)\cap S(1)} q_{0n}] \quad (120) \\ &= [\Sigma_{n\in S(0)\cap S(1)} R_{1n} / \Sigma_{n\in S(0)\cap S(1)} (1 - \Delta)^{A(0,n)+1}] / \\ &\quad [\Sigma_{n\in S(0)\cap S(1)} R_{0n} / \Sigma_{n\in S(0)\cap S(1)} (1 - \Delta)^{A(0,n)}] \\ &\quad \text{using definitions (115)} \\ &= [\Sigma_{n\in S(0)\cap S(1)} R_{1n} / (1 - \Delta)] / [\Sigma_{n\in S(0)\cap S(1)} R_{0n}] \\ &= P_{MO}/(1 - \Delta). \end{aligned}$$

Thus, under the geometric property depreciation assumptions, the maximum overlap unit value price index P_{MOUV} is also equal to the string of indices in (119) that are all equal to each other.

This analysis indicates a way forward to deal with the depreciation of the structure problem. With an appropriate estimate for the average *property* depreciation rate Δ, we need only apply a simple adjustment to the aggregate rent ratio for properties present in both periods under consideration. Of course, our assumptions about the form of depreciation may not be very accurate, *but making some adjustment for depreciation is better than making no adjustment at all.*

In order to deal with the problems arising from demolished and vacant rental units and newly constructed (or renovated) units, it is necessary to make more assumptions. The problem is "how can the quality of a new rental property relative to existing rental properties be determined in the period when the new property appears?" Similarly, in order to construct an estimate of the change in real rental services over the two periods under consideration, it is necessary to know what is the quality or utility of a rental unit that has disappeared relative to rental properties that continue to exist. In order to address these questions, the model of quality adjustment that is explained in Sections 3 and 4 of Chapter 8 will be applied.[127]

First, consider the case where there are only three rental properties in scope for periods 0 and 1. Property 1 is present in both periods, property 2 is present in period 0 but not in period 1 (a disappearing property), and property 3 is not present in period 0 but is present in period 1 (a new property).[128] Denote the real quantity of these three rental properties by q_C, q_D, and q_N, respectively, using definitions (115) for the three properties.[129] We assume that renters value the relative usefulness or utility of the various properties in scope by using the following *linear valuation* (or utility) *function*:

$$f(q_C,q_D,q_N) \equiv \alpha_C q_C + \alpha_D q_D + \alpha_N q_N, \quad (121)$$

where α_C, α_D, and α_N are positive constants that reflect the relative value to renters of the three properties in scope in periods 0 and 1, and q_C, q_D, and q_N are the real quantities for the three properties.

In period 0, suppose renters collectively maximize the utility function[130] $f(q_C,q_D,q_N)$ defined by (121) with respect to q_C, q_D, and q_N subject to the budget constraint $p_0Cq_C + p_0Dq_D = p_0Cq_0C + p_0Dq_0D$ and the non-availability constraint $q_N = 0$, where q_0C and q_0D are the property depreciation-adjusted quantities for the two properties that are available for rent in period 0. The first-order conditions for the observed (q_0C,q_0D) to solve this constrained utility maximization problem are as follows:

$$\partial f(q_0C,q_0D,0)/\partial q_C = \alpha_C = \lambda^0 p_0C; \quad (122)$$
$$\partial f(q_0C,q_0D,0)/\partial q_D = \alpha_D = \lambda^0 p_0D, \quad (123)$$

where $\lambda^0 > 0$ is the optimal Lagrange multiplier. It can be shown that $\lambda^0 = 1/P^0$, where P^0 can be interpreted as *the period 0 aggregate price level* for the active renters in period 0.[131] Equations (122) and (123) can be rewritten as follows:

$$p_0C = P^0\alpha_C; \quad (124)$$
$$p_0D = P^0\alpha_D. \quad (125)$$

In period 1, suppose renters again collectively maximize the utility function $f(q_C,q_D,q_N)$ defined by (121) with respect to q_C, q_D, and q_N subject to the period 1 budget constraint $p_1Cq_C + p_1Nq_N = p_1Cq_1C + p_1Nq_1N$ and the non-availability constraint $q_D = 0$, where q_1C and q_1N are the property depreciation-adjusted quantities for the two properties that are available for rent in period 1. The first-order conditions for the observed (q_1C,q_1N) to solve this constrained utility maximization problem are as follows:

$$\partial f(q_1C,0,q_N)/\partial q_C = \alpha_C = \lambda^1 p_1C; \quad (126)$$
$$\partial f(q_1C,0,q_1N)/\partial q_N = \alpha_N = \lambda^1 p_1N \quad (127)$$

where $\lambda^1 > 0$ is the optimal period 1 Lagrange multiplier. Again, it can be shown that $\lambda^1 = 1/P^1$, where P^1 can be interpreted as *the period 0 aggregate price level* for the active

[127] See Diewert (2021).

[128] The "new" property 3 may not be a truly new property; it may be the case that property 3 was temporarily vacant in period 1. Similarly, property 2 may not permanently disappear in period 1; it may reappear in a subsequent period.

[129] Thus, q_0C is set equal to $(1 - \Delta)^{A(0,C)}$; $q_1C \equiv (1 - \Delta)^{A(0,C)+1}$; $q_0D \equiv (1 - \Delta)^{A(0,D)}$; and $q_1N \equiv 1$. Thus, an estimate of the age of the rental properties is required along with an estimate for the geometric property depreciation rate Δ. The corresponding prices are defined as $p_0C \equiv R_0C/q_0C$, $p_0D \equiv R_0D/q_0D$, $p_1C \equiv R_1C/q_1C$, and $p_1N \circ R_1N/q_1N = R_1N$.

[130] Alternatively, assume that each renter has the same linear preferences over alternative rental properties. It turns out that equations (124) and (125) will still be satisfied.

[131] See Diewert (2021, 8–10).

renters in period 0. Equations (126) and (127) can be rewritten as follows:

$$p_1 C = P^1 \alpha_C; \quad (128)$$
$$p_1 N = P^1 \alpha_N. \quad (129)$$

Note that equations (124), (125), (128), and (129) are a special case of Court's (1939, 109–111) hedonic quality adjustment suggestion number two. He transformed these underlying equations by taking logarithms of both sides of these equations in order to obtain the *classic TPD hedonic regression model*.[132]

Looking at equations (124), (125), (128), and (129), it can be seen that we have four equations in five unknowns: the price levels P^0 and P^1 and the three relative quality parameters α_C, α_D, and α_N. Note each α_n measures the relative usefulness of an additional unit of product $n = C, D$, or N to purchasers of the three products. It can be seen that P^t and α_n cannot all be identified using observable data; that is, if P^0, P^1, α_C, α_D, and α_N satisfy equations (124), (125), (128), and (129) and λ is any positive number, then λP^0, λP^1, $\lambda^{-1}\alpha_C$, $\lambda^{-1}\alpha_D$, and $\lambda^{-1}\alpha_N$ will also satisfy these equations. Thus, it is necessary to place a normalization (like $P^0 = 1$ or $\alpha_C = 1$) on the five parameters that appear in these equations in order to obtain a unique solution. In the index number context, it is natural to set the price level for period 0 equal to unity, and so we impose the following normalization on the five unknown parameters that appear in equations (124), (125), (128), and (129):

$$P^0 = 1. \quad (130)$$

The unique solution to equations (124), (125), (128), (129), and (130) is

$$P^0 = 1;\ P^1 = p_1 C/p_0 C;\ \alpha_C = p_0 C;$$
$$\alpha_D = p_0 D;\ \alpha_N = p_1 N/(p_1 C/p_0 C) = p_1 N/P^1. \quad (131)$$

Note that the resulting *price index*, P^1/P^0, is equal to $p_1 C/p_0 C$, the price ratio for the commodity that is present in both periods. Thus, the price index for this very simple model turns out to be a *maximum overlap price index*.[133]

We now consider how companion quantity levels, Q^0 and Q^1, for the price levels, P^0 and P^1, can be determined. Define the *aggregate value of rents paid in period t* as V^t for $t = 0,1$. Making use of the fact that $R_0 N = 0$ and $R_1 D = 0$, we have the following expressions for V^0 and V^1:

$$V^0 \equiv R_0 C + R_0 D = p_0 C q_0 C + p_0 D q_0 D; \quad (132)$$
$$V^1 \equiv R_C{}^1 + R_N{}^1 = p_1 C q_1 C + p_1 N q_1 N. \quad (133)$$

The quantity level Q^t for period t can be determined *directly* by evaluating the linear utility function defined by (121) at the period t quantity data or *indirectly* by deflating the period t aggregate value of rents V^t by the period t estimated price level, P^t:

$$Q^0 \equiv \alpha_C q_0 C + \alpha_D q_0 D = p_0 C q_0 C + p_0 D q_0 D$$
$$= [p_0 C q_0 C + p_0 D q_0 D]/P^0 = V^0/P^0 = V^0; \quad (134)$$

$$Q^1 \equiv \alpha_C q_1 C + \alpha_N q_1 N = p_0 C q_1 C + [p_1 N/P^1] q_1 N$$
$$= [p_1 C/P^1] q_1 C + [p_1 N/P^1] q_1 N = V^1/P^1$$
$$= V^1/[P^1/P^0], \quad (135)$$

where the various equalities in (134) and (135) follow by substituting equations (131) –(133) into the direct definitions for Q^0 and Q^1. Thus, real rents in period 0, Q_0, are set equal to the aggregate value of rents in period 0, V_0. Real rents in period 1, Q_1, are set equal to the aggregate value of rents in period 1, V_1, and deflated by the maximum overlap rent price index, P_1/P_0. In this case, there is only one rental unit in scope that is occupied in both periods and is equal to the following expression:

$$P^1/P^0 = [R_1 C/R_0 C]/(1 - \Delta). \quad (136)$$

An interesting aspect of this rent model is that the aggregate price and quantity levels, P^0, P^1, Q^0, Q^1, and the price index, P^1/P^0, can all be determined by the national statistician using only information on collected rents (the R_{tn}) and an estimate for the appropriate monthly geometric property depreciation rate, Δ. Thus, detailed information on the characteristics of the rental dwelling units is not required in order to implement this very simple approach which is basically a modified repeat rents index.

It is useful to look at the quantity index, Q^1/Q^0, that is implied by this simple model.[134] Using the final expressions in (134) and (135) and definitions (132) and (133), we have:[135]

$$Q^1/Q^0 = [V^1/V^0]/[P^1/P^0] \quad (137)$$
$$= [(R_1 C + R_1 N)/(R_0 C + R_0 D)]/[P^1/P^0]$$
$$= (1 - \Delta)[(R_1 C + R_1 N)/(R_0 C + R_0 D)]/[R_1 C/R_0 C]$$
$$\text{using (136)}$$
$$= (1 - \Delta)[1 + (R_1 N/R_1 C)]/[1 + (R_0 D/R_0 C)].$$

[132] For more accessible sources on the log price TPD hedonic regression model, see Griliches (1971) and Aizcorbe (2014). Summers (1973) proposed the same model in the international comparisons context where it is known as the *country product dummy model*. This model can also be viewed as a *repeat rent* model that is analogous to the *repeat sales* model that dates back to Bailey, Muth, and Nourse (1963).

[133] Keynes (1930, 94) was an early author who advocated this method for dealing with new goods by restricting attention to the goods that were present in both periods being compared. He called his suggested method the *highest common factor method*. Marshall (1887, 373) implicitly endorsed this method. Triplett (2004, 18) called it the *overlapping link method*.

[134] It is important to construct companion aggregate quantity levels Q^t to complement the aggregate price levels P^t because the methodology outlined here will be applied to a local area or to a specific class of rental properties. These subindices will have to be aggregated into a national index and in order to do that, it is necessary to have information on expenditure or quantity weights for the various sub-national indices.

[135] Note that the decomposition given by (137) does not require a knowledge of A_{tn} or any other rental housing characteristic. But the assumption of a common property depreciation rate implicitly implies that the rental properties in scope should have similar characteristics in order to justify the assumption of a common depreciation rate.

Thus, there are three *growth factors* that determine the overall growth of real rentals:

- $(1 - \Delta)$, which is one minus the rental property geometric depreciation rate; this factor will reduce the overall growth of real rentals.
- $1 + (R_1N/R_1C)$, which is one plus the ratio of new rental value to continuing rental value in period 1; this growth factor will increase the overall growth of real rentals.
- $1 + (R_0D/R_0C)$, which is one plus the ratio of disappearing rental value to continuing rental value in period 0; this growth factor is in the denominator and hence will decrease the overall growth of real rentals.

In a growing economy with new rental units being added to the marketplace, we would expect the ratio R_1N/R_1C to exceed the ratio R_0D/R_0C; that is, the availability of new rental[136] units should normally offset the loss of existing rental units due to demolition and temporary vacancies.

A problem with this simple model is that there is only one product that is present in both periods. However, it is possible to generalize the present model to allow for multiple overlapping products and for many new and disappearing rental units; see the annex for this generalization.

In the period following period 1, the same methodology can be applied to a new bilateral data set where the set of common rental properties in periods 1 and 2 will in general be different. New chain link price and quantity indices can be calculated and linked up to previous price and quantity levels. Chain drift should not be a problem due to the fact that so many properties will be in the maximum overlap category, and price and quantity changes will not be large as we move from period to period.

Thus, this very simple rents model can in principle deal with the three big difficulties associated with the pure repeat rents model.

However, there are two main problems with this modified repeat rents model:

- *The model requires an appropriate geometric property depreciation rate.*
- The model ignores other important characteristics of rental housing *that may not remain constant over time*, such as renovations to the structure and changes in local amenities that affect the utility of the rental property.

Note that the geometric depreciation rate is applied to the entire property rent that has to cover the user cost of both the structure and the land. Thus, properties that have very different mixes of structure and land value will have different overall property depreciation rates. If the land structure mix were to remain constant over time, the assumption of a *property* depreciation rate may be adequate. But of course, the structure part of a property changes its *real value* due to depreciation, whereas land does not depreciate. Moreover, the structure-to-land *nominal value* ratio is likely to change over time.[137]

Thus, we turn to a hedonic regression model to address these difficulties.

15. Price Indices for Rental Housing: Hedonic Regression Approaches

The hedonic regression model that was explained in Section 13 can be applied to property rentals rather than property sales. Thus, we now assume that, in addition to the age of the structure on rental property n in period t, $A_{tn} = A(t,n)$, information on the land area and the floor space area of property n in period t, L_{tn} and S_{tn}, is also available. *Quality-adjusted structure floor space* for property n in period t, S_{tn}^*, is defined as follows:

$$S_{tn}^* \equiv S_{tn}(1 - \delta)^{A(t,n)};\ t = 0,1, \ldots, T;\ n \in S(t), \tag{138}$$

where δ is the one-period geometric depreciation rate for all structures for the rental properties in scope. The *utility* or *real quantity* of rental property n in period t, q_{tn}, is set equal to the following function of L_{tn} and S_{tn}^*:

$$\begin{aligned} q_{tn} &\equiv L_{tn}^{\alpha} S_{tn}^{*\beta}\ t = 0,1, \ldots, T;\ n \in S(t) \\ &= L_{tn}^{\alpha}\,[S_{tn}(1 - \delta)^{A(t,n)}]^{\beta} \text{ using definitions (138)} \\ &= L_{tn}^{\alpha}\, S_{tn}^{\beta}\,(1 - \delta)^{\beta A(t,n)} \\ &= L_{tn}^{\alpha}\, S_{tn}^{\beta}\,\varphi^{A(t,n)}, \end{aligned} \tag{139}$$

where α and β are positive parameters (which do not necessarily sum to one),[138] and the constant φ is defined as

$$\varphi \equiv (1 - \delta)^{\beta}. \tag{140}$$

The *constant quality price* of rental property n in period t, p_{tn}, is defined as rents paid, R_{tn}, divided by q_{tn}. The next assumption is that these constant quality prices move in a proportional manner (approximately). Thus, we have the following assumptions:

$$p_{tn} \equiv R_{tn}/q_{tn} \approx P^t;\ t = 0,1, \ldots, T;\ n \in S(t). \tag{141}$$

Thus, the constant quality rental prices p_{tn} move in an *approximately proportional manner over time*, with the period t factor of proportionality equal to the scalar P^t. Thus, P^t can be interpreted as the *price level* for rents in period t. The approximate equalities in equations (141) can be rewritten as the equalities $R_{tn} = P^t q_{tn} e_{tn}$, where e_{tn} is a positive

[136] A "new" rental unit includes a rental unit which was available in prior periods but vacant in period 0. Landlords sometimes circumvent local rent controls by renovating their properties so it may be prudent to use the above suggested quality adjustment procedure to capture such renovations rather than attempting to link the "new" rental unit to a prior period.

[137] For information on the increasing share of land in housing prices for many economies over the period 1870–2012, see the important paper by Knoll, Schularick, and Steger (2017).

[138] Thus, the utility function is a Cobb–Douglas function. The analysis in this section follows that of McMillen (2003, 289–290), Shimizu, Nishimura, and Watanabe (2010, 795), and Diewert, Huang, and Burnett-Issacs (2017). McMillen assumed that $\alpha + \beta = 1$. The above authors applied their models to the sales of properties but the same model can be applied to property rents. We follow Shimizu, Nishimura, and Watanabe in allowing α and β to be unrestricted.

error term with mean equal to 1. Taking logarithms of both sides of these equations leads to the following *time dummy hedonic regression*:

$$\begin{aligned} lnR_{tn} &= lnP^t + lnq_{tn} + \varepsilon_{tn};\ t = 0,1,\ldots,T;\ n\in S(t) \\ &= lnP^t + \alpha lnL_{tn} + \beta lnS_{tn} + (ln\varphi)A_{tn} + lne_{tn} \\ &= \rho^t + \alpha lnL_{tn} + \beta lnS_{tn} + \gamma A_{tn} + lne_{tn}, \end{aligned} \quad (142)$$

where $\rho^t \equiv \ln P^t$ for $t = 0,1,\ldots,T$ and $\gamma \equiv \ln\varphi = \beta\ln(1 - \delta)$. The unknown parameters in (142) are the constant quality log rental price levels, $\rho^0,\rho^1,\ldots,r^T$, and the taste parameters α, β, and γ. Once these parameters have been determined, the geometric depreciation rate δ that appears in equation (139) can be recovered from the regression parameter estimates (β^* and γ^*) as follows:

$$\delta^* \equiv 1 - e^{\gamma^*/\beta^*}. \quad (143)$$

An estimate for the property geometric depreciation rate Δ, which appeared in equation (115) in the previous section can be obtained using the estimated structure depreciation rate δ^* defined by (143); that is, solve the equation $(1 - \Delta)$ equal to $(1 - \delta^*)^{\beta^*}$ for Δ. The solution is

$$\Delta^* = 1 - (1 - \delta^*)^{\beta^*}. \quad (144)$$

If $\beta^* = 1$, then $\Delta^* = \delta^*$. Typically $0 < \beta^* < 1$, in which case, the property depreciation rate Δ^* will be less than the structure depreciation rate δ^*. Thus, it can be seen that the hedonic regression model approach to the construction of rental property indices is not a totally different approach to the earlier matched model approach. The weakness of the matched model approach is that it requires an estimate for the property depreciation rate. It can be seen that the hedonic regression approach can generate an estimate for the property depreciation rate. Thus, running an occasional hedonic regression of the form given by (142) will generate an estimate for the property depreciation rate Δ, which played a prominent role in the modified repeat rents model outlined in the previous section.

The estimated aggregate rental price levels for each period t, P^{t*}, generated by the hedonic regression defined by (142) are defined as the exponentials of the estimated ρ^{t*}:

$$P^{t*} \equiv \exp[\rho^{t*}];\ t = 0,1,\ldots,T. \quad (145)$$

The corresponding aggregate quantity levels Q^{t*} are defined as follows:

$$Q^{t*} \equiv \Sigma_{n\in S(t)}\ R_{tn}/P^{t*};\ t = 0,1,\ldots,T. \quad (146)$$

The corresponding *rental price indices* for periods $t = 0,1,\ldots,T$ are defined as P^{t*}/P^{0*}.

If there were only one stratum and one hedonic regression, then it would not be necessary to calculate the aggregate quantity index Q^{t*} defined by (146). But there will be many strata (classified by location, the type of structure, and other characteristics), and so to form an aggregate Laspeyres, Paasche, or Fisher index of rental prices, it will be necessary to calculate P^{t*} and Q^{t*} by stratum and then use two-stage aggregation to construct regional or national rental price indices. Since the Laspeyres and Paasche indices have an equal justification (and are the indices that use the most representative weights for the two periods being compared), the Fisher index is recommended. It is a symmetric average of the Paasche and Laspeyres indices that satisfies the time reversal test.

To explain in more detail how the TPD model works, exponentiate both sides of equations (142) and drop the error terms. Then for each rental property $n\in S(t)$ in scope for period t, we have the following expression for the rent for property n in period t, R_{tn}:

$$R_{tn} = P^tL_{tn}^{\ \alpha}\ S_{tn}^{\ \beta}(1 - \delta)^{\beta}A^{(t,n)};\ n\in S(t) = P^tq_{tn}, \quad (147)$$

where $q_{tn} \equiv L_{tn}^{\ \alpha}S_{tn}^{\ \beta}(1 - \delta)^{\beta}A^{(t,n)}$ is the *real quantity* or *utility of rental property n in period t*, $L_{tn} = L_n$ is the land area of property n, $S_{tn} = S_n$ is the floor space area of property n, δ is the common geometric structure depreciation rate, and $A(t,n) = A_{tn}$ is the age of the structure property n in period t. Thus, the model (without error terms) assigns the same price, P^t, to each rental property in scope in period t. *Hence, individual rental prices in this model will vary in a proportional manner over time*. Thus, any reasonable matched model index number formula for the period t index relative to period 0 will be equal to P^t/P^0. If $T = 1$, then it can be verified that the hedonic regression price index for period 1, P^1/P^0, will be equal to P_{MOL} defined by (117), P_{MOP} defined by (118), P_{MOF} defined by (119), and P_{MOUV} defined by (120).

It can be seen that the hedonic regression model approach to the construction of rental property indices is not a totally different approach to the earlier matched model approach. The weakness of the matched model approach is that it required an estimate for the property depreciation rate. It can be seen that the hedonic regression approach can generate an estimate for the property depreciation rate.

The problems associated with the hedonic regression approach are twofold:

- Information on the characteristics of the rental properties is required.
- The hedonic regression model may not fit the data very well, in which case we can conclude that the somewhat restrictive assumptions of the model do not provide an adequate approximation to reality.

If the sample of rental properties in scope is large enough, then set $T = 1$, and in this case, the hedonic regression model defined by (142) becomes a standard adjacent period time dummy hedonic regression. P^{1*}/P^{0*} can be used to update the period 0 index level. If T is greater than 1, then we have a rolling window hedonic regression.[139] In this case, there is a problem in determining exactly how to link the results of the new regression in say period $T + 1$ to the results of the previous regression for period T. A variety of linking methods have been suggested in the literature.[140] In the present

[139] See Shimizu, Nishimura, and Watanabe (2010) for a worked example of this type of regression model applied to sales of properties rather than to rentals.

[140] In the context of rolling window multilateral methods, Ivancic, Diewert, and Fox (2011) (IDF) suggested that the movement of the indices for the last two periods in the new window be linked to the last index value

context, it is likely that the choice of method will not make a material difference.

We indicate how this very simple log price hedonic regression model can be generalized to include additional (discrete) characteristics of the properties. Suppose that properties have been classified into six postal zones. If property n in period t belongs to postal zone j, then define the dummy variable $d_{j,tn}$ for observation n in period t to equal 1, and if property n in period t does not belong to postal zone j, then define the dummy variable $d_{j,tn}$ for observation n in period t to equal 0. Next, suppose that properties have been classified according to the number of bathrooms m in the structure where the maximum number of bathrooms is six. If property n in period t has i bathrooms, then define the dummy variable $d_{1i,tn}$ for observation n in period t to equal 1, and if property n in period t does not have i bathrooms, then define the dummy variable $d_{1i,tn}$ for observation n in period t to equal 0. Finally, suppose that the properties have been classified according to the number of bedrooms k in the structure where the number of bedrooms ranges from three to seven. If property n in period t has k bedrooms, then define the dummy variable $d_{2k,tn}$ for observation n in period t to equal 1, and if property n in period t does not have k bedrooms, then define the dummy variable $d_{2k,tn}$ for observation n in period t to equal 0. Now consider the following generalization of the hedonic regression model defined by (142):

$$\begin{aligned} lnR_{tn} &= \rho_t + \alpha lnL_{tn} + \beta lnS_{tn} + \gamma A_{tn} \\ &+ \Sigma_{j=1}^{6}\, \omega_j d_{j,tn} + \Sigma_{i=1}^{6}\, \eta_i d_{1i,tn} \\ &+ \Sigma_{k=3}^{7}\, \theta_k d_2 k_{,tn} + \varepsilon_{tn};\ t = 0,1,\ldots,T;\ n \in S(t). \end{aligned} \tag{148}$$

The ω_j parameters affect the quality of the land component of property value, while the last two sets of dummy variables affect the quality of the structure component of property value. Not all of the parameters ρ_t, ω_j, η_i, and θ_k can be identified; that is, there is exact multicollinearity associated with the dummy variables associated with these parameters. Thus, to identify all of the remaining parameters, we make the following normalizations:

$$\omega_4 \equiv 0;\ \eta_3 \equiv 0;\ \theta_5 = 0. \tag{149}$$

The model defined by (148) was applied to sales of properties in a suburb of Vancouver Canada, and it gave reasonable results for the implied structure depreciation rate; see Diewert, Huang, and Burnett-Issacs (2017). variants of this model should also work well for rentals of properties.

The results of the present section and the previous section can be summarized as follows:

- The repeat rents model studied in the previous section can be applied provided that some adjustment for the aging of the rental structure is made.
- Hedonic regressions that regress the logarithms of rents on the characteristics of the rental properties plus a time dummy variable can be run for various segments of the rental market. The estimated time dummy coefficients can be converted into period-by-period price levels that in turn can be converted into rental property price indices.

The underlying economic structure of the hedonic regression approach can be explained as follows. All renters in the segment of the rental market in scope have the same utility function, $u = f(S,L,A,X,Y,Z)$, where S, L, and A are the floor space area, land area, and age of the rental property and X, Y, and Z are the other characteristics of the property. In period t, renters compete with each other to equalize the observed rent to utility ratio for each property. Thus, we have the following approximate equalities for each rental property n in period t:

$$R_{tn}/f(S_{tn},L_{tn},A_{tn},X_{tn},Y_{tn},Z_{tn}) \approx P_t;\ t = 0,1,\ldots,T;\ n \in S(t), \tag{150}$$

where P_t is the common period t rent-to-utility ratio across the rental properties in scope. As was seen earlier, P_t can be interpreted as the period t price level for the properties in scope. It can be seen that equation (150) can be turned into the following (possibly nonlinear) regression model:

$$\begin{aligned} lnR_{tn} &= lnP_t + ln\, f(S_{tn},L_{tn},A_{tn},X_{tn},Y_{tn},Z_{tn}) + e_{tn}; \\ & t = 0,1,\ldots,T;\ n \in S(t), \end{aligned} \tag{151}$$

where e_{tn} are the error terms.

In the following section, we turn our attention to the problems associated with valuing the services of OOH.

16. Owner-Occupied Housing: The User Cost Perspective

Owner-occupied housing (OOH) is a consumer durable good, so the opportunity cost approach to the valuation of the services of a consumer durable that was explained in Section 5 could be applied to this valuation problem. Recall that the opportunity cost valuation of an owned consumer durable is simply the *maximum* of the foregone rental or leasing price for the services of the durable during a period of time and the corresponding user cost for the durable. In the previous two sections, the focus was on generating price indices for rental dwelling units. One approach to the valuation of the services of an owned dwelling unit is to *impute* a rent to it using the rent of a comparable rented unit. This is the *rental equivalence approach* to the valuation of the services of an owned dwelling unit. A second approach to this valuation problem is to construct *user costs* for owned dwelling units. The second approach will be explored in this section.

There are a number of difficulties in applying the usual durables user cost theory to housing:

generated by the previous window. Krsinich (2016) suggested that the movement of the indices generated by the new window over the entire new window period be linked to the previous window index value for the second period in the previous window. Krsinich called this a *window splice* as opposed to the IDF *movement splice*. De Haan (2015, 27) suggested that perhaps the linking period should be in the middle of the old window which the Australian Bureau of Statistics (2016, 12) termed a *half splice*. Finally IDF and Diewert and Fox (2020) suggested taking the geometric mean of all possible ways of linking the results of the new window to the results of the previous window. Diewert and Fox called this the *mean splice* and they thought that this would be the "safest" method of linking.

- Each owned dwelling unit is a unique good due to its unique location and the fact that the structure depreciates over time (and renovations may be undertaken over time).
- Each owned dwelling unit does not trade in each time period. Thus, precise period-by-period market opportunity costs are not readily available.
- An owned dwelling unit is a composite commodity made up of separate land and structure components. In general, the price trends in these two components will be different.

In order to deal with the aforementioned difficulties, typically, some form of econometric modeling will be required. Thus, suppose that some form of hedonic regression on sales of owned dwelling units in scope has been undertaken, such as the various builder's models explained in Sections 11 and 12. Suppose that we have information on a sample of owned properties in scope for periods t and $t + 1$ and there are N properties in the sample. We assume asset prices, P_{Ltn} and P_{Stn},[141] can be assigned to the land and structure areas, L_{tn} and S_{tn}, that can be imputed for rental dwelling n in period t. The aggregate user cost U_{tn} is approximated by the sum of the (end-of-period) user cost components for land and structures, u_{Ltn} and u_{Stn}, respectively. The geometric model of depreciation for structures is used, and the one-period depreciation rate is $0 < \delta < 1$. The depreciation rate for land is 0. The age of the structure for rental unit n in period t is $A(t,n)$ periods. Setting the *overall user cost value* of unit n in period t and $t + 1$ to the sum of the corresponding land and structure user costs leads to the following equations:

$$\begin{aligned} U_{tn} &= u_{Ltn}L_{tn} + u_{Stn}(1 - \delta)^{A(t,n)}S_{tn};\ n = 1, \ldots, N \\ &= [r_t - i_{Lt}]P_{Ltn}L_{tn} + [r_t - i_{St} + (1 + i_{St})\delta]P_{Stn}(1 - \delta)^{A(t,n)}S_{tn}; \qquad (152) \end{aligned}$$

$$\begin{aligned} U_{t+1}n &= u_{Lt+1}nL_{tn} + u_{St+1}n(1 - \delta)^{A(t \cdot n)+1} S_{tn};\ n = 1, \ldots, N \\ &= [r_{t+1} - i_{Lt+1}]P_{Lt+1}nL_{tn} + [r_{t+1} - i_{St+1} \\ &\quad + (1 + i_{St+1})\delta]P_{St+1}n(1 - \delta)^{A(t,n)+1}S_{tn}, \qquad (153) \end{aligned}$$

where r_t is the opportunity cost of capital for the owners of the owned properties in period t and i_{Lt} and i_{St} are the land and structure price inflation rates that owners expect at the beginning of period t. Note that the land and structure areas for unit n, L_{tn} and S_{tn}, typically do not change over time. It is well known in the housing literature that user costs for dwelling units are much more volatile than the corresponding rents for the same units.[142] Thus, in order for the user costs U_{tn} and $U_{t+1}n$ to approximate their market rents (if they were rented), it is necessary to use a nominal smoothed value for the nominal interest rates r_t and particularly for the expected asset inflation rates, i_{Lt} and i_{St}.[143] Note that the quantity of constant quality structure for property n in periods t and $t + 1$ are $S_{tn}^* \equiv (1 - \delta)^{A(t,n)}$ S_{tn} and $S_{t+1}n^* \equiv (1 - \delta)^{A(t,n)+1}$ S_{tn}; that is, the imputed constant quality amount of structure constant quality quantity declines as time increases. The corresponding constant quality amount of land rent, L_{tn}, remains constant over all periods. To form a constant quality overall price index for user costs, calculate Laspeyres, Paasche, or Fisher indices where the price data for periods t and $t + 1$ are the vectors $[u_{Lt1}, \ldots, u_{LtN}; u_{St1}, \ldots, u_{StN}]$ and $[u_{Lt+11}, \ldots, u_{Lt+1}N; u_{St+11}, \ldots, u_{St+1}N]$ and the quantity data for periods t and $t + 1$ are the vectors $[L_{t1}, \ldots, L_{tN}; (1 - \delta)^{A(t,1)}S_{t1}, \ldots, (1 - \delta)^{A(t,N)}S_{tN}]$ and $[L_{t1}, \ldots, L_{tN}; (1 - \delta)^{A(t,1)+1}S_{t1}, \ldots, (1\text{-d})^{A(t,N)+1} S_{tN}]$. Adjustments for new housing and demolitions can be made as well.

It can be seen that it is not a simple matter to implement the user cost approach to valuing the services of OOH. However, at the national level, it may be possible to use national balance sheet estimates for the value of OOH and for the value of OOH structures. Thus, the value of OOH land can be obtained by subtracting the value of OOH structures from the total OOH property value. A rough approximation to the price of OOH land can be obtained as the OOH value of land since the quantity of land in use for housing purposes will not change much from period to period.[144] Aggregate price and quantity indices for structures used by homeowners may be available from the national accounts of the country if the country has a system of total factor productivity accounts.[145] However, this information may only be available on a quarterly or annual basis and on a delayed basis, which limits the usefulness of this information for the construction of a monthly CPI.

However, monthly information on housing sales is often collected by private companies (such as real estate associations). This information usually includes information on housing characteristics. Thus, it becomes possible to implement hedonic regression models along the lines explained in Sections 11 and 12, and the information from these regressions can be used in order to implement simplified user cost approaches. It should be noted that Iceland has used a simplified user cost approach to value the services of OOH in its CPI for many years without encountering opposition to the use of user costs.[146]

17. Valuing the Services of OOH: User Costs versus Rental Equivalence

In this section, the various factors that cause the user cost of an owned dwelling unit to differ from a rental price for a comparable property are examined.[147] In addition, other

[141] P_{Stn} is the price of a square meter of *new* structure of the type used by owned unit n at the beginning of period t.

[142] On this point, see Genesove (2003), Verbrugge (2008), Shimizu, Nishimura, and Watanabe (2010b), Diewert and Nakamura (2011), Garner and Verbrugge (2011), and Suzuki, Asami, and Shimizu (2018).

[143] The expected land inflation rate i_{Lt} should be an average of land price inflation over the past 15 to 25 years to reflect the long holding periods that investors have for rental properties and the high transaction costs of buying and selling properties. Diewert and Fox (2018) used a rolling window annualized 25 year inflation rate for land for the 25 years prior to period t to generate very smooth estimates for the expected land inflation rate in their user costs for land in the United States.

[144] For an example of this methodological approach to obtaining housing land price indices, see Knoll, Schularick, and Steger (2017).

[145] The use of user costs to measure capital input in production accounts can be traced back to Dale Jorgenson and his coauthors; see Hall and Jorgenson (1967), Christensen and Jorgenson (1969), and Jorgenson (1989).

[146] See Guðnason and Jónsdóttir (2011). Simplified user costs are also discussed in Diewert (2005a), Verbrugge (2008), and Hill, Steurer, and Waltl (2020).

[147] Our discussion here is similar to that of Hill, Steurer, and Waltl (2020) who note that the services a household obtains from renting a dwelling are not necessarily the same as the services obtained by an owner-occu-

factors that affect user costs for house in general will be discussed.[148]

- Utilities such as electricity, water, and natural gas may be included in the rent for a dwelling unit that is similar to an owned unit. The user cost of an owned unit should exclude these costs since these expenditures are covered in other categories of a CPI.
- When calculating the user cost of the owner of a dwelling unit of renting the unit, there is the problem of determining what is the correct market rental opportunity cost. It turns out that all rents paid in say period t for comparable units to an owned unit can be classified into three categories: (i) the rental agreement is not being renegotiated during this period, (ii) the rental agreement is renegotiated during this period with the same tenants, and (iii) the rental agreement is a new one with new tenants. Typically, there are no escalations of rents for continuing tenants during the leasehold period, and often, renegotiated rents with continuing tenants are also sticky; that is, there is not much change in these renegotiated rents.[149] For purposes of measuring the user cost of an owner of renting an owned unit, category (iii) rents should be used as the appropriate comparable market rent.[150]
- Property taxes will be included in market rents, and they should also be included in an owner's user cost.
- Normal maintenance expenditures on the structure will be part of market rents. These expenditures should not be included in an owner's user cost for a dwelling unit that is being used by the owner since these expenditures by homeowners should already be included in other expenditure categories in the CPI. Landlords may also have considerable overhead expenses that are associated with the management of rental properties. These expenses can perhaps be grouped together with maintenance expenditures.
- The structure depreciation rate for rented dwelling units may be higher than the rate for comparable owned dwelling units, since owners are likely to take better care of their property and will avoid property damage. This expected difference in the value of depreciation should be deducted from the market rent that is applied to a comparable owned home.
- The owners of rental properties need to charge a small premium to the rents that they receive from rented units in order to cover the loss of rental income due to vacancies. This vacancy premium does not apply to the user cost of an owned unit, and thus the comparable market rent for an owned unit should be adjusted downward to account for this vacancy factor.
- Insurance payments are included in market rents. However, in the CPI, insurance payments made by owner occupiers of their dwelling units will typically be included in another category, so in this case, the imputed insurance premiums should be deducted from the market rent that is applied to a comparable owned home.
- The opportunity cost of capital for a landlord and for an owner living in a dwelling unit may be different. A landlord who rents properties to tenants may include a risk premium in his or her cost of capital to account for possible downturns in the rental market.
- It is likely that there is an *owner's premium* to owning rather than renting. A poor person may not qualify for a mortgage loan to purchase a dwelling unit so he or she is forced to rent rather than to purchase. A richer person has the choice between renting or owning a dwelling unit of the same quality. If the richer person is risk averse, he or she will probably prefer to own the same quality dwelling unit rather than renting to eliminate the transaction costs of moving if evicted. The risks of unforeseen increases in rents demanded by the landlord are also eliminated by owning rather than renting. This factor may help explain why property investors do not purchase high-end properties for rental purposes: There is a lack of demand to rent expensive properties, and thus user costs for the landlord cannot be covered by market rents for high-end properties.

Recall that the *total user cost* of dwelling unit n in period t was U_{tn} defined by (152).[151] Define period t property *value* of the same property n, V_{tn}, as the sum of its land value and structure value:

$$V_{tn} \equiv P_{Ltn}L_{tn} + P_{Stn}(1 - \delta)^{A(t,n)}S_{tn};\ n = 1, \ldots, N, \quad (154)$$

where P_{Ltn} is the price per square meter of a unit of land and P_{Stn} is the price per square meter of a unit of new structure of the type on property n for period t. Define the period t property *n land and structure shares of total property value* as

$$s_{Ltn} \equiv P_{Ltn}L_{tn}/V_{tn};\ s_{Stn} \equiv P_{Stn}(1 - \delta)^{A(t,n)}S_{tn}/V_{tn};\ n = 1, \ldots, N. \quad (155)$$

Then using (152) and the preceding definitions, the ratio of total user cost to property value for property n in period t can be written as follows:

$$U_{tn}/V_{tn} = [r_t - i_{Lt}]s_{Ltn} + [r_t - i_{St} + (1 + i_{St})\delta]s_{Stn};\ n = 1, \ldots, N. \quad (156)$$

pier. One difference between our analysis and their analysis it that their user cost formula is a single user cost formula that applies to the entire property. However, depreciation affects only the structure part of rents and if one attempts to adjust a market rent for this aging factor, it is necessary to apply the depreciation adjustment only to the structure part of rents.

148 There are many papers that compare user costs with equivalent rents. For US studies see Verbrugge (2008, 2012), Garner and Verbrugge (2009, 2011), and Adams and Verbrugge (2021). For comparisons, for Belgium, see Goeyvaerts and Buyst (2019) and for Ireland, see Coffey, McQuinn, and O'Toole (2020).

149 On the stickiness of rents, see Shimizu, Nishimura and Watanabe (2010b), Lewis and Restieaux (2015, 72–75), Gallin and Verbrugge (2019), Coffey, McQuinn and O'Toole and Suzuki, Asami and Shimizu (2021). Lewis and Restieaux label their three categories as (i) Occupied Let, (ii) Renewal, and (iii) New Let. Their category (i) is a stock measure that includes all occupied rental units while their categories (ii) and (iii) match up with categories (ii) and (iii) in the text above. Rents in categories (ii) and (iii) may be subject to rent controls which means that rents in these categories do not reflect current opportunity costs. The problems caused by rent controls are discussed by Díaz and Luengo-Prado (2008) and Coffey, McQuinn and O'Toole (2020).

150 However, when constructing a rental price index for renters, rents for all 3 categories should be used.

151 For convenience, we repeat this formula: $U_{tn} = [r_t - i_{Lt}]P_{Ltn}L_{tn} + [r_t - i_{St} + (1 + i_{St})\delta]P_{Stn}(1 - \delta)^{A(t,n)}S_{tn}$.

Recall that r_t is a smoothed longer-term opportunity cost of capital for period t, i_{Lt} is the long-term expected land price inflation rate, i_{St} is a long-term expected structure price inflation rate, and δ is the geometric structure depreciation rate. The *rent to capital value ratio* or *capitalization rate*[152] defined by (156) does not take into account the complications that were discussed earlier; that is, the *user cost* U_{tn} that would apply to an owner occupier of dwelling unit n in period t is not equal to the *rent* R_{tn} that a landlord would charge to a tenant for the same dwelling unit. Thus, it is necessary to modify (156) to take into account these complications. Define v_t as the period t rate of expected loss of rental income due to vacancies (as a fraction of period t capital value), define m_{tn} as expected period t maintenance and overhead expenditures for property n divided by the corresponding period t structure value,[153] define the land tax rate t_{Ltn} as the ratio of land taxes paid by the owners of property n in period t to the imputed land value $P_{Ltn}L_{tn}$ and the structure tax rate τ_{Stn} as the ratio of structure property taxes paid in period t for property n to imputed structure value $P_{Stn}(1-\delta)^{A(t,n)-1}S_{tn}$. Finally, define π_{tn} as the ratio of insurance payments made in period t by property n to imputed structure value $P_{Stn}(1-\delta)^{A(t,n)-1}S_{tn}$. Using the preceding discussion on complications to the standard user cost model, it can be seen that a more meaningful rent to value ratio decomposition for property n in period t is given by the following modification of (156) for $n = 1, \ldots, N$:

$$R_{tn}/V_{tn} = [r_t - i_{Lt} + v_t + \tau_{Ltn}]s_{Ltn} + [r_t - i_{St} + (1 + i_{St})\delta + v_t + \tau_{Stn} + m_{tn} + \pi_{tn}]s_{Stn}. \quad (157)$$

If property tax payments are not a separate category in the CPI, then the appropriate user cost for an owner of property n in period t, U_{tn}, as a fraction of property value, V_{tn}, is equal to the following expression:

$$U_{tn}/V_{tn} = [r_t - i_{Lt} + t_{Ltn}]s_{Ltn} + [r_t - i_{St} + (1 + i_{St})\delta + \tau_{Stn}]s_{Stn} \quad (158)$$

Note that the terms v_t, m_{tn}, and π_{tn} have been dropped from (158). Thus, the differences between (157) and (158) are equal to the following expressions for $n = 1, \ldots, N$:

$$R_{tn}/V_{tn} - U_{tn}/V_{tn} = v_t + [m_{tn} + \pi_{tn}]s_{Stn}. \quad (159)$$

It can be seen that simply applying the rent of a comparable rented dwelling unit to an owned unit will overstate the appropriate user cost that should be applied to the owned unit. The preceding computations did not take into account the possibility that the depreciation rate for a rental property is greater than the corresponding depreciation rate for a similar owned property.

The user cost formulae defined by (157)–(159) look rather complicated, and they require information that may not be available to the statistician. Thus, additional assumptions may have to be made that allow approximate user costs for owned dwelling units to be calculated. In situations where equivalent rental prices are not available, this may be the only feasible method to value the services of OOH. For example, the European Union issued the following regulation in 2005 that gives guidance in forming estimates of the services of OOH when equivalent rental prices are not available:

> Under the user-cost method, the output of dwelling services is the sum of intermediate consumption, consumption of fixed capital (CFC), other taxes less subsidies on production and net operating surplus (NOS). For owner occupied dwellings, no labour input is recorded for work done by the owners (1). Experience suggests that CFC and NOS are the two largest items, each representing 30 to 40 % of output.
>
> CFC should be calculated based on a perpetual inventory model (PIM) or other approved methods. A separate estimate for the owner-occupied residential buildings should be available. The net operating surplus should be measured by applying a constant real annual rate of return of 2.5% to the net value of the stock of owner-occupied dwellings at current prices (replacement costs). The real rate of return of 2.5% is applied to the value of the stock at current prices since the increase in current value of dwellings is already taken account of in the PIM. The same rate of return should be applied to the value of the land at current prices on which the owner-occupied dwellings are located.
>
> The value of land at current prices may be difficult to observe annually. Ratios of land value to the value of buildings in different strata may be derived from an analysis of the composition of the costs of new houses and associated land.
>
> Eurostat (2005).

To value the services of OOH in Iceland, the highly simplified user cost formula $U_t = (r_t^* + \delta)P_t$ was used, where U_t is the period t property user cost, r_t^* is a real interest rate (that varied between 3.6 and 4.3 percent), δ is an annual property depreciation rate (set equal to 1.25 percent), and P_t is a period t constant quality property price index.[154]

The Office for National Statistics in the United Kingdom used the user cost formula $U_t = (r + m + \delta - i)P_t$ to value the services of OOH, where r is a rate of return that includes a risk premium, δ is a depreciation rate, m is the maintenance rate, i is the expected capital appreciation rate of the unit, and P_t is

[152] Crone, Nakamura, and Voith (2000) used hedonic techniques to estimate both a rent index and a selling price index for housing in the United States. They also suggested that *capitalization rates* (that is, the ratio of the market rent of a housing property to its selling price) can be applied to an index of housing selling prices in order to obtain an imputed rent index for OOH. As will be shown below, capitalization rates are functions of many variables, some of which can change considerably over time. Also, it will be seen that capitalization rates for rented houses are not exactly appropriate as estimators for capitalization rates for owned houses.

[153] Older structures will probably have higher m_{tn} ratios.

[154] See Guðnason and Jónsdóttir (2011, 148). Note that as in the case of Iceland, the depreciation rate is applied to the total property value and not just to the structure value. This may be an acceptable approximation if the shares of land and structure in the total property value remain roughly constant over time. However, the empirical results of Knoll, Schularick, and Steger (2017) on house price inflation in 14 advanced economies indicate that the share of land has increased substantially in recent years.

a period t property price index.[155] For other simplified user cost formulae, see Verbrugge (2008) and Garner and Verbrugge (2009). When they set i equal to expected CPI inflation, reported rents approximated the corresponding user costs fairly well.

Returning back to the user cost formulae defined by (157) and (158), there is another factor that will tend to make the user cost valuation of the services of an owned dwelling unit much bigger than the corresponding actual rental price: Households that rent tend to be poorer than households that own. *Thus, renters simply cannot afford to rent high-end housing units.* High-end dwelling units that do rent will tend to rent for prices that are much less than their long-run user costs.[156] In advanced countries, the rent-to-property-value ratio for the more expensive properties tends to be about one half the rent-to-property-value ratio for the least expensive properties.[157] Thus, it is likely that the widespread use of the rental equivalence approach to the valuation of the services of OOH results in a measure of the value of housing services that are much lower than valuations based on long-run user costs.

There is one additional troublesome issue that has not been discussed thus far, and that is the issue of what to do with transfer costs. Transfer costs are the costs associated with the purchase of a dwelling unit. These costs include transaction taxes, legal fees, and real estate agent fees. These costs can be substantial. Thus, when a household purchases a dwelling unit, the final cost of the purchase should include all of the associated transfer costs. According to user cost theory, the appropriate valuation of the property at the end of the period should be the value of the sale of the house after transfer costs. This viewpoint suggests that the transaction costs of the purchaser should be immediately expensed in the period of purchase. However, from the viewpoint of a landlord who has just purchased a dwelling unit for rental purposes, it would not be sensible to charge the tenant the full cost of these transaction fees in the first month of rent. The landlord would tend to capitalize these costs and recover them gradually over the time period that the landlord expects to own the property. Thus, take the capitalized transfer costs that are charged to property n in period t and divide by the total property value V_{tn} to obtain the imputed property transfer cost ratio, λ_{tn}. The new rental cost formula for rented unit n in period t, the counterpart to (157), becomes the following formula:

$$R_{tn} = [r_t - i_{Lt} + v_t + \tau_{Ltn} + \lambda_{tn}]P_{Ltn}L_{tn} + [r_t - i_{St} + (1 + i_{St})\delta + v_t + \tau_{Stn} + m_{tn} + \pi_{tn} + \lambda_{tn}]P_{Stn}(1 - \delta)^{A(t,n)-1}S_{tn}. \quad (160)$$

From the viewpoint of an owner of a newly purchased dwelling unit, the owner does not actually sell the unit in the next period; the owner holds on to the dwelling unit for periods that range from 10 to 20 years on average. Thus, it is probably best to regard the transfer costs as a fixed cost that should be amortized over the expected holding period before the dwelling unit is sold again. If this amortization is appropriate, then the new user cost formula that is the counterpart to (158) is the following formula, which should be used to value the services of the owned unit if it is not rented out to tenants:

$$U_{tn} = [r_t - i_{Lt} + \tau_{Ltn} + \lambda_{tn}]P_{Ltn}L_{tn} + [r_t - i_{St} + (1 + i_{St})\delta + \tau_{Stn} + \lambda_{tn}]P_{Stn}(1-\delta)^{A(t,n)-1}S_{tn}. \quad (161)$$

The preceding discussion indicates that it is not a straightforward matter to determine the conceptually correct rental equivalent price to value the services of an owned dwelling unit.[158]

18. The Payments Approach and the Household Costs Index

A fifth possible approach to the treatment of OOH in a CPI, the *payments approach*, was described by Goodhart as follows:

> The second main approach is the payments approach, measuring actual cash outflows, on down payments, mortgage repayments and mortgage interest, or some subset of the above. . . . Despite its problems, such a cash payment approach was used in the United Kingdom until 1994 and still is in Ireland.
>
> Charles Goodhart (2001, F350–F351)

Thus, the *payments approach* to OOH is a modified *cash flow approach* to the *costs* of operating an owner-occupied dwelling.[159] It consists mainly of mortgage interest and principal payments along with property taxes. Imputations for capital gains, for the cost of capital tied up in house equity and depreciation, are ignored in this approach. This leads to the following objections to this approach; that is, it ignores the opportunity costs of holding the equity in the owner-occupied dwelling, it ignores depreciation, and it uses nominal interest rates without any offset for anticipated changes in the price of land and the structure over the accounting period. In general, due to its omission of depreciation, the payments approach will tend to lead to smaller monthly expenditures on OOH than the rental equivalence, user cost, and opportunity cost approaches, except during periods of

[155] See Lewis and Restieaux (2015, 156). We have changed their notation to match up with our notation.

[156] Often high-end houses that are not being used by their owners are rented out at prices that are far below their user costs just so someone will be in the house to maintain it and deter theft and vandalism. This is the "caretaker" explanation for falling ratios of rents to property value as property values increase.

[157] See Heston and Nakamura (2009, 2011). Aten (2018) found similar results for the United States. Shimizu, Diewert, Nishimura, and Watanabe (2012) found that user cost valuations for OOH in Tokyo were about 1.7 times as large as the equivalent rent estimates.

[158] For a more comprehensive decomposition of the user cost formula for an owned dwelling unit with a mortgage on the unit, see Díaz and Luengo-Prada (2008), Diewert, Nakamura, and Nakamura (2009), Diewert and Nakamura (2011), and Goeyvaerts and Buyst (2019).

[159] It is not a true cash flow approach because it omits the outlays for the purchase of a dwelling unit and it omits the potential benefits from the eventual sale of the unit. The Office for National Statistics (ONS) in the United Kingdom correctly labels this class of index as a *Household Costs Index* (HCI). The ONS describes this type of index as follows: "More specifically, they will aim to measure how much the nominal disposable income of different household groups would need to change, in response to changes in costs, to enable households to purchase the same quantity of goods and services of the same quality. Put simply, the broad approach of the HCI is to measure the outgoings of households" (ONS, 2017, 2).

high inflation, when the nominal mortgage rate term may become very large without any offsetting item for possible house price inflation.[160] This feature of the payments approach makes it unsuitable for measuring the services of OOH in a cost of living index.

The payments approach (like the acquisitions approach) is not a suitable approach if the goal of consumer price measurement is to measure the *flow* of consumption services. The rental equivalence, user cost, and opportunity cost approaches are useful for measuring the flow of consumption services. The acquisitions approach is useful for central bank monitoring of marketplace consumer price inflation due to its avoidance of imputations (except imputations for quality change are allowed).

The current corona virus pandemic has created an important use for the payments approach, which as indicated earlier, is essentially a cash flow approach; that is, how much money is required to allow a homeowner to cover the out-of-pocket costs associated with homeownership. For households who own their own home and lose their sources of income due to government-mandated lockdowns of sectors of the economy, it would be useful for the government to have estimates of the cash costs of keeping pandemic-affected homeowners in their dwelling units. However, note that what is required to meet this purpose are estimates of actual household costs rather than an index of their costs.

Another rationale for the payments approach has been developed by Astin and Leyland, and we outline it here.

Astin and Leyland (2015, 1) labeled their index version of the payments approach as a *Household Inflation Index* (HII), and they described it as a measure of "inflation as perceived and experienced by households in their role as consumers." Thus, broadly speaking, they wanted to produce a CPI that would more closely reflect consumer *experience* and *perceptions* of the inflation that they are experiencing. On page 3 of their paper, they outlined more specifically how their HII would differ from say the European Union's HICP, which Astin was instrumental in setting up:

- The HII would be a democratic index rather than a plutocratic index.[161]
- Interest paid on car loans, student loans, and credit cards are household expenditures that would be in scope for their index.
- The HII would include domestic household tourist expenditures abroad and exclude the consumption expenditures of foreign tourists in the home country.[162]
- The HII would include gross insurance premiums paid by households for cars, travel, and health.[163]

Astin and Leyland (2015) suggest that if the main purpose of a CPI is for the national indexation of pensions and only one CPI is available for this purpose, then a democratic CPI is better for this purpose than the usual plutocratic CPI.[164] Note that interest paid on car loans would be explicitly included in a user cost approach to household vehicle services and interest on capital tied up would be implicitly included in the monthly or annual fee for a leased car. Thus, interest payments made explicitly or implicitly by households appear in the non-payment approaches to the treatment of durables.

Astin and Leyland (2015, 3, 22) also made the following specific suggestions on how expenditures on OOH should be treated in their proposed HII; their proposed HII should include the following categories of household expenditure:

- Total mortgage payments (interest and principal) for the dwelling
- The transaction costs associated with the purchase of a house (transaction taxes, legal fees, and real estate agent fees)
- State and local property taxes
- Insurance
- Spending on renovations and extensions
- Minor repairs and maintenance

[160] See the comparison of alternative OOH price indices for the United Kingdom using the rental equivalence approach and the payments approach made by the ONS (2017, 10) (2018, 3). The latter publication also implements the acquisitions approach and compares the three indices for the United Kingdom. The payments approach index is much more volatile than the other two indices.

[161] This terminology dates back to Prais (1959). In practical terms, what the authors suggested is that national statistical agencies should construct separate CPIs for different groups of households that are demographically homogeneous. This is sensible advice. The demographic groups should be further classified into at least two subgroups depending on whether the households are renters or owners of dwelling units. The owners of dwelling units could be further decomposed into groups depending on the size of their mortgage debt. Owners of houses with no outstanding mortgages do not require the same compensation to maintain their level of housing service consumption as renters. As cash transactions become obsolete, banks and other financial institutions that issue household credit and debit cards will have information on household purchases at the individual household level. Thus in the future, it will become easier to construct CPIs for groups of households classified by their demographic characteristics and location.

[162] Including expenditures made by foreign visitors in a CPI is called the *domestic treatment* of household transactions and excluding foreign visitor expenditures while including national expenditures made by national residents abroad is called the *national treatment*. Thus, Astin and Leyland argued for the national treatment of tourist expenditures in their CPI concept. On the other hand, Astin (1999, 6–7) argued for the domestic treatment of tourist expenditures for the HICP, which is satisfactory if one wants an inflation index which is suitable for central bank monitoring of inflation. Diewert (2002, 595–596) argued that the domestic perspective was appropriate if one wanted a measure of consumer price inflation from a domestic producer perspective but the national perspective was preferred for a measure of consumer inflation faced by residents in the country under consideration.

[163] The gross premiums approach simply uses the total premium amount as the value of a property insurance policy held by a household. The net premium approach subtracts either actual claims or the expected value of payments for claims on the policies in force for the period under consideration. From a national accounts perspective, the net claims approach can be justified. But the gross claims approach can be justified on a consumer theory basis; see Diewert (1993, 415–423). However, in either case, the separation of the net or gross premium payments into price and quantity components is a complex matter where standard practice has not yet emerged. For example, suppose the risk associated with a claim increases over time. Should the price of the policy be quality adjusted downward which would be consistent with insurance services as a payment per unit risk?

[164] A plutocratic CPI implicitly gives a higher expenditure weight to the CPI of a well off household. In theory, a democratic CPI should give an equal weight to all households when forming the aggregate CPI. However, rather than producing a democratic CPI, if enough information on the spending habits of different groups is available, then it may be preferable to apply a separate CPI that reflected the spending habits of the particular group under consideration; that is, *it may be preferable to publish CPIs for different demographic groups.*

Typically, the payments approach applied to OOH would not include the principal component of mortgage payments, but Astin and Leyland properly note that these payments are *experienced* by households and hence they advocated including *total* mortgage payments in their HII.

The transaction costs associated with the purchase of a house should be in scope for an acquisitions CPI as well as in a CPI that was based on the user cost approach.[165] If the OOH component of the CPI were based on the rental equivalence approach, these transaction costs may be partially included in the imputed rent applied to the owned dwelling unit.[166]

State and local property taxes paid by homeowners on a continuing basis are definitely part of the costs of the services of owned housing and should be included in the user cost approach to housing. These costs are implicitly included in the rental equivalence approach.

Property insurance costs are imbedded in rents, and so these costs are included in market rents. Thus, using the rental equivalence approach to OOH, housing insurance payments should not be added to the equivalent rent. However, if the user cost approach is used for valuing the services of OOH, then housing insurance payments should be included in the user cost formula (along with property taxes). If insurance payments are a separate elementary category in the CPI, housing insurance payments could be included in the insurance subindex; that is, it is necessary to avoid double counting of household expenditures in constructing a CPI.

Household expenditures on renovations and extensions of an owned dwelling unit should be taken into account in a CPI. If a user cost approach is being used, then these expenditures should be applied to the structure component of the overall property user cost; that is, these expenditures should be deflated and added to the owned structure stock for the following period. Thus, a renovation to an owned property should lead to an increase in the real quantity of the structure on the property, but it may be difficult to capture this quality improvement using the rental equivalence approach. Depending on the details of how the rental equivalence approach to OOH is being implemented, it may be necessary to treat household expenditures on renovations of an owned dwelling unit as a separate category in the CPI. These expenditures should be amortized, but it may be acceptable to simply treat these expenditures as current expenditures instead of recognizing that the benefits of these renovation expenditures extend over time. Minor repairs and maintenance also have benefits that extend over time, but the time horizon of these benefits will tend to be relatively short, and so immediate expensing of these expenditures is an acceptable approximation.

The previous discussion of the Astin and Leyland proposal shows that many aspects of their suggested index are reasonable and not entirely inconsistent with the other approaches to the treatment of durables that we have considered in this chapter.[167] However, while their proposed HII is a reasonable index that can reflect household experience and perceptions of inflation, it is not an index that can measure household consumption of the services of durable goods because it focuses on the immediate costs associated with the purchase of durable goods and ignores possible future benefits of these purchases. Thus, the payments approach does not lead to indices which are suitable for indexation purposes.

The Office for National Statistics (ONS) in the United Kingdom has basically implemented much of the Astin and Leland proposed HII on an ongoing basis[168] and compared their new index with traditional acquisition and rental equivalence type CPIs; see the ONS (2018). However, the ONS (properly) recognized that the HII is focused on *costs*, and so they renamed the index as a Household Costs Index (HCI). The ONS describes their HCI in a methodology paper as follows:

> The Household Costs Indices (HCIs) are a set of experimental measures, currently in development 1, that aim to more closely reflect UK households' experience of changing prices and costs. More specifically, they will aim to measure how much the nominal disposable income of different household groups would need to change, in response to changes in costs, to enable households to purchase the same quantity of goods and services of the same quality. Put simply, the broad approach of the HCIs is to measure the outgoings of households.
>
> Office for National Statistics (2017, 2)

The ONS (2017, 2) noted that its HCI differs from a traditional CPI[169] that uses the rental equivalence approach to the treatment of OOH in the following four ways:

- The use of democratic weighting
- The use of a payments approach for measuring owner occupiers' housing costs (OOH)
- The inclusion of a measure of interest costs on credit card debt
- The use of gross expenditure to calculate the weight for insurance premiums

These dot points show that the ONS HCI is very similar to the Astin and Leyland HII. Both indices are versions of the payments approach. One major difference is that the ONS treatment of the payments approach includes mortgage interest on owned dwellings but excludes repayment of principal (whereas the HHI includes repayment of principal).[170]

[165] Conceptually, these transaction costs should be amortized over the expected holding period for a house purchase if one uses the user cost approach.

[166] However, the transaction costs of purchasing a rental property could have a longer amortization period if the rental property were held by the landlord for a longer time period than the average holding period for an owner of a property using the property to provide personal housing services.

[167] For a more complete discussion of the Astin and Leyland proposals, see ONS (2017).

[168] See the Office for National Statistics (2018).

[169] The traditional CPI that the ONS uses for comparison purposes (which they call the CPIH) is identical to Eurostat's HICP except that the services of OOH are measured by the rental equivalence approach plus local property taxes (Council Taxes); see ONS (2016, 3). The HICP simply omits the services of OOH.

[170] See ONS (2017, 8–9). The ONS payments approach to OOH is compared to the rental equivalence approach for the United Kingdom over the years 2006–2016. In the future, the ONS intends to produce HCIs with and without principal payments.

The ONS cautions users that there are problems with the use of the payments approach:

> Using a payments-based approach is commonly considered to be the best construct for assessing changes in net money incomes over time. This is in line with the stated aims of the HCIs, as briefly set out in section 1 of this article. However, the inclusion of nominal interest payments on mortgage debt is not without its problems conceptually. Its inclusion has been criticised as the treatment of interest flows is not consistent across persons (or households). For example, Charles Goodhart (2001) describes that if a borrower is worse off in some way when interest rates rise, then equivalently a lender owning an interest bearing asset is better off, and it may be analytically unsound to include one but not the other.
>
> Office for National Statistics (2017, 10)

The Goodhart objection to the payments approach is similar to our major objection: The approach measures the *costs* facing households but does not always recognize possible offsetting *benefits* that may accrue to households. However, a payments-type index can be useful as an index of household outlays and hence *perceptions of inflation*, which was the reason why Astin and Leyland introduced their version of the payments approach to the measurement of household inflation.

The ONS compares its versions of the rental equivalence, acquisitions, and payments approaches to the measurement of the services of owner-occupied dwellings on a regular ongoing basis; see ONS (2018, 3) for a chart of the three types of index for the United Kingdom over the years 2005–2018 on a quarterly basis. This chart shows the volatility of the payments-based index as compared to the other two indices. The rental equivalence index shows a steady upward growth with the net acquisitions index being slightly more volatile and finishing above the rental equivalence index. The payments index finished up far below the other two indices. This work by the ONS shows that the choice of methodology for the treatment of OOH in a CPI matters.

The ONS has provided a number of publications that explain in some detail both the rationale for the four main approaches to the treatment of OOH and data sources and methods; see ONS (2016, 2017, 2018). These publications should be useful for statistical agencies that are planning to offer alternative analytical indices for the treatment of OOH in a CPI. However, some comments on how the ONS constructs its rental equivalence and acquisitions indices for OOH may be useful.

The ONS (2016, 33) explains that it constructs its *net acquisitions approach index for OOH* as follows: Prices are based on a price index for new house sales, but the weights for these prices are set equal to the value of residential construction during the time period under consideration. The underlying price concept that the ONS would like to implement for its net acquisitions index is the price of the structure component of new dwelling unit sales to owners of houses who live in them. In other words, the land component of the selling price is to be stripped out of the sale price. The ONS recognizes that its empirical measures of price and expenditure are flawed for this treatment of OOH: The prices collected are sales of new dwelling units to *all* purchasers (purchasers who intend to live in the dwelling unit and hence are in scope and purchasers who plan to rent the dwelling unit to tenants and hence are not in scope for OOH), and more importantly, the selling prices of new dwelling units include a land component that is supposed to be excluded. The residential investment weights are also flawed because the investment includes investments in new rental units that should be excluded. The reason for the preceding desired treatment of the acquisitions approach applied to new dwelling units is that Eurostat would like to implement this net approach[171] to new house sales for its HICP.[172] A possible better solution to implementing this pricing concept is to simply use the deflator for residential building investments, which is already constructed by countries as part of their national accounts. This deflator could be improved if the residential building price index could be decomposed into two strata: one stratum for sales intended for purchasers who plan to live in the new residential structure and another stratum for investments in rental properties. But even if this latter decomposition of the residential construction price index were not made, using an overall residential construction price index along with estimates for the value of new rental buildings and for total residential construction[173] would lead to a price index that should be much closer to the desired (by Eurostat) price index for OOH. These limitations of the ONS acquisitions price index for OOH should be kept in mind when looking at their chart for the acquisitions, rental equivalence, and payments indices for OOH in the United Kingdom; see ONS (2018, 3).[174]

There are also problems with the ONS (2018, 3) rental equivalence price index series. ONS (2016, 21–23) explains how to construct its rental equivalence index. A sample of rental prices is collected across the United Kingdom, and then the prices are stratified based on the (i) type of dwelling unit,[175] (ii) postal code' (iii) number of bedrooms, and (iv) whether furnished or unfurnished. Given our earlier discussion of the application of hedonic regression models to the construction of house price indices and rental indices, it can be seen that the list of stratifying characteristics is not ideal. The number of bedrooms can act as a proxy for floor space area, but there is no information on land plot area and no information on the age of the structure. The latter omission is particularly important. The evidence from hedonic regressions for both selling prices and rental prices

[171] It is a net approach because the gross purchase price of a new dwelling unit is to be net of the land price component of the selling price. It is also a net approach because it excludes intra-household sales of residential housing units.

[172] There is already an EU regulation that requires member countries to produce such a monthly acquisitions-type index for OOH but since not all EU countries are yet able to comply with the regulation, the current HICP still ignores OOH.

[173] The OOH expenditure weight could be obtained by subtracting the value of rental residential investment from total residential investment value. A possible reason for not implementing this version of the net acquisitions approach to OOH is that national statistical agencies are not in a position to produce a monthly construction cost index in a timely manner.

[174] It is likely that the ONS (2018, 3) acquisitions index has an upward bias relative to the Eurostat target net acquisitions index because the ONS price index has a substantial land price component in it which will reflect rapidly increasing land prices in the UK over the sample period.

[175] The four categories are (i) detached house, (ii) semi-detached house, (iii) terraced house, and (iv) flat or maisonette.

indicates that the aging of the structure leads to a quality decline in structure service of about 1percent per year for a residential property. Thus, if the land and structure components of property value are equal, the neglect of structure depreciation could lead to a downward bias of about 0.5 percent per year in a rental price index that does not take into account the quality decline due to aging of the property. This is a substantial bias. The ONS should stratify rental properties according to the age of the structure in order to take this bias into account (or move to a hedonic regression framework with the age of the structure as an explanatory variable).

There is another potential bias in the ONS rental equivalence index for OOH. The rental equivalence approach to valuing the services of OOH is an *opportunity cost approach*. The choice to live in an owned dwelling unit rather than rent it out means that the owner of the structure is giving up *the current market rent* that the owner of the unit could get if the unit were rented. This is the appropriate opportunity cost from the viewpoint of the rental equivalence approach to valuing the services of an owned dwelling unit. Thus, the appropriate opportunity cost is the *current* rent for a property that is similar to the owned property to a new tenant, but the opportunity cost that the ONS (2016) uses is the average of all *existing* rental prices for similar properties.[176] The latter average will tend to be lower than new rents if there is rental price inflation and higher if there is rental price deflation.[177] Thus, the ONS procedures undervalue the rental opportunity costs of living in an owned dwelling unit under conditions of general inflation.[178]

Recall the discussion in the previous section that compared the rental equivalence approach to the opportunity cost approach to the valuation of owned housing services. The opportunity cost approach sets the true opportunity cost of living in an owned dwelling unit as the maximum of its market rental price and its user cost. In many countries, the ratio of house rent to property value approximately doubles as we move from less expensive to more expensive properties.[179] This means that, in general, the rental equivalence approach to the valuation of OOH will give a much smaller expenditure *weight* to the services of OOH as compared to the user cost and opportunity cost approaches.

The preceding limitations of the ONS rental equivalence price index for OOH should be kept in mind when looking at the ONS charts for the acquisitions, rental equivalence, and payments indices for OOH in the United Kingdom; see the charts in ONS (2018, 3).

We conclude this section by reviewing some issues concerning the timing of payments made by households for the consumption of durable goods. Consider the following quotation from the ONS:

> Consumption expenditure can be measured in three ways which it is important to distinguish. These ways are:
>
> *Acquisition* means that the total value of all goods and services delivered during a given period is taken into account, whether or not they were wholly paid for during the period.
>
> *Use* means that the total value of all goods and services consumed during a given period is taken into account.
>
> *Payment* means that the total payments made for goods and services during a given period is taken into account, whether or not they were delivered.
>
> For practical purposes, these three concepts cannot be distinguished in the case of non-durable items bought for cash, and they do not need to be distinguished for many durable items bought for cash. The distinction is, however, important for purchases financed by some form of credit, notably major durable goods, which are acquired at a certain point of time, used over a considerable number of years, and paid for, at least partly, some time after they were acquired, possibly in a series of instalments. Housing costs paid by owner-occupiers are an obvious example.
>
> Office for National Statistics (2010, 6)

In what follows, we will look at the problems associated with the three methods of valuation in a number of specific cases.[180]

Case 1: *The payment period coincides with the acquisition period*. Let P_1 be the acquisition price for such a unit of a durable good in period 1. Then the acquisition price in period 1 is obviously P_1, the payments price is also P_1, and the period 1 user cost price is p_1, and its exact form depends on the model of depreciation that is applicable for this particular durable good. In other words, there are no problems in sorting out the three methods of valuation in this case.

Case 2: *The initial payment period coincides with the acquisition period but payments for the purchase of the durable continue on for subsequent periods*. Suppose that payments must be made for T periods and the sequence of monetary payments is $\pi_1,\pi_2, \ldots,\pi_T$. Suppose also that the sequence of expected one-period financial opportunity costs of capital for the purchasing household is $r_1,r_2, \ldots,r_{T-1}$. Then the discounted stream of payments, P_1, is the period 1 (expected) cost of purchasing the good, where P_1 is defined as follows:

$$P_1 \equiv \pi_1 + (1 + r_1)^{-1}\pi_2 + (1 + r_1)^{-1}(1 + r_2)^{-1}\pi_3 + \ldots + (1 + r_1)^{-1}(1 + r_2)^{-1} \ldots (1 + r_{T-1})^{-1}\pi_T. \quad (162)$$

[176] Existing (contractual) rental prices are appropriate for valuing rental properties in a CPI. But they are not appropriate for use in the rental equivalence approach (except as an approximation): the rental equivalence approach requires the use of current opportunity costs, not historical costs.

[177] The ONS is well aware of this difference: “There is an important difference between newly let properties and existing tenants; price rises are highest when properties are newly let compared with existing tenants renewing a lease” (Office for National Statistics, 2016, 50).

[178] The use of all contract rents instead of renewal contract rents to value the services of a house will lead to a lower *weight* in the CPI (under conditions of general inflation) but it may not affect the corresponding rate of *change* in the price index.

[179] See footnotes 157 and 158 in the previous section.

[180] We will address the problems from the viewpoint of the approach to intertemporal consumption theory that dates back to Hicks (1946).

In this case, the acquisitions price for the durable good in period 1 is defined to be P_1, the payments price is π_1, and the user cost will be determined using the appropriate depreciation model, where P_1 is taken to be the beginning of the period price for the durable good. In a subsequent period $t \leq T$, the acquisitions price for the used durable good will be 0, the payments price will be π_t, and the period t user cost value v_t will be determined using the appropriate depreciation model for this type of durable good. *If* the useful life of the durable good happens to equal T and *if* the period t payment is equal to the corresponding period t user cost valuation v_t for $t = 1,2, . . .,T$, then obviously, the period t user cost valuation v_t will be equal to the observable period t payment π_t.[181]

There are problems associated with the computation of P_1 defined by (162); that is, in order to compute P_1 when the durable good is purchased during period 1, the sequence of future payments π_t has to be known, and guesses will have to be made on the magnitudes of the sequence of expected nominal interest rates r_t. However, the important point to be made here is that P_1 defined by (162) will be *less* than the simple sum of π_t, $\Sigma_{t=1}^{T} \pi_t$, provided that the nominal interest rates r_t are positive.

Case 3: *The full payment for the good (or service) is made in period 1 but the services of the commodity are not delivered until period t*. Let the period 1 payment be π_1 as usual. Thus, the sequence of payments associated with the purchase of the commodity under consideration is π_1 for period 1 and 0 for all subsequent periods. The acquisition of the commodity does not take place until period t, but the appropriate acquisition price P_t is not the period 1 payment, π_1, but the following *escalated period 1 price*:

$$P_t \equiv (1 + r_1)(1 + r_2) . . . (1 + r_{t-1})\pi_1. \qquad (163)$$

The logic behind this valuation is the following one. During period 1 when the product was paid for, the payment could have been used to pay down debt (at the interest rate r_1) or the payment could have been used to invest in an asset that earned the rate of return r_1. Thus, after one period, the opportunity cost of the investment in the pre-purchased product has grown to $\pi_1(1 + r_1)$; after two periods, the opportunity cost has grown to $\pi_1(1 + r_1)(1 + r_2)$, . . . ; and by period t, when the good or service is acquired, the opportunity cost has grown to $\pi_1(1 + r_1)(1 + r_2) . . . (1 + r_{t-1})$, which is (163). The important point to be made here is that P_t will be *greater* than the period 1 prepayment, π_1, provided that the nominal interest rates r_t are positive. Since the product has not been acquired by the household for periods 1,2, . . .,t–1, the corresponding user cost valuations, $v_1,v_2, . . .,v_{t-1}$, should be set equal to 0. However, when period t is reached, "normal" user costs can be calculated for durable goods using the P_t defined by (163) as the beginning of period t price of the durable, assuming that the form of depreciation is known.

Prepayment for services or durable goods is widespread; for example, trip and hotel reservations made in advance and paid for in advance are service examples, and prepayment for condominium units that are under construction is a durable good example.

Case 4: *The good or service is acquired in period 1 but is not paid for until period 2*. In this case, the sequence of payments is 0, π_2, 0, . . .,0. The commodity is acquired in period 1 and the appropriate period 1 acquisition price is P_1 defined as follows:

$$P_1 \equiv (1 + r_1)^{-1}\pi_2. \qquad (164)$$

The justification for this acquisition price runs as follows: The purchasing household lays aside the amount of money P_1 to buy the product in period 1. This money is invested and earns the one-period rate of return r_1. Thus, when period 2 comes along, the household has $P_1(1 + r_1) = \pi_2$, which is just enough money to complete the purchase in period 2. Thus, P_1 is an appropriate period 1 acquisitions price. If the commodity is a durable good, then assuming that the form of depreciation is known, P_1 defined by (164) can be used as the beginning of period 1 price for the period 1 user cost, and the entire sequence of user costs can be calculated.

This form of pricing is used as a way of offering lower prices for a wide variety of products. A particular application of this model to a service is the use of credit cards to purchase consumption items. A household that pays its balance owed on time can avoid interest charges and thus can postpone payment for its household purchases for up to one month in many cases.[182]

If interest rates are very low, then statistical agencies may well find it is not worth taking into account the preceding refinements. However, if nominal interest rates are high, it may be necessary to make some of the preceding adjustments.[183]

It can be seen that the durability of housing creates a host of measurement problems that statistical agencies are not well equipped to handle.

19. The Treatment of Household Monetary Balances in a CPI

The treatment of financial services in a CPI is a controversial topic. The academic literature has not come to a general consensus on how to model many financial services provided to households. However, given the importance of financial services in all economies, it may be useful to outline some of the issues surrounding this topic.

We will concentrate on household banking services in this section.[184] It is clear that many services that banks provide to households are reasonably simple to model; that is, it is

[181] The period t user cost valuation v_t for a unit of the durable good that is t periods old can be converted into an equivalent amount of a new unit of a durable good if the geometric or one-hoss-shay model of depreciation is applicable for the durable good under consideration. Otherwise, units of the durable goods of different ages at the same point in time need to be aggregated using an index number formula.

[182] However, a household that does not pay off its balance owed in a timely fashion will find itself in Case 3.

[183] We note that the above adjustments for the timing of payments have implications for the system of national accounts that have not been fully worked out.

[184] There are also important controversies surrounding the treatment of insurances services in a CPI.

straightforward to collect prices on the costs of using the services of a safety deposit box. It is not so straightforward to measure the services of bank household deposit services or bank loans to households. However, it is possible to adapt the basic user cost theory explained in Section 4 to model the services of household transferable deposits[185] and time or savings deposits held in banks or other financial institutions.

Recall from Section 4 that r^0 was the household's opportunity cost of financial capital at the beginning of period 0. In the national accounts banking literature, r^0 is called the *household reference rate of return on safe assets* for the period under consideration. We assume that the bank providing household deposit services pays the deposit holder an interest rate of $r_D{}^0$ on its holdings of bank deposits of the type under consideration at the end of the accounting period. For a checking account, $r_D{}^0$ will typically be equal to zero. For a savings or time deposit account, $r_D{}^0$ will typically be a number that is less than r^0.[186] Then the *beginning of the period user cost* $u_D{}^0$ of holding a dollar of deposits (on average) throughout period 0 is[187]

$$u_D{}^0 \equiv 1 - (1 + r_D{}^0)/(1 + r^0) = (r^0 - r_D{}^0)/(1 + r^0). \quad (165)$$

This user cost looks at the opportunity cost of holding a dollar of bank deposits at the beginning of the accounting period (as opposed to investing the dollar at the rate of return of r^0 or to paying off outstanding debts at the interest rate of r^0), but at the end of the accounting period, the deposit holder gets the dollar back plus interest $r_D{}^0$ earned in tying up that dollar for the period, but this amount, equal to $1 + r_D{}^0$, needs to be discounted by one plus the opportunity cost of capital, $1 + r^0$.

As usual, instead of discounting costs and benefits to the beginning of the accounting period, the costs and benefits can be anti-discounted to the end of the accounting period, which leads to the following *end-of-period user cost* $u_D{}^{0*}$ of holding a dollar of deposits throughout the period:

$$u_D{}^{0*} \equiv (1 + r^0)u_D{}^0 = (r^0 - r_D{}^0). \quad (166)$$

Define the household's nominal *asset value* of bank deposits held at the beginning of period 0 as $V_D{}^0$, and define the corresponding nominal value of *deposit services* for period 0 as $v_D{}^0$. Given the end-of-period user cost for a bank deposit, $p_D{}^0$, and the (asset) value of household bank deposits at the beginning of period 0, $V_D{}^0$, the *imputed (nominal) value of bank deposit services from the household perspective*, $v_D{}^0$, is defined as the product of $p_D{}^0$ and $V_D{}^0$:

$$v_D{}^0 \equiv u_D{}^{0*}V_D{}^0 = (r^0 - r_D{}^0)V_D{}^0. \quad (167)$$

The end-of-period user cost of holding a dollar's worth of bank deposits defined by (166) and the corresponding value of total deposit services defined by (167) are derived using a household opportunity cost perspective.

The question that now arises is: "What is the real value of deposit services to the household?"; that is, what is the appropriate deflator for the nominal service flow $v_D{}^0$ defined by (167)? The answer to this question is not clear cut.

In order to answer this question, it is necessary to ask what the *purpose* of the deposit holdings is. Feenstra (1986) and others provide an answer to this purpose question: Cash balances or their deposit equivalents are held in order to buy consumer goods and services. The idea here is that consumers receive income flows from selling their labor services or from dividend and bond interest payments at regular intervals. These income flows are converted into cash or bank deposits at the beginning of the payment period and then are spent over the course of the payment period in order to purchase consumer goods and services. This is termed a *cash in advance model*. Thus, if the household purpose in holding bank deposits is to buy consumer goods and services, then it seems reasonable to deflate $V_D{}^0$ by the corresponding period 0 aggregate consumer price level (excluding financial services), say $P_C{}^0$, to obtain the equivalent amount of real consumption that the nominal value of deposit balances, $V_D{}^0$, could purchase; that is, define the *consumption equivalent* of the household's nominal deposit balances, $q_D{}^0$, as follows:[188]

$$q_D{}^0 \equiv V_D{}^0/P_C{}^0. \quad (168)$$

Now deflate the value of household deposit *services*, $v_D{}^0$ defined by (167), by $q_D{}^0$ in order to obtain the *price for bank deposit services from the household perspective* $p_D{}^0$ defined as follows:

$$\begin{aligned} p_D{}^0 &\equiv v_D{}^0/q_D{}^0 \quad (169)\\ &= [(r^0 - r_D{}^0)V_D{}^0]/[V_D{}^0/P_C{}^0] \text{ using (167) and (168)}\\ &= (r^0 - r_D{}^0)P_C{}^0. \end{aligned}$$

Note that the price level for deposit services for period 0, $p_D{}^0$, is proportional to the consumer price level for goods and services in period 0, $P_C{}^0$. The corresponding real value of deposit services for period 0, $q_D{}^0$, is set equal to the period 0 nominal household stock of monetary balances, $V_D{}^0$, deflated by the consumer price level for period 0, $P_C{}^0$.[189] We note that the data variables which appear in equations (167)–(169) are all relatively easy to measure, with the exception of the reference rate or opportunity cost of financial capital

[185] Before internet banking became popular, these deposits were called checking deposits.

[186] Under current conditions, for some countries, r_D could be a small negative number. For most countries that exhibit low inflation, r_D will be a small positive number.

[187] This user cost of money dates back to Diewert (1974), who did not include the deposit interest rate term, $r_D{}^0$. This extra term was introduced by Donovan (1978) and Barnett (1978, 1980).

[188] Feenstra (1986) provided a formal model of a cash in advance economy that justifies the deflation of nominal household bank balances by a CPI. Alternatively, we can make a simple opportunity cost argument to justify deflating V_D by P_C: by holding deposits, the household gives up current consumption. Note that the conceptually correct CPI to do the deflation should be based on the *acquisition approach* to the construction of a CPI.

[189] This user cost approach to modeling the price, quantity, and value of household monetary services was developed by Donovan (1978), Barnett (1978, 1980), Fixler (2009), and Barnett and Chauvet (2011). For discussions on how the user cost approach to modeling monetary services in both the household and production accounts, see Fixler and Zieschang (1991, 1992, 1999), Diewert, Fixler, and Zieschang (2011, 2016), Diewert (2014), and Diewert and Fox (2018, 2019).

interest rate, r^0. There is no easy answer on how exactly to measure this interest rate.[190]

The cash in advance approach to modeling the demand for monetary services can be applied to the household demand to hold currency and transferable deposits. Since many time deposit bank accounts also allow households to use these deposits to buy goods and services, the preceding model could also be applied to these accounts. To get a rough idea of the relative size of these two types of monetary accounts and their relationship to total annual purchases of consumer goods and services, the data from the Integrated Macroeconomic Accounts for the United States for the year 2019 can be used; see the Bureau of Economic Analysis (2020). For 2019, final consumption expenditures were 14.56 trillion dollars; household holdings of currency and transferable deposits were 1.26 trillion dollars and holdings of time and savings deposits were 10.16 trillion dollars. It can be seen that these holdings of household monetary assets are much larger than the amounts that cash in advance models would predict. Thus, households are holding large amounts of bank deposits for reasons other than for the purpose of funding their normal purchases of consumer goods and services.

Monetary theory suggests several additional reasons for consumers to hold currency and bank deposits:

- As a *store of value*; that is, to save up funds for future major purposes such as buying an automobile or a house
- For *precautionary purposes*; that is, as a form of self-insurance against future income shocks
- For *portfolio balancing purposes*

These purposes reflect the fact that a large fraction of consumer holdings of currency and bank deposits are probably held for investment purposes broadly speaking, rather than as a means of facilitating current period purchases of consumer goods and services. Thus, statistical agencies constructing a CPI may want to rule holdings of currencies and deposits as being out of scope. On the other hand, it would be useful for statistical agencies to produce a supplementary CPI that includes the services of monetary deposits along the lines indicated previously because household holdings of monetary deposits have a direct opportunity cost in foregone consumption and including monetary services in a broader measure of consumption would be useful for some analytic purposes.

It should be mentioned that not all economists subscribe to the preceding user cost approach for modeling the household demand for monetary services. The Basu, Fernald, Inklaar, and Wang approach to modeling bank outputs and inputs is critical of the preceding deflation-based user cost approach to modeling the price and quantity of financial services presented in this section.[191] Rather than defining the real quantity of financial services as being proportional to suitably deflated stocks of financial assets held by banks or households, the aforementioned authors suggest that *direct measures* of the services rendered by consuming financial services be constructed (such as the number of transactions) and then the nominal service flows would be deflated by these direct measures, yielding an implicit price index for the services as an alternative to deflating nominal asset holdings by a price index.[192] We have two responses to this methodology:

- Direct transaction fees are taken into account separately in our suggested user cost approach (although some free services may be omitted in this approach)
- The transaction fee approach seems to be a cost of production approach that is not necessarily relevant for consumers of the service

However, economists have not settled on a universally accepted methodology for modeling the household demand to hold bank deposits, so statistical agencies need to keep this fact in mind.

20. Summary and Conclusion

It is clear that constructing constant quality price indices for consumer durables is not as conceptually simple as constructing price indices for nondurables and services where the matched model approach can guide index construction. The fundamental problem of accounting arises when constructing a price index for the services of a durable good: *Imputations will have to be made in order to decompose the initial purchase cost into period-by-period service flow components over the life time of the durable good.* The method of imputation will involve assumptions, which may not be accepted by all interested parties. In spite of this difficulty, it will be useful for statistical agencies to construct analytical series for the services of long-lived consumer durables that can be made available to the public. This will meet the needs of different users.[193]

When constructing property price indices based on sales of properties, there is another factor that reinforces the argument for multiple price indices: When transactions are sparse, property indices based on the sparse data can be very volatile. Thus, for some purposes, it may be useful to construct a smoothed index (that is revised for a certain number of months) in addition to a volatile real-time index.[194]

For non-housing consumer durables, at present, statistical agencies produce CPI based on the *acquisitions approach*. This type of index is useful for measuring consumer price inflation based on market transactions, with minimal imputations (except for possible quality change). In addition to this standard index, statistical agencies should produce supplementary indices based on the

[190] See the discussion between Fixler (2009), Basu (2009), and Wang, Basu, and Fernald (2009).

[191] See Wang (2003), Wang, Basu, and Fernald (2009), Basu (2009), Basu, Inklaar, and Wang (2011), Inklaar and Wang (2010), and Colangelo and Inklaar (2012).

[192] See Inklaar and Wang (2010) and Colangelo and Inklaar (2012) for empirical estimates of the differences between the demand-side deflation approach and an approach incorporating "engineering" indicators of financial service delivery.

[193] Hill, Steurer, and Waltl (2020), using Australian data, found substantial differences using the three main approaches to the valuation of OOH. This emphasizes the need for statistical agencies to produce estimates for all three approaches if possible.

[194] See Rambaldi and Fletcher (2014) on various smoothing methods that could be used. Diewert and Shimizu (2020) suggested a very simple method which worked well in their empirical application.

user cost approach in order to more accurately measure the flow of services generated by stocks of consumer durables.[195]

The valuation of the services of housing is very difficult due to the fact that housing services are unique: The location of each dwelling unit is unique and the location affects the land price component of the property and thus affects rents and user costs. Moreover, the structure component of housing does not remain constant over time due to depreciation of the structure and to renovation expenditures. various methods that can deal with these difficulties (to some degree at least) were explained in Sections 11–17. The details of the methods are too complex to summarize here, but the suggested methods based on various hedonic regression models have been applied and offer possible ways forward.

For OOH, the three main approaches should be implemented by statistical agencies to serve the needs of different users. There are two possible versions for the *acquisitions approach* depending on how the new dwelling purchase is treated: (i) construct a price index for the purchase of new dwelling units in an inclusive basis, including the price of land, or (ii) exclude land cost from the purchase cost. The latter index should be well approximated by a construction cost index (with appropriate margins added for developer margins). The inclusive index will be useful for new house buyers, who have to pay for the land plot as well as the new structure. A *rental equivalence price index* for the services of OOH should also be constructed. For many countries, such an (implicit) index is already available as part of the national accounts valuation for the services of OOH.[196] A *user cost index* for the services of OOH should also be constructed since the user cost valuation for the services of a high-end dwelling unit will typically be much greater than the corresponding price that the unit could rent for.[197] If the rental equivalent rent and user cost for an owned unit are constructed and are of the same quality, then applying the *opportunity cost approach* to the valuation of the services of the owned unit is appropriate.

For rented housing, the measurement problems are perhaps not so severe; monthly or weekly rents can be observed for the same rental unit, and so it would seem that the usual matched model methodology could be applied in this situation. However, an index based on the matched model methodology and normal index number theory will generally have an upward bias because of the neglect of depreciation or a lowering of quality due to the aging of the structure. In order to deal with this bias, it will in general require a hedonic regression approach with age as one of the explanatory variables.

We will conclude by noting some specific recommendations that emerge from this chapter:

- There are three main approaches for the treatment of consumer durables in a CPI: the acquisitions approach, the rental equivalence approach, and the user cost approach.
- The acquisitions approach is suitable (for most purposes) for durable goods with a relatively short expected useful life.
- The acquisitions approach is particularly useful for central bankers who want consumer inflation indices that are largely free from imputations.
- The acquisitions approach provides an index for purchases of a durable good, and this index is a required input into the construction of a user cost index.
- The remaining two approaches are useful for measuring the flow of services yielded by consumer durables over their useful lives.
- At present, only the flow of services for OOH is estimated by national statistical agencies (using the rental equivalence or user cost approaches) because this information is required for the international System of National Accounts; that is, the flow of services for other durable goods is not measured at present.
- The acquisitions approach will substantially understate the value of the service flow from consumer durables that have relatively long lives. Hence, at least one of the rental equivalence or user cost approaches should be implemented by statistical agencies for durables with long lives.[198] Examples of long-lived durables are automobiles and household furnishings.
- The rental equivalence approach to the valuation of the services provided by consumer durables is the preferred method of valuation (with the exception of OOH) when rental or leasing markets for the class of durables exist because, in principle, no imputations are required to implement this method.[199]
- However, when rental markets for the durable good under consideration are thin or do not exist, then the user cost approach should be used to value the services of the durable good.
- The user cost approach requires the construction of a price index for new acquisitions of the durable. It also requires a model of depreciation and assumptions about the opportunity cost of capital and about expected asset inflation rates. Thus, the user cost approach necessarily involves imputations.
- In order to avoid unnecessary volatility in the user costs, long-run expected asset inflation rates should be used in the user cost formula.[200]

[195] The rental equivalence approach could be used for durables that are rented or leased but typically, most consumer durables are not rented. Depreciation rates will in most cases be based on educated guesses. Durable stock estimates can be made once depreciation rates have been determined. The current value of household stocks of consumer durables should also be constructed and added to household balance sheets.

[196] However, if possible, the equivalent rents should be based on new contract rents in order to provide a current opportunity cost for using the services of an owned dwelling unit; recall the discussion on this point in Section 17.

[197] Recall the evidence on this point in Heston and Nakamura (2009, 2011) and others.

[198] If the acquisitions approach is used in the headline CPI, the alternative approaches can be published as experimental or supplementary series.

[199] However, for housing, the "comparable" rental property may not be exactly the same as the owned unit. Moreover, the observed rents may include insurance services and the services of some utilities and possibly furniture. It will be difficult to extract these costs from the observed rent.

[200] The long-run asset inflation rate over the past 20 or 25 years or the long-run rate of inflation in housing rents could be used to predict future asset inflation rates. Many other prediction methods could be used; see, for example, Verbrugge (2008). However, the focus should be on predicting long-run asset inflation rather than period-to-period inflation.

- Rental markets for high-end dwelling units are generally nonexistent or very thin and hence, it may not be possible to use the rental equivalence approach for high-end OOH. Even if some rental information on high-end housing units is available, usually these rents are far below the corresponding user costs.
- The "true" opportunity cost for using the services of a consumer durable is the maximum of its rental price (if it exists) and its user cost. Thus, the use of the rental equivalence approach to value the services of a high-end housing unit will understate the "true" service flow by a substantial amount.[201]
- In order to construct national balance sheets and to measure national multifactor productivity, it is necessary to decompose the selling prices of dwelling units into structure and land components. This can be done for both detached housing and condominium units using hedonic regression techniques; see Sections 11 and 12. This decomposition is also required in order to construct accurate user costs for housing units since depreciation applies to the structure but not to the land component of the property.
- When constructing price indices for rental housing, statistical agencies need to make an adjustment to observed rents for the same unit for depreciation of the structure and possible improvements to the structure.
- When using observed rents to measure the service flow for comparable owned properties, statistical agencies should use new contract rents to evaluate the service flow for the owned units since rents for continuing tenants may be sticky and not reflect current opportunity costs.
- When constructing user costs for OOH, statistical agencies need to avoid double counting of some housing-related costs that may appear elsewhere in the CPI such as insurance costs. Similar double counting problems may arise with housing rents, which may include the services of some utilities or furniture and of course, the housing rent will include insurance costs. In principle, these associated costs should be deducted from the observed rent and placed in the appropriate classification of the CPI. In practice, this is a difficult imputation problem.
- A variant of the acquisitions approach is sometimes applied to OOH. This variant excludes the land component of the purchase of a new house. As mentioned earlier, this variant reduces to a construction cost index for housing with some allowance made for builders' profit margins. This variant generates valuations for OOH that may be far below the comparable rental equivalent and user cost valuations. It is difficult to justify the use of this variant in a CPI.[202]
- A more comprehensive measure of the flow of consumption services would include estimates for the flow of services from storable goods and for household holdings of currency and transferable deposits.

Which of the three main methods for valuing the purchase of a consumer durable should be used for indexing pensions or indexing salaries for consumer inflation? This is a difficult question to answer. If we start out with the idea that we want a national CPI, then if there were no durable goods, a national acquisitions price index would be the target index. But it is not clear that this is the "correct" price index once we recognize the existence of consumer durables: An acquisitions index does not recognize the imputed costs of previously purchased consumer durable goods. Thus, in order to deal with this difficulty, we need to move to a rental equivalence index or a user cost index if rental markets are thin. But if a national index based on say the rental equivalence approach were used to determine pension payments for veterans or retired civil servants or for employees in an industry, the resulting payments do not take into account that different households have different holdings of consumer durables (housing in particular), and they do not need to be compensated for their consumption of existing holdings. There are additional complications that need to be addressed:

- If the goal is to maintain the purchasing power of a certain group of households (such as retirees or veterans), then an appropriate index needs to be constructed for the relevant group.
- The relevant group may live in different regions of the country and so, in principle, separate indices need to be constructed for each region by group.

[201] Long-run user costs and rents will tend to be approximately equal to each other for lower-end housing units since this type of housing unit will be built by property developers who provide rental housing and they need to set rents that are approximately equal to their long-run user costs. However, short-run dynamics can cause user costs and rents to diverge even for lower-end housing units.

[202] It is not a "true" acquisitions price that is observed in the marketplace since it involves imputations to subtract the land value from the property sale. The resulting acquisitions price obviously does not reflect the total services provided by the purchase.

Annex: Adjusting Housing Rental Price Indices for New and Disappearing Units

A problem with the simple repeat rents model that was proposed in Section 14 is that the model that extended the modified repeat rents index to deal with new and disappearing units was highly simplified. In this annex, this simple model is generalized to allow for multiple overlapping products and for many new and disappearing rental units.[203]

Suppose that there are M rental properties in scope for the rental price index that are present in periods 0 and 1. Suppose further that for rental property n in period t that has a structure on it of age $A(t,m)$, its utility or *real quantity* q_{tm} as a function of the structure age is defined as follows:

$$q_{tm} \equiv (1-\Delta)^{A(t,m)};\ t = 0,1;\ m = 1, \ldots, M, \qquad \text{(A.1)}$$

where Δ is the assumed common to all rental units geometric *property depreciation rate* that is due to structure depreciation. As in Section 14, the observed rent for property m in period t is R_{tm}. The constant quality price for property m in period t, p_{tm}, is defined as the observed rent R_{tm} divided by the corresponding real quantity q_{tm}:

$$p_{tm} \equiv R_{tm}/q_{tm} = R_{tm}/(1-\Delta)^{A(t,m)};\ t = 0,1;\ m = 1, \ldots, M. \qquad \text{(A.2)}$$

In period 0, there are also J rental properties that disappear in period 1. The observed rents, structure ages, and constant quality prices and quantities for period 0 for these disappearing rental units are R_{D0j}, $A^{\circ}(0,j)$, p_{D0j}, and q_{D0j}, respectively, for $j = 1, \ldots, J$. The constant quality prices and quantities for these units satisfy the following relationships:

$$q_{D0j} \equiv (1-\Delta)^{A^{\circ}(0,j)};\ j = 1, \ldots, J; \qquad \text{(A.3)}$$

$$p_{D0j} \equiv R_{D0j}/q_{D0j} = R_{D0j}/(1-\Delta)^{A^{\circ}(0,j)};\ j = 1, \ldots, J. \qquad \text{(A.4)}$$

In period 1, there are also K newly occupied rental properties that appear in period 1. The observed rents, structure ages, and constant quality prices and quantities for period 1 for these new rental units are $R_{N1}k$, $A^{*}(1,k)$, $p_{N1}k$, and $q_{N1}k$, respectively, for $k = 1, \ldots, K$. The constant quality prices and quantities for these units satisfy the following relationships:[204]

$$q_{N1}k \equiv (1-\Delta)^{A^{*}(1,k)};\ k = 1, \ldots, K; \qquad \text{(A.5)}$$

$$p_{N1}k \equiv R_{N1}k/q_{N1}k = R_{N1}k/(1-\Delta)^{A^{*}(1,k)};\ k = 1, \ldots, K. \qquad \text{(A.6)}$$

Thus, for each rental unit m that is rented in periods 0 and 1, the tenant occupying rental unit m experiences a utility decline going from period 0 to 1 that is equal to

$$\begin{aligned} q_1m/q_0m &= (1-\Delta)^{A(0,m)+1}/(1-\Delta)^{A(0,m)} \\ &= 1-\Delta;\ m = 1, \ldots, M. \end{aligned} \qquad \text{(A.7)}$$

Using definitions (A.2), the corresponding rates of price change are given by

$$\begin{aligned} p_{1m}/p_{0m} &= [R_{1m}/(1-\Delta)^{A(0,m)+1}]/[R_{0m}/(1-\Delta)^{A(1,m)}] \\ &= [R_{1m}/R_{0m}]/(1-\Delta);\ m = 1, \ldots, M. \end{aligned} \qquad \text{(A.8)}$$

The *maximum overlap Laspeyres rent index*, P_{MOL}, is defined as follows:

$$\begin{aligned} P_{MOL} &\equiv \Sigma_{m=1}^{M}\, p_1mq_0m/\Sigma_{m=1}^{M}\, p_0mq_0m \qquad \text{(A.9)} \\ &= \Sigma_{m=1}^{M}\, [R_1m/(1-\Delta)^{A(0,m)+1}][(1-\Delta)^{A(0,m)}]/ \\ &\quad \Sigma_{m=1}^{M}\, R_0m \text{ using (A.1) and (A.2)} \\ &= \Sigma_{m=1}^{M}\, [R_1m/(1-\Delta)]/\Sigma_{m=1}^{M}\, R_0m \\ &= [\Sigma_{m=1}^{M}\, R_1m/\Sigma_{m=1}^{M}\, R_0m]/(1-\Delta) \\ &= P_{RR}/(1-\Delta), \end{aligned}$$

where P_{RR} is the *repeat rent index* defined as

$$P_{RR} \equiv \Sigma_{m=1}^{M}\, R_1m/\Sigma_{m=1}^{M}\, R_0m. \qquad \text{(A.10)}$$

The *maximum overlap Paasche rent index*, P_{MOP}, is defined as follows:

$$\begin{aligned} P_{MOP} &\equiv \Sigma_{m=1}^{M}\, p_1mq_1m/\Sigma_{m=1}^{M}\, p_0mq_1m \qquad \text{(A.11)} \\ &= \Sigma_{m=1}^{M}\, R_1m/\, \Sigma_{m=1}^{M}\, [R_0m/(1-\Delta)^{A(0,m)}][(1-\Delta)^{A(0,m)+1}] \\ &\quad \text{using (A.1) and (A.2)} \\ &= \Sigma_{m=1}^{M}\, R_1m/\Sigma_{m=1}^{M}\, R_0m(1-\Delta) \\ &= [\Sigma_{m=1}^{M}\, R_1m/\Sigma_{m=1}^{M}\, R_0m]/(1-\Delta) \\ &= P_{RR}/(1-\Delta) \text{ using definition (A.10).} \end{aligned}$$

The *maximum overlap Fisher rent index*, P_{MOF}, is defined as the geometric mean of the maximum overlap Laspeyres and Paasche indices:

$$\begin{aligned} P_{MOF} &\equiv [P_{MOL}P_{MOP}]^{1/2} \\ &= P_{RR}/(1-\Delta) \text{ using (A.9) } \textit{and} \text{ (A.11).} \end{aligned} \qquad \text{(A.12)}$$

Thus, the maximum overlap Laspeyres, Paasche, and Fisher rent indices are all equal to the repeat rent index P_{RR} divided by $(1-\Delta)$, where Δ is the property rental geometric depreciation rate.

The property depreciation rate allows us to adjust the observed rent for each rental unit for quality changes due to the aging of the structure, but it does not allow us to compare the utility of each rental unit with an alternative rental unit. In order to form overall price and quantity indices that take into account the new and disappearing rental units, it is necessary to make some stronger assumptions. Thus, we assume that tenants evaluate the relative utility of the various rental units that are available according to the following utility function:

$$\begin{aligned} &f(q_1, \ldots, q_M;\ q_{D1}, \ldots, q_{DJ};\ q_{N1}, \ldots, q_{NK}) \\ &\equiv \Sigma_{m=1}^{M}\, \alpha_m q_m + \Sigma_{j=1}^{J}\, \beta_j q_{Dj} + \Sigma_{k=1}^{K}\, \gamma_k q_{Nk}, \end{aligned} \qquad \text{(A.13)}$$

[203] This more general model is based on Section 4 in Diewert (2021).

[204] If the new period 1 rental unit has a new structure, then $A^{*}(1,k)$ is set equal to 0; if the "new" rental unit consists of an old structure that was not rented in period 0, then $A^{*}(1,k)$ is set equal to the age of the structure in months if the index is a monthly index.

where α_m, β_j, and γ_k are the positive parameters that reflect the relative utilities of the various rental properties that are available in any given period. The "observed" quantities, q_0m, q_1m, q_{D0j}, and $q_{N1}k$, for the various available rental properties in periods 0 and 1 are defined by (A.1), (A.3), and (A.5).

In period t, a tenant occupying rental unit m incurs the rental cost $R_{tm} = p_{tm}q_{tm}$. The utility benefit to the tenant $B_{tm} = \alpha_m q_{tm}$. Since it is assumed that each tenant has the same preferences, the cost benefit ratios, $R_{tm}/B_{tm} = p_{tm}q_{tm}/a_m q_{tm} = p_{tm}/a_m$ should be approximately equal to a constant that we can interpret as an aggregate price level P^t; that is, utility-maximizing tenants should bid up rents for units m, where R_{tm}/B_{tm} is low and avoid rental units, where R_{tm}/B_{tm} is relatively high. Thus, for period 0, the following approximate equalities should hold:

$$R_0m/\alpha_m q_0m \approx P^0;\ m = 1, \ldots, M; \qquad \text{(A.14)}$$
$$R_{D0j}/\beta_j q_{D0j} \approx P^0;\ j = 1, \ldots, J. \qquad \text{(A.15)}$$

Now use definitions (A.1) and (A.3) to eliminate q_0m and q_{D0j} from (A.14) and (A.15). After suitable rearrangement, we obtain the following approximate equalities:

$$R_0m \approx P^0\alpha_m(1 - \Delta)^{A(0,m)};\ m = 1, \ldots, M; \qquad \text{(A.16)}$$
$$R_{D0j} \approx P^0\beta_j(1 - \Delta)^{A^*(0,j)};\ j = 1, \ldots, J. \qquad \text{(A.17)}$$

The same logic can be applied to the rental units that are available in period 1. Thus, for period 1, the following approximate equalities should hold:

$$R_1m/\alpha_m q_1m \approx P^1;\ m = 1, \ldots, M; \qquad \text{(A.18)}$$
$$R_{N1}k/\gamma_k q_{N1}k \approx P^1;\ k = 1, \ldots, K. \qquad \text{(A.19)}$$

Again use definitions (A.1) and (A.3) to eliminate q_1m and $q_{N1}k$ from (A.18) and (A.19) in order to obtain the following approximate equalities:

$$R_{1m} \approx P^1\alpha_m(1 - \Delta)^{A(1,m)};\ m = 1, \ldots, M; \qquad \text{(A.20)}$$
$$R_{N1k} \approx P^1\gamma_k(1 - \Delta)^{A^*(1,k)};\ k = 1, \ldots, K. \qquad \text{(A.21)}$$

If we take logarithms of both sides of equations (A.16), (A.17), (A.20), and (A.21), define $\varphi \equiv 1 - \Delta$, and add error terms to the resulting equations, it can be seen that we have an *adjacent period time dummy hedonic regression model*, which can be used to obtain estimates for the M unknown α_m, the J unknown β_j, the K unknown g_k, and the three unknown parameters, P^0, P^1, and Δ. There are $2M + J + K + 3$ degrees of freedom in the regression. However, it can be seen that not all parameters can be identified; it will be necessary to impose a normalization on the parameters such as $P^0 = 1$ or $\alpha_1 = 1$. The age of the structure on each rental property is the only rental property characteristic that is required to run the hedonic regression.[205]

[205] But typically, the properties in scope will have some similar characteristics; for example, they will be classified based on the type of property, whether furnished or unfurnished, and the presence in local neighborhood. The adequacy of the model should be judged by the fit of the regression.

Suppose the normalization $P^0 = 1$ is used in the hedonic regression. Denote the estimates for P^1 and Δ by P^{1*} and Δ^*. We need to define the resulting aggregate real rental quantities for the two periods under consideration. We first define some sub-aggregate rental values. Define the value of rents for units that are present in both periods as the continuing aggregate rents, $R_C{}^0$ and $R_C{}^1$, for periods 0 and 1 as follows:

$$R_C{}^0 \equiv \Sigma_{m=1}{}^M R_{0m};\ R_C{}^1 \equiv \Sigma_{m=1}{}^M R_{1m}. \qquad \text{(A.22)}$$

Define the aggregate rents for the units that are present in one period but absent in the other period as follows:

$$R_D{}^0 \equiv \Sigma_{j=1}{}^J R_{D0j};\ R_N{}^1 \equiv \Sigma_{k=1}{}^K R_{N1k}. \qquad \text{(A.23)}$$

Denote the aggregate price levels for the rental units in scope for periods 0 and 1 by P^0 and P^1 and the corresponding aggregate quantity levels by Q^0 and Q^1. These aggregates are defined as follows:

$$P^0 \equiv 1;\ P^1 \equiv P^{1*};\ Q^0 \equiv R_C{}^0 + R_D{}^0;\ Q^1 \equiv (R_C{}^1 + R_N{}^1)/P^{1*}. \qquad \text{(A.24)}$$

In order to justify the definitions for the period 0 and 1 aggregates, suppose the approximate equalities (A.14) and (A.15) hold exactly. Then it can be seen that

$$\begin{aligned} Q^0 &\equiv R_C{}^0 + R_D{}^0 && \text{(A.25)}\\ &= \Sigma_{m=1}{}^M R_{0m} + \Sigma_{j=1}{}^J R_{D0j} \text{ using (A.22) and (A.23)}\\ &= \Sigma_{m=1}{}^M P^0\alpha_m q_{0m} + \Sigma_{j=1}{}^J P^0\beta_j q_{D0j} \text{ using (A.14) and (A.15)}\\ &= \Sigma_{m=1}{}^M \alpha_m q_{0m} + \Sigma_{j=1}{}^J \beta_j q_{D0j} \text{ using } P^0 \equiv 1. \end{aligned}$$

Thus, Q^0 is equal to period 0 aggregate utility $\Sigma_{m=1}{}^M \alpha_m q_0m + \Sigma_{j=1}{}^J \beta_j q_{D0j}$. Now suppose the approximate equalities (A.18) and (A.19) hold exactly. Using (A.24), we have

$$\begin{aligned} Q^1 &\equiv (R_C{}^1 + R_N{}^1)/P^{1*} && \text{(A.26)}\\ &= (\Sigma_{m=1}{}^M R_1m + \Sigma_{k=1}{}^K R_{N1}k)/ P^{1*}\\ &\quad \text{using (A.22) and (A.23)}\\ &= (\Sigma_{m=1}{}^M P^{1*}\alpha_m q_1m + \Sigma_{k=1}{}^K P^{1*}\gamma_k q_{N1}k)/P^{1*}\\ &\quad \text{using (A.18) and (A.19)}\\ &= \Sigma_{m=1}{}^M \alpha_m q_1m + \Sigma_{k=1}{}^K \gamma_k q_{N1}k. \end{aligned}$$

Thus, Q^1 is equal to period 1 aggregate utility $\Sigma_{m=1}{}^M \alpha_m q_1m + \Sigma_{k=1}{}^K \gamma_k q_{N1}k$.

It is useful to analyze the factors that influence the growth of real aggregate rents. Using definitions (A.24), we have the following decomposition that is a counterpart to the decomposition (137) for real rents in Section 14 of the main text:

$$\begin{aligned} Q^1/Q^0 &= [(R_C{}^1 + R_N{}^1)/P^{1*}]/[R_C{}^0 + R_D{}^0] && \text{(A.27)}\\ &\quad \text{using definitions (A.24)}\\ &= [1/P^{1*}][R_C{}^1/R_C{}^0][1 + (R_N{}^1/R_C{}^1)]/[1 + (R_D{}^0/R_C{}^0)]\\ &= P_{RR}[1/P^{1*}][1 + (R_N{}^1/R_C{}^1)]/[1 + (R_D{}^0/R_C{}^0)], \end{aligned}$$

where $P_{RR} \equiv R_C^1/R_C^0$ is *the repeat rent price index* for the rental properties that are occupied in both periods.

If a reasonable estimate for the rental property depreciation rate Δ^* is available to the statistical agency, then there is an alternative to running the hedonic regression defined by the logarithms of equations (A.16), (A.17), (A.20), and (A.21). This alternative approach simply sets P^{1*}, which plays a crucial role in definitions (A.24), equal to the maximum overlap Fisher index P_{MOF} defined by (A.12). Thus, the definitions (A.24) are replaced by the following definitions:

$$P^0 \equiv 1;\ P^{1*} \equiv P_{MOF} \equiv P_{RR}/(1-\Delta^*);\ Q^0 \equiv R_C^{\,0} + R_D^{\,0};\ Q^1 \equiv (R_C^{\,1} + R_N^{\,1})/P^{1*}. \tag{A.28}$$

Under these conditions, the decomposition of Q^1/Q^0 becomes

$$Q^1/Q^0 = P_{RR}[1/P^{1*}][1 + (R_N^{\,1}/R_C^{\,1})]/[1 + (R_D^{\,0}/R_C^{\,0})] \text{ using (A.27)} \tag{A.29}$$
$$= [1-\Delta^*][1 + (R_N^{\,1}/R_C^{\,1})]/[1 + (R_D^{\,0}/R_C^{\,0})] \text{ using (A.28),}$$

which is analogous to (137) in the main text.

References

Adams, Brian, and Randal J. Verbrugge. 2021. "Location, Location, Structure Type: Rent Divergence within Neighborhoods." Manuscript, Washington, DC: Bureau of Labor Statistics.

Aizcorbe, Ana. 2014. *A Practical Guide to Price Index and Hedonic Techniques*. Oxford: Oxford University Press.

Astin, John. 1999. "The European Union Harmonized Indices of Consumer Prices (HICP)." *Statistical Journal of the United Nations ECE* 16(1999): 123–135.

Astin, John, and Jill Leyland. 2015. "Towards a Household Inflation Index." Unpublished paper prepared for the Royal Statistical Society.

Aten, Bettina H. 2018. "Valuing Owner-Occupied Housing: An Empirical Exercise Using the American Community Survey (ACS) Housing Files." Research Paper, Bureau of Economic Analysis, Washington, DC, March.

Australian Bureau of Statistics. 2016. "Making Greater Use of Transactions Data to Compile the Consumer Price Index." Information Paper 6401.0.60.003, November 29, Canberra: ABS.

Bailey, Martin J., Richard F. Muth, and Hugh O. Nourse. 1963. "A Regression Method for Real Estate Price Construction." *Journal of the American Statistical Association* 58: 933–942.

Bank for International Settlements. 2018. "BIS Property Prices." BIS Statistics Warehouse, Basel: BIS.

Barnett, William A. 1978. "The User Cost of Money." *Economics Letters* 2: 145–149.

———. 1980. "Economic Monetary Aggregates: An Application of Aggregation and Index Number Theory." *Journal of Econometrics* 14: 11–48.

Barnett, William A., and Marcelle Chauvet. 2011. "How Better Monetary Statistics Could Have Signalled the Financial Crisis." *Journal of Econometrics* 161: 6–23.

Basu, Susanto. 2009. "Incorporating Financial Services in a Consumer Price Index: Comment." In *Price Index Concepts and Measurement*, edited by W. Erwin Diewert, John Greenlees, and Charles R. Hulten, Studies in Income and Wealth, vol. 70, 266–271. Chicago: University of Chicago Press.

Basu, Susanto, Robert Inklaar, and J. Christina Wang. 2011. "The Value of Risk: Measuring the Services of U.S. Commercial Banks." *Economic Inquiry* 49(1): 226–245.

Beidelman, Carl R. 1973. *Valuation of Used Capital Assets*. Sarasota Florida: American Accounting Association.

———. 1976. "Economic Depreciation in a Capital Goods Industry." *National Tax Journal* 29: 379–390.

Böhm-Bawerk, Eugen V. 1891. *The Positive Theory of Capital*, W. Smart (translator of the original German book published in 1888). New York: G.E. Stechert.

Bostic, Raphael W., Stanley D. Longhofer, and Christian L. Readfearn. 2007. "Land Leverage: Decomposing Home Price Dynamics." *Real Estate Economics* 35(2): 183–2008.

Bureau of Economic Analysis. 2020. "Integrated Macroeconomic Accounts for the United States, Table S.3.a, Households and Nonprofit Institutions Serving Households." March 30, https://www.bea.gov/data/special-topics/integrated-macroeconomic-accounts

Bureau of Labor Statistics. 1983. "Changing the Home Ownership Component of the Consumer Price Index to Rental Equivalence." *CPI Detailed Report*, Washington, DC: BLS.

Burnett-Issacs, Kate, Ning Huang, and W. Erwin Diewert. 2021. "Developing Land and Structure Price Indexes for Ottawa Condominium Apartments." Discussion Paper 16–09, Vancouver School of Economics, University of British Columbia, Vancouver, BC, Canada; forthcoming in the *Journal of Official Statistics*.

Cairns, Robert D. 2013. "The Fundamental Problem of Accounting." *Canadian Journal of Economics* 46(2): 634–655.

Canning, John Bennet. 1929. *The Economics of Accountancy*. New York: The Ronald Press Co.

Christensen, Laurits R., and Dale W. Jorgenson. 1969. "The Measurement of U.S. Real Capital Input, 1929–1967." *Review of Income and Wealth* 15(4): 293–320.

Church, A. Hamilton. 1901. "The Proper Distribution of Establishment Charges, Part III." *The Engineering Magazine* 21: 904–912.

Clapp, John M. 1980. "The Elasticity of Substitution for Land: The Effects of Measurement Errors." *Journal of Urban Economics* 8: 255–263.

Coffey, Cathal, Kieran McQuinn, and Conor O'Toole. 2020. "Rental Equivalence, Owner-Occupied Housing and Inflation Measurement: Micro-level Evidence from Ireland." Economic and Social Research Institute Working Paper 685, Dublin, Ireland.

Colangelo, Antonio, and Robert Inklaar. 2012. Bank Output Measurement in the Euro Area: A Modified Approach." *The Review of Income and Wealth* 58(1): 142–165.

Court, Andrew T. 1939. "Hedonic Price Indexes with Automotive Examples." In *The Dynamics of Automobile Demand*, 98–117. New York: General Motors Corporation.

Crone, Theodore M., Leonard I. Nakamura, and Richard P. Voith. 2000. "Measuring Housing Services Inflation." *Journal of Economic and Social Measurement* 26: 153–171.

———. 2011. "Hedonic Estimates of the Cost of Housing Services: Rental and Owner Occupied Units." In *Price and Productivity Measurement: Volume 1: Housing*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 51–68. Victoria: Trafford Press.

Crosby, Neil, Steven Devaney, and Vicki Law. 2012. "Rental Depreciation and Capital Expenditure in the UK Commercial Real Estate Market, 1993–2009." *Journal of Property Research* 29(3): 227–246.

Davis, Morris A., and Jonathan Heathcote. 2007. "The Price and Quantity of Residential Land in the United States." *Journal of Monetary Economics* 54(8): 2595–2620.

de Haan, Jan. 2015. "Rolling Year Time Dummy Indexes and the Choice of Splicing Method." Room Document at the

14th meeting of the Ottawa Group, May 22, Tokyo, http://www.stat.go.jp/english/info/meetings/og2015/pdf/t1s3room
Díaz, Antonia, and Maria Jose Luengo-Prado. 2008. "On the User Cost and Homeownership." *Review of Economic Dynamics* 11(3): 584–613.
Diewert, W. Erwin. 1974. "Intertemporal Consumer Theory and the Demand for Durables." *Econometrica* 42: 497–516.
———. 1976. "Exact and Superlative Index Numbers." *Journal of Econometrics* 4: 114–145.
———. 1977. "Walras' Theory of Capital Formation and the Existence of a Temporary Equilibrium." In *Equilibrium and Disequilibrium in Economic Theory*, edited by Gerhard Schwödiauer, 73–126. Dordrecht: Reidel Publishing Co.
———. 1980. "Aggregation Problems in the Measurement of Capital." In *The Measurement of Capital*, edited by Dan Usher, 433–528. University of Chicago Press, Chicago.
———. 1993. "Symmetric Means and Choice under Uncertainty." In *Essays in Index Number Theory*, edited by W. Erwin Diewert and Alice O. Nakamura, vol. 1, 355–433. Amsterdam: North-Holland.
———. 2002. "Harmonized Indexes of Consumer Prices: Their Conceptual Foundations." *Swiss Journal of Economics and Statistics* 138(4): 547–637.
———. 2005a. "Issues in the Measurement of Capital Services, Depreciation, Asset Price Changes and Interest Rates." In *Measuring Capital in the New Economy*, edited by Carol Corrado, John Haltiwanger, and Daniel Sichel, 479–542. Chicago: University of Chicago Press.
———. 2005b. "Adjacent Period Dummy variable Hedonic Regressions and Bilateral Index Number Theory." *Annales d'Économie et de Statistique* 79/80: 759–786.
———. 2008. "OECD Workshop on Productivity Analysis and Measurement: Conclusions and Future Directions." In *Proceedings From the OECD Workshops on Productivity Measurement and Analysis*, 11–36. Paris: OECD.
———. 2010. "Alternative Approaches to Measuring House Price Inflation." Discussion Paper 10–10, Department of Economics, The University of British Columbia, Vancouver, Canada, V6T 1Z1.
———. 2014. "The Treatment of Financial Transactions in the SNA: A User Cost Approach." *Eurostat Review of National Accounts and Macroeconomic Indicators* 1: 73–89.
———. 2021. "Quality Adjustment Methods." In *Consumer Price Index Theory*, Draft Chapter 8. Washington, DC: International Monetary Fund, https://www.imf.org/en/Data/Statistics/cpi-manual.
Diewert, W. Erwin, Dennis Fixler, and Kimberly D. Zieschang. 2011. "The Measurement of Banking Services in the System of National Accounts." Discussion Paper 11–04, Department of Economics, University of British Columbia, Vancouver, Canada.
———. 2016. "Problems with the Measurement of Banking Services in a National Accounting Framework." In *National Accounting and Economic Growth*, edited by John M. Hartwick, 117–176. Cheltenham UK: Edward Elgar Publishing.
Diewert, W. Erwin, and Kevin J. Fox. 2016. "Sunk Costs and the Measurement of Commercial Property Depreciation." *Canadian Journal of Economics* 49(4): 1340–1366.
———. 2018. "Alternative User Costs, Productivity and Inequality in US Business Sectors." In *Productivity and Inequality*, edited by William H. Greene, Lynda Khalaf, Paul Makdissi, Robin Sickles, Michael Veall, and Marcel Voia, 21–69. New York: Springer.
———. 2019. "Money and the Measurement of Total Factor Productivity." *Journal of Financial Stability* 42: 84–89.
———. 2020. "Substitution Bias in Multilateral Methods for CPI Construction Using Scanner Data." *Journal of Business and Economic Statistics*, https://doi.org/10.1080/07350015.2020.1816176
Diewert, W. Erwin, Jan de Haan, and Rens Hendriks. 2011. "The Decomposition of a House Price Index into Land and Structures Components: A Hedonic Regression Approach." *The Valuation Journal* 6: 58–106.
———. 2015. "Hedonic Regressions and the Decomposition of a House Price index into Land and Structure Components." *Econometric Reviews* 34: 106–126.
Diewert, W. Erwin, and Robert J. Hill. 2010. "Alternative Approaches to Index Number Theory." In *Price and Productivity Measurement*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 263–278. Victoria Canada: Trafford Press.
Diewert, W. Erwin, Ning Huang, and Kate Burnett-Issacs. 2017. "Alternative Approaches for Resale Housing Price Indexes." Discussion Paper 17–05, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.
Diewert, W. Erwin, and Denis A. Lawrence. 2000. "Progress in Measuring the Price and Quantity of Capital." In *Econometrics Volume 2: Econometrics and the cost of Capital: Essays in Honor of Dale W. Jorgenson*, edited by Lawrence J. Lau, 273–326. Cambridge, MA: The MIT Press.
Diewert, W. Erwin, and Alice O. Nakamura. 2011. Accounting for Housing in a CPI." In W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, *Price and Productivity Measurement: Volume 1: Housing*, 7–32. Victoria: Trafford Press.
Diewert, W. Erwin, Alice O. Nakamura, and Leonard I. Nakamura. 2009. "The Housing Bubble and a New Approach to Accounting for Housing in a CPI." *Journal of Housing Economics* 18(3): 156–171.
Diewert, W. Erwin, Kiyohiko G. Nishimura, Chihiro Shimizu, and Tsutomu Watanabe. 2020. *Property Price Index: Theory and Practice*. Tokyo: Springer Japan.
Diewert, W. Erwin, and Chihiro Shimizu. 2015. "Residential Property Price Indexes for Tokyo." *Macroeconomic Dynamics* 19: 1659–1714.
———. 2016. "Hedonic Regression Models for Tokyo Condominium Sales." *Regional Science and Urban Economics* 60: 300–315.
———. 2017. "Alternative Approaches to Commercial Property Price Indexes for Tokyo." *Review of Income and Wealth* 63(3): 492–519.
———. 2019. "Residential Property Price Indexes: Spatial Coordinates versus Neighbourhood Dummy variables." Discussion Paper 19–09, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.
———. 2020. "Alternative Land Price Indexes for Commercial Properties in Tokyo." *Review of Income and Wealth* 66(4): 784–824.
Diewert, W. Erwin, and Hui Wei. 2017. "Getting Rental Prices Right for Computers: Reconciling Different Perspectives on Depreciation." *Review of Income and Wealth*, 63(S1):149–168.
Donovan, Donal J. 1978. "Modeling the Demand for Liquid Assets: An Application to Canada." *IMF Staff Papers* 25: 676–704.
European Central Bank. 2018. *Residential Property Price Index Statistics*, Statistical Data Warehouse. Frankfurt: European Central Bank.
Eurostat. 2001. *Handbook on Price and Volume Measures in National Accounts*. Luxembourg: European Commission.
———. 2005. "On the principles for estimating dwelling services for the purpose of Council Regulation (EC, Euratom) No 1287/2003 on the harmonisation of gross national income at market prices." *Official Journal of the European Union*, Commission Regulation (EC) No 1722/2005, October 20.

———. 2013. *Handbook on Residential Property Prices Indices (RPPIs)*. Luxembourg: Publications Office of the European Union.
———. 2017. *Technical Manual on Owner-Occupied Housing and House Price Indices*. Brussels: European Commission.
Eurostat, IMF, OECD, UN and World Bank. 1993. *System of National Accounts 1993*. United Nations: New York, New York.
Feenstra, Robert C. 1986. "Functional Equivalence between Liquidity Costs and the Utility of Money." *Journal of Monetary Economics* 17(2): 271–291.
Fenwick, David. 2009. "A Statistical System for Residential Property Price Indices." Eurostat-IAOS-IFC Conference on Residential Property Price Indices, Bank for International Settlements, November.
———. 2012. "A Family of Price Indices." The Meeting of Groups of Experts on Consumer Price Indices, UNECE/ILO, United Nations Palais des Nations, Geneva Switzerland, May 30-June 1.
Fisher, Irving. 1922. *The Making of Index Numbers*, Boston: Houghton-Mifflin.
Fixler, Dennis. 2009. "Incorporating Financial Services in a Consumer Price Index." In *Price Index Concepts and Measurement*, edited by W. Erwin Diewert, John Greenlees, and Charles R. Hulten, Studies in Income and Wealth, vol. 70, 239–266. Chicago: University of Chicago Press.
Fixler, Dennis, and Kimberly D. Zieschang. 1991. "Measuring the Nominal Value of Financial Services in the National Income Accounts." *Economic Inquiry*, 29: 53–68.
———. 1992. "User Costs, Shadow Prices, and the Real Output of Banks." In *Output Measurement in the Service Sector*, edited by Zvi Griliches, 219–243. Chicago: University of Chicago Press.
———. 1999. "The Productivity of the Banking Sector: Integrating Financial and Production Approaches to Measuring Financial Service Output." The *Canadian Journal of Economics* 32: 547–569.
Francke, Marc K. 2008. "The Hierarchical Trend Model." In *Mass Appraisal Methods: An International Perspective for Property Valuers*, edited by Tom Kauko and Maurizio Damato, 164–180. Oxford: Wiley-Blackwell.
Francke, Marc K., and Gerjan A. Vos. 2004. "The Hierarchical Trend Model for Property Valuation and Local Price Indices." *Journal of Real Estate Finance and Economics* 28: 179–208.
Gallin, Joshua, and Randal J. Verbrugge. 2019. "A Theory of Sticky Rents: Search and Bargaining with Incomplete Information." *Journal of Economic Theory* 183: 478–519.
Garcke, Emile, and John M. Fells. 1893. *Factory Accounts: Their Principles and Practice*, 4th ed. (1st ed. 1887). London: Crosby, Lockwood and Son.
Garner, Thesia I., and Randal J. Verbrugge. 2009. "Reconciling User Costs and Rental Equivalence: Evidence from the U.S. Consumer Expenditure Survey." *Journal of Housing Economics* 18(3): 172–192.
———. 2011. "The Puzzling Divergence of Rents and User Costs, 1980–2004: Summary and Extensions." In *Price and Productivity Measurement: Volume 1: Housing*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 125–146. Victoria: Trafford Press.
Genesove, David. 2003. "The Nominal Rigidity of Apartment Rents." *The Review of Economics and Statistics* 85(4): 844–853.
Gilman, Stephen. 1939. *Accounting Concepts of Profit*, New York: The Rolland Press Co.
Goeyvaerts, Geert, and Erik Buyst. 2019. "Do Market Rents Reflect User Costs?." *Journal of Housing Economics* 44(1): 112–130.
Goodhart, Charles. 2001. "What Weights should be Given to Asset Prices in the Measurement of Inflation?." *The Economic Journal* 111(June), F335–F356.
Griliches, Zvi. 1971. "Introduction: Hedonic Price Indexes Revisited." In *Price Indexes and Quality Change*, edited by Zvi Griliches, 3–15. Cambridge, MA: Harvard University Press.
Guðnason, Rósmundur, and Guðrún R Jónsdóttir. 2011. "Owner Occupied Housing in the Icelandic CPI." In W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, *Price and Productivity Measurement: Volume 1: Housing*, 147–150. Victoria: Trafford Press.
Hall, Robert E. 1971. "The Measurement of Quality Change from Vintage Price Data." In *Price Indexes and Quality Change*, edited by Zvi Griliches, 240–271. Cambridge, MA: Harvard University Press.
Hall, Robert E., and Dale W. Jorgenson. 1967. "Tax Policy and Investment Behavior." *American Economic Review* 57: 391–414.
Harper, Michael J., Ernst R. Berndt, and David O. Wood. 1989. "Rates of Return and Capital Aggregation Using Alternative Rental Prices." In *Technology and Capital Formation*, edited by Dale W. Jorgenson and Ralph Landau, 331–372. Cambridge, MA: The MIT Press.
Heston, Alan, and Alice O. Nakamura. 2009. "Questions about the Equivalence of Market Rents and User Costs for Owner Occupied Housing." *Journal of Housing Economics* 18: 273–279.
———. 2011. "Reported Prices and Rents of Housing: Reflections of Costs, Amenities or Both?." In *Price and Productivity Measurement: Volume 1: Housing*, edited by W. Erwin Diewert, Bert M. Balk, Dennis Fixler, Kevin J. Fox, and Alice O. Nakamura, 117–124. Victoria: Trafford Press.
Hicks, John R. 1946. *Value and Capital*, 2nd ed. Oxford: Clarendon Press.
Hill, Robert J. 2013. "Hedonic Price Indexes for Residential Housing: A Survey, Evaluation and Taxonomy." *Journal of Economic Surveys* 27: 879–914.
Hill, Robert J., and Michael Scholz. 2018. "Can Geospatial Data Improve House Price Indexes? A Hedonic Imputation Approach with Splines." *Review of Income and Wealth* 64(4): 737–756.
Hill, Robert J., Michael Scholz, Chihiro Shimizu, and Miriam Steurer. 2018. "An Evaluation of the Methods used by European Countries to compute their Official House Price Indices." *Economie et Statistique/Economics and Statistics* Numbers 500–502: 221–238.
Hill, Robert J., Miriam Steurer, and Sofie R. Waltl. 2020. "Owner Occupied Housing, Inflation and Monetary Policy." Graz Economic Papers 2020–18, Department of Public Economics, University of Graz, Graz, Austria.
Hoffmann, Johannes, and Claudia Kurz. 2002. "Rent Indices for Housing in West Germany: 1985 to 1998." Discussion Paper 01/02, Economic Research Centre of the Deutsche Bundesbank, Frankfurt.
Hotelling, Harold. 1925. "A General Mathematical Theory of Depreciation." *Journal of the American Statistical Association* 20: 340–353.
Hulten, Charles R. 1990. "The Measurement of Capital." in *Fifty Years of Economic Measurement*, edited by Ernst R. Berndt and Jack E. Triplett, 119–158. Chicago: The University of Chicago Press.
———. 1996. "Capital and Wealth in the Revised SNA." In *The New System of National Accounts*, edited by John W. Kendrick, 149–181. New York: Kluwer Academic Publishers.
Hulten, Charles R., and Frank C. Wykoff. 1981a. "The Estimation of Economic Depreciation using Vintage Asset Prices." *Journal of Econometrics* 15: 367–396.
———. 1981b. "The Measurement of Economic Depreciation." In *Depreciation, Inflation and the Taxation of Income from Capital*, edited by Charles R. Hulten, 81–125. Washington, DC: The Urban Institute Press.
———. 1996. "Issues in the Measurement of Economic Depreciation: Introductory Remarks." *Economic Inquiry* 34: 10–23.

ILO, Eurostat, IMF, OECD, UNECE and the World Bank. 2004. *Consumer Price Index Manual: Theory and Practice*, edited by Peter Hill. Geneva: International Labour Office.

Inklaar, Robert, and J. Christina Wang. 2010. "Real Output of Bank Services: What Counts is What Banks Do, Not What They Own." Paper presented at the 31st General Conference of the International Association for Research in Income and Wealth at St. Gallen, Switzerland, August 22–28.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2011. "Scanner Data, Time Aggregation and the Construction of Price Indexes." *Journal of Econometrics* 161: 24–35.

Jorgenson, Dale W. 1989. "Capital as a Factor of Production." In *Technology and Capital Formation*, edited by Dale W. Jorgenson and Ralph Landau, 1–35. Cambridge, MA: The MIT Press.

———. 1996. "Empirical Studies of Depreciation." *Economic Inquiry* 34: 24–42.

Karlin, Samuel. 1959. *Mathematical Methods and Theory in Games, Programming and Economics*, vol. 1. Reading, MA: Addison-Wesley Publishing Co.

Katz, Arnold J. 1983. "Valuing the Services of Consumer Durables." *The Review of Income and Wealth* 29: 405–427.

Keynes, John M. 1930. *Treatise on Money*, vol. 1. London: Macmillan.

Knoll, Katharina, Moritz Schularick, and Thomas Steger. 2017. "No Price Like Home: Global House Prices, 1870–2012." *American Economic Review* 107(2): 331–353.

Koev, E., and J.M.C. Santos Silva. 2008. "Hedonic Methods for Decomposing House Price Indices into Land and Structure Components." Unpublished Paper, Department of Economics, University of Essex, England, October.

Konüs, Alexander A., and Sergei S. Byushgens. 1926. "K Probleme Pokupatelnoi Cili Deneg." (On the problem of the purchasing power of money) *Voprosi Konyunkturi* 2: 151–172.

Krsinich, Frances. 2016. "The FEWS Index: Fixed Effects with a Window Splice." *Journal of Official Statistics* 32: 375–404.

Kuhn, Harold W., and Albert W. Tucker. 1951. "Nonlinear Programming." In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 481–492. Berkeley CA: University of California Press.

Lebow, David E., and Jeremy B. Rudd. 2003. "Measurement Error in the Consumer Price Index: Where do we Stand?." *Journal of Economic Literature* 41: 159–201.

Lewis, Rhys, and Ainslie Restieaux. 2015. *Improvements to the Measurement of Owner Occupiers' Housing Costs and Private Housing Rental Prices*. Newport: Office for National Statistics.

Malpezzi, Stephen, Larry Ozanne, and Thomas G. Thibodeau. 1987. "Microeconomic Estimates of Housing Depreciation." *Land Economics* 63: 372–385.

Marshall, Alfred. 1887. "Remedies for Fluctuations of General Prices." *Contemporary Review* 51: 355–375.

———. 1898. *Principles of Economics*, 4th ed. London: The Macmillan Co.

Matheson, Ewing. 1910. *The Depreciation of Factories and their Valuation*, 4th ed. London: E. & F.N. Spon.

McMillen, Daniel P. 2003. "The Return of Centralization to Chicago: Using Repeat Sales to Identify Changes in House Price Distance Gradients." *Regional Science and Urban Economics* 33: 287–304.

OECD. 2001. *Measuring Productivity: Measurement of Aggregate and Industry-Level Productivity Growth*. Paris: OECD.

Office for National Statistics. 2010. *Consumer Price Indexes Technical Manual: 2010 Edition*. Newport: Office for National Statistics.

———. 2016. *CPIH Compendium*. Newport: Office for National Statistics.

———. 2017. *Household Costs Indices: Methodology*. Newport: Office for National Statistics.

———. 2018. *Measures of Owners Occupiers' Housing Costs, UK: April to June 2018*. Newport: Office for National Statistics.

Peasnell, Ken V. 1981. "On Capital Budgeting and Income Measurement." *Abacus* 17(1): 52–67.

Prais, Sigbert J. 1959. "Whose Cost of Living?." *The Review of Economic Studies* 26: 126–134.

Rambaldi, Alicia N., and Cameron S. Fletcher. 2014. "Hedonic Imputed Property Price Indexes: The Effects of Econometric Modeling Choices." *Review of Income and Wealth* 60(S2): S423–S448.

Rambaldi, Alicia N., Ryan J. McAllister, Kerry Collins, and Cameron S. Fletcher. 2010. "Separating Land from Structure in Property Prices: A Case Study from Brisbane Australia." School of Economics, The University of Queensland, St. Lucia, Queensland 4072, Australia.

Schreyer, Paul. 2001. *OECD Productivity Manual: A Guide to the Measurement of Industry-Level and Aggregate Productivity Growth*. Paris: OECD.

———. 2009. *Measuring Capital*, Statistics Directorate, National Accounts, STD/NAD(2009)1. Paris: OECD.

Schwann, Gregory M. 1998. "A Real Estate Price Index for Thin Markets." *Journal of Real Estate Finance and Economics* 16(3): 269–287.

Shimizu, Chihiro, Hideoki Takatsuji, Hiroya Ono, and Kiyohiko G. Nishimura. 2010. "Structural and Temporal Changes in the Housing Market and Hedonic Housing Price Indices." *International Journal of Housing Markets and Analysis* 3(4): 351–368.

Shimizu, Chihiro, Kiyohiko G. Nishimura, and Tsutomu Watanabe. 2010a. "Housing Prices in Tokyo: A Comparison of Hedonic and Repeat Sales Measures." *Journal of Economics and Statistics* 230: 792–813.

———. 2010b. "Nominal Rigidity of Housing Rent." *Financial Review* 106(1): 52–68.

Shimizu, Chihiro, and Tsutomu Watanabe. 2011. "Nominal Rigidity of Housing Rent." *Financial Review* 106(1): 52–68.

Shimizu, Chihiro, W. Erwin Diewert, Kiyohiko G. Nishimura, and Tsutomu Watanabe. 2012. "The Estimation of Owner Occupied Housing Indexes using the RPPI: The Case of Tokyo." Meeting of the Group of Experts on Consumer Price Indices, Geneva, May 28.

Silver, Mick. 2018. "How to Measure Hedonic Property Price Indexes Better." *EURONA* 1/2018: 35–66.

Solomons, David. 1961. "Economic and Accounting Concepts of Income." *The Accounting Review* 36: 374–383.

Statistics Portugal (Instituto Nacional de Estatistica). 2009. "Owner-Occupied Housing: Econometric Study and Model to Estimate Land Prices, Final Report." Paper presented to the Eurostat Working Group on the Harmonization of Consumer Price Indices, March 26–27, Luxembourg: Eurostat.

Summers, Robert. 1973. "International Comparisons with Incomplete Data." *Review of Income and Wealth* 29(1): 1–16.

Suzuki, Masatomo, Yasushi Asami, and Chihiro Shimizu. 2021. "Housing Rent Rigidity under Downward Pressure: Unit-level Longitudinal Evidence from Tokyo." *Journal of Housing Economics* 52, Available Online.

Triplett, Jack E. 2004. *Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes*, Directorate for Science, Technology and Industry, DSTI/DOC(2004)9. Paris: OECD.

Triplett, Jack E., and Barry P. Bosworth. 2004. *Productivity in the U.S. Services Sector: New Sources of Economic Growth*. Washington, DC: Brookings Institution Press.

Ueda, Kozo, Kota Watanabe, and Tsutomu Watanabe. 2020. "Consumer Inventory and the Cost of Living Index: Theory and Some Evidence from Japan." Unpublished Paper, Waseda University (E-mail: kozo.ueda@waseda.jp).

Verbrugge, Randal J. 2008. "The Puzzling Divergence of Rents and User Costs, 1980–2004." *Review of Income and Wealth* 54(4): 671–699.

———. 2012. "Do the Consumer Price Index's Utilities Adjustments for Owners' Equivalent Rent Distort Inflation Measurement?." *Journal of Business & Economic Statistics* 30(1): 143–148.

Walras, Léon. 1954. *Elements of Pure Economics*, translated by William Jaffe (first published in 1874). London: George Allen and Unwin.

Wang, J. Christina. 2003. "Loanable Funds, Risk and Bank Service Output." Federal Reserve Bank of Boston, Working Paper Series No. 03–4, http://www.bos.frb.org/economic/wp/wp2003/wp034.htm

Wang, J. Christina, Susanto Basu, and John G. Fernald. 2009. "A General Equilibrium Asset Approach to the Measurement of Nominal and Real Bank Output." In *Price Index Concepts and Measurement*, edited by W. Erwin Diewert, John Greenlees, and Charles R. Hulten, Studies in Income and Wealth, vol. 70, 273–328. Chicago: University of Chicago Press.

White, Kenneth J. 2004. *Shazam: User's Reference Manual*, version 10. Vancouver: Northwest Econometrics Ltd.

Wold, Herman. 1944. "A Synthesis of Pure Demand Analysis, Part 3." *Skandinavisk Aktuarietidskrift* 27: 69–120.

LOWE, YOUNG, AND SUPERLATIVE INDICES: EMPIRICAL STUDIES* 11

1. Introduction

This chapter summarizes the results of calculations of Lowe, Young, and superlative price indices based on data from the Danish CPI. Section 2 lists the Lowe and Young indices for 2014–2019. Section 3 presents estimates for annual superlative indices for 2012–2018, and Section 4 compares annual superlative indices with the corresponding Lowe and Young indices for 2014–2018. Section 5 provides an overview of this and other empirical studies on Lowe, Young, and superlative indices. Lowe and Young indices are the "practical" indices that are used by most National Statistical Offices to produce their CPIs . They utilize current monthly price indices for the main categories of household consumption (called elementary indices) and annual household expenditure weights for the same categories from a previous year. These data can be used retrospectively to construct annual superlative indices. A superlative index is approximately free from substitution bias. Thus, taking the difference between a superlative index and the "practical" index is a measure of upper-level substitution bias for the practical index.

The data set consists of the weights and price indices for 402 elementary aggregates used for calculating the Danish CPI for the period 2012–2019. The elementary price indices cover the period January 2012–December 2019. The annual expenditure weights are available for the years 2010–2018. The data set excludes elementary indices that were not compiled throughout the period. The weight of the excluded elementary indices amounts to approximately 5 percent of the total weighting basis.

The annex uses the Danish data to compute some additional indices, including several multilateral indices that use bilateral superlative indices as building blocks.

2. Lowe and Young Price Indices

Most countries calculate the CPI as an expenditure-weighted arithmetic average of the elementary aggregate indices that make up the CPI. Expenditure weights usually are only available with a time lag so that the weight reference period precedes the price reference period when the weights are introduced into the CPI. If the weights are price-updated from the weight reference period to the price reference period, the resulting index will correspond to a Lowe price index. If the weights are used without price-updating, it will correspond to a Young price index.[1]

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[1] See definitions (1)–(3). The Lowe, Young, and geometric Young indices will be defined in more detail in the annex. Diewert (2021) provides a detailed discussion of these indices and their properties.

Figure 11.1 shows the monthly Lowe, Young, and geometric Young price indices for 2014–2019. They are defined below by (1)–(3). The indices are calculated as annually chained indices with December as the link month.[2] Expenditure weights are introduced with a lag of two years so that indices for year t are based on expenditure weights for year $t - 2$.[3] Hence, indices for 2014 are based on weights for 2012; indices for 2015 are based on weights for 2013, and so on.

Table 11.1 shows the *annual* Lowe and Young price indices for 2014–2019 and the annual rates of change. The annual price indices are calculated as the arithmetic average of the monthly series. The annual rate of change is the rate of change between the annual indices.

From 2014 to 2019, the average annual rate of change of the Young index is 0.63 percent against 0.69 percent for Lowe index. Hence, the price-updating of weights from $t - 2$ to December $t - 1$ on average increases the annual rate of change of the index by 0.06 percentage points. The geometric Young index (GeoYoung) is below the Lowe and Young indices, showing an annual rate of change of 0.51 percent.

The Lowe price index is calculated by weighting together the elementary indices, P^i, with the price-updated weights:

$$P_{0,t}^{Lo} \equiv \sum_{i=1}^{402} w_{b(0)}^i \, P_{0,t}^i, \text{ where } w_{b(0)}^i \equiv w_b^j \, P_{b,0}^i / \sum_{j=1}^{402} w_b^j \, P_{b,0}^i \text{ for } i = 1, \ldots, 402. \quad (1)$$

$w_{b(0)}^i$ are the weights from the weight reference period (b) price-updated to the price reference period (0), when the weights are introduced into the Lowe index. The weights are price-updated from average of year $t - 2$ to December $t - 1$ and applied for the index calculations for year t. For example, the Lowe index from January to December 2014 is calculated based on the weights from 2012 price-updated from average 2012 to December 2013.[4]

[2] In the annex, January is used as the link month.

[3] The annex also computes "true" Lowe and Young indices as well as Lowe and Young indices that use weights that are lagged one and two years.

[4] Basically, the Lowe index is a fixed basket index that uses approximations to annual quantities as the "basket" in the numerator and denominator of the index. The basket is priced out at the prices of the current month in the numerator of the index and at the prices of the base period month in the denominator of the index. The price updating procedure deflates the annual weights by an annual price in order to obtain the annual "quantity" basket up to a factor of proportionality. The details of the updating procedure are explained more fully in the annex.

Figure 11.1 Monthly Lowe and Young Indices That Are Chained Annually, 2014–2019 (2014 = 100)

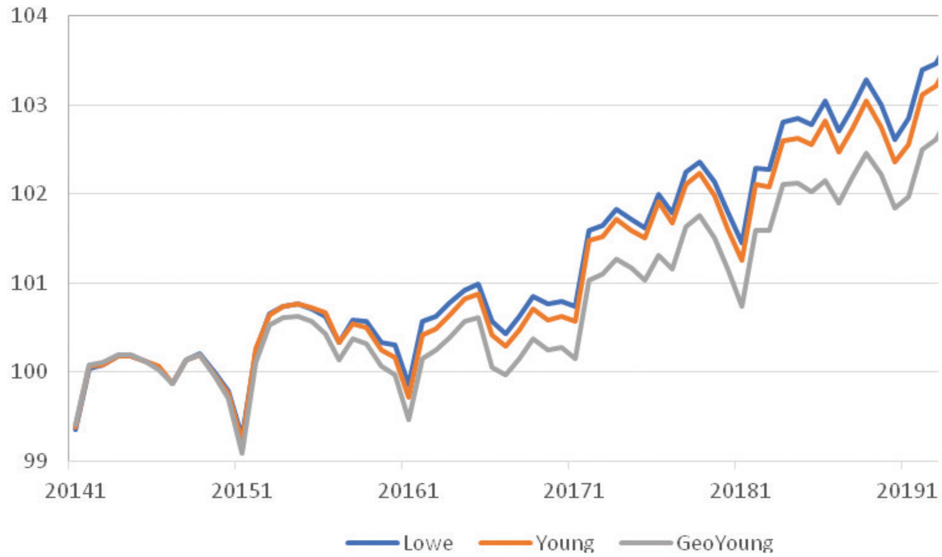

Table 11.1 Lowe and Young Annual Price Indices, 2014–2019

Annual chained indices (2014 = 100)						% change
	2015	2016	2017	2018	2019	2014–2019
Lowe	100.43	100.65	101.79	102.67	103.48	3.48
Young	100.40	100.51	101.66	102.45	103.21	3.21
Young*	100.44	100.65	101.81	102.64	103.42	3.42
GeoYoung	100.24	100.21	101.19	101.91	102.59	2.59
Annual rate of change (%)						Av. annual % change
	2015	2016	2017	2018	2019	2014–2019
Lowe	0.43	0.22	1.13	0.87	0.79	0.69
Young	0.40	0.11	1.15	0.78	0.74	0.63
Young*	0.44	0.21	1.15	0.82	0.76	0.67
GeoYoung	0.24	−0.03	0.98	0.71	0.66	0.51

The Young index and the geometric Young index are calculated as the expenditure-weighted arithmetic and geometric averages of the elementary price indices:

$$P^{Yo}_{0,t} \equiv \Sigma_{i=1}^{402} w^{i}_{b} P^{i}_{0,t}; \qquad (2)$$

$$P^{GYo}_{0,i} \equiv \Pi_{i=1}^{402} P_{0,t}^{w^{i}_{b}}. \qquad (3)$$

Both the Young index and the geometric Young index are calculated based on the weights from year $t - 2$ as they stand, without price-updating.

The index links from December to December are chained (multiplied) annually onto each other using the overlapping December as link month to obtain chained index series with a fixed index reference period. For example, the Young index for May 2017 with 2014 as the index reference period is calculated as

$$\begin{aligned} P^{Yo}_{14:May17} &\equiv P^{Yo}_{14:Dec14} \times P^{Yo}_{Dec14:Dec15} \times P^{Yo}_{Dec15:Dec16} \times P^{Yo}_{Dec16:May17} \\ &= \Sigma_{i=1}^{402} w^{i}_{12} P^{i}_{14:Dec14} \times \Sigma_{i=1}^{402} w^{i}_{13} P^{i}_{Dec14:Dec15} \\ &\times \Sigma_{i=1}^{402} w^{i}_{14} P^{i}_{Dec15:Dec16} \times \Sigma_{i=1}^{402} w^{i}_{15} P^{i}_{Dec16:May17}. \end{aligned} \qquad (4)$$

In Table 11.1, the index labeled Young* is based on the weights from year $t - 2$ price-updated from average year $t - 1$ to December $t - 1$. For instance, weights from 2014 are price-updated from average of 2015 to December 2015 and used for the calculation of the index from January to December 2016. The Young* index lies between the Young and Lowe indices, as could be expected. This approach is applied for calculating the Danish CPI. From 2014 to 2019, the Danish CPI increased by 3.47 percent over the six years with an average annual increase of 0.68 percent, compared to 3.42 percent and 0.67 percent for the Young* index calculated in this analysis.

The Young* index follows the requirement for the HICP of the European Union.[5] The HICP is defined as an annually chain-linked Laspeyres-type index using December as the link month. The weights should reflect the consumption pattern of year $t - 1$. However, in practice, year $t - 1$ expenditure data are not available for the calculation of the index from January year t. To obtain the best possible estimate of the weights for year $t - 1$, these should be derived from consumption data for year $t - 2$, the weight reference period. It is up to countries to decide whether to price-update the weights from $t - 2$ to $t - 1$, depending on which approach is considered to give the best estimate of the expenditure shares in year $t - 1$. In either case, the weights must be price-updated from year $t - 2$ to December $t - 1$.

3. Superlative Price Indices

Following the 2004 *CPI Manual*, the Fisher, Walsh, and Törnqvist price indices are the preferred target indices for the CPI and usually give very similar results:

> Fisher, Walsh and Törnqvist price indices approximate each other very closely using "normal" time series data. This is a very convenient result since these three index number formulae repeatedly show up as being "best" in all the approaches to index number theory. Hence, this approximation result implies that it normally will not matter which of these indices is chosen as the preferred target index for a consumer price index.[6]

Fisher, Walsh, and Törnqvist are superlative price indices[7] that require weights from both the price reference period and the current period. When annual weights become available, it is possible to estimate a superlative CPI by aggregating the elementary indices using weights from both periods.

Table 11.2 shows the annual Fisher, Walsh, and Törnqvist price indices for 2012–2018. The three indices give almost identical results; all three show an average annual rate of change of 0.50 percent over the period 2012–2018, which is approximately 0.1 to 0.2 percentage points per year below the "practical" indices that were calculated in the previous section. Thus, annual upper-level substitution bias for the practical Danish indices was fairly low over the sample period.

The Fisher, Walsh, and Törnqvist price indices are estimated using the following formulae, where w^i are the weights and P^i are the price indices for the elementary aggregates:

$$P_{0,t}^{F} \equiv \left[\left(\Sigma_{i=1}^{402} w_0^i P_{0,t}^{t}\right)\left(S_{i=1}^{402} w_t^i \left(P_{0,t}^{t}\right)^{-1}\right)^{-1}\right]^{1/2}; \tag{5}$$

$$P_{0,t}^{W} \equiv \Sigma_{i=1}^{402} w_W^i P_{0,t}^i \text{ where } w_W^i \equiv \left[w_0^i w_t^i / P_{0,t}^i\right]^{1/2} / \Sigma_{j=1}^{402} \left[w_0^j w_t^j / P_{0,t}^j\right]^{1/2} \text{ for } i = 1, \ldots, 402; \tag{6}$$

$$P_{0,t}^{T} \equiv \Pi_{i=1}^{402} \left(P_{0,t}^i\right)^{1/2} \left(w_0^i + w_t^i\right). \tag{7}$$

When calculating the superlative indices, the monthly elementary indices are aggregated into annual averages (by taking the arithmetic mean)[8] to align with the annual weight reference periods. The chained superlative indices are calculated by multiplying the annual links of the indices.[9] For example, the chained Walsh index from 2012 to 2015 is calculated as

$$P_{1215}^{W} \equiv P_{1213}^{W} \times P_{1314}^{W} \times P_{1415}^{W}. \tag{8}$$

The direct superlative indices in Table 11.2 are calculated based on the expenditure weights for 2012 and 2018 and the chain-linked annual elementary indices with 2012 = 100.

Table 11.2 Annual Superlative Price Indices, 2012–2018

	Annual chained indices (2012 = 100)						Direct index[1]
	2013	2014	2015	2016	2017	2018	2012–18
Fisher	100.67	101.14	101.33	101.43	102.41	103.06	103.31
Walsh	100.67	101.14	101.32	101.43	102.41	103.07	103.40
Törnqvist	100.66	101,14	101.32	101.42	102.41	103.06	103.38
	Annual rate of change (%)						Av. annual % change
	2013	2014	2015	2016	2017	2018	2012–18
Fisher	0.67	0.47	0.18	0.10	0.97	0.64	0.50
Walsh	0.67	0.47	0.18	0.10	0.97	0.64	0.50
Törnqvist	0.66	0.47	0.18	0.10	0.97	0.64	0.50

[1] Direct Paasche and Laspeyres indices for 2012–2018 are 102.9 and 104.23, respectively.

[5] See the *Harmonized Index of Consumer Prices (HICP) Methodological Manual,* Section 3.5.

[6] ILO/IMF/OECD/UNECE/Eurostat/The World Bank (2004): *Consumer Price Index Manual: Theory and Practice*. International Labour Office, Geneva, p. 313.

[7] The theory and advantages of superlative indices were developed by Diewert (1976).

[8] In the annex, there is some discussion on the problems associated with aggregating monthly price indices into annual indices.

[9] The details associated with forming the annual Fisher indices are explained in the annex.

Table 11.3 Comparing Superlative, Lowe, and Young Indices, 2014–2018

Annual chained indices (2014 = 100)					% change	Av. annual % change
	2015	2016	2017	2018	2014–2018	2014–2018
Fisher	100.18	100.28	101.26	101.90	1.90	0.47
Walsh	100.18	100.28	101.26	101.90	1.90	0.47
Törnqvist	100.18	100.28	101.25	101.90	1.90	0.47
Lowe	100.43	100.65	101.79	102.67	2.67	0.66
Young	100.40	100.51	101.66	102.45	2.45	0.61
GeoYoung	100.24	100.21	101.19	101.91	1.91	0.47

4. Comparing Lowe, Young, and Superlative Indices

Table 11.3 shows the superlative and Lowe and Young price indices for the period 2014–2018. The Fisher, Walsh, and Törnqvist indices are almost identical, all with an average annual rate of change of 0.47 percent over this period. The Lowe and Young indices are calculated, as explained earlier, as annually chained indices with December as the link month and with a two-year lag in the weight reference period, that is, indices for year t are based on consumption expenditure data for year $t - 2$.

Over the period 2014–2018, the Lowe index exceeds the Young index, but the differences are small. The average annual rate of change of the Young index is 0.61 percent against 0.66 percent for the Lowe index.

Compared to a superlative index, the Lowe index shows an upward bias of 0.19 percentage points per year, and the Young index shows an upward bias of 0.14 percentage points per year. The geometric Young index gives similar results to the superlative indices; that is, for the particular data set used in this chapter, the geometric Young index essentially eliminates upper-level substitution bias. Since this index can be compiled using the same information that is used in compiling the Lowe and Young indices, it would be of interest for other National Statistical Offices to carry out similar comparisons in order to determine whether upper-level substitution bias was substantially reduced using the geometric Young index formula. The results to be presented in the following section indicate that there is a tendency for the geometric Young index formula to underestimate inflation as measured by a superlative index.

5. Overview of Empirical Studies on Substitution Bias

Table 11.4 summarizes the results of this and six other studies of retrospective calculations comparing superlative price indices to the Lowe and Young indices. More details about these studies are provided here.[10]

Based on the studies presented in Table 11.4 some general conclusions may be drawn:

- Lowe exceeds Young – price-updating expenditure shares increases the rate of change of the CPI.
- The arithmetic Young index exceeds the geometric Young index.
- The Fisher, Walsh, and Törnqvist indices give very similar results under normal conditions.
- The Lowe and Young indices are biased upward compared to a superlative price index, with the Lowe index being more biased than the Young index. There is one exception (New Zealand, 2006–2008), where the Young index is below the superlative index.
- The geometric Young is biased downward compared to a superlative index, with one exception (Denmark, 2014–2018) where it equals the superlative indices.

(1) ***Recalculations of the Danish CPI, 1996–2006***. Carsten Boldsen Hansen. Paper Presented at the 2007 Ottawa G roup Meeting

This study uses the elementary indices and weights for the Danish CPI to calculate the Lowe, Young, and superlative indices. The Fisher, Walsh, and Törnqvist indices are almost identical. Over the period 1996–2003, the Walsh and Törnqvist indices showed an average annual rate of change of 2.28 percent, while the Fisher annual rate was 2.27 percent. For the Lowe and Young indices, the weights were updated every third year; new weights were introduced with a varying lag of two to three years. Based on the series for 1996–2003, the annual Lowe index exceeded the corresponding Young index by 0.06 percentage points on average. The Lowe and Young indices, on average, exceeded the annual rate of change of the Walsh index by 0.11 percentage points and 0.05 percentage points, respectively. The geometric Young index underestimated the annual rate of change of the Walsh index by 0.07 percentage points.

(2) ***Impact of the Price-Updating Weights Procedure on the Canadian Consumer Price Index***. Ning Huang, Statistics Canada. Room Document at the 2011 Ottawa Group Meeting

This study was based on data from the Canadian CPI for the period 1996–2005. In this period, the Canadian CPI was calculated as a chained index where weights were updated with intervals of four and five years with lags in the weight reference period of two years. For the period 1996–2005, the average annual rates of change for the Fisher, Walsh, and Törnqvist indices were 1.77 percent, 1.86 percent, and 1.90

[10] Papers from the Ottawa Group are available from www.ottawagroup.org

Table 11.4 Comparing the Empirical Studies of the Superlative, Lowe, and Young Indices

	Average annual rate of change (%)				Differences in annual rate of change (% point)			
	Lowe	Young	geometric Young	Superlative index	Lowe–Young	Lowe–superlative index	Young–superlative index	geometric Young–superlative index
Denmark 2014–2008	0.66	0.61	0.47	0.47	0.05	0.19	0.14	0.00
Denmark 1996–2003 (1)	2.39	2.33	2.21	2.28	0.06	0.11	0.05	−0.07
Canada 1996–2005 (2)	2.08	1.99	1.80	1.86	0.09	0.21	0.12	−0.06
Canada 2003–2011 (3)	1.84	1.81	1.65	1.70	0.03	0.15	0.12	−0.05
USA 2001–2007 (4)	2.50	2.42	2.12	2.24	0.08	0.26	0.18	−0.12
USA 2002–2010 (5)	2.49	2.35	2.15	2.31	0.14	0.18	0.04	−0.16
New Zealand 2006–2008 (6)	3.08	2.76	2.39	2.83	0.32	0.25	−0.07	−0.44

percent, respectively (Table 11.4). The significant difference between the Fisher and the two other indices was explained to be caused by the sub-index for computers. When this sub-index was removed from the calculations, the three superlative indices gave similar results. For the same period, the average annual rate of change of the Lowe index was 2.08 percent against 1.99 percent for the Young index and 1.80 percent for the geometric Young index (Table 11.4).[11]

(3) *Choice of Index Number Formula and the Upper-Level Substitution Bias in the Canadian CPI.* Ning Huang, Waruna Wimalaratne, and Brent Pollard. Paper Presented at the 2015 Ottawa Group Meeting

Based on Canadian data for 2003–2011, this chapter examines superlative indices and other symmetrically weighted indices. Lowe and Young indices are also compiled and the effect of different lags in the implementation of the expenditure weights in the calculation of the CPI are analyzed. Table 5.4 of the chapter compares a chained annual Fisher index to the chained annual Lowe and Young indices, compiled with a lag of one year in the introduction of the expenditure weights. The results of these calculations are reproduced in Table 11.4.

(4) *Reconsideration of Weighting and Updating Procedures in the US CPI.* John S. Greenlees and Elliot Williams. BLS Working Paper 431, 2009[12]

The study was based on the data from the US Urban CPI for 2001–2007. In this period, the Urban CPI was calculated with biannual links and with a two-year lag in the weight reference period. Based on data for the US Urban CPI for 2001–2007, the annual rate of change of the Törnqvist index was 2.24 percent. For the same period, the Young index showed an annual rate of change of 2.42 percent and the Lowe rate was 2.50 percent (Table 11.4). The geometric Young index showed an annual rate of change of 2.12 percent, and hence it was well below the superlative index.

[11] For the follow up-studies on the Canadian CPI, see Huang, Wimalaratne, and Pollard (2015, 2017).
[12] https://www.bls.gov/pir/journal/gj14.pdf.

(5) *Post-Laspeyres: The Case for a New Formula for Compiling Consumer Price Indices.* Paul Armknecht and Mick Silver. Paper Presented at the 2013 Ottawa Group Meeting

Based on data from the US Urban CPI, this study calculated superlative indices and alternative formulae for 2002–2010. In this period, the US CPI was calculated with biannual links. The weights covered two-year periods, were updated every second-year, and were two years old when introduced into the CPI. The Fisher and the Törnqvist tracked each other very closely. Over the period 2002–10, the Fisher price index increased by an annual average rate of change of 2.31 percent, compared with 2.49 percent for the Lowe index and 2.35 percent for the Young index (page 13).

(6) *New Zealand 2006 and 2008 Consumers Price Index Reviews: Price Updating. Chris Pike et al.* Room document at the 2009 Ottawa Group Meeting

The study was based on quarterly New Zealand CPI data for June 2006 to June 2008 with weights of 2003/04 and 2006/07, respectively, implemented in June 2006 and June 2008 quarters. For this period the average annual rate of change of the Lowe index was 3.08 percent against 2.76 percent for the Young index and 2.39 percent for geometric Young index (as shown in Table 11.4). A Fisher index for the same period showed an average annual rate of change of 2.83 percent (page 24). This is the only study in which the Young index underestimated the superlative index.

The overall conclusion that can be drawn from this chapter is that it would be useful for National Statistical Offices to undertake similar retrospective studies in order to obtain approximate numerical estimates of the upper-level substitution bias that might have been present in their CPIs.

References

Armknecht, Paul, and Mick Silver. 2013. "Post-Laspeyres: The Case for a new Formula for Compiling Consumer Price Indices." Room document at the 2013 Ottawa Group meeting. Available at: www.ottawagroup.org

Boldsen Hansen, Carsten. 2007. "Recalculations of the Danish CPI 1996–2006." Paper presented at the 2007 Ottawa Group meeting. Available at: www.ottawagroup.org

Diewert, W. Erwin. 1976. "Exact and Superlative Price Index Numbers." *Journal of Econometrics* 4: 114–145.

———. 2021. "Basic Index Number Theory." In *Consumer Price Index Theory,* Draft Chapter 2. Washington D.C.: International Monetary Fund. Available at: https://www.imf.org/en/Data/Statistics/cpi-manual.

Eurostat. 2018. *Harmonized Index of Consumer Prices (HICP) Methodological Manual*. Available at: https://ec.europa.eu/eurostat/web/products-manuals-and-guidelines/-/KS-GQ-17–015

Greenlees, John S., and Elliot Williams. 2009. "Reconsideration of Weighting and Updating Procedures in the U.S. CPI." BLS Working Paper 431, October 2009. Available at: https://www.bls.gov/pir/journal/gj14.pdf

Huang, Ning. 2011. "Impact of the Price-Updating Weights Procedure on the Canadian Consumer Price Index." Room document at the 2011 Ottawa Group meeting. Available at: www.ottawagroup.org

Huang, Ning, Waruna Wimalaratne, and Brent Pollard. 2015. "Choice of Index Number Formula and the Upper-Level Substitution Bias in the Canadian CPI." Presented at 14th Ottawa group meeting. Available at: www.ottawagroup.org.

———. 2017. "The Effects of the Frequency and Implementation Lag of Basket Updates on the Canadian CPI." *Journal of Official Statistics* 33(4): 1979–1004.

ILO, IMF, OECD, UNECE, Eurostat, The World Bank. 2004. *Consumer Price Index Manual. Theory and Practice*. Geneva: International Labour Office.

Pike, Chris, Ben Nimmo, and Ludeth Mariposa. 2009. "New Zealand 2006 and 2008 Consumers Price Index Review: Price Updating." Room document at the 2009 Ottawa Group meeting. Available at: www.ottawagroup.org

Annex: Supplementary Indices for Denmark

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A.11.1 Introduction

The main text of this chapter used monthly price indices for 402 monthly elementary aggregates that are components of the Danish CPI for the seven years 2012–2018. Annual expenditure weights for these 402 aggregates were also available for these years. various monthly chained Young and Lowe indices were calculated using annual weights lagging one or two years since these types of indices are used by National Statistical Offices to calculate their CPIs. The monthly price data were aggregated into yearly price data, and then, along with the annual expenditure information, the annual Lowe and Young indices along with the annual superlative Fisher, Törnqvist, and Walsh indices were calculated. It was found that the three superlative indices were very close to each other, which is typically the case if the price and quantity data do not fluctuate too much.[13] The difference between these superlative indices and the Lowe or Young indices was used to form estimates of *upper-level substitution bi*as for a national CPI that is based on the use of these monthly indices that use lagged annual weights. The main text also reviewed recent studies on the magnitude of upper-level substitution bias.[14]

The present annex uses the same data set to calculate various supplementary indices. In Section A.2, various monthly indices that aggregate the 402 elementary indices without using the annual weights are calculated. Thus, these indices use only monthly price information. It is of interest to calculate these unweighted indices to see if weighting really matters. If unweighted indices can adequately approximate an appropriate-weighted index, then National Statistical Offices would not have to go to the expense of collecting household expenditure information. The three main unweighted indices that are used at lower levels of aggregation by statistical offices in recent times are the Jevons, Dutot, and Carli indices.[15] These indices will be defined here along with other indices that will be discussed subsequently.[16] Comparing these indices that do not use expenditure weights with indices that do use weights will give readers some idea of the importance of weighting.

In Section A.3, the monthly price indices are aggregated into annual price indices for the 402 classes of consumer goods and services. The annual expenditure shares for the 402 products are divided by the corresponding annual prices in order to generate 402 annual "quantities" or volumes for the seven years of annual data. Using these 402 annual "prices" and "quantities," *annual standard fixed-base and chained Laspeyres, Paasche, and Fisher indices* are calculated. Two multilateral indices are also calculated: the *GEKS* and the *Relative Price Similarity-Linked Predicted Share indices*.[17]

Sections A.4 and A.5 calculate various *weighted month-to-month* using the same Danish data set. As was noted in the main text, National Statistical Offices cannot calculate month-to-month CPIs in real time using annual weights for the current year since these weights are only available with a lag of one or two years. However, annual weights for the current year can be used in retrospective index number studies, so in Section A.4, *Lowe, Young, and geometric Young indices* are calculated using (i) current year expenditure weights; (ii) weights lagged one year; and (iii) weights lagged two years. These indices that use lagged expenditure weights are "practical" CPIs.

Finally, in Section A.5, the assumption is made that the annual expenditure shares can provide an approximation to monthly expenditure shares. Using this (problematic) assumption, monthly "quantities" or "volumes" can be computed and can be combined with the monthly price information to produce approximate *month-to-month fixed-base and chained Laspeyres, Paasche, and Fisher indices*. These indices can then be compared with the "practical" indices calculated in Section A.4. We also compute some multilateral indices using the monthly price indices and volume indices.

Section A.6 draws some tentative conclusions from these computations.

A.11.2 Month-to-Month Aggregate Unweighted Indices

The monthly CPIs for 402 aggregate product classes for Denmark for the years 2012–2018 were provided by Statistics Denmark. These indices were normalized so that the price for each product class for January of 2012 was set equal to unity; that is, each price index was divided by the corresponding index value for January of 2012. The resulting *normalized price for product class n in month t* is denoted by $p_{t,n}$ for $t = 1, \ldots, 84$. Thus, $t = 1$ identifies the data for January of 2012, $t = 2$ corresponds to the data for February of 2012, and so on. Statistics Denmark also provided annual expenditure shares for each product class for the years 2012–2018, but this information will not be used in this section.

In the following definitions, $N = 402$ in our particular application. The *Jevons index* for month t, $P_J^{\,t}$, is defined as follows:

[13] See Diewert (1978), who showed that these superlative indices numerically approximate each other to the second order around an equal price and quantity point.

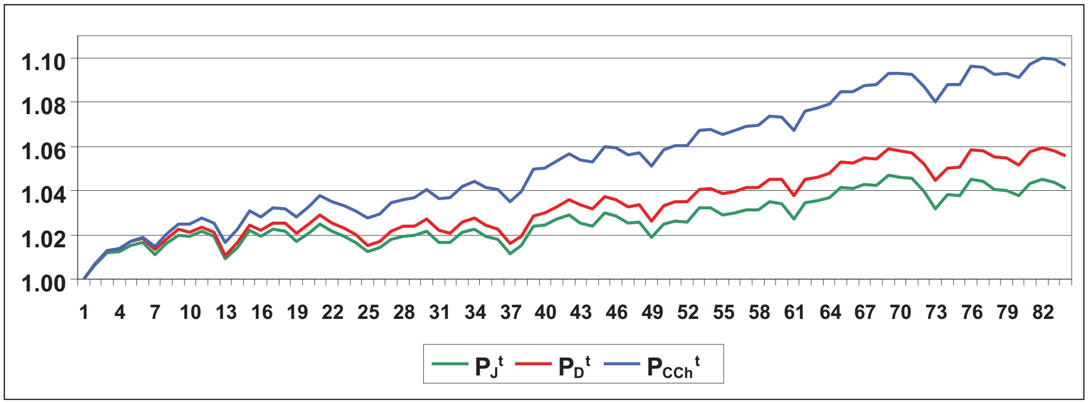
[14] A path-breaking study on types of bias that might be associated with the Lowe-type CPIs and the possible magnitude of these types of bias was the Boskin Report; see Boskin et al. (1996). See Diewert (1998) for a follow-up study on possible methods for measuring the various sources of bias.

[15] National Statistical Offices use unweighted indices (which are called elementary indices by NSOs) at the initial stages of aggregation. At the final stage of aggregation, NSOs always use price and expenditure weight information. In this annex, unweighted indices are computed purely for illustrative purposes in order to see how close indices that are computed using only price information can approximate various weighted indices which are considered in the main text and in this annex.

[16] The history and properties of these indices are discussed in Diewert (2021a).

[17] These indices are defined and discussed in Diewert (2021b).

Figure A11.1 Jevons, Dutot, and Chained Carli Month-to-Month Indices



$$P_J^t \equiv \prod_{n=1}^{N} (p_{t,n}/p_{1,}n)^{1/N} \qquad t = 1, \ldots, 84 \qquad \text{(A1)}$$
$$= \prod_{n=1}^{N} (p_{t,n})^{1/N},$$

where the second equality follows from the fact that all prices have been normalized so that $p_{1,}n = 1$ for $n = 1, \ldots, N$. Thus, the Jevons fixed-base index for month t is defined to be the geometric mean of the price ratios $p_{t,n}/p_{1,}n$. Since there are no missing products in this Danish data set, the fixed-base and chained Jevons indices are identical.

The fixed-base *Dutot index* for month t, P_D^t, is defined as the arithmetic average of the prices in month t divided by the arithmetic average of the prices in month 1:

$$P_D^t \equiv \Sigma_{n=1}^{N} (1/N)p_{t,n}/S_{n=1}^{N} (1/N)p_{1,n} \qquad t = 1, \ldots, 84 \qquad \text{(A2)}$$
$$= \Sigma_{n=1}^{N} (1/N)p_{t,n}/S_{n=1}^{N} (1/N)1 \text{ since } p_{1,}n = 1 \text{ for all } n$$
$$= \Sigma_{n=1}^{N} (1/N)p_{t,n}.$$

Again, since there are no missing products in the data set, the fixed-base and chained Dutot index are identical.

The third commonly used elementary index is the *Carli index*. The *fixed-base version* of this index for month t, P_C^t, is defined as the arithmetic mean of the long-term relative prices, $p_{t,n}/p_{1,n}$:

$$P_C^t \equiv \Sigma_{n=1}^{N} (1/N)(p_{t,n}/p_{1,}n) \qquad t = 1, \ldots, 84 \qquad \text{(A3)}$$
$$= \Sigma_{n=1}^{N} (1/N)(p_{t,n}) \text{ since } p_{1,}n = 1 \text{ for all } n$$
$$= P_D^t \text{ using the third line in (A2).}$$

Thus, if there are no missing prices for the window of data under consideration and all prices are normalized to equal 1 in the base month, then the fixed-base Carli index for month t, P_C^t, is equal to the fixed-base (and chained) Dutot index, P_D^t.

The definition of the chained Carli index for month t, P_{CCh}^t, is more complicated. First, define the *Carli chain link index* between months $t-1$ and t, P_{CLink}^t, as follows:

$$P_{CLink}^t \equiv \Sigma_{n=1}^{N} (1/N)(p_{t,n}/p_{t-1,n}) \qquad t = 2,3, \ldots, 84. \qquad \text{(A4)}$$

Using definition (A4), the *Carli chain linked indices* for all months t in scope, P_{CCh}^t, are defined as follows:

$$P_{CCh}^1 \equiv 1;\ P_{CCh}^t \equiv P_{CCh}^{t-1} \times P_{CLink}^t; \qquad t = 2,3, \ldots, 84. \qquad \text{(A5)}$$

The indices P_J^t, P_D^t, and P_{CCh}^t for $t = 84$ are 1.04091, 1.05581, and 1.09678, respectively.[18] Figure A11.1 indicates that the chained Carli index finishes substantially above the Dutot index and the Dutot index finishes above the Jevons index. The choice of an elementary index number formula does matter.

It is not surprising that the chained Carli index finishes above the Jevons index (which is also a chained Jevons index) because the geometric mean of N price ratios will always be equal to or less than the arithmetic mean of the same N price ratios.[19] It is also the case that the geometric mean of N prices will always be equal to or less than the corresponding arithmetic mean of the same N prices, and this explains why the Jevons index is less that the Dutot index when all prices are normalized to equal one in the base period.

The Jevons, Dutot, and chained Carli indices for our Danish CPI data are plotted in Figure A11.1.

When there are no missing prices, the Jevons and Dutot indices both satisfy Walsh's (1901, 389, 1921, 540) multiperiod identity test. This test is explained as follows: if the prices in period t are identical to the prices in period 1, then the index number formula should register a value of 1 to indicate that there is no change in the price level going from period 1 to t. The fixed-base Carli index also satisfies this test, but the chained

[18] The corresponding annualized average geometric growth rates for these indices are as follows: $(P_J 84)^{1/6} = 1.00671$, $(P_D^{84})^{1/6} = 1.00909$, and $(P_{CCh}^{84}) = 1.0155$.

[19] This follows from Schlömilch's inequality; see Hardy, Littlewood, and Pólya (1934, 26) or Diewert (2021a).

Carli index does not. Thus, the chained Carli index number formula is said to suffer from a *chain drift problem*.[20]

The economic approach to index number theory can be applied to bilateral indices that utilize *both* price and quantity information; it cannot be applied if only price information is available. Thus, the economic approach cannot determine which elementary index that utilizes only price information is the "best." However, the test or axiomatic approach to index number theory can be applied to elementary indices that utilize only price information. Since the chained Carli index does not satisfy the multiperiod identity test but the Jevons and Dutot indices do satisfy this important test, the Jevons and Dutot indices are favored over chained Carli indices. However, since the Jevons index is invariant to changes in the units of measurement, while the Dutot index does not satisfy this important test, the Jevons index probably emerges as a "best" index from the viewpoint of the test approach to index number theory when only price information is available.[21]

In addition to showing that the choice of an index number formula matters, Figure A11.1 shows that the Danish CPI data indicates the presence of a considerable amount of seasonality in the pattern of prices. Prices are generally very low in January and very high in October or November of each year.[22]

A.11.3 Standard Annual Indices

Statistics Denmark has provided estimated annual expenditure shares for the 402 elementary aggregates for the years 2012–2018. We will denote these years as years $y = 1–7$ in what follows. Denote the *annual expenditure share* for product class n in year y as $S_{y,n}$ for $y = 1, \ldots, 7$ and $n = 1, \ldots, N = 402$. We need to define *annual prices* for the 402 products, $p_{y,n}^*$, that will match up with these annual expenditure shares. It turns out that it is not a trivial matter to construct annual prices from monthly prices.

If monthly price and quantity (or volume) information is available and there is seasonality in prices and quantities, then Mudgett (1955) and Stone (1956) recommended that an annual index should treat each product in each season as a separate product in the annual index number formula.[23] Diewert et al. (2022) showed how this suggestion could be implemented for various index number formulae, provided that monthly price and quantity information is available.[24] Since monthly quantity or expenditure information on the 402 product classes is not available, this suggestion cannot be implemented using the Danish data.

Another approach to the problem of aggregating data over months to form annual indices is to form annual unit value prices for each product. Purchases of a product over a time period may take place at different prices, so the following question arises: How should these possibly different prices be aggregated into a single price that is representative of all transaction prices made during the period? Walsh (1901, 96; 1921, 88) was the first to provide an answer to this question: He suggested that the appropriate price was the *unit value price*, which is equal to the total value of transactions for the product under consideration divided by the total quantity transacted. The advantage of using a unit value price as the representative price is that the corresponding aggregate quantity is equal to the total quantity transacted during the period. This same aggregation strategy can be applied to the problem of aggregating over months. Thus, let $p_{y,m,n}$ be the monthly unit value price for product n in month m of year y and let $q_{y,m,n}$ be the corresponding monthly total quantity transacted for product n in month m of year y. Then the corresponding *annual unit value price* for product n in year y, $p_{y,n}$, is defined as follows:

$$P_{y,n} \equiv \Sigma_{m=1}^{12} p_{y,m,n} q_{y,m,n} / \Sigma_{m=1}^{M} q_{y,m,n} = \Sigma_{m=1}^{12} p_{y,m,nqy,m,n} / Q_{y,n}; \quad y = 1, \ldots, 7;\ n = 1, \ldots, 402. \qquad \text{(A5)}$$

The *aggregate annual quantity* for product n in year y is defined as

$$Q_{y,n} \equiv \Sigma_{m=1}^{12} q_{y,m,n}; \qquad y = 1, \ldots, 7;\ n = 1, \ldots, 402. \qquad \text{(A6)}$$

(A5) and (A6) define theoretical annual prices and quantities for each year y and each product n. Define the *annual price and quantity vectors* for year y as $P^y \equiv [P_{y,1}, \ldots, P_{y,}]$ and $Q^y \equiv [Q_{y,1}, \ldots, Q_{y,N}]$ for $N = 402$. Define total consumption for year y as $P^y \cdot Q^y \equiv \Sigma_{n=1}^{N} P_{y,n} Q_{y,n}$, and define the *annual sh*are for product n of total consumption in year y as

$$S_{y,n} \equiv P_{y,n} Q_{y,n} / P^y \cdot Q^y; \qquad y = 1, \ldots, 7;\ n = 1, \ldots, 402. \qquad \text{(A7)}$$

Using definitions (A5)–(A7), it can be seen that if we divide each annual expenditure share $S_{y,n}$ by the corresponding annual unit value price $P_{y,n}$ defined by (A5), we obtain the annual quantity $Q_{y,n}$ defined by (A6) divided by total year y consumption, $P^y \cdot Q^y$; that is, we have the following relationships:

$$S_{y,n} / P_{y,n} = [P_{y,n} Q_{y,n} / P^y \cdot Q^y] / P_{y,n} = Q_{y,n} / P^y \cdot Q^y; \quad y = 1, \ldots, 7;\ n = 1, \ldots, 402. \qquad \text{(A8)}$$

This algebra shows that deflating an annual expenditure share by an appropriate annual price will lead to a "quantity" that is equal to the "true" annual quantity transacted divided by total annual consumption. The problem with the aforementioned algebra starts at definition (A5), which defined the annual unit value price for each product. In order to actually calculate these annual prices, $P_{y,n}$, it is necessary to have information on the corresponding annual quantities transacted, $Q_{y,n}$. But this information is not available.

[20] For additional material and references to the literature on the chain drift problem, see Diewert (2021b).

[21] See Diewert (2021a, 2021b) for more complete discussions of the test approach to index number theory.

[22] However, other European countries (such as Belgium, Italy, and the Netherlands) also have CPIs which exhibit similar amounts of seasonality.

[23] Diewert (1983) showed how this approach to the construction of Mudgett Stone annual indices could be extended to provide an annualized price comparison of the data for a current rolling year (12 consecutive months of data) to a base year.

[24] Using their Israeli data set, these authors showed that different methods of aggregation over months gave rise to substantially different annual indices. The Mudgett-Stone approach to forming annual indices is our preferred approach from a theoretical point of view. However, this approach needs some modification if there is substantial price change *within* the year as might be caused by a hyperinflation; see Hill (1996).

In order to form approximations to the "true" annual product prices and quantities, some additional assumptions must be made. Our *first additional assumption* is that for each product, purchases are distributed evenly over each month in each year. This assumption implies the following equations:

$$q_{y,m,n}/Q_{y,n} = 1/12;$$
$$y = 1,\ldots,7;\ m = 1,\ldots,12;\ n = 1,\ldots,402. \quad \text{(A9)}$$

Upon substituting assumptions (A9) into definitions (A5), we obtain the following equations:

$$P_{y,n} = \Sigma_{m=1}^{12} p_{y,m,n} q_{y,m,n}/Q_{y,n};$$
$$y = 1,\ldots,7;\ n = 1,\ldots,402 \quad \text{(A10)}$$
$$= \Sigma_{m=1}^{12} (1/12) p_{y,m,n}.$$

Thus, under assumption (A9), the annual unit value price for product n is simply the arithmetic average of the monthly unit value prices.

Our *second additional assumption* is that the monthly elementary price indices that have been constructed by Statistics Denmark (the observable $p_{y,m,n}$) are adequate approximations to the monthly unit value prices (normalized to equal unity in month 1).[25] This assumption along with our previous assumption (A9) that implied equations (A10) means that taking the arithmetic average of the monthly Danish elementary indices is an appropriate annual price index. In fact, many statistical agencies (including Statistics Denmark) use simple averages of their monthly elementary indices as appropriate annual elementary indices. Our discussion here simply indicates to readers that these annual indices are not necessarily accurate approximations to "true" annual indices that are based on alternative methodologies. In any case, in this section, we will construct *annual product prices* using the prices $p_{y,n}^*$ defined by the second line (A10). Thus, define the year y annual price for product n, $p_{y,n}^*$, as follows:

$$p_{y,n}^* \equiv \Sigma_{m=1}^{12} (1/12) p_{y,m,n};\quad y = 1, 7;\ n = 1,\ldots,402. \quad \text{(A11)}$$

The corresponding *annual product quantities* (or volumes) $q_{y,n}^*$ that will be used in this section are defined as follows:

$$q_{y,n}^* \equiv S_{y,n}/p_{y,n}^* \qquad y = 1,\ldots,7;\ n = 1,\ldots,402. \quad \text{(A12)}$$

Using equations (A8) and our assumptions, it can be seen that these annual "quantities" $q_{y,n}^*$ defined by (A12) are approximately equal to the true quantities transacted in year y divided by total consumption in year y, $P^y \cdot Q^y$.[26]

In the indices and tables that follow, the underlying annual price and quantity data used to generate the indices will be $p_{y,n}^*$ and $q_{y,n}^*$ defined by (A11) and (A12). The year y price and quantity vectors are defined as $p^{y*} \equiv [p_{y,1}^*,\ldots, p_{y,402}^*]$ and $q^{y*} \equiv [p_{y,1}^*,\ldots,q_{y,402}^*]$ for $y = 1,\ldots,7$.

In making a price comparison between two periods, the Laspeyres and Paasche indices are fundamental because they simply do a ratio comparison of the cost of a fixed reference quantity vector at the prices of the comparison period in the numerator and at the base period prices in the denominator. The Laspeyres index chooses the quantity vector that was consumed in the base period as the reference quantity vector, and the Paasche index chooses the comparison period quantity vector. These indices are both meaningful and easy to explain to the public. In general they will give different answers. If it is necessary to give a single estimate for inflation over the two periods being compared, then it is useful to take a symmetric average of the Laspeyres and Paasche indices as the single estimate. It turns out that the geometric average of these two indices has the "best" properties from the viewpoint of the test approach to index number theory, which is the Fisher (1922) ideal index.[27] The Fisher index also has good properties from the viewpoint of the economic approach to index number theory. Thus, in this section, we use the Danish CPI data to calculate annual Laspeyres, Paasche, and Fisher indices using p_y^* and q_y^* as the underlying price and quantity data.[28]

The *fixed-base Laspeyres, Paasche, and Fisher indices* for year y, P_L^y, P_P^y, and P_F^y are defined as follows:

$$P_L^y \equiv p^{y*}\cdot q^{1*}/p^{1*}\cdot q^{1*};\qquad y = 1,\ldots,7; \quad \text{(A13)}$$
$$P_P^y \equiv p^{y*}\cdot q^{y*}/p^{1*}\cdot q^{y*};\qquad y = 1,\ldots,7; \quad \text{(A14)}$$
$$P_F^y \equiv [P_L^y P_P^y]^{1/2};\qquad y = 1,\ldots,7. \quad \text{(A15)}$$

These indices are listed in Table A2.

In order to define chained indices, it is useful to define the following *Laspeyres, Paasche, and Fisher bilateral annual indices* that compare the prices of year y relative to the base year z as follows:

$$P_L(y/z) \equiv p^{y*}\cdot q^{z*}/p^{z*}\cdot q^{z*};\quad y = 1,\ldots,7;\ z = 1,\ldots,7; \quad \text{(A16)}$$
$$P_P(y/z) \equiv p^{y*}\cdot q^{y*}/p^{z*}\cdot q^{y*};\quad y = 1,\ldots,7;\ z = 1,\ldots,7; \quad \text{(A17)}$$
$$P_F(y/z) \equiv [P_L(y/z)P_P(y/z)]^{1/2};\quad y = 1,\ldots,7;\ z = 1,\ldots,7. \quad \text{(A18)}$$

The *annual chained Laspeyres, Paasche, and Fisher indices* for year 1 are defined as follows:

$$P_{LCH}^{1*} \equiv 1;\ P_{PCH}^{1*} \equiv 1;\ P_{FCH}^{1*} \equiv 1. \quad \text{(A19)}$$

For years y following year 1, these indices are defined recursively using the bilateral maximum overlap annual indices defined earlier by (A16)–(A19) as follows:

$$P_{LCH}^{y} \equiv P_{LCH}^{y-1}\, P_L(y/(y-1));\qquad y = 2,\ldots,7; \quad \text{(A20)}$$

[25] These normalizations simply change the units of measurement for the product groups.

[26] Note that the annual share vectors that are generated by the price and quantity vectors py^* and qy^* are equal to the Statistics Denmark share vectors $Sy \equiv [S_{y,1},\ldots,S_{y,402}]$ for $y = 1,\ldots,7$.

[27] See Diewert (1997, 138).

[28] Since the underlying price and quantity data are not actual annual unit value prices or actual total annual quantities, it is more correct to say that we are calculating various annual indices using py^* and qy^* as the underlying price and quantity data and the Laspeyres, Paasche, and Fisher formulae applied to these data.

Table A1 Predicted Share Measures of Price Dissimilarity for Denmark for Years 1–7

Year	y = 1	y = 2	y = 3	y = 4	y = 5	y = 6	y = 7
z = 1	0.000000	0.000033	0.000090	0.000209	0.000363	0.000482	0.000575
z = 2	0.000033	0.000000	0.000023	0.000111	0.000234	0.000322	0.000403
z = 3	0.000090	0.000023	0.000000	0.000049	0.000137	0.000202	0.000265
z = 4	0.000209	0.000111	0.000049	0.000000	0.000033	0.000081	0.000134
z = 5	0.000363	0.000234	0.000137	0.000033	0.000000	0.000021	0.000058
z = 6	0.000482	0.000322	0.000202	0.000081	0.000021	0.000000	0.000016
z = 7	0.000575	0.000403	0.000265	0.000134	0.000058	0.000016	0.000000

$$P_{PCH}^{\ y} \equiv P_{PCH}^{\ y-1}\, P_P(y/(y-1)); \qquad y = 2,\ldots,7; \qquad (A21)$$
$$P_{FCH}^{\ y} \equiv P_{FCH}^{\ y-1}\, P_F(y/(y-1)); \qquad y = 2,\ldots,7. \qquad (A22)$$

The chained Laspeyres, Paasche, and Fisher indices are also plotted in Figure A11.2.

A problem with the chained indices is that in general, they will not satisfy Walsh's multiperiod identity test, and hence they may be subject to a certain amount of chain drift. On the other hand, fixed-base indices compare the prices of all periods with the prices of period 1, and hence the prices of period 1 play an asymmetric role. Gini (1924, 1931) showed how to solve these problems with fixed-base and chained indices by introducing the GEKS index. This index is equal to the normalization of all possible "star" indices; that is, each period is chosen as the base period, and the final index is the geometric mean of the star indices. Formally, the *annual GEKS price level*s, $p_{GEKS}^{\ y}$, are defined as follows:

$$p_{GEKS}^{\ y} \equiv [\textstyle\prod_{z=1}^{7} P_F\,(y/z)]^{1/7}; \qquad y = 1,\ldots,7. \qquad (A23)$$

The *annual GEKS price index* $P_{\text{GEKS}}^{\ y*}$ is defined as the following normalization of the above GEKS price levels:

$$P_{GEKS}^{\ y} \equiv p_{GEKS}^{\ y}/p_{GEKS}^{\ 1}; \qquad y = 1,\ldots,7. \qquad (A24)$$

The GEKS index is also shown in Figure A11.2.

The final annual "standard" index that will be calculated in this section is another multilateral index: the *predicted share relative price similarity-linked price index*, $P_S^{\ y}$. The idea behind this index is to use the Fisher index to link any two periods in the available data sample. However, rather than picking the first year in the sample as the base year and computing fixed-base Fisher indices or using chained Fisher indices, a set of bilateral links is chosen to link pairs of observations that have the most similar structure of relative prices. The most similar price pairs of observations are combined to construct an overall price index. If prices in any two years are equal or proportional to each other, then any "reasonable" bilateral index number will register the value 1 if prices are equal and will register the proportionality factor if prices are proportional to each other. But if prices are not proportional, then how exactly should the lack of price proportionality be measured?

Recall that $S_{y,n}$ is the Statistics Denmark annual share of household consumption for product class n in year y. The annual prices and quantities for year y, $p_{y,n}^{\ *}$ and $q_{y,n}^{\ *}$ defined by (A11) and (A12), satisfy the following equations:

$$S_{y,n} = p_{y,n}^{\ *}q_{y,n}^{\ *}/p^{y*}\cdot q^{y*}; \quad y = 1,\ldots,7;\ n = 1,\ldots,402. \qquad (A25)$$

Now think of using the *prices* of year z, p^{z*}, and the *quantities* of year y, q^{y*}, to *predict* the actual year y, product n expenditure share $S_{y,n}$ given by (A25) for $n = 1,\ldots,402$. Denote this *predicted share* by $S_{z,y,n}$, which is defined as follows:

$$S_{z,y,n} \equiv p_{z,n}^{\ *}q_{y,n}^{\ *}/p^{z*}\cdot q^{y*}; \ z = 1,\ldots,7; \ y = 1,\ldots,7;\ n = 1,\ldots,402. \qquad (A26)$$

If the prices in year y are proportional to the prices of year z so that $p^{z*} = \lambda p^{y*}$, where λ is a positive number, then it can be verified that the predicted shares defined by (A26) will be equal to the actual expenditure shares defined by (A25) for year y; that is, for the two years defined by y and z, we will have $S_{y,n} = S_{z,y,n}$ for $n = 1,\ldots,N$. The following *predicted share measure of relative price dissimilarity* between the prices of year y and the prices of year z, $\Delta_{PS}(p^{z*},p^{y*},q^{z*},q^{y*})$, is well defined even if some product prices and shares in the two years being compared are equal to 0:[29]

$$\begin{aligned}\Delta_{PS}(p^{z*},p^{y*},q^{z*},q^{y*}) &\equiv \textstyle\sum_{n=1}^{402} [S_{y,n} - S_{z,y,n}]^2 + \sum_{n=1}^{402} [S_{z,n} - S_{y,z,n}]^2 \qquad (A27)\\ &= \textstyle\sum_{n=1}^{402} [(p_{y,n}^{\ *}q_{y,n}^{\ *}/p^{y*}\cdot q^{y}) - (p_{z,n}^{\ *}q_{y,n}^{\ *}/p^{z*}\cdot q^{y})]^2 \\ &\quad + \textstyle\sum_{n=1}^{402} [(p_{z,n}^{\ *}q_{z,n}^{\ *}/p^{z*}\cdot q^{z}) - (p_{y,n}^{\ *}q_{z,n}^{\ *}/p^{y*}\cdot q^{z})]^2\end{aligned}$$

In general, $\Delta_{PS}(p^{z*},p^{y*},q^{z*},q^{y*})$ takes on values between 0 and 2. If $\Delta_{PS}(p^{z*},p^{y*},q^{z*},q^{y*}) = 0$, then it must be the case that relative prices are the same for years z and y; that is, we have $p^{z*} = \lambda p^{y*}$ for some $\lambda > 0$. A bigger value of $\Delta_{PS}(p^{z*},p^{y*},q^{z*},q^{y*})$ generally indicates bigger deviations from price proportionality.

To see how this predicted share measure of annual relative price dissimilarity turned out for our Danish annual data, see Table A1.

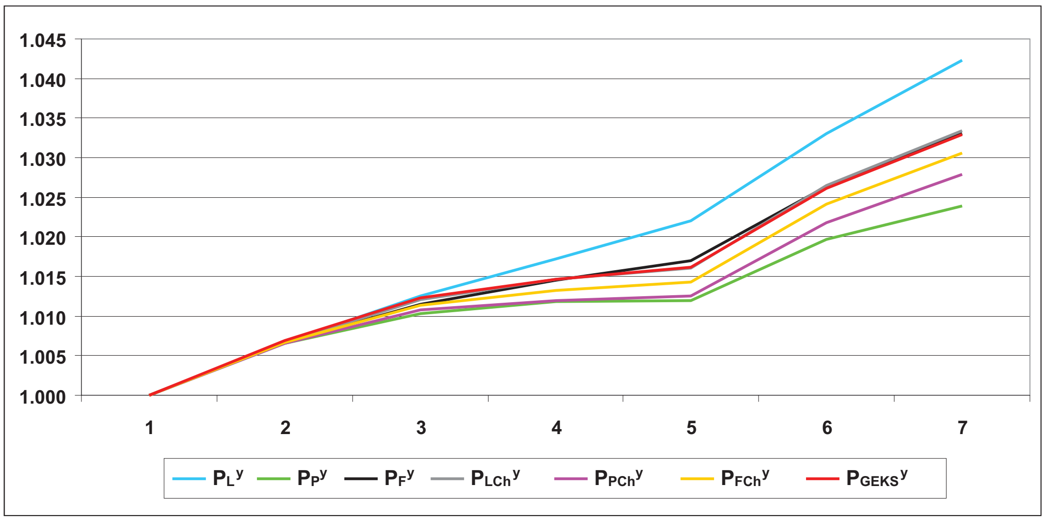
This matrix is used to construct $P_S^{\ y}$, the real-time *similarity-linked price index for the Danish annual data*. This index is constructed as follows. Set $P_S^{\ 1} \equiv 1$. The bilateral Fisher

[29] For information on the properties of this measure of relative price dissimilarity, see Diewert (2021b).

Table A2 Annual Fixed-Base and Chained Laspeyres, Paasche, and Fisher Indices, GEKS Indices and Real-Time Similarity-Linked Indices

Year *y*	P_L^y	P_P^y	P_F^y	P_{LCh}^y	P_{PCh}^y	P_{FCh}^y	P_{GEKS}^y	P_S^y
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.00672	1.00659	1.00666	1.00672	1.00659	1.00666	1.00688	1.00666
3	1.01253	1.01033	1.01143	1.01209	1.01075	1.01142	1.01226	1.01142
4	1.01727	1.01186	1.01456	1.01466	1.01191	1.01329	1.01459	1.01329
5	1.02198	1.01195	1.01695	1.01606	1.01255	1.01430	1.01621	1.01430
6	1.03302	1.01974	1.02636	1.02645	1.02181	1.02413	1.02615	1.02413
7	1.04230	1.02390	1.03306	1.03338	1.02791	1.03064	1.03292	1.03064
G. Rate	1.00693	1.00394	1.00544	1.00549	1.00460	1.00504	1.00541	1.00504

Figure A11.2 Annual Fixed-Base and Chained Laspeyres, Paasche, and Fisher Indices and the GEKS Index



index linking year 2 to year 1, $P_F(2/1)$[30] is set equal to P_S^2. Now look down the $y = 3$ column in Table A1. We need to link year 3 to either year 1 or year 2. The dissimilarity measures for these two years are 0.000090 and 0.000023, respectively. The degree of relative price dissimilarity is far smaller for the link to year 2 than it is to year 1 (year 3 prices are much closer to being proportional to year 2 prices than to year 1 prices), so we use the Fisher link from period 2 to period 3, $P_F^1(3/2)$, to link period 3 to period 2. Thus, the final year 3 similarity-linked index for $y = 3$ is $P_S^3 \equiv P_S^2 \times P_F(3/2)$. Now we need to link year 4 to year 1, 2, or 3. Look down the $y = 4$ column in Table A1 to find the lowest dissimilarity measure above the main diagonal of the matrix. The smallest of the 3 numbers 0.000209, 0.000111, and 0.000049 is 0.00049. Thus, we link the year 4 data to the year 3 data using the Fisher link from year 3 to year 4, $P_F^1(4/3)$, and the year 4 similarity-linked final index value is $P_S^4 \equiv P_S^1 \times P_F^1(4/3)$. *Thus, for each year, as the new data become available, we use the Fisher bilateral index that links the new period to the previous period that has the lowest measure of relative price dissimilarity*. The final two bilateral links are year 5 to year 4 and year 6 to year 5. The resulting year 5 and 6 similarity-linked index values are $P_S^{5,} \equiv P_S^4 \times P_F(5/4)$ and $P_S^6 \equiv P_S^6 \times P_F(6/5)$. The optimal set of *bilateral links* for the real-time similarity-linked indices can be summarized as follows:

$$1 - 2 - 3 - 4 - 5 - 6.$$

Thus, for the Danish annual data, *the real-time similarity-linked indices coincide with the Fisher chained indices*; that is, we have $P_S^y = P_{FCh}^y$ for $y = 1, \ldots, 7$.

The annual fixed-base Laspeyres, Paasche, and Fisher indices, P_L^y, P_P^y, P_F^y, the chained Laspeyres, Paasche, and Fisher indices, P_{LCh}^y, P_{PCh}^y, P_{FCh}^y, the GEKS index P_{GEKS}^y, and the predicted share similarity-linked index P_S^y are listed in Table A1 and are plotted in Figure A11.2.

The last row in Table A2 lists the geometric average rate of growth of the relevant index over the seven-year period; that is, the average geometric growth rate for the fixed-base Laspeyres index, P_L^y, was $1.00693 = 1.04230^{1/6}$, which translates into an average inflation rate of 0.693 percent per year.

[30] $P_F(2/1)$ is defined by (A18), $P_F(y/z) \equiv [P_L(y/z)P_P(y/z)]^{1/2}$, with $y = 2$ and $z = 1$.

Table A3 Annual Jevons, Dutot, Fixed-Base, and Chained Carli Indices and the Annual Predicted Share Index

Year y	$P_J^{\,y}$	$P_D^{\,y}$	$P_C^{\,y}$	$P_{CCh}^{\,y}$	$P_S^{\,y}$
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.00495	1.00650	1.00595	1.00595	1.00666
3	1.00418	1.00682	1.00631	1.00590	1.01142
4	1.00999	1.01462	1.01376	1.01302	1.01329
5	1.01478	1.02252	1.02171	1.01917	1.01430
6	1.02543	1.03520	1.03446	1.03094	1.02413
7	1.02608	1.03815	1.03758	1.03240	1.03064
G. rate	1.00430	1.00626	1.00617	1.00533	1.00504

The similarity-linked index $P_S^{\,y}$ turned out to equal the chained Fisher index $P_{FCh}^{\,t}$. $P_S^{\,y}$ is a preferred index since it satisfies the multiperiod identity test, and it (theoretically) can be implemented in real time provided that household expenditure information is available in real time. The GEKS index $P_{GEKS}^{\,y}$ also satisfies the multiperiod identity test, and it also does not depend on the choice of a base period. It cannot be implemented in real time, but rolling window versions of this index can be implemented in real time.[31] Note that $P_S^{\,y}$ lies in the middle of the various indices that are plotted in Figure A11.2, and $P_{GEKS}^{\,y}$ lies slightly above $P_S^{\,y}$. The Fisher fixed-base index $P_F^{\,y}$ can hardly be distinguished from $P_{GEKS}^{\,y}$. The outlier indices are the fixed-base Laspeyres and Paasche indices; $P_L^{\,y}$ is on average 0.693 – 0.504 = 0.189 percentage points above our preferred chained Fisher and similarity-linked indices, while $P_P^{\,y}$ is on average 0.110 percentage points below $P_{FCh}^{\,y}$ and $P_S^{\,y}$.

The average difference between the growth rates for the fixed-base Laspeyres and the chained Fisher indices is 0.189 percentage points, while the difference between the chained Laspeyres and the chained Fisher indices is only 0.045 percentage points. Thus, substitution bias using the fixed-base Laspeyres formula is much larger than the substitution bias using the chained Laspeyres index.

Finally, note that the average difference between the fixed-base Laspeyres and Paasche annual growth rates is 0.299 percentage points, while the average difference between the chained Laspeyres and Paasche growth rates is only 0.089 percentage points. Thus, for the Danish data, chaining reduces the spread between the Laspeyres and Paasche formulae. This is an indication that it is probably preferable to use chained Fisher indices rather than fixed-base Fisher indices.[32]

To conclude this section, we use the annual price data $p_{y,n}^{\,*}$ to calculate annual fixed-base Jevons, Dutot, and Carli indices, $P_J^{\,y}$, $P_D^{\,y}$, and $P_C^{\,y}$. Since there are no missing observations, the fixed-base Jevons and Dutot indices coincide with their chained counterparts. However, since the annual average product prices no longer equal 1 for year 1, it is no longer the case that $P_D^{\,y} = P_C^{\,y}$. Thus, the fixed-base annual Carli index, $P_C^{\,y}$, must be calculated separately. The chained annual Carli index for year y is denoted by $P_{CCh}^{\,y}$. These indices are listed in Table A3 and are plotted (along with $P_S^{\,y}$ for comparison purposes) in Figure A11.3.

It can be seen that the growth rate for the annual Jevons index $P_J^{\,y}$ is on average 0.074 percentage points below the growth rate of our preferred similarity-linked index $P_S^{\,y}$, while the growth rates for the Dutot index $P_D^{\,y}$ and the fixed-base Carli index $P_C^{\,y}$ are about 0.12 percentage points above the growth rate for $P_S^{\,y}$ on average.[33] The growth rate for the chained Carli index is only about 0.03 percentage points above the $P_S^{\,y}$ growth rate on average. However, for several years, the chained Carli differed substantially from the similarity-linked index. Thus, it can be seen that weighting does matter: The unweighted indices are not completely reliable, but they can approximate trend inflation.

As was noted in Section A.2, the month-to-month Jevons index ended up at 1.04091, and as shown in Table A3, the annual Jevons index ended up much lower at 1.02608, a gap of 1.5 percentage points. This large difference is due to the substantial seasonality in the monthly prices: The January prices were always unusually low relative to average prices for the year, and this seasonality in prices is what explains the large difference.

It can be seen that all four elementary indices capture the trend in the similarity-linked indices $P_S^{\,y}$ fairly well. It also can be seen that the Dutot indices are quite close to the fixed-base Carli indices; this is to be expected since the year one annual prices for the 402 products are fairly close to unity. While none of the annual elementary indices were very close to our best-weighted index $P_S^{\,y}$ (which was also equal to the chained Fisher index) for all years, it can be seen that the Jevons index is reasonably close at the end of the sample period and probably provides the best approximation to $P_S^{\,y}$.

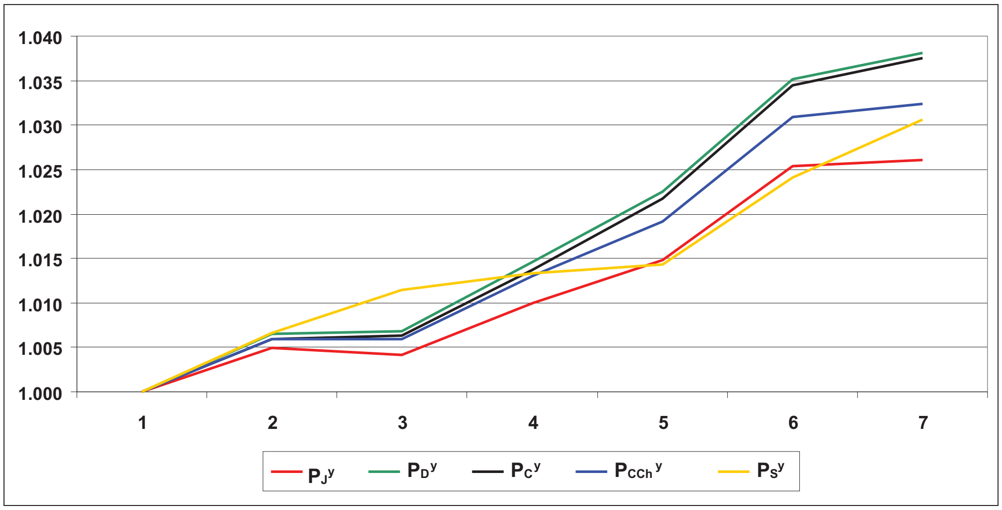
In the following section, we return to the construction of month-to-month indices that use the annual expenditure shares to weight the 402 elementary prices.

[31] See Ivancic, Diewert, and Fox (2011) on rolling window GEKS. The pros and cons of various multilateral index number formulae are discussed by Diewert (2021b).

[32] Chaining tends to be the preferred option if the underlying data have smooth trends; see Diewert (1978, 2021b) and Hill (1988).

[33] Since the annual prices were not normalized to equal one in the first year, the fixed-base Carli index is no longer exactly equal to the Dutot index. However, since the annual product prices for the first year are approximately equal to one, the fixed-base Carli index is approximately equal to the fixed-base (and chained) Dutot index.

Figure A11.3 Jevons, Dutot, Carli, Chained Carli, and Similarity-Linked Indices



A.11.4 Month-to-Month Indices Using Annual Weights

National Statistical Offices in general do not calculate their CPI using the standard index number formulae that are listed in the previous sections. They use annual expenditure shares $S_{y,n}$ or annual "quantities" $q_{y,n}^*$ like those defined by (A11) in the previous section along with monthly prices. They use these prices and quantities in modifications of what are called Lowe (1823) or Young (1812) indices in the index number literature. The modifications involve a mixture of the use of fixed-base and chained indices as was explained in the main text. In this Annex, we will explain in more detail how exactly these "practical" indices are constructed.

The basic *Lowe index* is similar to the Laspeyres index in that it prices out a basket of goods and services at the prices of month t in the numerator of the index and divides by the value of the same basket valued at the prices of month 1. It is different from the Laspeyres index because the quantity basket is not necessarily equal to the basket that was consumed in month 1.

Recall that the price of product n in month t for the Danish data was denoted by $p_{t,n}$ for $t = 1, \ldots, 84$ and $n = 1, \ldots, 402$, and the vector of month t prices was defined as $p^t \equiv [p_{t,1}, \ldots, p_{t,402}]$ for $t = 1, \ldots, 84$. The expenditure share for product n in year y was defined as $S_{y,n}$ for $y = 1, \ldots, 7$ and $n = 1, \ldots, 402$. In the previous section, $S_{y,n}$ was deflated by the corresponding annual price $p_{y,n}^*$ to form the annual "quantity" $q_{y,n}^*$. Define the annual quantity vector for year y as $q^{y*} \equiv [q_{y,1}^*, \ldots, q_{y,402}]$ for years $y = 1, \ldots, 7$.

The monthly *Lowe price index*, $P_{Lo}^{\ t}$, for the first 13 months in the data set is defined as follows:[34]

$$P_{Lo}^{\ t} = p^t \cdot q^{1*} / p^1 \cdot q^{1*};\ t = 1, \ldots, 13. \qquad \text{(A27)}$$

Thus, the cost of the year 1 annual basket of commodities q^{1*} valued at the prices of month t, $p^t \cdot q^{1*}$, is divided by the cost of the year 1 annual basket valued at the prices of January in year 1, $p^1 \cdot q^{1*}$, to give us the Lowe index for month t, $P_{Lo}^{\ t}$, for the first 13 months in the data window.[35]

In earlier years, many National Statistical Offices did not change the annual basket for their Lowe indices for many years. However, in recent times, most countries using the Lowe index methodology for their CPIs update their annual baskets every year. Thus, their Lowe indices are a mixture of fixed-base and chained Lowe indices. For the version of the Lowe index used in this Annex, the annual basket will be changed in January of each year. Thus, (A27) defines our Lowe index for Denmark for the first 13 months in our data window. For the remaining months, $P_{Lo}^{\ t}$ is defined as follows:

$$\begin{aligned}
P_{Lo}^{\ t} &= P_{Lo}^{\ 13}\, p^t \cdot q^{2*} / p^{13} \cdot q^{2*} && t = 13, \ldots, 25; \qquad \text{(A28)}\\
P_{Lo}^{\ t} &= P_{Lo}^{\ 25}\, p^t \cdot q^{3*} / p^{25} \cdot q^{3*}; && t = 25, \ldots, 37;\\
P_{Lo}^{\ t} &= P_{Lo}^{\ 37}\, p^t \cdot q^{4*} / p^{37} \cdot q^{4*}; && t = 37, \ldots, 49;\\
P_{Lo}^{\ t} &= P_{Lo}^{\ 49}\, p^t \cdot q^{5*} / p^{49} \cdot q^{5*}; && t = 49, \ldots, 61;\\
P_{Lo}^{\ t} &= P_{Lo}^{\ 61}\, p^t \cdot q^{6*} / p^{61} \cdot q^{6*} && t = 61, \ldots, 73;\\
P_{Lo}^{\ t} &= P_{Lo}^{\ 73}\, p^t \cdot q^{7*} / p^{73} \cdot q^{7*}; && t = 73, \ldots, 84.
\end{aligned}$$

[34] Note that these Lowe indices can be interpreted as weighted Dutot indices.

[35] The Lowe index is not as fundamental as the Laspeyres or Paasche indices: households in month t do not (in general) consume the annual basket; they consume an appropriate monthly basket. If seasonality in prices and quantities is moderate and if consumption growth over the year is relatively even, then the Lowe index can provide an adequate approximation to the Laspeyres and Paasche indices between months 1 and t. However, as was seen in the main text and in Section A.2 above, there is a great deal of seasonality in the Danish price data, and so it is likely that there is a considerable amount of seasonality in consumption as well, and hence the Lowe index may not provide a very good approximation to the underlying monthly Laspeyres and Paasche indices.

These Lowe indices could be constructed by national offices retrospectively, but they cannot be calculated in real time. Thus, in practice, the annual baskets used in the Lowe formula are lagged one or two years. For illustrative purposes, we will use the year 1 basket as in definitions (A27) for the first year of our data set and then lag the annual basket by one year in subsequent years. Thus, our Lowe indices using one-year lagged annual weights, $P_{Lo1}{}^{t}$, are defined as follows:

$$
\begin{aligned}
P_{Lo1}{}^{t} &= p^{t}\cdot q^{1*}/p^{1}\cdot q^{1*}; & t &= 1,\ldots,13; \quad \text{(A29)}\\
P_{Lo1}{}^{t} &= P_{Lo1}{}^{13}\, p^{t}\cdot q^{1*}/p^{13}\cdot q^{1*}; & t &= 13,\ldots,25;\\
P_{Lo1}{}^{t} &= P_{Lo1}{}^{25}\, p^{t}\cdot q^{2*}/p^{25}\cdot q^{2*}; & t &= 25,\ldots,37;\\
P_{Lo1}{}^{t} &= P_{Lo1}{}^{37}\, p^{t}\cdot q^{3*}/p^{37}\cdot q^{3*}; & t &= 37,\ldots,49;\\
P_{Lo1}{}^{t} &= P_{Lo1}{}^{49}\, p^{t}\cdot q^{4*}/p^{49}\cdot q^{4*}; & t &= 49,\ldots,61;\\
P_{Lo1}{}^{t} &= P_{Lo1}{}^{61}\, p^{t}\cdot q^{5*}/p^{61}\cdot q^{5*}; & t &= 61,\ldots,73;\\
P_{Lo1}{}^{t} &= P_{Lo1}{}^{73}\, p^{t}\cdot q^{6*}/p^{73}\cdot q^{6*}; & t &= 73,\ldots,84.
\end{aligned}
$$

Our illustrative Lowe indices using two-year lagged annual weights, $P_{Lo2}{}^{t}$, are defined as follows:

$$
\begin{aligned}
P_{Lo2}{}^{t} &= p^{t}\cdot q^{1*}/p^{1}\cdot q^{1*}; & t &= 1,\ldots,13; \quad \text{(A30)}\\
P_{Lo2}{}^{t} &= P_{Lo2}{}^{13}\, p^{t}\cdot q^{1*}/p^{13}\cdot q^{1*}; & t &= 13,\ldots,25;\\
P_{Lo2}{}^{t} &= P_{Lo2}{}^{25}\, p^{t}\cdot q^{1*}/p^{25}\cdot q^{1*}; & t &= 25,\ldots,37;\\
P_{Lo2}{}^{t} &= P_{Lo2}{}^{37}\, p^{t}\cdot q^{2*}/p^{37}\cdot q^{2*}; & t &= 37,\ldots,49;\\
P_{Lo2}{}^{t} &= P_{Lo2}{}^{49}\, p^{t}\cdot q^{3*}/p^{49}\cdot q^{3*}; & t &= 49,\ldots,61;\\
P_{Lo2}{}^{t} &= P_{Lo2}{}^{61}\, p^{t}\cdot q^{4*}/p^{61}\cdot q^{4*}; & t &= 61,\ldots,73;\\
P_{Lo2}{}^{t} &= P_{Lo2}{}^{73}\, p^{t}\cdot q^{5*}/p^{73}\cdot q^{5*}; & t &= 73,\ldots,84.
\end{aligned}
$$

For years 1 and 2, the annual weights of year 1 are used in these definitions. Starting at year 3, the annual weights are lagged by two years. The Lowe indices $P_{Lo}{}^{t}$, $P_{Lo1}{}^{t}$, and $P_{Lo2}{}^{t}$ defined above by (A27)–(A30) are plotted in Figure A11.4.[36]

The *Young index* $P_{Y}{}^{t}$ for the first 13 months uses the annual expenditure shares of year 1, $S_{1,n}$ for $n = 1,\ldots,402$ as weights for the monthly prices of month t divided by the price of month t for each product n, the $p_{t,n}/p_{1,n}$, as follows:[37]

$$P_{Y}{}^{t} \equiv \Sigma_{n=1}{}^{402}\, S_{1,n}(p_{t,n}/p_{1,n});\ t = 1,\ldots,13. \quad \text{(A31)}$$

Thus, for the first 13 month in our window of observations, the Young price index is equivalent to a weighted fixed-base Carli index. For the version of the Young index used in this annex, the annual share weights will be changed in January of each year. Thus, (A31) defines our Young index for Denmark for the first 13 months in our data window. For the remaining months, $P_{Y}{}^{t}$ is defined as follows:

$$
\begin{aligned}
P_{Y}{}^{t} &= P_{Y}{}^{13}\, \Sigma_{n=1}{}^{402}\, S_{2,}n(p_{t,n}/p_{13,}n) & t &= 13,\ldots,25; \quad \text{(A32)}\\
P_{Y}{}^{t} &= P_{Y}{}^{25}\, \Sigma_{n=1}{}^{402}\, S_{3,}n(p_{t,n}/p_{25,}n); & t &= 25,\ldots,37;\\
P_{Y}{}^{t} &= P_{Y}{}^{37}\, \Sigma_{n=1}{}^{402}\, S_{4,}n(p_{t,n}/p_{37,}n); & t &= 37,\ldots,49;\\
P_{Y}{}^{t} &= P_{Y}{}^{49}\, \Sigma_{n=1}{}^{402}\, S_{5,}n(p_{t,n}/p_{49,}n); & t &= 49,\ldots,61;\\
P_{Y}{}^{t} &= P_{Y}{}^{61}\, \Sigma_{n=1}{}^{402}\, S_{6,}n(p_{t,n}/p_{61,}n); & t &= 61,\ldots,73;\\
P_{Y}{}^{t} &= P_{Y}{}^{73}\, \Sigma_{n=1}{}^{402}\, S_{7,}n(p_{t,n}/p_{73,}n); & t &= 73,\ldots,84.
\end{aligned}
$$

As was the case with the Lowe index, the Young index cannot be calculated in real time. Thus, real-time Young indices cannot use current year expenditure weights but must use weights that are lagged one or two years. In order to calculate Young indices using one-year lagged weights, we will use the year 1 basket as in definitions (A30) for the first year of our data set and then lag the annual basket by one year in subsequent years. Thus, our Young indices using one-year lagged annual weights, $P_{Y1}{}^{t}$, are defined as follows:

$$
\begin{aligned}
P_{Y1}{}^{t} &= \Sigma_{n=1}{}^{402}\, S_{1,}n(p_{t,n}/p_{1,}n); & t &= 1,\ldots,13; \quad \text{(A33)}\\
P_{Y1}{}^{t} &= P_{Y1}{}^{13}\, \Sigma_{n=1}{}^{402}\, S_{1,}n(p_{t,n}/p_{13,}n); & t &= 13,\ldots,25;\\
P_{Y1}{}^{t} &= P_{Y1}{}^{25}\, \Sigma_{n=1}{}^{402}\, S_{2,}n(p_{t,n}/p_{25,}n); & t &= 25,\ldots,37;\\
P_{Y1}{}^{t} &= P_{Y1}{}^{37}\, \Sigma_{n=1}{}^{402}\, S_{3,}n(p_{t,n}/p_{37,}n); & t &= 37,\ldots,49;\\
P_{Y1}{}^{t} &= P_{Y1}{}^{49}\, \Sigma_{n=1}{}^{402}\, S_{4,}n(p_{t,n}/p_{49,}n); & t &= 49,\ldots,61;\\
P_{Y1}{}^{t} &= P_{Y1}{}^{61}\, \Sigma_{n=1}{}^{402}\, S_{5,}n(p_{t,n}/p_{61,}n); & t &= 61,\ldots,73;\\
P_{Y1}{}^{t} &= P_{Y1}{}^{73}\, \Sigma_{n=1}{}^{402}\, S_{6,}n(p_{t,n}/p_{73,}n); & t &= 73,\ldots,84.
\end{aligned}
$$

Our illustrative Young indices using two-year lagged annual weights, $P_{Y2}{}^{t}$, are defined as follows:

$$
\begin{aligned}
P_{Y2}{}^{t} &= \Sigma_{n=1}{}^{402}\, S_{1,}n(p_{t,n}/p_{1,}n); & t &= 1,\ldots,13; \quad \text{(A34)}\\
P_{Y2}{}^{t} &= P_{Y2}{}^{13}\, \Sigma_{n=1}{}^{402}\, S_{1,}n(p_{t,n}/p_{13,}n); & t &= 13,\ldots,25;\\
P_{Y2}{}^{t} &= P_{Y2}{}^{25}\, \Sigma_{n=1}{}^{402}\, S_{1,}n(p_{t,n}/p_{25,}n); & t &= 25,\ldots,37;\\
P_{Y2}{}^{t} &= P_{Y2}{}^{37}\, \Sigma_{n=1}{}^{402}\, S_{2,}n(p_{t,n}/p_{37,}n); & t &= 37,\ldots,49;\\
P_{Y2}{}^{t} &= P_{Y2}{}^{49}\, \Sigma_{n=1}{}^{402}\, S_{3,}n(p_{t,n}/p_{49,}n); & t &= 49,\ldots,61;\\
P_{Y2}{}^{t} &= P_{Y2}{}^{61}\, \Sigma_{n=1}{}^{402}\, S_{4,}n(p_{t,n}/p_{61,}n); & t &= 61,\ldots,73;\\
P_{Y2}{}^{t} &= P_{Y2}{}^{73}\, \Sigma_{n=1}{}^{402}\, S_{5,}n(p_{t,n}/p_{73,}n); & t &= 73,\ldots,84.
\end{aligned}
$$

For years 1, 2, and 3, the annual weights of year 1 are used in these definitions. Starting at year 3, the annual weights are lagged by two years. The Young indices $P_{Y}{}^{t}$, $P_{Y1}{}^{t}$, and $P_{Y2}{}^{t}$ defined by (A31)–(A33) are plotted in Figure A11.4.

In the main text, the Lowe and Young indices using expenditure weights lagged two years were calculated since these indices are frequently used by national statistical agencies. The geometric Young index has also been used by some Caribbean countries using lagged expenditure weights, so this index was also considered in the main text. The logarithm of the *geometric Young index*, $lnP_{GY}{}^{t}$, using current annual expenditure weights for year 1, is defined as follows:[38]

$$lnP_{GY}{}^{t} \equiv \Sigma_{n=1}{}^{402}\, S_{1,}n\ ln(p_{t,n}/p_{1,}n);\ t = 1,\ldots,13. \quad \text{(A35)}$$

[36]These partially chained Lowe indices are chained every January. As was explained in the main text, Statistics Denmark does the annual chaining every December. Thus, the Lowe indices in this annex will not be equal to the Lowe indices computed in the main text.

[37]Note that these Young indices can be interpreted as weighted Carli indices.

[38]Note that these geometric Young indices can be interpreted as weighted Jevons indices.

For the version of the geometric Young index used in this annex, the annual share weights will be changed in January of each year. Thus, (A35) defines the logarithm of our *geometric Young index* for Denmark for the first 13 months in our data window. For the remaining months, $lnP_Y^{\,t}$ is defined as follows:

$$lnP_{GY}^{\,t} = lnP_{GY}^{\,13} + \Sigma_{n=1}^{402} S_{2,n} ln(p_{t,n}/p_{13,n}); \quad t = 13, \ldots, 25; \qquad \text{(A36)}$$

$$lnP_{GY}^{\,t} = lnP_{GY}^{\,25} + \Sigma_{n=1}^{402} S_{3,n} ln(p_{t,n}/p_{25,n}); \; t = 25, \ldots, 37;$$

$$lnP_{GY}^{\,t} = lnP_{GY}^{\,37} + \Sigma_{n=1}^{402} S_{4,n} ln(p_{t,n}/p_{37,n}); \; t = 37, \ldots, 49;$$

$$lnP_{GY}^{\,t} = lnP_{GY}^{\,49} + \Sigma_{n=1}^{402} S_{5,n} ln(p_{t,n}/p_{49,n}); \; t = 49, \ldots, 61;$$

$$lnP_{GY}^{\,t} = lnP_{GY}^{\,61} + \Sigma_{n=1}^{402} S_{6,n} ln(p_{t,n}/p_{61,n}); \; t = 61, \ldots, 73;$$

$$lnP_{GY}^{\,t} = lnP_{GY}^{\,73} + \Sigma_{n=1}^{402} S_{7,n} ln(p_{t,n}/p_{73,n}); \; t = 73, \ldots, 84.$$

As was the case with the Lowe and Young indices, the geometric Young index cannot be implemented in real time. Our illustrative version of the *geometric Young index that uses expenditure weights lagged one year*, P_{GY1}, has logarithms that are defined by (A35) and (A36) except the expenditure share weights in lines 1–6 of equations (A36) are replaced by the following annual weights: $S_{1,n}$, $S_{2,n}$, $S_{3,n}$, $S_{4,n}$, $S_{5,n}$, and $S_{6,n}$. Our version of the *geometric Young index that uses expenditure weights lagged two years*, P_{GY2} has logarithms that are defined by (A35) and (A36) except the expenditure share weights in lines 1–6 of equations (A36) are replaced by the following annual weights: $S_{1,n}$, $S_{1,n}$, $S_{2,n}$, $S_{3,n}$, $S_{4,n}$, and $S_{5,n}$. The geometric Young indices $P_{GY}^{\,t}$, $P_{GY1}^{\,t}$, and $P_{GY2}^{\,t}$ are listed in Table A4 and are plotted in Figure A11.4. In addition to these nine indices, real-time similarity-linked monthly price indices, $P_S^{\,t}$, are also listed in Table A4 and are plotted in Figure A11.4. These indices are an approximation to "true" month-to-month similarity-linked indices, which have good axiomatic

Table A4 Lowe, Young, geometric Young, and Similarity-Linked Monthly Indices

Month	$P_{Lo}^{\,t}$	$P_{Lo1}^{\,t}$	$P_{Lo2}^{\,t}$	$P_Y^{\,t}$	$P_{Y1}^{\,t}$	$P_{Y2}^{\,t}$	$P_{GY}^{\,t}$	$P_{GY1}^{\,t}$	$P_{GY2}^{\,t}$	$P_S^{\,t}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.01157	1.01157	1.01157	1.01208	1.01208	1.01208	1.01171	1.01171	1.01171	1.01171
3	1.01484	1.01484	1.01484	1.01577	1.01577	1.01577	1.01513	1.01513	1.01513	1.01514
4	1.01422	1.01422	1.01422	1.01540	1.01540	1.01540	1.01457	1.01457	1.01457	1.01458
5	1.01468	1.01468	1.01468	1.01595	1.01595	1.01595	1.01506	1.01506	1.01506	1.01507
6	1.01295	1.01295	1.01295	1.01427	1.01427	1.01427	1.01336	1.01336	1.01336	1.01337
7	1.01209	1.01209	1.01209	1.01310	1.01310	1.01310	1.01213	1.01213	1.01213	1.01211
8	1.01547	1.01547	1.01547	1.01666	1.01666	1.01666	1.01574	1.01574	1.01574	1.01573
9	1.01791	1.01791	1.01791	1.01953	1.01953	1.01953	1.01823	1.01823	1.01823	1.01823
10	1.01671	1.01671	1.01671	1.01830	1.01830	1.01830	1.01707	1.01707	1.01707	1.01707
11	1.01578	1.01578	1.01578	1.01728	1.01728	1.01728	1.01615	1.01615	1.01615	1.01616
12	1.01289	1.01289	1.01289	1.01432	1.01432	1.01432	1.01311	1.01311	1.01311	1.01312
13	1.01030	1.01030	1.01030	1.01117	1.01117	1.01117	1.00986	1.00986	1.00986	1.01059
14	1.02195	1.02152	1.02152	1.02369	1.02303	1.02303	1.02176	1.02115	1.02115	1.02252
15	1.02224	1.02268	1.02268	1.02453	1.02473	1.02473	1.02233	1.02251	1.02251	1.02309
16	1.02008	1.02022	1.02022	1.02255	1.02231	1.02231	1.02017	1.01991	1.01991	1.02092
17	1.02167	1.02190	1.02190	1.02404	1.02394	1.02394	1.02177	1.02165	1.02165	1.02253
18	1.02114	1.02132	1.02132	1.02336	1.02305	1.02305	1.02124	1.02091	1.02091	1.02200
19	1.01826	1.01809	1.01809	1.02044	1.01954	1.01954	1.01806	1.01714	1.01714	1.01879
20	1.01796	1.01854	1.01854	1.02054	1.02036	1.02036	1.01790	1.01768	1.01768	1.01864
21	1.02156	1.02218	1.02218	1.02460	1.02441	1.02441	1.02148	1.02123	1.02123	1.02222
22	1.02238	1.02269	1.02269	1.02535	1.02488	1.02488	1.02240	1.02188	1.02188	1.02314
23	1.02025	1.02104	1.02104	1.02311	1.02301	1.02301	1.02023	1.02010	1.02010	1.02097
24	1.01949	1.02031	1.02031	1.02221	1.02215	1.02215	1.01945	1.01936	1.01936	1.02019
25	1.01863	1.01926	1.01926	1.02080	1.02076	1.02076	1.01835	1.01828	1.01828	1.01830
26	1.02623	1.02669	1.02641	1.02875	1.02842	1.02798	1.02598	1.02563	1.02523	1.02593
27	1.02745	1.02650	1.02678	1.03037	1.02835	1.02846	1.02734	1.02541	1.02553	1.02730
28	1.02928	1.02804	1.02782	1.03237	1.02994	1.02948	1.02916	1.02682	1.02635	1.02912
29	1.02833	1.02787	1.02778	1.03135	1.02973	1.02945	1.02822	1.02665	1.02638	1.02818
30	1.02814	1.02773	1.02719	1.03115	1.02964	1.02880	1.02795	1.02639	1.02556	1.02791

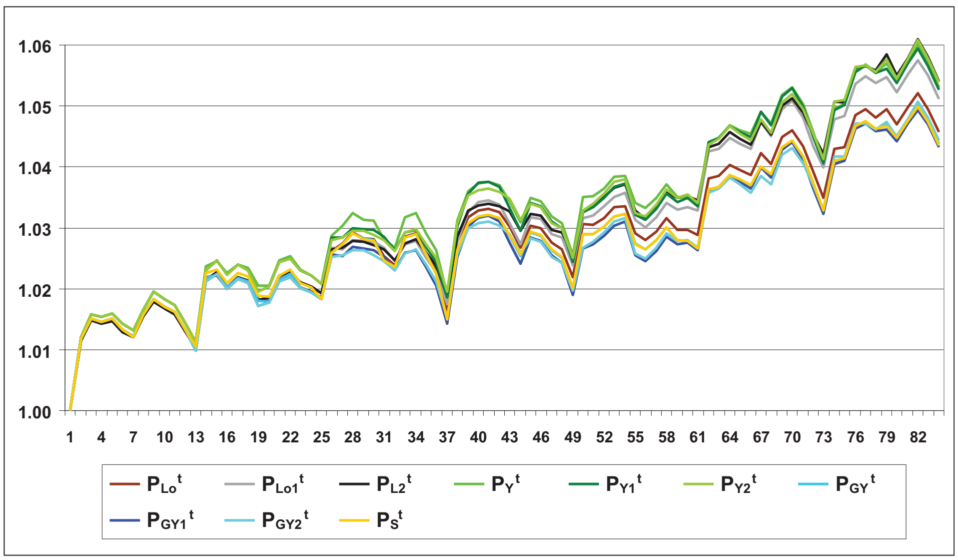
Month	P_{Lo}^t	P_{Lo1}^t	P_{Lo2}^t	P_Y^t	P_{Y1}^t	P_{Y2}^t	P_{GY}^t	P_{GY1}^t	P_{GY2}^t	P_S^t
31	1.02534	1.02678	1.02646	1.02817	1.02860	1.02795	1.02474	1.02522	1.02457	1.02470
32	1.02378	1.02450	1.02463	1.02668	1.02631	1.02617	1.02353	1.02312	1.02297	1.02349
33	1.02850	1.02723	1.02740	1.03179	1.02926	1.02918	1.02839	1.02590	1.02584	1.02835
34	1.02911	1.02779	1.02811	1.03237	1.02972	1.02973	1.02898	1.02639	1.02640	1.02895
35	1.02590	1.02513	1.02586	1.02909	1.02685	1.02727	1.02572	1.02352	1.02394	1.02568
36	1.02321	1.02273	1.02381	1.02635	1.02434	1.02507	1.02251	1.02055	1.02127	1.02251
37	1.01632	1.01743	1.01871	1.01909	1.01863	1.01955	1.01474	1.01432	1.01521	1.01494
38	1.02765	1.02772	1.02860	1.03123	1.03024	1.03071	1.02614	1.02516	1.02567	1.02633
39	1.03179	1.03253	1.03281	1.03597	1.03564	1.03527	1.03056	1.03022	1.02997	1.03077
40	1.03287	1.03427	1.03368	1.03721	1.03737	1.03620	1.03164	1.03176	1.03077	1.03184
41	1.03313	1.03448	1.03400	1.03749	1.03752	1.03645	1.03194	1.03194	1.03101	1.03214
42	1.03252	1.03375	1.03348	1.03698	1.03672	1.03584	1.03135	1.03106	1.03031	1.03156
43	1.03021	1.03074	1.03264	1.03440	1.03321	1.03474	1.02868	1.02753	1.02897	1.02887
44	1.02672	1.02724	1.02958	1.03087	1.02952	1.03137	1.02542	1.02410	1.02591	1.02561
45	1.03037	1.03172	1.03222	1.03486	1.03411	1.03396	1.02918	1.02840	1.02832	1.02939
46	1.02990	1.03150	1.03199	1.03432	1.03358	1.03336	1.02866	1.02789	1.02774	1.02887
47	1.02751	1.02902	1.02964	1.03186	1.03109	1.03076	1.02629	1.02549	1.02524	1.02649
48	1.02642	1.02835	1.02924	1.03073	1.02996	1.02987	1.02502	1.02424	1.02420	1.02522
49	1.02196	1.02387	1.02561	1.02549	1.02441	1.02514	1.02009	1.01903	1.01971	1.02010
50	1.03064	1.03161	1.03280	1.03504	1.03258	1.03283	1.02896	1.02657	1.02677	1.02897
51	1.03045	1.03203	1.03341	1.03524	1.03342	1.03401	1.02902	1.02727	1.02777	1.02903
52	1.03163	1.03349	1.03509	1.03650	1.03490	1.03561	1.03020	1.02867	1.02929	1.03021
53	1.03337	1.03510	1.03653	1.03836	1.03667	1.03754	1.03192	1.03030	1.03107	1.03193
54	1.03360	1.03579	1.03719	1.03853	1.03728	1.03793	1.03223	1.03102	1.03161	1.03224
55	1.02907	1.03126	1.03285	1.03407	1.03218	1.03249	1.02737	1.02553	1.02580	1.02733
56	1.02793	1.03001	1.03139	1.03328	1.03133	1.03181	1.02647	1.02455	1.02495	1.02644
57	1.02942	1.03174	1.03343	1.03496	1.03316	1.03381	1.02801	1.02625	1.02681	1.02798
58	1.03154	1.03411	1.03577	1.03712	1.03560	1.03626	1.03006	1.02855	1.02912	1.03003
59	1.02964	1.03301	1.03488	1.03510	1.03426	1.03497	1.02806	1.02723	1.02783	1.02803
60	1.02962	1.03345	1.03522	1.03509	1.03485	1.03543	1.02785	1.02758	1.02805	1.02782
61	1.02877	1.03279	1.03446	1.03375	1.03357	1.03402	1.02657	1.02630	1.02668	1.02666
62	1.03813	1.04260	1.04329	1.04406	1.04403	1.04365	1.03630	1.03617	1.03570	1.03636
63	1.03854	1.04293	1.04382	1.04471	1.04476	1.04468	1.03666	1.03661	1.03640	1.03672
64	1.04039	1.04482	1.04571	1.04685	1.04676	1.04663	1.03862	1.03843	1.03822	1.03869
65	1.03956	1.04376	1.04463	1.04604	1.04576	1.04550	1.03786	1.03749	1.03717	1.03793
66	1.03871	1.04298	1.04360	1.04540	1.04494	1.04432	1.03698	1.03649	1.03578	1.03705
67	1.04230	1.04733	1.04744	1.04910	1.04900	1.04774	1.04006	1.03985	1.03859	1.04007
68	1.04045	1.04510	1.04532	1.04716	1.04684	1.04564	1.03868	1.03830	1.03708	1.03875
69	1.04487	1.04951	1.04997	1.05180	1.05152	1.05061	1.04314	1.04283	1.04193	1.04322
70	1.04603	1.05087	1.05123	1.05306	1.05293	1.05198	1.04423	1.04404	1.04313	1.04432
71	1.04344	1.04805	1.04889	1.05026	1.04987	1.04941	1.04166	1.04120	1.04065	1.04172
72	1.03924	1.04339	1.04535	1.04570	1.04494	1.04545	1.03733	1.03650	1.03694	1.03742
73	1.03497	1.03987	1.04216	1.04085	1.04055	1.04134	1.03268	1.03231	1.03299	1.03286
74	1.04292	1.04782	1.05074	1.04965	1.04939	1.05076	1.04085	1.04050	1.04174	1.04095
75	1.04324	1.04830	1.05064	1.05027	1.05012	1.05101	1.04128	1.04098	1.04172	1.04140
76	1.04847	1.05368	1.05608	1.05575	1.05557	1.05646	1.04659	1.04629	1.04705	1.04671

(Continued)

Table A4 (Continued)

Month	P_{Lo}^{t}	P_{Lo1}^{t}	P_{Lo2}^{t}	P_{Y}^{t}	P_{Y1}^{t}	P_{Y2}^{t}	P_{GY}^{t}	P_{GY1}^{t}	P_{GY2}^{t}	P_{S}^{t}
77	1.04946	1.05485	1.05660	1.05688	1.05662	1.05673	1.04744	1.04709	1.04717	1.04756
78	1.04813	1.05382	1.05588	1.05559	1.05537	1.05562	1.04613	1.04589	1.04621	1.04625
79	1.04940	1.05477	1.05850	1.05697	1.05614	1.05763	1.04696	1.04611	1.04743	1.04667
80	1.04699	1.05228	1.05505	1.05457	1.05371	1.05445	1.04506	1.04416	1.04484	1.04477
81	1.04969	1.05519	1.05784	1.05750	1.05690	1.05765	1.04782	1.04713	1.04784	1.04753
82	1.05209	1.05757	1.06093	1.06008	1.05941	1.06078	1.05012	1.04938	1.05066	1.04983
83	1.04951	1.05497	1.05809	1.05718	1.05660	1.05784	1.04765	1.04697	1.04814	1.04735
84	1.04572	1.05119	1.05407	1.05312	1.05265	1.05384	1.04377	1.04322	1.04433	1.04347
G. Rate	1.00748	1.00836	1.00882	1.00866	1.00859	1.00878	1.00716	1.00708	1.00726	1.00712

Figure A11.4 Monthly Lowe, Young, geometric Young, and Similarity-Linked Indices



and economic properties. The term P_S^t will be defined formally in the following section.

All of these indices capture the trend in Danish CPI inflation reasonably well. However, the three "true" indices that use current year annual weights do differ considerably at times. If we take the geometric average annual growth rates for "true" Lowe, Young, and geometric Young indices, P_{Lo}^t, P_Y^t, P_{GY}^t, 1.00748,[39] 1.00866, and 1.00716 , and subtract the average annual growth rate for the similarity-linked indices P_S^t, 1.00712, we find that the approximate annual substitution bias in the three "true" indices over the entire sample period is 0.038, 0.154, and 0.004 percentage points per year, respectively.[40]

Looking at Table A4, it can be seen that the average annual inflation rate of the three Lowe indices increase as the lag in the annual weights increases. The average growth rates of P_{Lo}^t, P_{Lo1}^t, and P_{Lo2}^t are 1.00748, 1.00836, and 1.0866. Thus, the average annual substitution bias for the Lowe indices increases from 0.038 percentage points per year for the current weight Lowe index to 0.124 and 0.172 percentage points per year for the practical Lowe indices that use weights that are one and two years old. The geometric average annual growth rates of P_Y^t, P_{Y1}^t, and P_{Y2}^t less the corresponding average of the real-time similarity-linked indices P_S^t are 0.154, 0.147, and 0.164 percentage points, respectively. Finally, the average annual geometric growth rates of the geometric Young indices, P_{GY}^t, P_{GY1}^t, and P_{GY2}^t, less the corresponding annual average of the real-time similarity-linked indices are 0.004, –0.004, and 0.014 percentage points, respectively. It can be seen that the three geometric Young indices are close to each other and have the smallest approximate substitution bias.

At the end of the sample period, the highest line corresponds to P_{Lo2}^t followed by the three Young indices, P_{Y2}^t, P_Y^t,

[39] To be precise, the geometric annual average growth rate for the "true" Lowe index $P_{Lo}t$ is defined as $(P_{Lo}^{84})^{1/6} = 1.00748$, so $P_{Lo}^{84} = 1.04572 = (1.00748)^6$.

[40] The lagged indices are only approximations to the "true" lagged indices since we use the year 1 expenditure weights in place of the lagged expenditure weights for years 1 and 2.

and $P_{YI}^{\,t}$. These four high inflation indices are tightly clustered and difficult to distinguish. These indices are followed by the Lowe index that uses weights lagged one year, $P_{LoI}^{\,t}$. There is a gap between these five indices and the next index, which is the "true" Lowe index $P_{Lo}^{\,t}$. The final four indices, $P_{GY2}^{\,t}$, $P_{GY}^{\,t}$, $P_{S}^{\,t}$, and $P_{YI}^{\,t}$, are tightly clustered and difficult to distinguish in Figure A11.4. The seasonality in the monthly data is again apparent.

For the Danish data under consideration, there appears to be upward substitution biases in the lagged Lowe and Young indices, while the lagged geometric Young indices appear to be largely free from substitution bias. These results are in agreement with the results provided in the main text.

In the following section, the construction of the similarity-linked indices $P_S^{\,t}$ will be explained.

A.11.5 Month-to-Month Approximate Fisher and Similarity-Linked Indices

In this section, standard weighted month-to-month price indices for Denmark, such as the Laspeyres, Paasche, and Fisher indices, are constructed.[41] However, as was noted in earlier sections of this chapter, monthly information on quantities or expenditures on consumer goods and services is not available. Thus, we use the available annual expenditure information as *approximations* to actual monthly expenditures. Recall that the annual expenditure share on product n in year y was defined as $S_{y,n}$ for $y = 1, \ldots, 7$. The approximate *monthly expenditure share for product n in month t*, $s_{t,n}$, is defined as follows:

$$s_{t,n} \equiv S_{1,}n;\ t = 1, \ldots, 12;\ n = 1, \ldots, 402; \quad \text{(A37)}$$
$$s_{t,n} \equiv S_{2,}n;\ t = 13, \ldots, 24;\ n = 1, \ldots, 402;$$
$$s_{t,n} \equiv S_{3,}n;\ t = 25, \ldots, 36;\ n = 1, \ldots, 402;$$
$$s_{t,n} \equiv S_{4,}n;\ t = 37, \ldots, 48;\ n = 1, \ldots, 402;$$
$$s_{t,n} \equiv S_{5,}n;\ t = 49, \ldots, 60;\ n = 1, \ldots, 402;$$
$$s_{t,n} \equiv S_{6,}n;\ t = 61, \ldots, 72;\ n = 1, \ldots, 402;$$
$$s_{t,n} \equiv S_{7,}n;\ t = 73, \ldots, 84;\ n = 1, \ldots, 402.$$

Recall that the official month t price index for product n (normalized to equal 1 in month 1) was defined as $p_{t,n}$ in Section 2. This monthly price index is used to deflate the corresponding monthly expenditure to form an approximate month t, product n "quantity" (or volume), $q_{t,n}$; that is, we have the following definitions:

$$q_{t,n} \equiv s_{t,n}/p_{t,n};\ t = 1, \ldots, 84;\ n = 1, \ldots, 402. \quad \text{(A38)}$$

Define the month t price and quantity vectors as $p^t \equiv [p_{t,1}, \ldots, p_{t,402}]$ and $q^t \equiv [q_{t,1}, \ldots, q_{t,402}]$ for $t = 1, \ldots, 84$. Now repeat definitions (A13)–(A24) in Section 3 to define the fixed-base monthly Laspeyres, Paasche, and Fisher indices $P_L^{\,t}$, $P_P^{\,t}$, and $P_F^{\,t}$, the chained monthly Laspeyres, Paasche, and Fisher indices $P_{LCh}^{\,t}$, $P_{PCh}^{\,t}$, and $P_{FCh}^{\,t}$, and the monthly GEKS index $P_{GEKS}^{\,t}$. In forming these indices using definitions (A13)–(A24), the monthly price vector p^t replaces the annual price vector p^{y*}, the monthly quantity vector q^t replaces the annual quantity vector q^{y*} and $t = 1, \ldots, 84$ replaces $y = 1, \ldots, 7$. These monthly indices are listed in Table A6 and are plotted in Figure A11.5.

The task of defining monthly relative price similarity-linked indices remains. The definitions for the real-time predicted share similarity-linked monthly price index $P_S^{\,t}$ is similar to the earlier definition of these indices for the annual indices. We use the *prices* of month r, p^r, and the *quantities* of month t, q^t, to *predict* the actual month t, product n expenditure shares $s_{t,n}$ defined by (A37) for $n = 1, \ldots, 402$. Denote this *predicted share* by $s_{r,t,n}$, which is defined as follows:

$$s_{r,}t_{,}n \equiv p_{r,n}q_{t,n}/p^r{\cdot}q^t;\ r = 1, \ldots, 84;\ t = 1, \ldots, 84;\ n = 1, \ldots, 402. \quad \text{(A39)}$$

If the prices in month r are proportional to the prices in month t so that $p^r = \lambda p^t$, where λ is a positive number, then it can be verified that the predicted shares defined by (A39) will be equal to the actual expenditure shares defined by (A37) for month t; that is, for the two months defined by r and t, we have $s_{t,n} = s_{r,t,n}$ for $n = 1, \ldots, 402$. The following *predicted share measure of relative price dissimilarity* between the prices of month r and the prices of month t, $\Delta_{PS}(p^r,p^t,q^r,q^t)$, is defined as follows:

$$\begin{aligned}\Delta_{PS}(p^r,p^t\, q^r,q^t) &\equiv \Sigma_{n=1}^{402}\, [s_{t,n} - s_{r,t,n}]^2 + \Sigma_{n=1}^{402}\, [s_{r,n} - s_{t,r,n}]^2 \quad \text{(A40)}\\ &= \Sigma_{n=1}^{402}\, [(p_{t,n}q_{t,n}/p^t{\cdot}q^t) - (p_{r,n}q_{t,n}/p^r{\cdot}q^t)]^2\\ &+ \Sigma_{n=1}^{402}\, [(p_{r,n}q_{r,n}/p^r{\cdot}q^r) - (p_{t,n}q_{r,n}/p^t{\cdot}q^r)]^2.\end{aligned}$$

To see how this predicted share measure of monthly relative price dissimilarity for months 1 to 12 turned out for our Danish data, refer to Table A5.[42]

This matrix can be used to construct the real-time *similarity-linked price index for the Danish monthly data* $P_S^{\,t}$ for the first 12 months. This index is constructed in the same way as the annual indices. Thus, set $P_S^{\,1} \equiv 1$. The bilateral Fisher index linking month 2 to month 1, $P_F(2/1)$, is set equal to $P_S^{\,2}$. Now look down the $t = 3$ column in Table A5. We need to link month 3 to either month 1 or month 2. The dissimilarity measures for these two months are 0.00016 and 0.00006, respectively. The degree of relative price dissimilarity is far smaller for the link to month 2 than for the link to month 1, so we use the Fisher link from month 2 to month 3, $P_F^{\,1}(3/2)$, to link month 3 to month 2. The final month 3 similarity-linked index for $t = 4$ is $P_S^{\,3} \equiv P_S^{\,2}{\times}P_F(3/2)$. The first three measures of dissimilarity in column 4 of Table A5 are 0.00021, 0.00012, and 0.00005. Thus, it is optimal to link month 4 to month 3 and so on. The optimal set of *bilateral links* for the real-time similarity-linked indices for months 1 to 12 can be summarized as follows:

1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 - 10 – 11 – 12.

[41] It would be more accurate to call these indices approximations to standard monthly indices since accurate monthly quantity or expenditure information is not available.

[42] In order to fit all 12 columns of dissimilarity measures for months 1–12 on a single page, the actual dissimilarity measures have been multiplied by 10 in Table A5.

Table A5 Predicted Share Measures of Price Dissimilarity for Denmark for Months 1–12

Month t	1	2	3	4	5	6	7	8	9	10	11	12
1	0.00000	0.00017	0.00016	0.00021	0.00020	0.00024	0.00027	0.00024	0.00027	0.00028	0.00028	0.00036
2	0.00017	0.00000	0.00006	0.00012	0.00014	0.00016	0.00012	0.00017	0.00021	0.00020	0.00022	0.00028
3	0.00016	0.00006	0.00000	0.00005	0.00009	0.00014	0.00015	0.00012	0.00012	0.00013	0.00018	0.00026
4	0.00021	0.00012	0.00005	0.00000	0.00003	0.00008	0.00015	0.00008	0.00007	0.00009	0.00013	0.00019
5	0.00020	0.00014	0.00009	0.00003	0.00000	0.00004	0.00014	0.00009	0.00008	0.00008	0.00008	0.00012
6	0.00024	0.00016	0.00014	0.00008	0.00004	0.00000	0.00010	0.00009	0.00009	0.00007	0.00005	0.00006
7	0.00027	0.00012	0.00015	0.00015	0.00014	0.00010	0.00000	0.00006	0.00012	0.00012	0.00015	0.00018
8	0.00024	0.00017	0.00012	0.00008	0.00009	0.00009	0.00006	0.00000	0.00004	0.00006	0.00010	0.00015
9	0.00027	0.00021	0.00012	0.00007	0.00008	0.00009	0.00012	0.00004	0.00000	0.00002	0.00006	0.00011
10	0.00028	0.00020	0.00013	0.00009	0.00008	0.00007	0.00012	0.00006	0.00002	0.00000	0.00003	0.00007
11	0.00028	0.00022	0.00018	0.00013	0.00008	0.00005	0.00015	0.00010	0.00006	0.00003	0.00000	0.00002
12	0.00036	0.00028	0.00026	0.00019	0.00012	0.00006	0.00018	0.00015	0.00011	0.00007	0.00002	0.00000

Thus, for the Danish monthly data*, the real-time similarity-linked indices coincide with the Fisher chained indices* for months 1–12, that is, we have $P_S{}^t = P_{FCh}{}^t$ for $t = 1, \ldots, 12$.

It turns out that using the monthly Danish data, we found that most bilateral links were chain links. There were only 10 links that were not chained: 31 linked to 26, 33 linked to 28, 55 linked to 50, 59 linked to 57, 64 linked to 62, 67 linked to 55, 68 linked to 66, 74 linked to 68, 79 linked to 67, and 83 linked to 81. Two of these links (67–55 and 79–67) were year-over-year links. The *real-time similarity-linked price index* $P_S{}^t$ is listed in Table A6 and are plotted in Figure A11.5.

There is one more month-to-month similarity-linked index that is listed in Table A6: the *modified predicted share similarity-linked index*, $P_{SM}{}^t$. This index is an index that can be constructed in real time after one year of price and quantity data have been collected. Instead of using real-time linking in the first year, Hill's (2001) spanning tree method of linking the first 12 months is used. Basically, this method looks at the first 12 months of data as a whole and finds the path linking all 12 months that generates the lowest sum of bilateral measures of price dissimilarity. Thus, this method of linking requires that the first 12 months of data be used as a "training" set of data where an initial set of bilateral links is determined simultaneously using already available historical data.[43] Using the information in Table A5, we find that the optimal path that makes simultaneous use of the data is the following set of bilateral links:

```
        1
        |
2 – 3 – 4 – 5 – 6
                |
7 – 8 – 9 – 10 – 11 – 12.
```

Thus, month 1 is linked to month 3, month 3 is linked to months 2 and 4, month 5 is linked to months 6 and 10, month 10 is linked to months 9 and 11, month 12 is linked to month 11, month 9 is linked to month 10, month 8 is linked to month 9, and finally month 7 is linked to month 8. Once the first 12 observations have been linked, we use real-time linking to calculate the remainder of the bilateral links for the modified similarity-linked index, $P_{SM}{}^t$. The bilateral links for months 13 to 84 are exactly the same as the corresponding links for $P_S{}^t$ for $t = 13, \ldots, 84$. The modified similarity-linked index $P_{SM}{}^t$ is listed in Table A6. It is not shown in Figure A11.5 because $P_{SM}{}^t$ cannot be distinguished from the real-time similarity-linked index $P_S{}^t$ defined earlier.

It can be seen that the two similarity-linked indices, $P_S{}^t$ and $P_{SM}{}^t$, approximate each other to the fourth decimal place. These indices end up at 1.0435 (to four decimal places) and should have the least amount of upper-level substitution bias for the Danish monthly data set. The average annual geometric growth of the real-time monthly similarity-linked indices $P_S{}^t$ is 1.00712 or 0.712 percentage points per year. The fixed-base and chained monthly Laspeyres indices, $P_L{}^t$ and $P_{LCh}{}^t$, have average annual geometric growth rates equal to 1.00961 and 1.01229, respectively, which indicate average upward biases of 0.249 and 0.517 percentage points per year relative to the preferred similarity-linked index $P_S{}^t$. Thus, the behavior of the monthly chained Laspeyres index is very different from the behavior of the annual chained Laspeyres index: The annual fixed-base and chained Laspeyres indices, $P_L{}^y$ and $P_{LCh}{}^y$, had geometric average annual growth rates equal to 1.00693 and 1.00549 compared to 1.00504, the average annual growth rate for the annual similarity-linked indices, which indicate much smaller average annual upward biases of 0.189 and 0.045 percentage points, respectively, in the annual Laspeyres indices. The monthly chained Laspeyres has a very large upward chain drift, whereas the annual chained Laspeyres index has a very moderate upward chain drift. The monthly fixed-base and chained Paasche indices, $P_P{}^t$ and $P_{PCh}{}^t$, have annual average growth rates equal to 1.00554 and 1.00214, which indicate average downward biases of 0.158 and 0.498 percentage points, respectively, relative to the growth rate for $P_S{}^t$. The

[43] Hill's method can be particularly useful if the monthly data exhibit substantial seasonal fluctuations.

Table A6 Laspeyres, Paasche, Fisher Fixed-Base, and Chained Indices, GEKS Index, and Similarity-Linked Indices

Month *t*	P_L^t	P_P^t	P_{LCh}^t	P_{PCh}^t	P_F^t	P_{FCh}^t	P_{GEKS}^t	P_S^t	P_{SM}^t
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.01208	1.01134	1.01208	1.01134	1.01171	1.01171	1.01193	1.01171	1.01173
3	1.01577	1.01456	1.01586	1.01443	1.01516	1.01514	1.01570	1.01514	1.01516
4	1.01540	1.01381	1.01550	1.01367	1.01460	1.01458	1.01518	1.01458	1.01460
5	1.01595	1.01422	1.01610	1.01403	1.01509	1.01507	1.01569	1.01507	1.01509
6	1.01427	1.01248	1.01463	1.01211	1.01337	1.01337	1.01390	1.01337	1.01339
7	1.01310	1.01116	1.01411	1.01012	1.01213	1.01211	1.01209	1.01211	1.01214
8	1.01666	1.01481	1.01809	1.01337	1.01573	1.01573	1.01580	1.01573	1.01575
9	1.01953	1.01694	1.02095	1.01552	1.01824	1.01823	1.01906	1.01823	1.01826
10	1.01830	1.01587	1.01992	1.01423	1.01708	1.01707	1.01788	1.01707	1.01710
11	1.01728	1.01507	1.01914	1.01319	1.01617	1.01616	1.01696	1.01616	1.01618
12	1.01432	1.01189	1.01622	1.01003	1.01311	1.01312	1.01381	1.01312	1.01314
13	1.01117	1.01012	1.01382	1.00738	1.01064	1.01059	1.01067	1.01059	1.01062
14	1.02286	1.02159	1.02637	1.01868	1.02223	1.02252	1.02282	1.02252	1.02254
15	1.02436	1.02213	1.02734	1.01886	1.02325	1.02309	1.02371	1.02309	1.02312
16	1.02198	1.01975	1.02534	1.01653	1.02087	1.02092	1.02162	1.02092	1.02095
17	1.02360	1.02145	1.02712	1.01796	1.02253	1.02253	1.02310	1.02253	1.02256
18	1.02300	1.02080	1.02678	1.01723	1.02190	1.02200	1.02240	1.02200	1.02202
19	1.01949	1.01727	1.02429	1.01332	1.01838	1.01879	1.01864	1.01879	1.01882
20	1.02019	1.01698	1.02449	1.01282	1.01858	1.01864	1.01882	1.01864	1.01866
21	1.02419	1.02009	1.02842	1.01605	1.02214	1.02222	1.02274	1.02222	1.02224
22	1.02462	1.02126	1.02962	1.01670	1.02294	1.02314	1.02397	1.02314	1.02317
23	1.02277	1.01911	1.02765	1.01433	1.02094	1.02097	1.02172	1.02097	1.02099
24	1.02202	1.01834	1.02701	1.01341	1.02018	1.02019	1.02078	1.02019	1.02022
25	1.02057	1.01739	1.02635	1.01031	1.01898	1.01830	1.01886	1.01830	1.01832
26	1.02791	1.02461	1.03434	1.01759	1.02626	1.02593	1.02653	1.02593	1.02596
27	1.02865	1.02548	1.03619	1.01849	1.02706	1.02730	1.02771	1.02730	1.02733
28	1.02982	1.02700	1.03818	1.02015	1.02841	1.02912	1.02926	1.02912	1.02915
29	1.02972	1.02616	1.03738	1.01906	1.02794	1.02818	1.02850	1.02818	1.02821
30	1.02908	1.02580	1.03750	1.01840	1.02744	1.02791	1.02809	1.02791	1.02793
31	1.02794	1.02257	1.03507	1.01441	1.02525	1.02469	1.02556	1.02470	1.02473
32	1.02616	1.02164	1.03422	1.01285	1.02390	1.02348	1.02433	1.02349	1.02352
33	1.02938	1.02612	1.03966	1.01716	1.02775	1.02835	1.02879	1.02835	1.02838
34	1.03005	1.02655	1.04041	1.01760	1.02830	1.02894	1.02931	1.02895	1.02897
35	1.02775	1.02305	1.03733	1.01414	1.02540	1.02567	1.02621	1.02568	1.02570
36	1.02554	1.01928	1.03473	1.01043	1.02241	1.02251	1.02323	1.02251	1.02254
37	1.01998	1.01314	1.02765	1.00238	1.01655	1.01493	1.01592	1.01494	1.01496
38	1.02982	1.02483	1.03990	1.01293	1.02733	1.02632	1.02714	1.02633	1.02636
39	1.03496	1.02911	1.04502	1.01670	1.03203	1.03076	1.03171	1.03077	1.03079
40	1.03570	1.03018	1.04629	1.01758	1.03293	1.03183	1.03272	1.03184	1.03187
41	1.03621	1.03041	1.04673	1.01775	1.03330	1.03213	1.03307	1.03214	1.03217
42	1.03567	1.02941	1.04637	1.01695	1.03253	1.03156	1.03239	1.03156	1.03159
43	1.03425	1.02645	1.04430	1.01366	1.03034	1.02886	1.03000	1.02887	1.02890
44	1.03176	1.02306	1.04132	1.01013	1.02740	1.02561	1.02687	1.02561	1.02564
45	1.03487	1.02626	1.04566	1.01336	1.03056	1.02938	1.03059	1.02939	1.02941

(Continued)

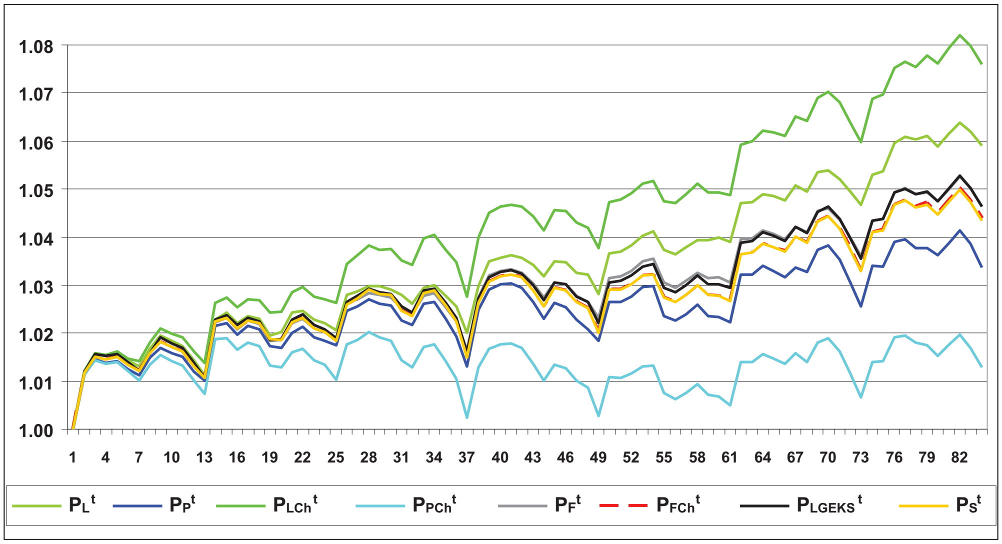
Table A6 (Continued)

Month t	$P_L^{\,t}$	$P_P^{\,t}$	$P_{LCh}^{\,t}$	$P_{PCh}^{\,t}$	$P_F^{\,t}$	$P_{FCh}^{\,t}$	$P_{GEKS}^{\,t}$	$P_S^{\,t}$	$P_{SM}^{\,t}$
46	1.03482	1.02545	1.04535	1.01265	1.03012	1.02887	1.03013	1.02887	1.02890
47	1.03261	1.02277	1.04311	1.01013	1.02768	1.02649	1.02763	1.02649	1.02652
48	1.03220	1.02081	1.04199	1.00872	1.02649	1.02522	1.02646	1.02522	1.02525
49	1.02809	1.01841	1.03768	1.00281	1.02324	1.02009	1.02211	1.02010	1.02012
50	1.03660	1.02648	1.04734	1.01092	1.03153	1.02897	1.03060	1.02897	1.02900
51	1.03701	1.02657	1.04774	1.01066	1.03178	1.02903	1.03084	1.02903	1.02906
52	1.03830	1.02765	1.04911	1.01165	1.03296	1.03021	1.03204	1.03021	1.03024
53	1.04025	1.02957	1.05107	1.01314	1.03489	1.03193	1.03385	1.03193	1.03196
54	1.04119	1.02983	1.05160	1.01323	1.03549	1.03224	1.03430	1.03224	1.03227
55	1.03739	1.02355	1.04752	1.00754	1.03045	1.02734	1.02936	1.02733	1.02735
56	1.03645	1.02261	1.04709	1.00622	1.02950	1.02645	1.02843	1.02644	1.02647
57	1.03787	1.02394	1.04886	1.00753	1.03088	1.02799	1.02996	1.02798	1.02800
58	1.03934	1.02599	1.05115	1.00936	1.03264	1.03004	1.03199	1.03003	1.03006
59	1.03935	1.02346	1.04936	1.00715	1.03137	1.02804	1.03025	1.02803	1.02805
60	1.04000	1.02333	1.04936	1.00674	1.03163	1.02783	1.03023	1.02782	1.02785
61	1.03901	1.02218	1.04880	1.00500	1.03056	1.02667	1.02935	1.02666	1.02668
62	1.04704	1.03213	1.05927	1.01397	1.03956	1.03637	1.03875	1.03636	1.03639
63	1.04728	1.03218	1.06001	1.01397	1.03970	1.03674	1.03913	1.03672	1.03675
64	1.04883	1.03405	1.06224	1.01568	1.04142	1.03870	1.04100	1.03869	1.03872
65	1.04851	1.03296	1.06173	1.01468	1.04071	1.03794	1.04021	1.03793	1.03796
66	1.04755	1.03162	1.06109	1.01357	1.03955	1.03706	1.03915	1.03705	1.03707
67	1.05076	1.03365	1.06514	1.01573	1.04217	1.04014	1.04210	1.04007	1.04010
68	1.04944	1.03268	1.06415	1.01398	1.04102	1.03877	1.04074	1.03875	1.03878
69	1.05360	1.03726	1.06904	1.01804	1.04540	1.04323	1.04528	1.04322	1.04324
70	1.05397	1.03821	1.07033	1.01896	1.04606	1.04433	1.04634	1.04432	1.04434
71	1.05210	1.03527	1.06798	1.01614	1.04365	1.04174	1.04372	1.04172	1.04175
72	1.04938	1.03060	1.06388	1.01164	1.03995	1.03743	1.03964	1.03742	1.03744
73	1.04663	1.02558	1.05980	1.00663	1.03605	1.03287	1.03553	1.03286	1.03289
74	1.05304	1.03401	1.06876	1.01403	1.04348	1.04103	1.04346	1.04095	1.04098
75	1.05372	1.03375	1.06962	1.01407	1.04369	1.04148	1.04381	1.04140	1.04142
76	1.05965	1.03894	1.07525	1.01908	1.04925	1.04679	1.04927	1.04671	1.04673
77	1.06088	1.03960	1.07650	1.01955	1.05018	1.04764	1.05004	1.04756	1.04758
78	1.06027	1.03769	1.07542	1.01802	1.04892	1.04633	1.04883	1.04625	1.04628
79	1.06109	1.03763	1.07773	1.01752	1.04930	1.04719	1.04947	1.04667	1.04669
80	1.05888	1.03626	1.07620	1.01527	1.04751	1.04529	1.04741	1.04477	1.04479
81	1.06152	1.03874	1.07939	1.01763	1.05007	1.04806	1.05026	1.04753	1.04755
82	1.06384	1.04138	1.08203	1.01962	1.05255	1.05036	1.05272	1.04983	1.04986
83	1.06199	1.03855	1.07976	1.01694	1.05021	1.04788	1.05024	1.04735	1.04738
84	1.05906	1.03372	1.07603	1.01292	1.04632	1.04400	1.04637	1.04347	1.04350
G. Rate	1.00961	1.00554	1.01229	1.00214	1.00757	1.00720	1.00758	1.00712	1.00712

monthly chained Fisher index, $P_{FCh}^{\,t}$, is very close to the two monthly similarity-linked indices with an annual average growth rate of 1.00720. The annual average growth rates for the monthly fixed-base Fisher index and the monthly GEKS index, 1.00757 and 1.00758, respectively, are a bit above the chained monthly fixed-base Fisher growth rate.

In Figure A11.5, the top line is the monthly chained Laspeyres index $P_{LCh}^{\,t}$, followed by the fixed-base Laspeyres index $P_L^{\,t}$. The black line is the fixed-base Fisher index that lies a bit above the real-time similarity-linked index $P_S^{\,t}$, which can barely be distinguished from the chained Fisher index. The lowest line corresponds to the monthly chained Paasche

Figure A11.5 Monthly Laspeyres, Paasche, and Fisher Indices, GEKS Index, and Similarity-Linked Indices



index, which lies below the fixed-base Paasche index. The seasonal fluctuations in the Danish data are substantial.

A.11.6 Conclusion

The main findings in Section A.11.2 were as follows:

- In situations where there were no missing prices and no expenditure or quantity information was available, the monthly Jevons index $P_J{}^t$ was the preferred index.
- The upper-level monthly price data from Denmark for the years 2012 to 2018 exhibited substantial seasonal fluctuations.

The main findings in Section A.11.3 were as follows:

- It is not a trivial matter to aggregate monthly consumer prices into annual prices. The usual National Statistical Office practice of forming annual prices as the arithmetic average of monthly prices is not consistent with theoretical approaches to index number theory and is likely to be particularly inaccurate if there are strong seasonal fluctuations in monthly prices and quantities. Since the Danish monthly price data does exhibit strong seasonal fluctuations, it is likely that the corresponding monthly expenditure data also exhibits strong seasonal fluctuations.
- In this section, (approximate) Laspeyres, Paasche, and Fisher fixed-base and chained annual indices, $P_L{}^y$, $P_P{}^y$, and $P_F{}^y$ and $P_{LCh}{}^y$, $P_{PCh}{}^y$, and $P_{FCh}{}^y$, were computed for the seven years in the sample. The multilateral GEKS and predicted share similarity-linked annual indices, $P_{GEKS}{}^y$ and $P_S{}^y$, were also computed. The similarity-linked annual indices $P_S{}^y$ have good properties from the viewpoint of both the economic and test approaches to index number theory, and so the bias in the remaining indices was measured relative to this index. The annual chained Fisher indices $P_F{}^y$ were found to be identical to the annual similarity-linked indices $P_F{}^y$.
- The fixed-base annual Laspeyres index $P_L{}^y$ was on average 0.19 percentage points above our preferred chained Fisher and similarity-linked indices, while the fixed-base Paasche annual index $P_P{}^y$ was on average 0.045 percentage points below $P_{FCh}{}^y$ and $P_S{}^y$.
- The average difference between the fixed-base Laspeyres and the chained Fisher indices was 0.19 percentage points, while the difference between the chained Laspeyres and the chained Fisher indices was only 0.045 percentage points. Thus, annual substitution bias using the fixed-base Laspeyres formula is much larger than the substitution bias using the chained Laspeyres index.

The real-time "practical" month-to-month CPIs that National Statistical Offices are able to calculate at higher levels of aggregation use annual expenditure shares (or annual quantities) from a previous year and their monthly price indices. The three main monthly indices of this type that are used are the Lowe, Young, and geometric Young indices. If current annual expenditure or quantity weights are used, these indices are denoted by $P_{Lo}{}^t$, $P_Y{}^t$, and $P_{GY}{}^t$, respectively.[44] If the annual weights are lagged one year, these indices are denoted by $P_{Lo1}{}^t$, $P_{Y1}{}^t$, and $P_{GY1}{}^t$, respectively. If the annual weights are lagged two years, these indices are denoted by $P_{Lo2}{}^t$, $P_{Y2}{}^t$, and $P_{GY2}{}^t$, respectively. The upper-level substitution bias in these indices is measured relative to the monthly similarity-linked indices $P_S{}^t$, which were defined in Section A.11.5. The main findings in Section A.4 were as follows:

- The monthly average upward substitution bias for the "true" Lowe indices was close to 0.04 percentage points per year; for the "true" Young index, it was 0.15 percentage points per year, while the "true" geometric Young indices had a tiny upward substitution bias equal to 0.004 percentage points per year on average.

[44] Of course, these indices cannot be calculated in real time so they are not really "practical."

- The means of the three monthly Lowe indices increased as the lag in the annual weights increased. The average substitution bias for the Lowe indices increased from 0.04 percentage points per year for the current weight Lowe index to 0.17 percentage points per year for the practical Lowe index that uses weights that are two years old.
- The average substitution bias for the monthly Young indices increased from 0.15 percentage points per year for the Young index that uses current expenditure weights to 0.17 percentage points per year for the practical Young index that uses weights that are two years old.
- The average substitution bias for the monthly geometric Young indices increased from 0.004 percentage points per year for the geometric Young index that uses current expenditure weights to 0.014 percentage points per year for the practical geometric Young index that uses weights that are two years old.
- The three monthly geometric Young indices were close to each other and had the smallest approximate substitution bias.

In Section A.11.5, (approximate) Laspeyres, Paasche, and Fisher fixed-base and chained monthly indices, $P_L^{\,t}$, $P_P^{\,t}$, and $P_F^{\,t}$ and $P_{LCh}^{\,t}$, $P_{PCh}^{\,t}$, and $P_{FCh}^{\,t}$, were computed for the 84 months in the sample. The multilateral GEKS and predicted share similarity-linked monthly indices, $P_{GEKS}^{\,t}$ and $P_S^{\,t}$, were also computed. The main findings in Section A.5 were as follows:

- The monthly chained Fisher indices $P_F^{\,t}$ were not identical to the monthly similarity-linked indices $P_F^{\,t}$, but they are so close to each other that they cannot be distinguished from each other on a chart.
- The monthly fixed-base and chained Laspeyres indices, $P_L^{\,t}$ and $P_{LCh}^{\,t}$, had an average upward bias (relative to our preferred similarity-linked indices) of 0.25 and 0.52 percentage points per year over the sample period, respectively. Thus, the behavior of the monthly chained Laspeyres index is very different from the behavior of the annual chained Laspeyres index: The monthly chained Laspeyres had a very large upward chain drift, whereas the annual chained Laspeyres index had a much smaller upward chain drift.
- The monthly fixed-base and chained Paasche indices, $P_P^{\,t}$ and $P_{PCh}^{\,t}$, had an average downward bias of 0.16 and 0.50 percentage points, respectively.
- The monthly chained Fisher index, $P_{FCh}^{\,t}$, was very close to the monthly similarity-linked indices. Thus, chained Fisher indices performed well for this particular data set, both in the annual context and in the monthly context.
- The monthly fixed-base Fisher index, $P_F^{\,t}$, was very close to the monthly GEKS index, $P_{GEKS}^{\,t}$, and these indices are slightly above our preferred similarity-linked indices.

Some overall conclusions are as follows:

- National Statistical Offices could consider computing geometric Young indices for their official CPIs in place of the Lowe and Young indices that are presently widely used. From the main text and this annex, it appears that the lagged Lowe and Young indices have some measurable upward substitution bias, while the lagged geometric Young index has perhaps a smaller amount of downward substitution bias.
- For countries that have substantial seasonal fluctuations in prices and quantities, the use of annual expenditure weights will lead to inaccurate monthly CPIs. Moreover, taking an arithmetic average of monthly prices will lead to inaccurate annual prices, which in turn will lead to inaccurate estimates of household consumption.[45] Thus, it would be very useful if countries would attempt to estimate monthly expenditure weights.
- It will not be possible to obtain current expenditure information by month for all categories of consumption. But typically, some expenditure information can be obtained on a delayed basis. Thus, it would be useful for Statistical Offices to produce an *analytical CPI* that could be revised as more information becomes available. The US Bureau of Labor Statistics produces alternative CPIs on a regular basis which indicates that it is possible to produce multiple CPIs without confusing the public.

The last point is an important one, particularly in recent times when all economies have been affected by the COVID pandemic and developments in the Ukraine. A Lowe or Young CPI is a very useful measure of consumer inflation provided that relative quantities grow in a proportional manner or provided that consumer expenditure shares are approximately constant across recent months and years. However, substantial changes in consumer expenditure shares can occur rather suddenly, which greatly strengthens the case for having alternative, revisable CPIs that make use of weight information, which is available on a delayed basis.[46]

Additional References

Diewert, W. Erwin. 1983. “The Treatment of Seasonality in a Cost of Living Index.” In *Price Level Measurement*, edited by W. Erwin Diewert and Claude Montmarquette, 1019–1045. Ottawa: Statistics Canada.

———. 1978. “Superlative Index Numbers and Consistency in Aggregation.” *Econometrica* 46: 883–900.

———. 1997. “Commentary on Mathew D. Shapiro and David W. Wilcox: Alternative Strategies for Aggregating Prices in the CPI.” *The Federal Reserve Bank of St. Louis R*eview 79(3): 127–137.

———. 1998. “Index Number Issues in the Consumer Price Index.” *Journal of Economic Perspectives* 12(1): 47–58.

———. 2021a. “Elementary Indexes.” In *Consumer Price Index Theory*, Draft Chapter 6. Washington D.C.: International Monetary Fund. https://www.imf.org/en/Data/Statistics/cpi-manual.

———. 2021b. “The Chain Drift Problem and Multilateral Indexes.” In *Consumer Price Index The*ory, Draft Chapter 7. Washington D.C.: International Monetary Fund. https://www.imf.org/en/Data/Statistics/cpi-manual.

Diewert, W. Erwin, Yoel Finkel, Doron Sayag, and Graham White. 2022. “Seasonal Products.” In *Consumer Price Index The*ory, Draft Chapter 9. Washington D.C.: International Monetary Fund. https://www.imf.org/en/Data/Statistics/cpi-manual.

[45] See Diewert et al. (2022) for details on this point.

[46] See Diewert and Fox (2022a, 2022b) for details on this point.

Diewert, W. Erwin, and Kevin J. Fox. 2022a. "Measuring Inflation under Pandemic Conditions." *Journal of Official Statistics* 38(1): 1–34.

———. 2022b. "Measuring Real Consumption and Consumer Price Index Bias under Lockdown Conditions." *Canadian Journal of Economics*, forthcoming.

Fisher, Irving. 1922. *The Making of Index Numbers*. Boston: Houghton Mifflin Co.

Gini, Corrado. 1924. "Quelques considérations au sujet de la construction des nombres indices des prix et des questions analogues." *Metron* 4(1): 3–162.

———. 1931. "On the Circular Test of Index Numbers." *Metron* 9(9): 3–24.

Hardy, Godfrey Harold, John Edensor Littlewood, and György Pólya. 1934. *Inequalities*. Cambridge: Cambridge University Press.

Hill, Peter. 1988. "Recent Developments in Index Number Theory and Practice." *OECD Economic Studies* 10: 123–148.

———. 1996. *Inflation Accounting: A Manual on National Accounting under Conditions of High Inflation*. Paris: OECD.

Hill, Robert J. 2001. "Measuring Inflation and Growth Using Spanning Trees." *International Economic Review* 42: 167–185.

Ivancic, Lorraine, W. Erwin Diewert, and Kevin J. Fox. 2011. "Scanner Data, Time Aggregation and the Construction of Price Indexes." *Journal of Econometrics* 161: 24–35.

Lowe, Joseph. 1823. *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition. London: Longman, Hurst, Rees, Orme and Brown.

Mudgett, Bruce D. 1955. "The Measurement of Seasonal Movements in Price and Quantity Indexes." *Journal of the American Statistical Association* 50: 93–98.

Stone, John Richard Nicholas. 1956. *Quantity and Price Indexes in National Accounts*. Paris: OECD.

Walsh, Correa Moylan. 1901. *The Measurement of General Exchange Value*. New York: Macmillan and Co.

———. 1921. *The Problem of Estimation*. London: P.S. King & Son.

Young, Arthur. 1812. *An Inquiry into the Progressive Value of Money in England as Marked by the Price of Agricultural Products*. London: McMillan.

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