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Stress Tests of Euro Area Banks with Skewed Normal Credit Risk Distributions

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Stress Tests of Euro Area Banks with Skewed Normal Credit Risk Distributions
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ABSTRACT: This paper describes the 2023 euro area consultation top-down stress test that focused on the resilience of 91 systemically important banks' capital buffers as of end-2022 to macro baseline and adverse scenarios over the period 2023-25. As a result, the paper is an illustration of a top-down stress test framework with an application to euro are banks. The 2023 euro area consultation top-down stress test included unbiased dynamic panel data estimators based on Lancaster (2002) for projecting profitability components and information on Pillar 3 disclosures (exposure-at-default, probability of default, loss-given-default, expected losses). The paper also expands the 2023 euro area consultation top-down stress test by considering risk-weight functions with Skew-Normal and Transmuted-Normal probability distributions for the idiosyncratic and systemic risk factors. The results of the stress test with both distributions indicate that most euro area banks were resilient under the 2023 euro area consultation baseline and adverse scenarios as of July 2023 (publication of the Staff report).

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WORKING PAPERS

Stress Tests of Euro Area Banks with Skewed Normal Credit Risk Distributions

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Glossary

ASFM: Asymptotic Single-Factor Model

CCB: Capital Conservation Buffer

CET1: Common Equity Tier 1

CIPS: Cross-sectional augmented Im–Pesaran–Shin panel unit root test

EAD: Exposure at Default

EBA: European Banking Authority

ECB: European Central Bank

AIRB: Advanced Internal Ratings-Based

FIRB: Foundation Internal Ratings-Based

IFRS: International Financial Reporting Standards

IPS: Im–Pesaran–Shin panel unit root test

IRB: Internal Ratings-Based

LGD: Loss Given Default

MDA: Minimum Distributable Amount

NPL: Nonperforming Loan

PD: Probability of Default

ROAA: Return on Average Asset

RW: Risk Weight

RWA: Risk Weighted Assets

SME: Small and Medium Enterprise

SSM: Single Supervisory Mechanism

WEO: World Economic Outlook

Executive Summary

As euro area economies faced higher interest rates and severer financial conditions following tighter monetary policy starting in July 2022, stress tests have played an essential role in assessing banks' readiness for an eventual rise in corporate and household defaults. In particular, macro-based top-down stress tests have been widely used by supervisors and central banks to assess resilience of banks' capital ratios to baseline and adverse scenarios. In a similar vein, this paper describes the 2023 euro area consultation top-down stress test that focused on the resilience of 91 euro area systemically important banks' Common Equity Tier 1 capital and capital conservation buffer as of end-2022 to baseline and adverse scenarios over the period 2023-25. The stress test relied on Ayar et al. (2021), with the baseline and adverse scenarios based on the April 2023 WEO and the Global Financial Model (Vitek, 2018) driving projected changes in capital ratios through retained earnings in the numerator (the profitability channel) and through risk-weighted assets in the denominator (the credit risk channel) over the period 2023-25. However, modifications and adjustments to Aiyar et al (2021) stress test included: (i) unbiased dynamic panel data estimators based on Lancaster (2002) for projecting profitability components that were a function of bank specific and macro variables (growth, unemployment, long-term interest rate, and inflation); and (ii) information on Pillar 3 disclosures, such as exposure at default, probability of default, loss-given-default, expected losses for projecting risk-weighted assets.

The paper also expands the credit risk analysis in the 2023 euro area consultation top-down stress test. It contributes to the stress testing literature by considering an idiosyncratic risk factor that deviates from the Normality assumption in Vasicek's asymptotic single-factor model (ASFM) underpinning the risk weight function under Basel's Internal Ratings-Based approach. While it extends Vasicek's ASFM by incorporating idiosyncratic and systemic risk factors with Skew-Normal and Transmuted-Normal probability distributions, the calibration of risk-weighted assets in the empirical part assumes that only the idiosyncratic risk factor is a Transmuted-Normal random variable while the systemic risk factor is a Normal random variable. The relaxation of the Normality assumption for the idiosyncratic risk factor is motivated by the reduction in capital charges for SME and infrastructure exposures allowed by the Capital Requirements Regulation and the probability-of-default substitution effect on risk-weighted assets enabled by credit guarantees.

The results of the stress tests indicate that most euro area banks were resilient under the 2023 euro area consultation baseline and adverse scenarios as of July 2023 (publication of the Staff report). The profitability channel implied that capital would respond not only to the persistence displayed by profitability components but also to changes in macro variables. The credit risk channel entailed risk-weighted assets calibrated with a risk weight function based on a Skew-Normal or a Transmuted Normal distribution with a negative shape parameter or a negative weight parameter, respectively. Overall, the stress test results suggest that banks were likely to remain resilient under the baseline scenario, with the strength of the net interest margin under an environment of expected higher interest rates underlying the stronger capital ratios. On the other hand, the adverse scenario would have entailed moderate capital pressures. Both higher risk-weighted assets and negative return on average asset would have explained the reduction in capital adequacy ratios. As a result, some banks would have experienced a Common Equity Tier 1 capital shortfall relative to the combined minimum Common Equity Tier 1 capital ratio and capital conservation buffer requirement of 7 percent. Finally, the difference between the reduction in Common Equity Tier 1 capital ratios resulting from plugging in EBA's macro assumptions under the adverse scenario into the satellite and risk-weighted-asset models and the one published in the 2023 EBA-SSM bottom-up stress test results stemmed from the role of the different profitability drivers. The 2023 EBA-SSM bottom-up stress test implied lower positive contribution from the net interest income, a larger negative

contribution from the net non-interest income, a lower negative contribution from loan loss provisions, and a slightly higher contribution from risk-weighted assets than the ones estimated in the paper. Overall, the difference between changes in risk-weighted assets based on Skew-Normal and Transmuted-Normal probability distributions were negligible. The 2025 FSAP stress test of euro area banks provides an opportunity to assess the resilience of euro area banks with confidential supervisory data.

I. Introduction

Bottom-up stress tests, where banks incorporate macro scenarios into their risk management models in exercises coordinated by authorities, have become an important supervisory tool. In the case of the Single Supervisory Mechanism (SSM), the bottom-up stress tests are an essential part of the supervisory process. While supervisors and regulators provide banks with key macro assumptions and shocks, banks use their risk management models to assess the impact of macro scenarios on bank asset quality, profitability, and solvency. The bottom-up stress tests are an input to the Supervisory Review and Evaluation Process (SREP), with the qualitative results affecting Pillar 2 capital requirements (a capital add-on). The quantitative results are, in turn, also used to set the Pillar 2 capital guidance, which banks are expected to maintain to withstand stress. As a result, the bottom-up stress tests offer insights on whether banks can withstand adverse shocks.

As euro area economies faced higher interest rates and severer financial conditions following tighter monetary policy starting in July 2022, macro-based top-down stress tests have played an important role in assessing banks' readiness for an eventual rise in corporate and household defaults. At the same time, top-down stress tests have also been widely used by supervisors and central banks to assess banks' resilience to adverse scenarios. In a similar vein, this paper describes the 2023 euro area consultation top-down stress test that focused on the resilience of 91 euro area systemically important banks' Common Equity Tier 1 capital and capital conservation buffer as of end-2022 to baseline and adverse scenarios over the period 2023-25. These scenarios were based on the April 2023 WEO and the Global Financial Model (Vitek, 2018) assumptions, respectively. The assessment relied on the stylized top-down stress test outlined in Aiyar et al. (2021), with capital in the numerator of capital ratios being driven by changes in retained earnings (or the profitability channel) and risk-weighted assets in the denominator being driven by asset quality changes (or the credit risk channel). However, modifications and adjustments to Aiyar et al (2021) were made to directly incorporate: (i) unbiased dynamic panel data estimators for the determinants of profitability; and (ii) information on Pillar 3 disclosures, such as exposure at default, probability of default, loss given default, and expected losses.

The paper also expands the credit risk analysis in the 2023 euro area consultation top-down stress test. It contributes to the stress testing literature by considering an idiosyncratic risk factor that deviates from the Normality assumption underpinning Basel's Internal Ratings-Based (IRB) approach. In particular, it extends the Vasicek's asymptotic single-factor model (ASFM) by assuming that systemic and idiosyncratic risk factors are either Skew-Normal or Transmuted-Normal random variables as in Azzalini (1985) and Shaw and Buckley (2007), respectively. The relaxation of the Normality assumption is motivated by the reduction of capital charges for SME and infrastructure exposures (entities operating or financing physical structures or facilities, systems, and networks) allowed by the Capital Requirements Regulation[1] and the effect of probability of default substitution on capital charges enabled by credit guarantees. This is equivalent to skewing the underlying idiosyncratic distribution to the left and lowering associated risk-weighted assets.[2] As a result, the extension to the asymptotic single-factor model with an idiosyncratic risk factor that is a Skew-Normal or Transmuted-Normal random variable, allows a direct calibration of Pillar 3 disclosed risk-weighted assets without strong adjustments in credit risk parameters in the empirical section to net out the reduction effects from SME and infrastructure exposures and probability of default substitution from credit guarantees.

[1] As detailed in the Capital Requirements Regulations (CRR, Regulations (EU) nos. 575/2013, article 501(1), and 2019/876, article 501a(1))

[2] European Banking Authority (EBA, 2020) reports that 44 percent of the loans subject to public guarantee schemes with residual maturities of between 2 and 5 years in Europe amounted to €181 billion as of June 2020.

The results of the stress tests indicate that euro area systemically important banks were likely to remain resilient under the 2023 euro area consultation baseline and adverse scenarios as of July 2023 (publication of the Staff report). The profitability channel implied that capital levels would respond not only to the persistence displayed by profitability components but also to changes in macro variables. The credit risk channel entailed risk-weighted assets calibrated with a risk-weight function based on a Skew-Normal and Transmuted-Normal distributions for the idiosyncratic risk factor with a negative weight parameter. Overall, the stress test results implied that banks were likely to remain resilient under the baseline scenario as of July 2023, with the strength of the net interest margin under an environment of higher interest rates underlying higher capital ratios. On the other hand, the adverse scenario would have entailed moderate capital pressures, with higher loan loss provisions significantly weighing on profitability and risk-weighted assets. As a result, some banks would have experienced a capital shortfall relative to the combined minimum Common Equity Tier 1 capital ratio and capital conservation buffer requirement of 7 percent. Moreover, some banks would also have breached their MDA threshold under the adverse scenario, with negative consequences for dividend and coupon payments to hybrid capital.

Finally, the difference between the reduction in CET capital ratios obtained by plugging in the adverse scenario macro assumptions under the 2023 EBA-SSM stress test into the satellite and risk-weighted-asset models and the one published the 2023 EBA-SSM stress test results stemmed from the role of different drivers. The 2023 EBA-SSM stress test results implied a lower positive contribution from the net interest income, a larger negative contribution from the net non-interest income, and a lower negative contribution from loan loss provisions than the one estimated in the paper. Overall, the difference between changes in risk-weighted assets based on Skew-Normal and Transmuted-Normal probability distributions were negligible. Finally, the 2025 FSAP stress test of euro area banks provides an opportunity to assess the resilience of euro area banks with confidential supervisory data.

The paper is organized in 6 sections. The literature review in section II is followed with detailed discussion of the profitability satellite models and the risk-weight functions underlying the stress test in the subsequent sections III and IV. Key parameters under the two IRB approaches published in Pillar 3 disclosures are then presented in section V before the discussion of the stress results. Finally, the conclusion summarizes key results in the paper and draws some policy conclusions.

II. Literature Review

Solvency stress tests have been substantially expanded since the GFC. Different approaches by supervisory authorities and international organizations are summarized in Foglia (2008), Buncic et al. (2019), and Pliszka (2021) among others. A common approach relies on quantifying the impact on profitability, asset quality, and solvency of baseline and adverse scenarios. A reduction in profitability results in lower retained earnings and lower internally generated organic capital while higher probabilities of default and/or loss given default imply higher risk weighted assets. This approach is further detailed in Schmieder, Puuhr, and Hasan (2011) and is used in Aiyar et al. (2021) (and others) to stress test the resilience of European banks to a deterioration in credit risk during the pandemic. Finally, Henry and Kok (2013) provide details of a macro stress testing framework for assessing systemic risk in the euro area banking sector.

The satellite models in stress tests have relied on the vast literature on the determinants of bank profitability to identify bank-specific indicators and macro variables driving profitability, loan loss reserves, and nonperforming loans. Staikouras and Wood (2004) and Elekdag et al. (2019) review the literature on the determinants of profitability in European and euro area banks. As most analysis involves different banks in different time periods, panel data methods are used to estimate the effects of bank-specific performance indicators and macro variables on profitability. In particular, dynamic panel models with lagged dependent variables are helpful to capture persistence in bank profits as indicated in Goddard et al. (2011) and Gugler and Peev. (2018).[3] However, it can also result in biased estimators in small-size samples as pointed out by Nickell (1981). Instrumental variable and generalized method of moment estimators as in Arellano and Bond (1991) and Blundell and Bond (1998) have been used to overcome the estimation bias in FE methods.[4] However, Dang et al. (2015) cautions that instrumental variable and generalized method of moment estimators are sensitive to unobserved heterogeneity, residual serial correlation, and changes in control parameters that are all common in corporate finance. Kiviet (1995) and Bruno (2005), Everaert and Pozzi (2007), and Gouriéroux et al. (2010) propose analytical or simulation-based bias-correction methods to overcome the shortcomings while Lancaster (2002) proposes consistent estimators based on the orthogonal re-parametrization of the fixed effects and their integration from the likelihood with respect to an adequately selected prior density.[5]

Most macro stress tests also specify a credit risk model to convert changes in the probability of default into risk-weighted asset changes. The asymptotic single-factor model (ASFM) by Vasicek (1991) underlies Basel's capital requirements where banks should hold enough capital against unexpected losses.[6] Gordy (2003) shows that Basel's capital requirements and other ratings-based capital rules are consistent with the general class of credit value-at-risk models. Schonbucher (2000) generalizes Vasicek's ASFM to accommodate different probability distributions for the systemic and idiosyncratic risk factors, including probability distributions with higher skewness and kurtosis than a Normal distribution.[7] Along this line, Arellano-Valle et al. (2006) reviews the different approaches to expand the Normal distribution to incorporate skewness. Among them, Azzalini et al. (2003) consider the Skew-Normal (Skew-Student's) distribution with its probability density function resulting from the product of probability and cumulative density functions for a Normal (Student's) probability distribution with an extra parameter capturing the intensity of the modulation. Batiz-Zuk et al. (2013) and Kohlscheen et al. (2015) extend Vasicek's ASFM by accommodating systemic and idiosyncratic factors with Skew-Normal and Skew-Student's t distributions. In addition, Shaw and Buckley (2007) propose the Transmuted-Normal distribution as the result of mapping a base Normal distribution through a quadratic rank transmutation with a weight that captures the degree of skewness with respect to Normality. Ieren and Abdullhi (2020) further list its properties while Kozubowski and Podgorski (2016) show that the Transmuted-Normal distribution is a special case of extremal probability distributions.

[3] See Finkel (2008).

[4] Okui (2021) reviews the literature on linear dynamic panel data models based on generalized method of moments and the bias-corrected fixed-effect estimator.

[5] Pickup et al. (2017) note that the Lancaster (2002) reparameterization of incidental (fixed effect) parameters assumes that incidental parameters are information-orthogonal to other common parameters of interest. As a result, this yields consistent estimates of common parameters independent from incidental parameters.

[6] Saunders, Xiourous, and Zenios (2007) provide an extensive review of factor credit risk models.

[7] Navas-Palencia (2016) provides an example of Schonbucher's generalization to a logistic-factor model approximation.

III. The Profitability Channel

The profitability channel assumes that macro conditions and bank specific variables drive bank profitability and organic capital accumulation. Table 1 summarizes the effect of macro variables on profitability and nonperforming loans:

- Higher GDP growth captures larger volumes of transactions across different business lines in a bank. It has a positive effect on interest income and expenses and other income while its effect on loan loss provisions and the nonperforming loans ratio is negative due to a reduction in credit risk. Its effect on non-interest income and expenses is ambiguous. While higher GDP growth discourages banks from increasing non-interest income (e.g., fees and commissions), given the rise in interest income, higher customer activity has a positive effect on it. At the same time, higher GDP growth not only increases non-interest expenses (e.g., staff costs) to capture interest and non-interest income from better economic activity but also improves credit quality and lowers loan recovery expenses.

- Similarly, higher unemployment reduces transactions associated, for instance, with residential real estate, consumer loans, and deposits, negatively affecting interest income and expenses while loan loss provisions and nonperforming loans ratio rise in tandem with higher credit risk.

- Higher long-term interest rates also have a positive impact on the interest income and expenses and loan loss provisions due to higher loan and deposit pricing and higher credit risk, respectively, while it has a negative effect on non-interest income through the rise in marked-to-market losses associated with available-for-sale and trading securities.

- Higher inflation implies larger consumer loans and larger payments related to inflation-protected products, with a positive effect on interest income and expenses, respectively. It has an ambiguous effect on non-interest income and expenses. While higher inflation reduces non-interest income and non-interest expenses associated with capital market products as bonds and stocks become less attractive, it also increases demand for investment products as cash loses value. Finally, it also heightens credit risk and, as result, nonperforming loans as real income falls and leverage rises.[8]

As a profitability driver, higher nonperforming loans reduce interest income as defaulting borrowers struggle to make their debt payments.[9] Moreover, they have an ambiguous effect on non-interest income. While they reduce interest income from loans and associated non-interest income (e.g., commission and fees), they encourage non-interest income diversification. They also increase non-interest expenses, especially those related to loan recovery expenses, and loan loss provisions.

Finally, bank-specific variables have a bearing on profitability. These include the cost-to-income, deposits-to-total assets, equity-to-total assets, loans-to-total assets ratios, and the amount of total assets. However, they are not part of projections with a constant balance sheet assumption and are not further discussed in this section.

[8] See Hahm (2008).

[9] For simplicity, the analysis assumes that profitability is impacted only by loans transitioning from performing to nonperforming status and not by changes within the performing loan category.

Table 1. Euro Area Banks: Expected Signs Associated with the Determinants of Profitability

Dependent Variables:	Interest Income to Average	Interest Expenses to Average	Non-interest Income to Average	Non-interest Expenses to Average	Loan Loss Provisions to Average	Other Net Income to Average	Nonperforming Loans Ratio
Model:	(1)	(2)	(3)	(4)	(5)	(6)	(7)
GDP growth	+	+	+ / -	+ / -	-	+	-
Inflation rate	+	+	+ / -	+ / -		-	+
Long-term interest rate	+	+	+		+		+
Unemployment	-	-			+	-	+
Nonperforming loans	-		+	+	+		

IV. The Credit Risk Channel

The credit risk channel affecting banks' capital ratios works through probabilities of default driving risk-weighted assets. Though initially aimed at internationally active banks, the Basel Committee on Banking Supervision capital standards have become a benchmark for prudential regulation of minimum capital requirements. The framework contains capital definitions and minimum capital requirements addressing different risks, including the calculation of risk-weighted assets. The 2013 Capital Requirements Regulation (CRR) and posterior amendments transposed the Basel framework to European legislation. It includes capital requirements for credit risk under the Standardized and the Internal Ratings-Based (IRB) approaches.

Standardized Approach

The standardized approach is based on assigning risk weights to risk exposures as a function of external ratings or collateral availability. In the Capital Requirements Regulation, the Standardized risk-weighted assets are calculated as the product of the standardized risk weights and the exposure amount (remaining accounting value) net of specific credit risk and additional value adjustments. For corporate exposures, banks use external ratings recognized as eligible for capital purposes by national supervisors to set the risk weights. Banks can also tap other alternatives—e.g., country risk scores assigned by Export Credit Agencies in the case of sovereign and central bank exposures or sovereign credit ratings in the case of public sector enterprises. Table 2 indicates that risk weights associated with exposures to sovereigns, central banks, public sector enterprises, covered bonds, and corporates can range from zero up to 150 percent.

Table 2. Capital Requirements Regulation: Standardized Approach Risk Weights

(In percent)

Exposure	Criteria	1	2	3	4	5	6
Sovereigns, central banks	External rating	0	20	50	100	100	150
Public sector enterprises	Sov. rating	20	50	100	100	100	150
	PSE rating	20	50	50	100	100	150
Institutions	Sov. rating	20	50	100	100	100	150
	External rating, less than 3 Mo.	20	20	20	50	50	150
	External rating, more than 3 Mo.	20	50	50	100	100	150
Covered bonds	External rating	10	20	20	50	50	100
Corporate	External rating	20	50	100	100	150	150

Source: European Banking Authority

On the other hand, risk weights for non-default retail exposures are, to a certain extent, associated with the nature of collateral availability. Overall, risk weights can range as low as 30 percent for real estate exposures with loan-to-values below 30 percent and up to 110 percent for exposures materially dependent on the cash flows generated by real estate property. For exposures associated with revolving facilities and overdrafts, these are mostly risk weighted at 75 percent, except revolving facility and overdraft exposures meeting some repayment or usage conditions with lower risk profiles that are risk weighted at 45 percent. Finally, all other retail exposures are risk-weighted at 100 percent. If the exposures are in default, the unsecured part of the exposure is risk weighted either at 100 percent or 150 percent according to whether specific credit risk adjustments are higher or lower than 20 percent of the unsecured exposure, respectively.

Basel’s Internal-Risk Based Approach to Risk-Weighted Assets

Under the Internal Ratings-Based approach in the Capital Requirement Regulation, banks have two options to set their risk-weighted assets as a function of probabilities of default:

- The Foundation Internal Ratings-Based (FIRB) approach in which banks can use their own probability of default, exposure at default, and maturity adjustment in conjunction with a prescribed loss given default;[10] and

- The Advanced Internal Ratings-Based (AIRB) approach that allows banks to use their own internal calculation of the probability of default, exposure at default, maturity adjustment, and loss given default.

Under the IRB approach, risk-weighted assets are defined as follows:

$$RWA_i = (LGD_i\ F(0.999, PD_i, \rho_i)\ EAD_i - LGD_i\ PD_i\ EAD_i) \times MAdj \times 1.06 \times 12.5, \tag{1}$$

where i stands for a firm or a household within a homogenous asset class while the first and second terms between parathesis represent the total loan loss and expected losses (ELs), respectively.[11] The single-risk factor and the large homogeneous portfolio (LHP) assumptions yield a risk weight function (or the quantile function at a 99.9 percent level) as follows:

$$F(0.999, PD_i, \rho_i) = \Phi\left(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i}\Phi^{-1}(0.999)}{\sqrt{1-\rho_i}}\right) \tag{2}$$

where $\Phi(.)$ and $\Phi^{-1}(.)$ stand for the Normal cumulative distribution function and its inverse, respectively, while ρ_i is the correlation coefficient between the common and idiosyncratic risk factors. The first inverse of the cumulative density function $\Phi^{-1}(PD_i)$ of a Normal probability distribution is equivalent to the default barrier K_i:

[10] Loss given default values range from 11.25 percent for senior exposures to 100 percent for subordinated exposures in the case of exposures to central governments, central banks, institutions, and corporates and no less than 10 percent in the case of retail exposures).

[11] The CRR includes a 1.06 factor in RW function that is not part of Basel.

$$K_i = \Phi^{-1}(PD_i),$$

(3)

while the inverse of the cumulative density function $\Phi^{-1}(0.999)$ of the Normal probability distribution is the fraction x of the portfolio associated with the β-quantile function set at the 99.9 percent confidence level.

The maturity adjustment term $MAdj$ makes a distinction between corporates and retail as follows:

$$MAdj = \begin{cases} \dfrac{1 + (M - 2.5)b(PD_i)}{1 - 1.5b(PD_i)} \; if \; i = corporate \\ 1 \; if \; i = retail \end{cases}$$

(4)

where *M* stands for the average maturity and the term $b(PD)$ is as follows:

$$b(PD_i) = (0.11852 - 0.05478 \;\; ln(PD_i))^2.$$

(5)

Finally, the correlation coefficient ρ_i for all exposures except the retail ones, assumes the following values in the Capital Requirement Regulation:[12]

$$\rho_i = \begin{cases} 0.12\dfrac{1-e^{-50PD_i}}{1-e^{-50}} + 0.24\left(1 - \dfrac{1-e^{-50PD_i}}{1-e^{-50}}\right) if \; i = sovereign, corporate, or \; bank \\ 1.25\left[0.12\dfrac{1-e^{-50PD_i}}{1-e^{-50}} + 0.24\left(1 - \dfrac{1-e^{-50PD_i}}{1-e^{-50}}\right)\right] if \; i = unregulated \;\; financial \; institution \\ 0.12\dfrac{1-e^{-50PD_i}}{1-e^{-50}} + 0.24\left(1 - \dfrac{1-e^{-50PD_i}}{1-e^{-50}}\right) - 0.04\left(1 - \dfrac{\min(\max(5,S),50)-5}{45}\right) if \; i = SME \end{cases}$$

(6)

while the correlation coefficient ρ_i for retail exposures is set at:

$$\rho_i = \begin{cases} 0.15 \; if \; i = immovable \; property \; collateral \\ 0.04 \; if \; i = qualiying \; revolving \; retail \\ 0.03\dfrac{1-e^{-35PD_i}}{1-e^{-35}} + 0.16\left(1 - \dfrac{1-e^{-35PD_i}}{1-e^{-35}}\right) if \; i = other \; retail \end{cases}$$

(7)

Credit Risk Models with Skewed Normal Distributions

Batiz-Zuk et al. (2013) and Lee and Poon (2015) extend Vasicek's ASFM by incorporating systemic and idiosyncratic risk factors with Skew-Normal distributions. Batiz-Zuk et al. (2013) show that, if the borrower i's

[12] The correlation coefficient ρ for SMEs differs from the Basel framework by an additional term to account for annual sales. The remaining correlation coefficients ρ for corporate and retail exposures are the same as in the Basel framework.

value is a combination of a systemic risk factor with a Skew-Normal (Normal) distribution and a Normal (Skew-Normal) distribution, the borrower i's value also has a Skew-Normal distribution. However, they rely on a Cornish-Fisher Expansion and the adaptative Simpson quadrature to numerically evaluate the double integral and approximate the default barrier when one of the risk factors has a Skew-Normal distribution. On the other hand, Lee and Poon (2015) note that, conditional on the systemic risk and an additional factor that has a truncated Normal distribution, the probability of default corresponds to a cumulative Normal density function while the credit loss distribution involves the cumulative density function of a Skew-Normal distribution.

Extensions to Schonbucher (2000) generalizing Vasicek's ASFM can also accommodate skewed probability distributions for the systemic and idiosyncratic risk factors. This paper incorporates different probability distribution assumptions for idiosyncratic and systemic risk factors as in Schonbucher (2000). In particular, both the Skew-Normal and the Transmuted-Normal probability distributions allow for deviations in skewness and kurtosis from a Normal probability distribution. However, instead of relying on a Cornish-Fisher Expansion and the adaptative Simpson quadrature to numerically evaluate double integrals to estimate the default barrier as in Batiz-Zuk et al. (2013), this paper uses formulas in the table of Normal integrals described in Owen (1980) to derive closed-form solutions for the conditional probability of default.

The Skew-Normal Credit Risk Model

If the systemic and the idiosyncratic risk factors are, respectively, Normal ($\lambda_{Y,i} = 0$) and Skew-Normal ($\lambda_{\varepsilon_i} \neq 0$) random variables, the risk-weight function (or the β-quantile) obtained from the general credit loss distribution in Schonbucher (2000) can be expressed as:

$$F_1(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}) = \Phi\left(\sqrt{1 - \frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1 + {\lambda_{\varepsilon_i}}^2}}\,\frac{K_i - \sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1 + {\lambda_{\varepsilon_i}}^2}}\right) - 2T\left(\sqrt{1 - \frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1 + {\lambda_{\varepsilon_i}}^2}}\,\frac{K_i - \sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1 + {\lambda_{\varepsilon_i}}^2}}, \lambda_{\varepsilon_i}\right), \quad (8)$$

where λ_{ε_i} is the shape parameter of a Skew-Normal probability distribution capturing skewness in the idiosyncratic risk factor and $T(.,.)$ is Owen's T function. While reliance on a risk weight function with a Skew-Normal idiosyncratic risk factor and a Normal systemic risk factor does not capture contagion and sectoral spillover, the model can be modified to include a Skew-Normal systemic risk factor capturing contagion under extreme conditions (under the tail of the distribution) with a higher probability than a Normal distribution.[13]

Based on formulas 10,010.8 and c00,010.3 in Owen (1980), the barrier K_i for each borrower can be associated with its probability PD_i as follows:

[13] See Annexes II and III for more detailed discussion of the different models.

$$E\big(p_i(y)\big) = \Phi\left(\frac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,K_i + \sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\right)$$

$$-\,2T\left(\begin{array}{c}\dfrac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,K_i + \sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}, \\ \lambda_{\varepsilon_i}^{*} = \dfrac{\sqrt{1-\rho_i}\,\lambda_{\varepsilon_i}}{\sqrt{1-\rho_i+\rho_i\left(1+{\lambda_{\varepsilon_i}}^2\right)\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\end{array}\right) = PD_i. \tag{9}$$

The Transmuted-Normal Credit Risk Model

Systemic and Idiosyncratic Risk Factors with Transmuted-Normal Distributions

When the systemic and the idiosyncratic risk factors are, respectively, Normal ($\lambda_{Y,i} = 0$) and Transmuted-Normal ($\lambda_{\varepsilon_i} \neq 0$) random variables, the respective risk-weight function (or the β-quantile) is as follows:

$$F_1\big(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}\big) = \big(1+\lambda_{\varepsilon_i}\big)\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\,\frac{K_i-\sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right)$$

$$-\,\lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\,\frac{K_i-\sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right)\right]^2. \tag{10}$$

where -1 ≤ $\lambda_{\varepsilon_i} \leq 1$ is the weight associated with the quadratic term.

Based on formulas 10,010.8 and 20,010.4 in Owen (1980), the barrier K_i for each borrower can be associated with its probability PD_i as follows:

$$E\big(p_i(y)\big) = \Phi\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\,K_i - \sqrt{1-\rho_i}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right) + 2\lambda_{\varepsilon_i}T\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\,K_i - \sqrt{1-\rho_i}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}, \frac{\sqrt{1-\rho_i}}{\sqrt{1-\rho_i+2\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right) = PD_i. \tag{11}$$

Adjusting Loss Given Default and Probabilities of Default

As banks obtain total expected losses (ELs) by summing up expected losses in each asset-class, reported total expected losses cannot be easily replicated by the product of total exposure at default (EAD), weighted-average loss given default (LGD), and weighted-average probability of default (PD). Hunt and Taplin (2019) propose a three-step procedure to overcome this inconsistency:

1. Average LGDs and PDs in different asset classes with weights given by EAD.
2. Average LGDs and PDs with weights given by the product of EAD x PD and EAD x LGD, respectively.
3. Obtain the geometric average of LGDs and PDs in the two previous steps.

The resulting expression for total expected losses is as follows:

$$EL_T = \sqrt{\left(\frac{\sum_{i=1}^{n} PD_i\, EAD_i}{\sum_{i=1}^{n} EAD_i}\right)\left(\frac{\sum_{i=1}^{n} PD_i\, LGD_i EAD_i}{\sum_{i=1}^{n} LGD_i EAD_i}\right)}\sqrt{\left(\frac{\sum_{i=1}^{n} LGD_i\, EAD_i}{\sum_{i=1}^{n} EAD_i}\right)\left(\frac{\sum_{i=1}^{n} LGD_i EAD_i PD_i}{\sum_{i=1}^{n} PD_i EAD_i}\right)}\, EAD_T, \tag{12}$$

where the first term in parenthesis under each of the square roots is associated with step 1, the second term with step 2, and the square root of the product with step 3. As a result, total expected losses obtained from the three-step procedure across all asset classes should match total expected losses reported by banks. However, even when probabilities of default and loss given default are adjusted, there can remain a small difference between the adjusted and disclosed expected losses.

Dynamics for the Probabilities of Default

Finally, as historical probabilities of defaults cannot be easily gathered, changes in probabilities of default are simply assumed to be proportional to changes in nonperforming loans as follows:

$$PD_{t+1} - PD_t = \frac{NPL_{t+1} - NPL_t}{Loans_t - NPL_t} - \frac{NPL_t - NPL_{t-1}}{Loans_{t-1} - NPL_{t-1}}, \tag{13}$$

where NPL_t stands for nonperforming loans at time t.[14]

V. Assumptions, Data, and Results

As detailed above, changes in regulatory bank capital were driven by the profitability and credit risk channels. The constant balance sheet assumption implied that banks did not re-optimize their balance sheet in reaction to shocks in the baseline and the adverse scenarios. As a result, only macro variables determined profitability and risk weighted assets in the stress test. The worsened macroeconomic conditions implied a deterioration in profitability and credit quality, affecting business lines (lending and nonlending activities), products, securities, and loan portfolios.

[14]Though this probability-of-default specification overlooks vintage effects (long-term stock of nonperforming loans, cures, and write-offs), IFRS9 stagging, and borrower heterogeneity, it substantially simplifies the stress test computations.

Assumptions

The 2023 euro area consultation stress test included a period of economic slowdown under the baseline and a mild recession under the adverse scenario. As shown in Table 3 and Figure 1, country assumptions, when aggregated, would have the following implications for the euro area:

- GDP growth would have slowed down to 0.8 percent in 2023 from 3.4 in 2022 but would have recovered to 1.4 percent in 2024 and 1.9 percent in 2025 under the baseline scenario. Unemployment would have remained at 6.8 percent during 2023-24 but would have declined to 6.6 percent in 2025. Long-term interest rates would have remained above 2.4 percent over 2023-25 while inflation would have persistently slowed down from 8.4 percent in 2022 to 5.3 percent in 2023, 2.9 percent in 2024, and 2.2 percent in 2025.

- On the other hand, GDP growth would have contracted to -3.0 percent in 2023 but would have recovered to -0.3 percent in 2024 and 2.1 percent in 2025 under the mild adverse scenario, with unemployment rate increasing from 6.7 percent in 2022 to 8.1 percent in 2025. Long-term interest rate would have declined to 2.7 percent in 2025 after peaking at 6.1 percent in 2023 while inflation would have slowed down at lower pace to 5.6 percent in 2023, 3.3 percent in 2024, and 2.6 percent in 2025.

Finally, the euro area economy would have significantly contracted under the 2023 EBA adverse scenario. GDP growth would have declined to -3.4 percent in 2023 and -4.1 percent in 2024 but would have recovered to 1.6 percent in 2025. Unemployment would have spiked to 12.4 percent in 2025 while long-term interest rates would have remained high, above 4 percent, throughout the period 2023-25. Despite a large GDP contraction, inflation would have spiked to 9.2 percent in 2023 but would have reversed to 5.2 percent in 2024 and 3.7 percent in 2025.

Table 3. Euro Area: Macro Assumptions for Stress Tests

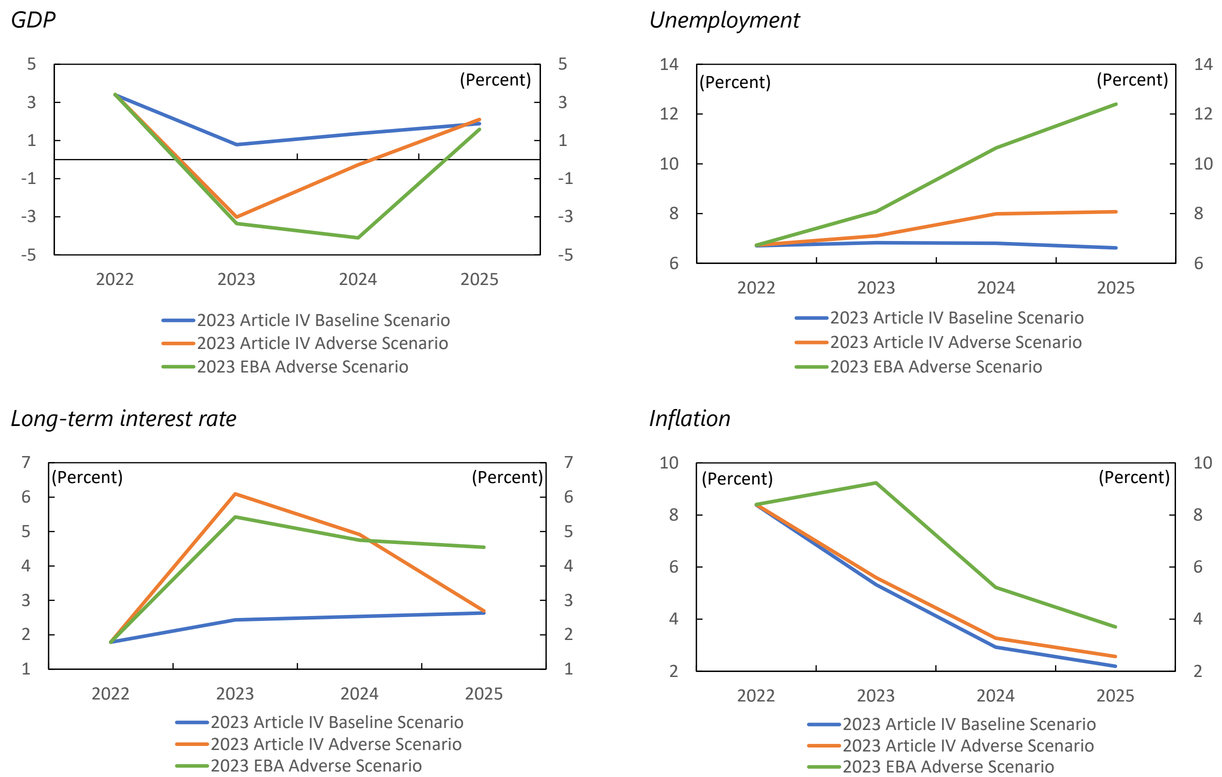
	2022	2023	2024	2025
	2023 Article IV Baseline Scenario			
GDP growth (April 2023 WEO, percent)	3.4	0.8	1.4	1.9
Unemployment rate (April 2023 WEO, percent)	6.7	6.8	6.8	6.6
Long-term interest rate 1/	1.8	2.4	2.5	2.6
Inflation rate	8.4	5.3	2.9	2.2
	2023 Article IV Adverse Scenario			
GDP growth 1/	3.4	-3.0	-0.3	2.1
Unemployment rate 1/	6.7	7.1	8.0	8.1
Long-term interest rate 1/	1.8	6.1	4.9	2.7
Inflation rate	8.4	5.6	3.3	2.6
	2023 EBA Adverse Scenario			
GDP growth	3.4	-3.4	-4.1	1.6
Unemployment rate	6.7	8.1	10.6	12.4
Long-term interest rate	1.8	5.4	4.8	4.5
Inflation rate	8.4	9.2	5.2	3.7

Source: European Banking Authority, Vitek (2023) and WEO (2023).

Note:

1/ Vitek (2023).

Figure 1. Euro Area: Macro Assumptions for Stress Tests



Source: European Banking Authority, Vitek (2023) and WEO (2023).

The estimated coefficients for the different satellite models and the different macro scenarios were then used to project the different profitability components and nonperforming loans. The sum of the different profitability components yielded net operating income, with retained earnings obtained by subtracting: (i) taxes, which were calculated according to the 2022 effective tax rate if the projected net operating income was positive or at zero otherwise, with a cap of 30 percent; and (ii) dividend payouts, which were assumed at 50 percent throughout the period if the projected net operating income was positive, or zero otherwise. Changes in probabilities of default were then obtained from changes in nonperforming loans and were translated into changes in risk-weighted assets with the calibrated Transmuted Normal risk-weight function. The latter replicated disclosed risk-weighted assets assuming a correlation coefficient that diverged from the prescribed one in the Capital Requirements Regulation to accommodate the diversification effect from the combination of different asset classes into only one. Finally, the disclosed risk-weighted assets under the Standardized approach were assumed to remain constant throughout the projection period 2023-25.[15]

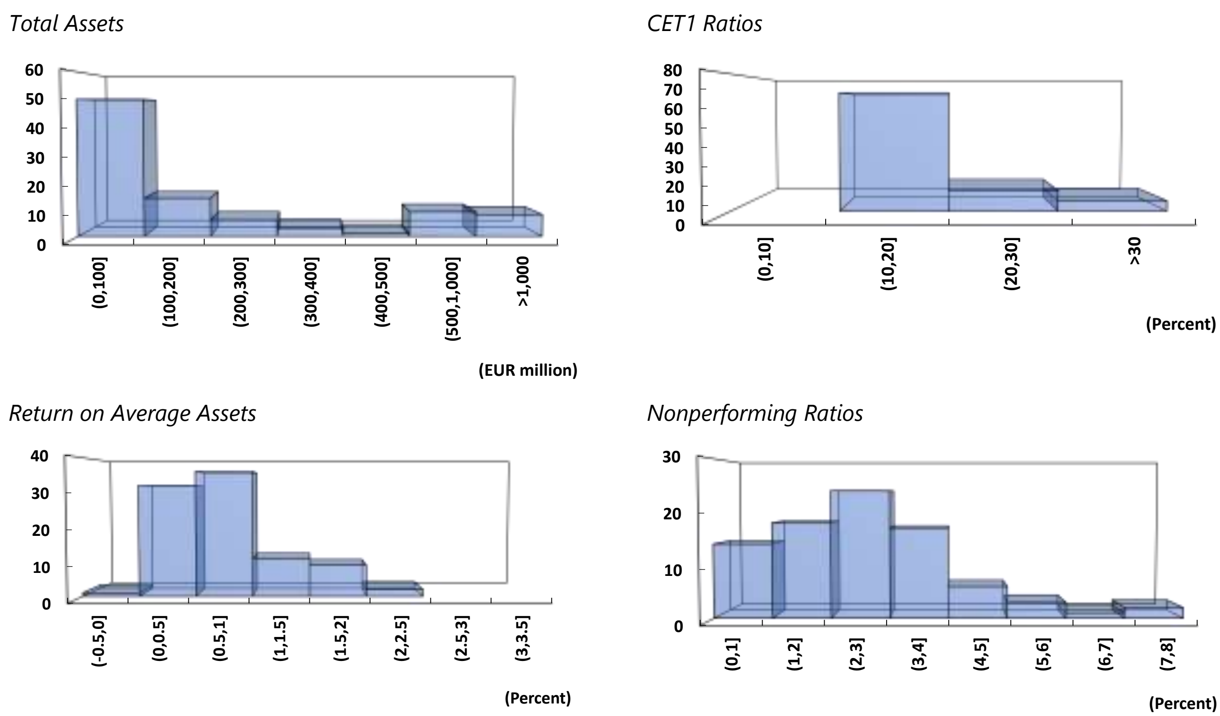
Key Performance Indicators

Overall, systemically important banks in the euro area displayed large capital buffers, improved but still weak profitability, and healthy asset quality at end-2022. The sample of systemically important banks in the paper consisted of 91 banks out of the 111 systemically important banks under the Single Supervisory Mechanism

[15] The analysis assumes regulatory stability and, therefore, does not consider regulatory changes for the sake of simplicity.

(SSM) at end-2022. Figure 2 indicates that 50 banks (or 55 percent of the sample banks) reported total assets below €100 million. However, nine banks controlled total assets between €500 million and €1 trillion while eight banks held total assets above €1 trillion, of which seven banks (Deutsche Bank, Groupe BPCE, Groupe Crédit Agricole, ING, Santander, Société Générale, and UniCredit) were considered global systemically important banks (GSIBs) in 2022. Banks were well capitalized on average, with 73 banks (79 percent of the 91 sample banks) reporting Common Equity Tier 1 capital ratios between 10 percent and 20 percent and the remaining 18 banks with CET1 ratios above 20 percent. Average profitability was weak, with 69 banks disclosing return on assets below 1 percent (70 percent of 91 sample banks) and only 22 displaying return on assets above 1 percent. The asset quality of banks' loan portfolio was high on average, with 56 banks (66 percent of the 85 sample banks) reporting NPL ratios below 3 percent, though some 29 banks reported ratios between 3 percent and 8 percent.

Figure 2. Euro Area Banks: Distribution of Total Assets, Profitability, Capital, and Asset Quality of Euro Area Banks, 2022



Source: Fitch Connect and author's calculations.

Credit Risk Parameters

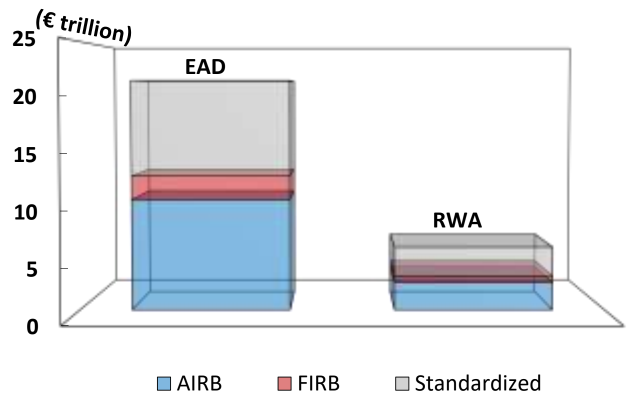
The end-year 2022 credit risk parameters under the AIRB and FIRB approaches were extracted from the 91 sample banks' pillar 3 disclosures. These included each bank's total exposures at default to the central governments and central banks, financial institutions, corporates (SME, specialized lending, and other), and retail (secured by immovable property SMEs and non-SMEs, qualifying revolving, and other SME and non-SME), their associated total risk-weighted assets and expected losses, their weighted-average probabilities of default, loss given default, and maturity. However, few banks reported total exposure at the default, weighted average probability of default, and average loss given default that were fully consistent with total expected losses, requiring the adjustment as outlined in Hunt and Taplin (2019) for consistency.

On average, euro area banks held portfolios with low-risk exposures at end-2022. Banks used different approaches to risk-weight their exposures. Figure 3, top chart, indicates that most euro area banks relied on both the AIRB and the Standardized approaches to risk weight their exposures at default. The total sample exposure at default amounted to €21 trillion at end-2022, of which €10 trillion consisted of exposures at default under the AIRB approach, €8 trillion under the Standardized approach, and only €2 trillion under the FIRB approach. The risk-weighted assets under the three approaches reached €5.8 trillion, with a similar breakdown under all three approaches. The risk weighted-asset density for all exposures was 27 percent, with the risk weighted-asset density slightly higher for exposures at default under the Standardized approach (30 percent of exposures at default) than under the FIRB (27 percent) and the AIRB (25 percent) approaches, reflecting different portfolio compositions and risks.

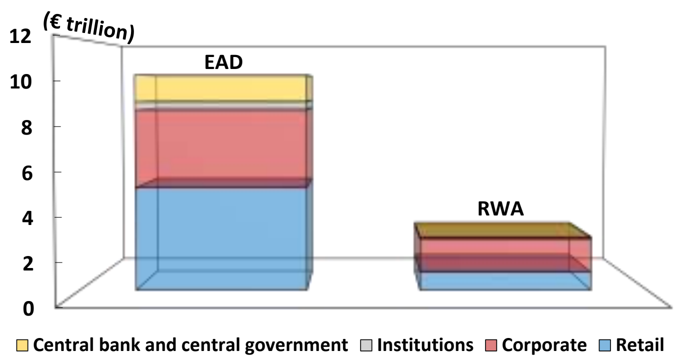
Most euro area banks reported loan portfolios with low probabilities of default but with a wider range of loss given default. Figure 4, left charts show that probability-of-default distributions under the FIRB and AIRB approaches were right-skewed with long tails, though average probability of default was below 3 percent for exposures to retail, and financial institutions at end-2022. However, only few banks displayed probabilities of default above 12 percent in all asset classes. These high probabilities of default could be sensitive to a change in financial and economic conditions facing borrowers. Though FIRB exposures had left-skewed loss-given-default distributions, the mode of the loss-given-default distributions was around 40 percent. On the other hand, AIRB portfolios had more dispersed distributions, indicating that banks had different views on the loss recovery across asset classes. As retail exposures were mostly loans collateralized by immovable property, their loss given default was lower than 30 percent while most loss given default for corporate exposures was higher than 30 percent.

Figure 3. Euro Area Banks: Distribution of Risk-Weighted Asset Densities for Different Asset Classes under the AIRB and FIRB Approaches, 2022

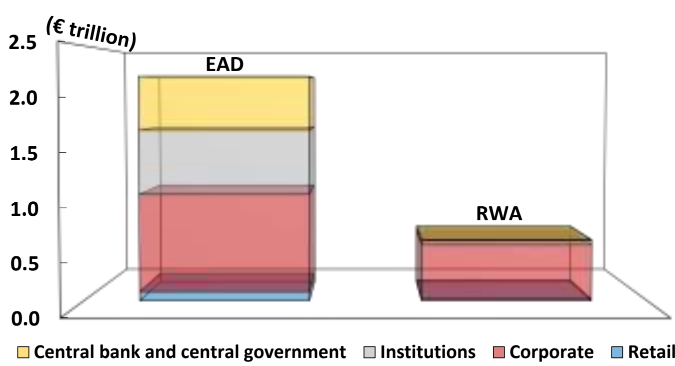
Total Exposure at Default and Risk-Weighted Assets



Exposure at Default and Risk Weighted Assets under the AIRB Approach



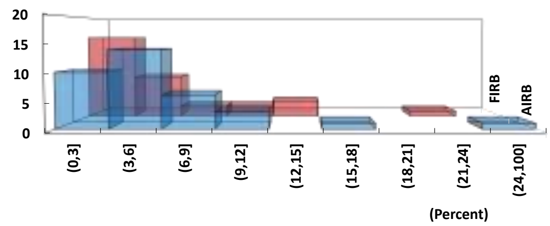
Exposure at Default and Risk Weighted Assets under the FIRB Approach



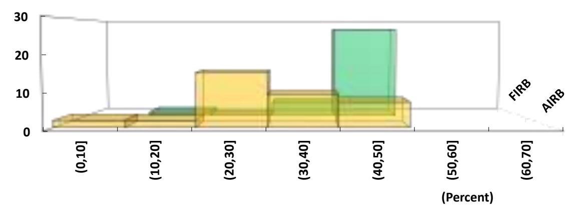
Source: Banks' Pillar 3 disclosures and author's calculations.

Figure 4. Euro Area Banks: Distribution of Probabilities of Default and Loss-Given-Default for Different Asset Classes under the AIRB and FIRB Approaches, 2022

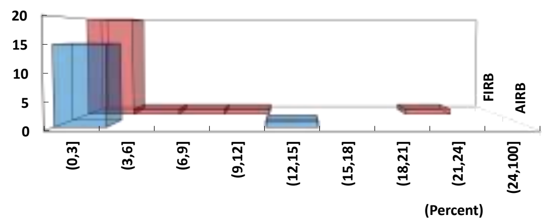
Corporate: Probabilities of Default under the AIRB and FIRB Approaches



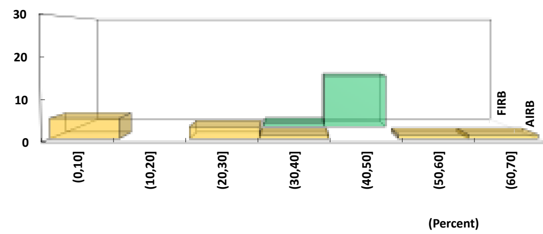
Corporate: Loss Given Default under the AIRB and FIRB Approaches



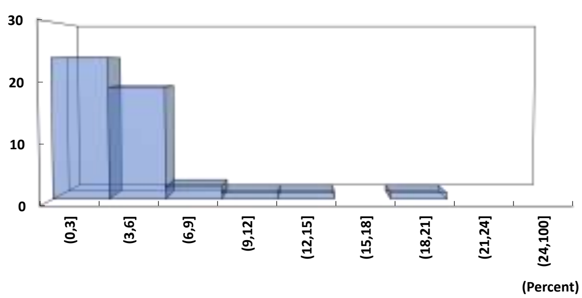
Financial Institution: Probabilities of Default under the AIRB and FIRB Approaches



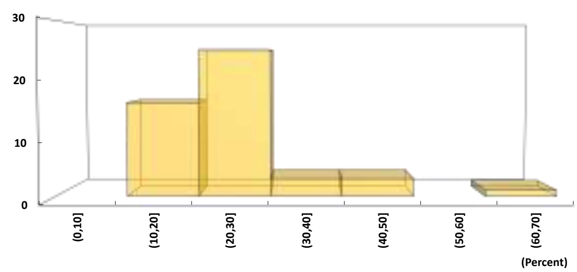
Financial Institution: Loss Given Default under the AIRB and FIRB Approaches



Retail: Probabilities of Default under the AIRB Approach



Retail: Loss Given Default under the AIRB Approach

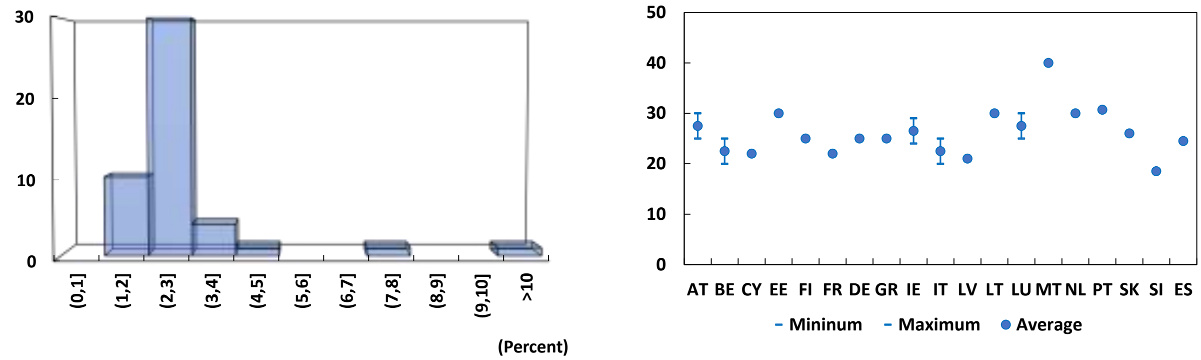


Source: Banks' Pillar 3 disclosures and author's calculations.

Finally, as there is no maturity adjustment for retail exposures under the AIRB approach, most banks did not report information on the average maturity of retail exposures. Figure 5, left chart, shows a right-skewed histogram of disclosed maturities for corporate exposures. Most of them were short term, ranging from 1 to 5 years as of end-2022. Few banks reported corporate exposures with maturities above 7 years. As most retail exposures consisted of mortgage loans secured by real estate, their maturity was proxied by the maturity in mortgage loan originations in the euro area. Figure 5, right chart, indicates that most mortgage loan maturities could have been above 20 years as of end-2022.

Figure 5. Euro Banks: Distribution of Maturities under the IRB Approach, 2022

Corporate Exposures under the FIRB Approach *Mortgage Loans in Origination*



Source: Banks' Pillar 3 disclosures, Hypostat 2023, European Mortgage Federation, and author's calculations.

Results

Variables in the satellite models need to be stationary to preclude spurious relationships. Table 4 shows panel unit root tests for growth, inflation, long term interest rate, unemployment, and nonperforming loans ratios. Both IPS and the CIPS unit root test results for inflation, long term interest rate, and unemployment ratios in levels are mixed when the model in levels includes either a constant or a constant and a trend while the IPS and CIPS unit root panel test results for growth and the NPL ratio with both model specifications suggest a rejection of the null hypothesis of non-stationarity. However, when all variables are in first difference, both IPS and CIPS unit root test results indicate that they are stationary when the model includes either a constant or a constant and a trend.

Table 4. Euro Area: Panel Unit Root Tests

Variable	IPS		CIPS	
	Constant	Constant and Trend	Constant	Constant and Trend
Levels				
Growth	-12.6	-11.1	-2.2	-3.1
	(0.00)	(0.00)	(0.06)	(<0.01)
Inflation	-3.2	2.8	-2.1	-2.4
	(0.00)	(1.00)	(>0.10)	(>0.10)
LT interest rate	-1.6	0.3	-1.0	-3.2
	(0.06)	(0.61)	(>0.10)	(<0.01)
Unemployment	-2.4	-2.3	-2.2	-3.0
	(0.01)	(0.01)	(0.05)	(<0.01)
NPL ratio	-24.9	-297.2	-3.1	-2.5
	(0.00)	(0.00)	(<0.01)	(0.10)
First difference				
Growth	-18.2	-16.3	-2.7	-2.7
	(0.00)	(0.00)	(<0.01)	(0.08)
Inflation	-9.1	-7.8	-2.8	-2.9
	(0.00)	(0.00)	(<0.01)	(0.03)
LT interest rate	-9.1	-6.6	-3.2	-3.3
	(0.00)	(0.00)	(<0.01)	(<0.01)
Unemployment	-9.3	-8.0	-2.8	-3.0
	(0.00)	(0.00)	(<0.01)	(<0.01)
NPL ratio	-45.9	-1095.0	-2.5	-2.4
	(0.00)	(0.00)	(<0.01)	(>0.10)
ST interest rate	-11.9	-9.4	-1.7	-1.6
	(0.00)	(0.00)	(>0.10)	(>0.10)

Source: Author's calculations.

1/ Red figures in the table represent coeficients that are statistically signficantly different from zero at 5 percent.

Sources: Fitch Connect, European Central Bank, and author's calculations.

The profitability channel affecting capital ratios works through retained earnings and the accumulation of organic capital.[16] Table 5 indicates that bank profitability for the 91 systemically important banks under the SSM supervision can be associated with bank-specific and macroeconomic variables in levels and first difference. The satellite models for the interest and non-interest income and expenses, loan loss provisions, other net income, and the NPL ratio are estimated with dynamic panel models relying on the orthogonal reparameterization of fixed effects as in Lancaster (2002) available in the R library *OrthoPanels*. The results of the satellite models indicate that: (i) interest and non-interest income and expenses and the NPL ratio have a strong persistence component; and (ii) all profitability components, except the non-interest income, and the NPL ratio are determined by real GDP growth, long-term interest rate, inflation, and the unemployment rate, along with the NPLs-to-gross loans ratio while their respective coefficients—except for the inflation rate and the GDP growth rate as a determinant of non-interest income—are mostly statistically significantly different from zero and display the expected signs. Moreover, the scale and signs of the estimated GDP growth, long-term interest rate, and unemployment coefficients for the interest income and expenditure satellite models are similar. Finally, as the long-term interest rate coefficient for the interest income is slightly lower than the one for the interest expense, net income slightly declines as long-term interest rate rises. This might reflect the fact

[16] All the estimations, optimizations, and simulations are coded in R.

that, as banks generally lend over the long term and fund themselves over the short term, banks` fixed-rate loan portfolios are repriced with higher long-term interest rates only when new loans are extended while all short-term funding is repriced on shorter period (repricing mismatch).

Table 5. Euro Area Banks: Satellite Models for Profitability and the NPL Ratio 1/ 2/

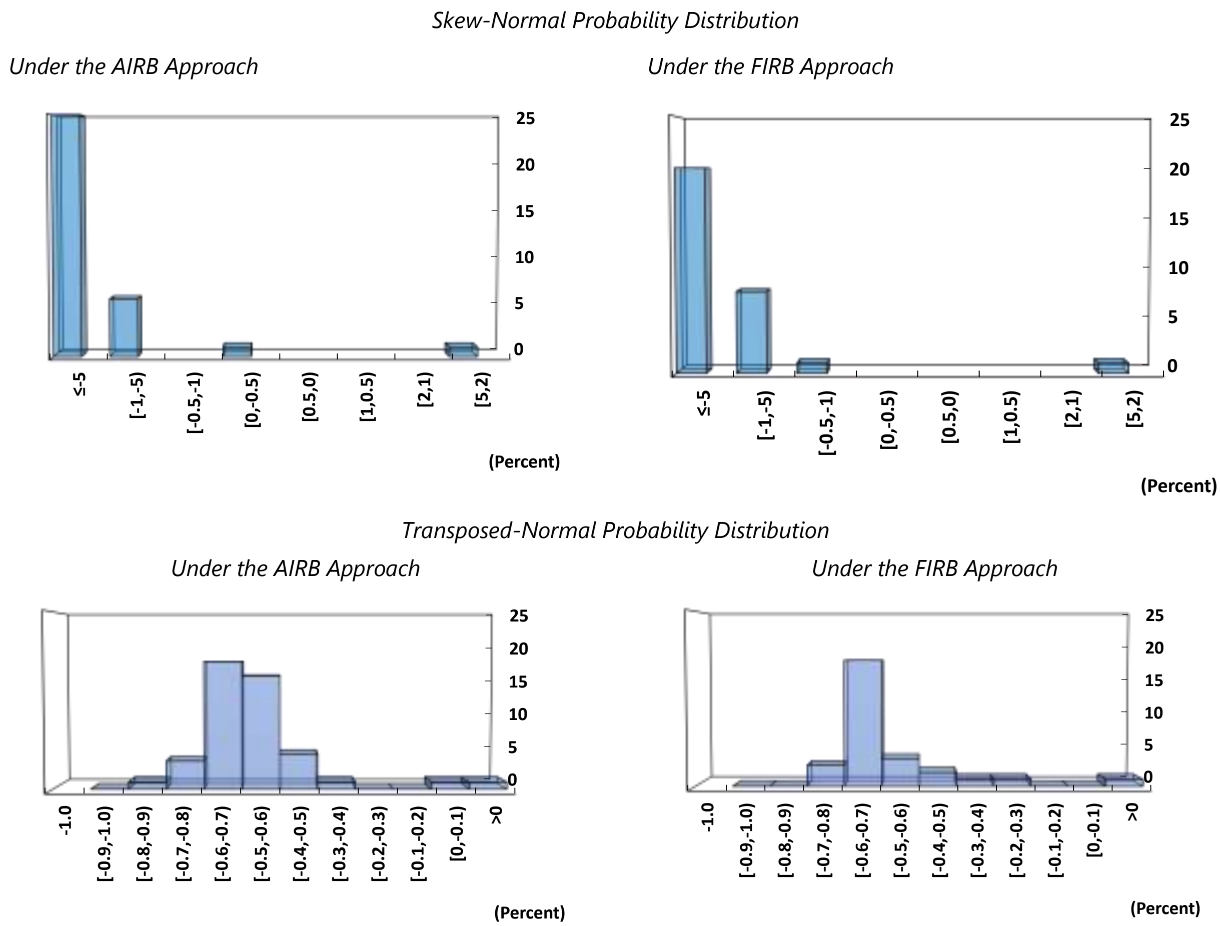
Dependent Variables: / Model:	Interest Income to Average Assets (1)	Interest Expenses to Average Assets (2)	Non-interest Income to Average Assets (3)	Non-interest Expenses to Average Assets (4)	Loan Loss Provisions to Average Assets (5)	Other Net Income to Average (6)	Nonperforming Loans Ratio (7)
ρ	0.88	0.84	0.39	0.78	0.21	0.00	0.97
	(0.81 ,0.96)	(0.80 ,0.89)	(0.32 ,0.46)	(0.72 ,0.84)	(0.14 ,0.29)	(-0.07 ,0.06)	(0.93 ,0.99)
σ^2	0.46	0.44	0.12	0.04	0.27	0.26	7.89
	(0.41 ,0.51)	(0.40 ,0.48)	(0.11 ,0.13)	(0.04 ,0.04)	(0.24 ,0.29)	(0.24 ,0.28)	(7.19 ,8.67)
GDP growth	0.03	0.03	0.00	-0.01	-0.04	0.02	-0.09
	(0.01 ,0.05)	(0.01 ,0.05)	(-0.00 ,0.01)	(-0.01 ,-0.00)	(-0.05 ,-0.02)	(0.01 ,0.03)	(-0.17 ,-0.01)
Inflation rate (first difference)	0.06	0.04	-0.01	0.00		-0.02	0.07
	(0.03 ,0.10)	(0.02 ,0.06)	(-0.02 ,0.01)	(-0.01 ,0.01)		(-0.03 ,-0.00)	(-0.05 ,0.18)
LT interest rate (first difference)	0.09	0.10	0.01		0.04		0.05
	(0.06 ,0.13)	(0.08 ,0.13)	(-0.01 ,0.02)		(0.02 ,0.07)		(-0.08 ,0.19)
Unemployment (first difference)	-0.10	-0.05			0.07	-0.07	0.44
	(-0.15 ,-0.05)	(-0.09 ,-0.02)			(0.04 ,0.11)	(-0.10 ,-0.04)	(0.25 ,0.62)
Cost-to-income ratio (lag)	0.06	0.05	-0.02	-0.01	-0.02	0.00	0.05
	(0.01 ,0.10)	(0.00 ,0.09)	(-0.04 ,0.00)	(-0.02 ,0.01)	(-0.05 ,0.02)	(-0.03 ,0.04)	(-0.14 ,0.23)
Deposits-to-total assets ratio (lag)	-1.37	-1.61	0.18	-0.14	-0.23	0.15	-4.23
	(-2.10 ,-0.60)	(-2.13 ,-1.09)	(-0.15 ,0.52)	(-0.33 ,0.05)	(-0.71 ,0.25)	(-0.23 ,0.53)	(-6.53 ,-1.92)
Equity-to-total assets ratio (lag)	-0.04	-0.03	-0.01	0.01	-0.03	0.00	-0.23
	(-0.08 ,-0.00)	(-0.05 ,-0.01)	(-0.03 ,0.00)	(-0.00 ,0.02)	(-0.06 ,-0.00)	(-0.01 ,0.02)	(-0.38 ,-0.09)
Loans-to-total assets ratio (lag)	1.00	0.58	-0.05	-0.04	0.90	-0.10	7.90
	(0.22 ,1.78)	(0.04 ,1.12)	(-0.44 ,0.34)	(-0.26 ,0.19)	(0.30 ,1.49)	(-0.51 ,0.31)	(5.06 ,10.76)
NPL ratio (lag)	-0.01		0.00	0.00	0.02		
	(-0.02 ,0.00)		(-0.00 ,0.01)	(-0.00 ,0.00)	(0.01 ,0.03)		
Total assets (log, lag)	-0.07	-0.05	-0.26	-0.09	0.01	-0.01	0.14
	(-0.20 ,0.07)	(-0.14 ,0.05)	(-0.33 ,-0.20)	(-0.13 ,-0.04)	(-0.08 ,0.09)	(-0.06 ,0.05)	(-0.27 ,0.56)
FixedEffects							
Bank							
Loglikelihood	1.2	-2.1	325.7	672.1	99.6	136.4	-935.9
Degrees of freedom	595	833	596	597	596	828	708

1/ Satellite models estimated with dynamic panel models relying on the orthogonal reparameterization of fixed effects as in Lancaster (2002).
2/ Equal-tailed credible intervals at the 5 percent level (except when idicated otherwise) obtained with the quantile interval of the posterior sample method.
3/ Red figures in the table represent coeficients that are statistically signficantly different from zero at 5 percent.

Sources: Fitch Connect, European Central Bank, and author's calculations.

Calibration of total risk-weighted assets with total exposure at default, average loss given default, average probability of default, and a Transmuted Normal idiosyncratic risk factor yields left-skewed risk-weight functions. Figure 6 plots weight parameters λ in the Transmuted-Normal distribution calibrated to replicate disclosed risk-weighted assets under the AIRB and FIRB approaches and obtained with the R libraries *nleqslv* and *OwenQ*. The distribution of weight parameters λ is centered around the interval [-0.7,-0.5) and [-0.6,-0.5) for the AIRB and FIRB approaches, respectively, implying that the Transmuted-Normal risk-weight function is more skewed to the left than the Normal-based IRB one. On the other hand, weight parameters close to zero would imply idiosyncratic risk factors with a Normal probability distribution.

Figure 6. Euro Area Banks: Distribution of Risk Weight λ Parameters in the Skew-Normal and Transmuted-Normal Distributions



Source: Author's calculations.

Overall, the stress test results suggest that banks were likely to remain resilient under the 2023 euro area consultation baseline scenario as of July 2023. Figure 7 indicates that, under the baseline scenario, the average Common Equity Tier 1 capital ratio for large euro area banks in the sample would have slightly risen to 15.6 and 15.3 percent at end-2025 from 14.7 percent at end-2022 when the idiosyncratic risk factor is a Skew-Normal or a Transmuted-Normal random variable, respectively. The positive contribution from the net interest and other income would not have been fully offset by the negative contribution from net non-interest income, loan loss provisions, taxes, dividends, and risk-weighted assets during 2023-25, underlying the strength of the net interest margin under an environment of higher expected interest rates. These findings under the baseline scenario were not only the result of benign macroeconomic conditions over the forecast horizon but also of healthier bank balance sheets at end-2022. Banks that benefited from the 2021-22 period to improve their asset quality, profitability, and capital ratios would have experienced smaller reductions in their Common Equity Tier 1 capital ratios. While there would be only one bank with a capital shortfall relative to the minimum 4.5 percent Common Equity Tier 1 capital requirement under the baseline, stronger economic activity during 2024-25 would have helped banks restore capital buffers to initial levels.

Figure 7. Euro Area Banks: CET1 Capital Ratios Under the April 2023 Baseline Scenario

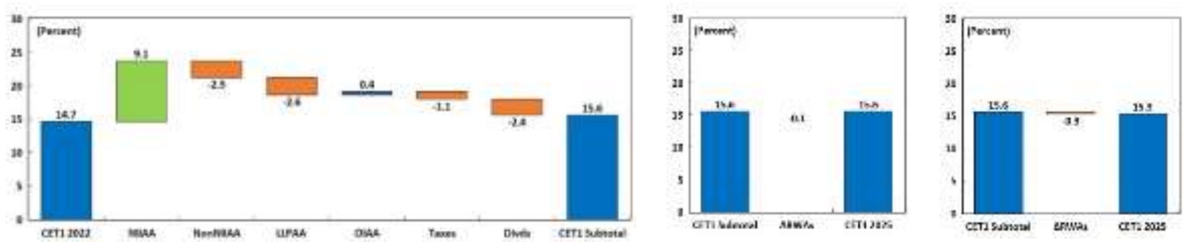
Drivers of CET1 Capital Ratios

A. Profitability Channel

B. Credit Risk Channel

Skew-Normal Distribution

Transmuted-Normal Distribution



Sources: Fitch Connect, S&P Market Intelligence, banks' pillar 3 disclosures, and staff's calculations.

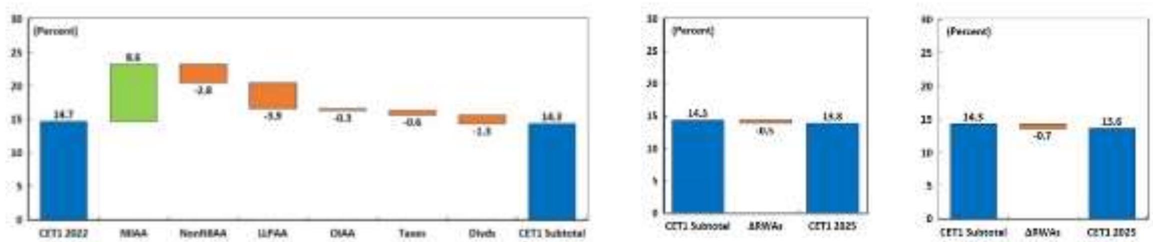
While the capital impact under the 2023 euro area consultation baseline scenario would have been manageable, the adverse scenario would have entailed moderate capital pressures. The adverse scenario consisted of a 1.2 percent cumulative GDP contraction over 2023–25. Figure 8 suggests that the average Common Equity Tier 1 capital ratio would have declined 1.1 percentage points to 13.8 and 13.6 percent by 2025 when the idiosyncratic risk factor is a Skew-Normal or a Transmuted-Normal random variable, respectively. Both higher risk-weighted assets and negative return on average asset would have explained most of the capital reduction. The positive contribution from the net interest income would have been more than offset by the negative contribution from net non-interest and other income, higher loan loss provisions, tax and dividend payments, and risk-weighted assets during 2023-25. Out of the many components, loan loss provisions would have significantly weighed on profitability. Moreover, higher probabilities of default would have also affected risk-weighted assets. In total, two banks would have experienced a capital shortfall relative to the 4.5 percent Common Equity Tier 1 capital requirement and three banks would have been below the combined Common Equity Tier 1 and capital conservation buffer requirement of 7 percent. In addition, about ten banks would have seen their Common Equity Tier 1 capital ratio drop below their MDA threshold by 2025. This projected capital erosion in the downturn scenario underscores the need to conserve capital and further build bank capital organically by having prudent capital planning.

Figure 8. Euro Area Banks: CET1 Capital Ratios Under the April 2023 Adverse Scenario

Drivers of CET1 Capital Ratios

A. Profitability Channel | *B. Credit Risk Channel*

Skew-Normal Distribution | *Transmuted-Normal Distribution*



Sources: Fitch Connect, S&P Market Intelligence, banks' pillar 3 disclosures, and staff's calculations.

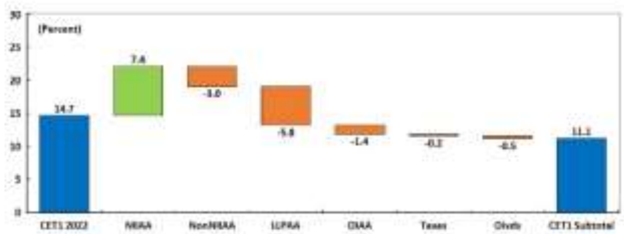
A comparison of the results from the paper with the ones from the 2023 EBA-SSM bottom-up stress test is useful. Figure 9 suggests that further refinements of the satellite and credit risk models would increase reliability and consistency of the stress test. Though the 2023 EBA-SSM bottom-up stress test sample included banks not headquartered in the euro area, the overall results should have been substantially influenced by the largest euro area banks. The decline in average Common Equity Tier 1 capital ratios resulting from plugging in EBA's growth, unemployment, inflation, and interest rate assumptions under the adverse scenario into the satellite and risk-weighted-asset models would have amounted to 4.4 and 4.5 percentage points when the idiosyncratic risk factor is a Skew-Normal or a Transmuted-Normal random variable, respectively. While this result is comparable to the 4.7 percent points average reduction in the Common Equity Tier1 capital ratios published by the EBA, visual inspection of the charts indicates key differences associated with the profitability drivers.[17] The 2023 EBA-SSM bottom-up stress test predicted a lower positive contribution from the net interest income, a larger negative contribution from the net non-interest income, and a lower negative contribution from loan loss provision than the ones estimated in the paper.

[17] European Banking Authority (2023).

Figure 9. Euro Area Banks: CET1 Capital Ratios Under the Under the EBA Adverse Scenario

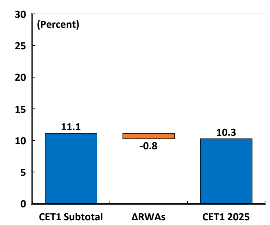
Drivers of CET1 Capital Ratios

A. Profitability Channel

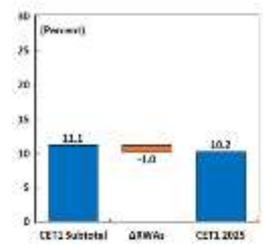


B. Credit Risk Channel

Skew-Normal Distribution

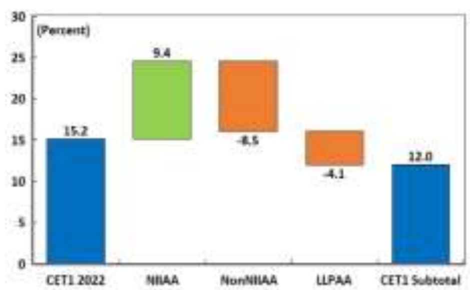


Transmuted-Normal Distribution



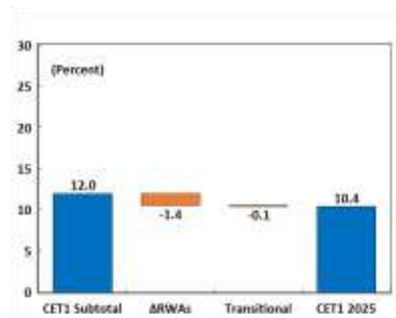
Drivers of CET1 Capital Ratios in the 2023 EBA-SSM bottom-up stress test

Profitability Channel



Credit Risk Channel

Normal Distribution



Sources: Fitch Connect, S&P Market Intelligence, banks' pillar 3 disclosures, and staff's calculations.

Conclusion

Though initially aimed at internationally active banks, the Basel Committee on Banking Supervision capital standards have become a benchmark for prudential regulation of minimum capital requirements. The framework contains standards addressing different risks and capital definitions, including the calculation of credit risk-weighted assets. The 2013 Capital Requirements Regulation (CRR) and posterior amendments transposed the Basel framework to European legislation. Both include capital requirements under two different approaches: (i) the Standardized approach that sets risk weights as a function of external ratings; and (ii) the Internal Ratings-Based (IRB) approach, with risk weights as a function of probabilities of default and loss given default.

The 2023 euro area consultation top-down stress test was based on changes in banks' capital requirement ratios driven by the profitability and credit risk channels. The profitability channel drove changes in capital resulting from the net income and retained earnings as a source of organic capital. The stress test combined bank-specific variables with macroeconomic data—namely real GDP growth, the unemployment rate, the long-term interest rate, and inflation—in dynamic panel data satellite models to project the interest and non-interest income and expenditures, other income, loan loss provisions, and nonperforming loans for the period 2023-25.[18] While dynamic panel data models with individual and time fixed effects control for cross sectional- and time-varying factors across banks, their ordinary least square estimators are biased in small sample sizes as indicated by Nickell (1981). To overcome the small sample bias, this paper relied on the orthogonal reparameterization approach by Lancaster (2002) for the estimation of dynamic panel models.

While the profitability channel worked through capital levels in the numerator of capital ratios, the credit risk channel affecting banks' capital ratios worked through probabilities of default embedded in risk-weighted assets in the denominator. Banks' pillar 3 disclosures provided key parameters—total exposures at default, average loss given default, average probabilities of default—to estimate the credit risk channel affecting risk-weighted assets given a worsening in economic conditions. This paper assumed that deviations from the IRB risk weight function were the result of relaxing the Normality assumption for the idiosyncratic factor driving default due to supporting factors for SMEs and infrastructure exposures and probability-of-default substitution effect on risk-weighted assets. The deviations were then captured by a negative shape parameter capturing skewness in a Skew-Normal probability distribution and a negative weight assigned to a quadratic term in a Transmuted-Normal probability distribution calibrated to replicate disclosed IRB risk weight assets.

The results of the stress tests indicate that most euro area banks were resilient under the 2023 euro area consultation baseline as July 2023 (publication of Staff report). The profitability channel implied that capital ratios would have responded not only to the persistence displayed by some profitability components but also to changes in macro variables. The credit risk channel implied that risk-weighted assets relying on a Transmuted-Normal distribution with a negative risk-weight parameter would have replicated the banks' IRB disclosed ones. Overall, the stress test results suggest that banks were likely to remain resilient under 2023 euro area consultation baseline scenario due to the strength of their net interest margin under an environment of higher expected interest rates. Finally, the difference between changes in risk-weighted assets based on Skew-Normal and Transmuted-Normal probability distributions were negligible.

[18] A finer breakdown including trading and fee incomes, for instance, would also be possible but would also require more granular data.

On the other hand, the 2023 euro area consultation adverse scenario would have entailed moderate capital pressures. Both higher risk-weighted assets and negative return on average assets would have explained the total capital depletion. Out of the many components, loan loss provisions would have significantly weighed on profitability. Some banks would have experienced a capital shortfall relative to the combined minimum Common Equity Tier 1 capital ratio and capital conservation buffer requirement of 7 percent. This projected capital erosion under the adverse scenario underscores the need to conserve capital and further build bank capital organically by having prudent capital planning. Finally, the difference between the projected Common Equity Tier 1 capital ratios resulting from plugging in EBA's macro assumptions under the adverse scenario into the satellite and risk-weighted-asset models and the Common Equity Tier 1 Capital ratios published by the EBA stemmed from the role of the different profitability drivers. The 2023 EBA-SSM stress test implied lower positive contribution from the net interest income, a larger negative contribution from the net non-interest income, and a lower negative contribution from loan loss provisions than the ones estimated in the paper. The 2025 FSAP stress test of euro area banks provided an opportunity, and a slightly higher contribution from risk-weighted assets to assess the resilience of euro area banks with confidential supervisory data.

The depth and reliability of top-down stress tests depend not only on data availability but also on the quality of their disclosures. Financial statements and Pillar 3 disclosures related to risks are mandated by European regulations. Though banks broadly comply with the mandated tables on different risks in their Pillar 3 disclosures, their presentation varies substantially. Moreover, some banks make their Pillar 3 disclosures available only in other languages than in English. As banks calculate FIRB and AIRB risk weights on a bottom-up basis and report risk parameters—including exposures at default, loss given default, probabilities of default—for different asset classes, the resulting total expected losses and risk-weighted assets cannot be easily replicated by using average loss given default and probabilities of default even after adjustments as suggested by Hunt and Taplin (2019). The new Pillar III data hub by the EBA is a welcome step in making Pillar III disclosures easily, consistently, and widely accessible.

Annex I. The Skew-Normal Risk Weight Function

The Skew-Normal Distribution

The probability density function for a Skew-Normal random variable is defined in Azzalini (1985) as follows:

$$f_1^{SN}(Z,\lambda) = \frac{2}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Phi(\lambda Z) \tag{I.1}$$

while its cumulative density function is:

$$\mathrm{SN}(Z,\lambda) = \Phi(Z) - 2T(Z,\lambda), \tag{I.2}$$

where λ a parameter capturing skewness, $\Phi(Z)$ is a cumulative density function for a standard Normal random variable:

$$\Phi(Z) = \int_{-\infty}^{Z} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt \tag{I.3}$$

and $T(.,.)$ is the Owen's T function:

$$T(x,a) = \frac{1}{2\pi} \int_0^a \frac{e^{-(1+x^2)\frac{h^2}{2}}}{1+x^2} dx. \tag{I.4}$$

Azzalini and Dalla Valle (1996) show that the mean of a Skew-Normal random variable is:

$$E(Z) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{I.5}$$

and its variance is:

$$Var(Z) = \left(1 - \frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}\right). \tag{I.6}$$

The density function of a Skew-Normal random variable can then be standardized with the following linear transformation:

$$Y = \sqrt{1 - \frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}}\, Z + \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1+\lambda^2}} \tag{I.7}$$

This implies that the probability density function for a standardized Skew-Normal random variable is given by:

$$f_1(Z,\lambda) = \frac{2}{\sqrt{2\pi}} \sqrt{1 - \frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}}\; e^{-\frac{\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}}\, Z + \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\right)^2}{2}}\; \Phi\left(\lambda\left(\sqrt{1 - \frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}}\, Z + \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\right)\right) \tag{I.8}$$

while its cumulative density function is as follows:

$$SN_1(Z,\lambda) = \Phi\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}}\,Z + \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}\right) - 2T\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2}}\,Z + \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}, \lambda\right). \tag{I.9}$$

The estimation of lambda in a Skew-Normal distribution is described, for instance, in Bayes and Branco (2007).

Idiosyncratic and Systemic Risk Factors with Skew-Normal Distributions

Following notation in Schonbucher (2000), borrower i's value $V_i(T)$ at the end-period T is driven by a systemic and an idiosyncratic risk factor:

$$V_i(T) = \sqrt{\rho_i}Y + \sqrt{1-\rho_i}\varepsilon_i, \tag{I.10}$$

where Y and ε_i stand for the systemic and idiosyncratic risk factors, respectively. These are assumed to be standardized Transmuted-Normal random variables with mean and variance equal to zero and one with a correlation coefficient ρ_i.

Based on the general credit loss distribution in Schonbucher (2000), a credit loss distribution incorporating Transmuted-Normal systemic and idiosyncratic risk factors is as follows:

$$F(x) \coloneqq P[X \le x] = 1 - SN_Y\left(\frac{K_i}{\sqrt{\rho_i}} - \sqrt{\frac{1-\rho_i}{\rho_i}}SN_{\varepsilon_i}{}^{-1}(x), \lambda_{Y,i}\right), \tag{I.11}$$

where $SN_Y(.)$ is the cumulative density function for the systemic risk factor Y with a Skew-Normal Distribution and a shape parameter $\lambda_{Y,i}$ capturing its skewness, K_i: is the default barrier for borrower i; and $SN_{\varepsilon_i}^{-1}(.)$ is the inverse of the cumulative density function for the idiosyncratic factor ε_i with a Skew-Normal distribution and a shape parameter λ_{ε_i} capturing its skewness.

The risk-weight function (or the the β-quantile) can be written as:

$$F_1(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}, \lambda_{Y,i}) = SN_{\varepsilon_i}\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}}SN_Y{}^{-1}(1-\beta), \lambda_{\varepsilon_i}\right), \tag{I.12}$$

Substituting the definition of the cumulative density function yields the following:

$$F_1\big(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}, \lambda_{Y,i}\big)$$
$$= \Phi\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\frac{K_i-\sqrt{\rho_i}{SN_Y}^{-1}(1-\beta)}{\sqrt{1-\rho_i}}+\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}\right)$$
$$-2T\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\frac{K_i-\sqrt{\rho_i}{SN_Y}^{-1}(1-\beta)}{\sqrt{1-\rho_i}}+\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}, \lambda_{\varepsilon_i}\right),$$

(I.13)

where the inverse of ${SN_Y}^{-1}(1-\beta)$ can be found by numerically solving the expression below for y given $\lambda_{Y,i}$

$$1-\beta = \Phi\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}\,y+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+{\lambda_{Y,i}}^2}}\right)-2T\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}\,y+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+{\lambda_{Y,i}}^2}}, \lambda_{Y,i}\right).$$

(I.14)

The barrier K_i for each borrower can be associated with a probability of default PD_i by taking expectation of the conditional probability $p_i(y)$:

$$E\big(p_i(y)\big) = E\left(SN_{\varepsilon_i}\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}, \lambda_{\varepsilon_i}\right)\right) = PD_i.$$

(I.15)

This is equivalent to:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty} SN_{\varepsilon_i}\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}, \lambda_{\varepsilon_i}\right) f_1^{SN}\big(y, \lambda_y\big)\,dy = PD_i.$$

(I.16)

Substituting the definition of the cumulative and probability density functions for a Skew-Normal random variable in (I.16) yields:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty}\left\{\Phi\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}}+\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}{}^2}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\ y\right)\right.$$
$$-2T\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}}+\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}{}^2}}\right.$$
$$\left.\left.-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\ y,\lambda_{\varepsilon_i}\right)\right\}\left\{\frac{2}{\sqrt{2\pi}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\ e^{-\frac{\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\ y+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)^2}{2}}\ \Phi\left(\lambda_{Y,i}\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\ y\right.\right.\right.$$
$$\left.\left.\left.+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)\right)\right\}dy = PD_i. \tag{I.17}$$

This expression can be integrated by parts. The first one I_1 is given by:

$$I_1 = \int_{-\infty}^{\infty}\Phi\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}}+\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}{}^2}}\right.$$
$$\left.-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\ y\right)\left\{\frac{2}{\sqrt{2\pi}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\ e^{-\frac{\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\ y+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)^2}{2}}\ \Phi\left(\lambda_{Y,i}\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\ y\right.\right.\right.$$
$$\left.\left.\left.+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)\right)\right\}dy. \tag{I.18}$$

If $u = \sqrt{1-2\,\lambda_{Y,i}{}^2/\left[\pi\left(1+\lambda_{Y,i}{}^2\right)\right]}\,y+\sqrt{2}\lambda_{Y,i}/\sqrt{\pi\left(1+\lambda_{Y,i}{}^2\right)}$, a change of variables in the expression above yields the following:

$$I_1 = \int_{-\infty}^{\infty} \Phi\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,\frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,\frac{\left(u-\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+{\lambda_{Y,i}}^2}}\right)}{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}}\right)\frac{2}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\Phi(\lambda_{Y,i}u)du. \tag{I.19}$$

The integral of I_1 is based on formula 20,010.3 in Owen (1980) as follows:

$$I_1 = 2BvN\left(\frac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,\frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}} + \frac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+{\lambda_{Y,i}}^2}}}{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}}}{\sqrt{1+\left(\frac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}}\right)^2}}, 0; \rho_{1,2} = -\frac{\lambda_{Y,i}}{\sqrt{1+{\lambda_{Y,i}}^2}}\frac{\frac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}}}{\sqrt{1+\left(\frac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{Y,i}}^2}{1+{\lambda_{Y,i}}^2}}}\right)^2}} \right) \tag{I.20}$$

Simplifying the expression above yields the following:

$$I_1 = 2BvN\left(\frac{\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}}K_i+\sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}^{\;2}}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}}+\sqrt{\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^{\;2}}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}}}{\sqrt{(1-\rho_i)\left(1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}\right)+\rho_i\left(1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}\right)}}, 0; \; \rho_{1,2} = -\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^{\;2}}}\sqrt{\frac{\rho_i\left(1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}\right)}{(1-\rho_i)\left(1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}\right)+\rho_i\left(1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}\right)}}\right) \tag{I.21}$$

The first ratio can be interpreted as the default barrier adjusted by the weighted cross-products of the mean and standard deviation of the systemic and idiosyncratic risk factors and normalized by the squared root of the weighted sum of the variances associated with the idiosyncratic and systemic risk factors, where the weights are given by $1-\rho_i$ and ρ_i, respectively. The correlation coefficient $\rho_{1,2}$is given by the product of the adjusted shape parameter capturing skewness in the systemic risk factor and the squared root of the share of weighted variance of idiosyncratic risk in the weighted sum of the variances associated with the idiosyncratic and systemic risk factors, with the weights given by $1-\rho_i$ and ρ_i, respectively.

The second part of the integration I_2 is as follows:

$$I_2 = -2\int_{-\infty}^{\infty} T\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}}\frac{K_i}{\sqrt{1-\rho_i}}+\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}^{\;2}}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}^{\;2}}{1+\lambda_{\varepsilon_i}^{\;2}}}\,y, \lambda_{\varepsilon_i}\right)\left\{\frac{2}{\sqrt{2\pi}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}}\,e^{-\frac{\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}}y+\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^{\;2}}}\right)^2}{2}}\Phi\left(\lambda\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^{\;2}}{1+\lambda_{Y,i}^{\;2}}}\,y + \sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^{\;2}}}\right)\right)\right\}dy. \tag{I.22}$$

Given that $u=\sqrt{1-2\,\lambda_{Y,i}^{\;2}/[\pi(1+\lambda_{Y,i}^{\;2})]}\,y+\sqrt{2}\lambda_{Y,i}/\sqrt{\pi(1+\lambda_{Y,i}^{\;2})}$, a change of variables in the expression above yields:

$$I_2 = -2\int_{-\infty}^{\infty} T\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\,\frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}{}^2}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\,\frac{\left(u-\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)}{\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}}, \lambda_{\varepsilon_i}\right)\frac{2}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\Phi(\lambda_{Y,i}u)du.$$

(I.23)

The expression I_2 above does not have a closed form solution and needs to be numerically evaluated.

Once the coefficient $\lambda_{Y,i}$ capturing skewness in the systemic risk factor is estimated by Maximum Likelihood and the inverse of the cumulative density function for the systemic risk factor with a Skew-Normal probability distribution, the total risk-weighted asset in expression (1) with the Skew-Normal risk-weight function (I.13) and the sum of $I_1$and I_2 in expressions (I.21) and (I.23) above can be jointly numerically solved for the shape parameter λ_{ε_i} and the default threshold K_i assuming that $E(p_i(y)) = PD_i$.

Idiosyncratic Risk Factor with a Skew-Normal Distribution

If the systemic and the idiosyncratic risk factors are, respectively, Normal ($\lambda_{Y,i} = 0$) and Skew-Normal ($\lambda_{\varepsilon_i} \neq 0$) random variables, the risk-weight function (or the β-quantile) can then be simplified to:

$$F_1(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}) = SN_{\varepsilon_i}\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}}N^{-1}(1-\beta), \lambda_{\varepsilon_i}\right).$$

(I.24)

Expanding the cumulative density function for the idiosyncratic risk factor with a Skew-Normal probability distribution yields:

$$F_1(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}) = \Phi\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\,\frac{K_i-\sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}{}^2}}\right) - 2T\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{\varepsilon_i}{}^2}{1+\lambda_{\varepsilon_i}{}^2}}\,\frac{K_i-\sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+\lambda_{\varepsilon_i}{}^2}}, \lambda_{\varepsilon_i}\right).$$

(I.25)

The barrier K_i for each borrower can be associated with PD_i by taking expectation of $p_i(y)$:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty} SN_{\varepsilon_i}\left(\frac{K_i - \sqrt{\rho_i}y}{\sqrt{1-\rho_i}}, \lambda_{\varepsilon_i}\right) f_1^N(y)\, dy = PD_i.$$

(I.26)

where $f_1^N(.)$ is the probability density function for the systemic risk factor with a Normal probability distribution.

Substituting the definition of the cumulative density function for a Skew-Normal random variable in (I.9) yields:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty}\left\{\Phi\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ y\right)\right.$$
$$\left. - 2T\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ y, \lambda_{\varepsilon_i}\right)\right\}\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy$$
$$= PD_i.$$

(I.27)

This expression can also be integrated by parts. The first one I_1 is given by:

$$I_1 = \int_{-\infty}^{\infty}\Phi\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ y\right)\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy.$$

(I.28)

The integral of I_1 is based on formula 10,010.8 in Owen (1980):

$$I_1 = \Phi\left(\frac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ \frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\right),$$

(I.29)

which can then be simplified to:

$$I_1 = \Phi\left(\frac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\ K_i + \sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\right),$$

(I.30)

The ratio can be interpreted as the default barrier adjusted by the mean of the idiosyncratic risk factor and normalized by the squared root of the weighted sum of the variances of the systemic risk and idiosyncratic factors, with weights given by $1-\rho_i$ and ρ_i, respectively.

The second part I_2 contains the integration of Owen's T function:

$$I_2 = -2\int_{-\infty}^{\infty} T\left(\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,\frac{K_i}{\sqrt{1-\rho_i}} + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,y, \lambda_{\varepsilon_i}\right)\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy. \tag{I.31}$$

The integral of I_2 is based on formula c00,010.3 in Owen (1980):

$$I_2 = -2T\left(\begin{matrix}\dfrac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,K_i + \sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}, \\ \lambda^*_{\varepsilon_i} = \dfrac{\sqrt{1-\rho_i}\lambda_{\varepsilon_i}}{\sqrt{1-\rho_i+\rho_i\left(1+{\lambda_{\varepsilon_i}}^2\right)\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\end{matrix}\right). \tag{I.32}$$

The second ratio given by $\lambda^*_{\varepsilon_i}$ is the adjusted shape parameter capturing skewness in the idiosyncratic factor normalized by the squared root of the weighted sum of variances associated with the systemic and idiosyncratic risk factors, where weights are given by the correlation coefficient $1-\rho_i$ and ρ_i, respectively, while the variance of the idiosyncratic risk factor is adjusted by $\left(1+{\lambda_{\varepsilon_i}}^2\right)$.

Finally, the conditional expectation of $p_i(y)$ is given by the sum of I_1 and I_2:

$$E\left(p_i(y)\right) = \Phi\left(\frac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,K_i + \sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\right)$$
$$-2T\left(\begin{matrix}\dfrac{\sqrt{1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}}\,K_i + \sqrt{1-\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{\varepsilon_i}}{\sqrt{1+{\lambda_{\varepsilon_i}}^2}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}, \\ \lambda^*_{\varepsilon_i} = \dfrac{\sqrt{1-\rho_i}\lambda_{\varepsilon_i}}{\sqrt{1-\rho_i+\rho_i\left(1+{\lambda_{\varepsilon_i}}^2\right)\left(1-\frac{2}{\pi}\frac{{\lambda_{\varepsilon_i}}^2}{1+{\lambda_{\varepsilon_i}}^2}\right)}}\end{matrix}\right) = PD_i, \tag{I.33}$$

The total risk-weighted asset expression (1) with the Skew-Normal risk-weight function in expression (I.25) and the expression (I.33) above can be jointly numerically solved for the shape parameter λ_{ε_i} and the default threshold K_i.

Systemic Risk Factor with a Skew-Normal Distribution

If the systemic and the idiosyncratic risk factors are, respectively, Skew-Normal ($\lambda_{Y,i} \neq 0$) and Normal ($\lambda_{\varepsilon_i} = 0$) random variables, the credit loss function is simplified to:

$$F_1(\beta, K_i, \rho_i, \lambda_{Y,i}) = \Phi\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}} SN_Y^{-1}(1-\beta)\right). \tag{I.34}$$

where the inverse of $SN_Y{}^{-1}(1-\beta)$ is given by:

$$1-\beta = \Phi\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\, y + \sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right) - 2T\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\, y + \sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}, \lambda_{Y,i}\right). \tag{I.35}$$

The barrier K_i is for each borrower can also be associated with its probability PD_i by taking the expectation of $p_i(y)$:

$$E(p_i(y)) = \int_{-\infty}^{\infty} \Phi\left(\frac{K_i - \sqrt{\rho_i} y}{\sqrt{1-\rho_i}}\right) f_1^{SN}(y, \lambda_{Y,i}) = PD_i. \tag{I.36}$$

Substituting the definition of the probability density function for the idiosyncratic risk factor results in:

$$E(p_i(y)) = \int_{-\infty}^{\infty} \Phi\left(\frac{K_i - \sqrt{\rho_i} y}{\sqrt{1-\rho_i}}\right) \left\{ \frac{2}{\sqrt{2\pi}} \sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\, e^{-\frac{\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\, y + \sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)^2}{2}} \Phi\left(\lambda_{Y,i}\left(\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}{}^2}{1+\lambda_{Y,i}{}^2}}\, y + \sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}{}^2}}\right)\right)\right\} dy = PD_i. \tag{I.37}$$

If $u = \sqrt{1 - 2\,\lambda_{Y,i}{}^2/[\pi(1+\lambda_{Y,i}{}^2)]}\, y + \sqrt{2}\lambda_{Y,i}/\sqrt{\pi(1+\lambda_{Y,i}{}^2)}$, a change of variable in the expression above yields:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty} \Phi\left(\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\left(\frac{u - \sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^2}}}{\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}}}\right)\right)\frac{2}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\Phi\big(\lambda_{Y,i}u\big)dy = PD_i. \tag{I.38}$$

The integration of this expression is based on formula 20,010.3 in Owen (1980):

$$E\big(p_i(y)\big) = 2BvN\left(\begin{matrix}\dfrac{\dfrac{K_i}{\sqrt{1-\rho_i}} + \dfrac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\left(\dfrac{\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^2}}}{\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}}}\right)}{\sqrt{1+\dfrac{\rho_i}{(1-\rho_i)}\dfrac{1}{\left(1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}\right)}}}, 0; \\ \rho_{12} = -\dfrac{\dfrac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\dfrac{1}{\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}}}\lambda_{Y,i}}{\sqrt{1+\dfrac{\rho_i}{(1-\rho_i)}\dfrac{1}{\left(1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}\right)}}\sqrt{1+\lambda_{Y,i}^2}}\end{matrix}\right) = PD_i \tag{I.39}$$

Simplifying the expression above yields:

$$E\big(p_i(y)\big) = 2BvN\left(\begin{matrix}\dfrac{K_i\sqrt{1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}} + \sqrt{\rho_i}\sqrt{\frac{2}{\pi}}\frac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^2}}}{\sqrt{(1-\rho_i)\left(1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}\right)+\rho_i}}, 0; \\ \rho_{12} = -\dfrac{\lambda_{Y,i}}{\sqrt{1+\lambda_{Y,i}^2}}\dfrac{\sqrt{\rho_i}}{\sqrt{(1-\rho_i)\left(1-\frac{2}{\pi}\frac{\lambda_{Y,i}^2}{1+\lambda_{Y,i}^2}\right)+\rho_i}}\end{matrix}\right) = PD_i \tag{I.40}$$

The first ratio can be interpreted as the default barrier adjusted by the mean of the systemic risk factor and normalized by the squared root of a weighted sum of the variances of the systemic and idiosyncratic risk factors, where the weights are given by $1-\rho_i$ and ρ_i. The second ratio ρ_{12} is given by the squared root of the correlation coefficient ρ_i adjusted by the ratio $\lambda_{Y,i}/\sqrt{1+\lambda_{Y,i}^2}$ related to mean of the systemic risk factor and normalized by the squared root of the weighted sum of the variances associated with the systemic risk and idiosyncratic risk factors, with the weights also given by $1 - \rho_i$ and by ρ_i.

Finally, the nonlinear system of equations (1) complemented by (I.34), (I.35), and (I.40) can be jointly solved for K_i, $TN_Y^{-1}(1 - \beta)$, and $\lambda_{Y,i}$.

Annex II. The Transmuted-Normal Risk Weight Function

The Transmuted-Normal Distribution

The probability density function for a Transmuted Normal random variable is obtained by a quadratic-rank transmutation map (QRTM) as described in Shaw and Buckley (2007) as follows:

$$f_1^{TN}(Z,\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\big(1+\lambda-2\lambda\Phi(Z)\big) \tag{II.1}$$

while its cumulative density function is:

$$TN(Z,\lambda) = (1+\lambda)\Phi(Z) - \lambda\Phi^2(Z), \tag{II.2}$$

where -1 $\leq \lambda \leq 1$ is the weight associated with the quadratic term and $\Phi(Z)$ is a cumulative density function for a standard Normal random variable:

$$\Phi(Z) = \int_{-\infty}^{Z} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt \tag{II.3}$$

Shaw and Buckley (2007) also show that the mean of a Transmuted Normal random variable is:

$$E(Z) = -\frac{\lambda}{\sqrt{\pi}} \tag{II.4}$$

and its variance is:

$$Var(Z) = \left(1-\frac{\lambda^2}{\pi}\right). \tag{II.5}$$

The density function of a Transmuted Normal random variable can then be standardized by the following linear transformation:

$$Y = \sqrt{1-\frac{\lambda^2}{\pi}}\, Z - \frac{\lambda}{\sqrt{\pi}} \tag{II.6}$$

This implies that the probability density function for a standardized Transmuted Normal random variable is given by:

$$f_1^{TN}(Z,\lambda) = \frac{1}{\sqrt{2\pi}}\sqrt{1-\frac{\lambda^2}{\pi}}\,e^{-\frac{\left(\sqrt{1-\frac{\lambda^2}{\pi}}Z-\frac{\lambda}{\sqrt{\pi}}\right)^2}{2}}\left(1+\lambda-2\lambda\Phi\left(\sqrt{1-\frac{\lambda^2}{\pi}}Z-\frac{\lambda}{\sqrt{\pi}}\right)\right) \quad \text{(II.7)}$$

while its cumulative density function is as follows:

$$TN_1(Z,\lambda) = (1+\lambda)\Phi\left(\sqrt{1-\frac{\lambda^2}{\pi}}Z-\frac{\lambda}{\sqrt{\pi}}\right)-\lambda\left[\Phi\left(\sqrt{1-\frac{\lambda^2}{\pi}}Z-\frac{\lambda}{\sqrt{\pi}}\right)\right]^2 \quad \text{(II.8)}$$

The estimation of lambda in a Transmuted-Normal distribution is described in Ieren and Abdullahi (2020).

Idiosyncratic and Systemic Risk Factors with Transmuted-Normal Distributions

Following notation in Schonbucher (2000), borrower i's value $V_i(T)$ at the end-period T is driven by a systemic and an idiosyncratic risk factor:

$$V_i(T) = \sqrt{\rho_i}Y + \sqrt{1-\rho_i}\varepsilon_i, \quad \text{(II.9)}$$

where Y and ε_i stand for the systemic and idiosyncratic risk factors, respectively. These are assumed to be standardized Transmuted-Normal random variables with mean and variance equal to zero and one, respectively, and with a correlation coefficient ρ_i.

Based on the general credit loss distribution in Schonbucher (2000), a credit loss distribution incorporating systemic and idiosyncratic risk factors with Transmuted-Normal distributions is as follows:

$$F(x) := P[X \le x] = 1 - TN_Y\left(\frac{K_i}{\sqrt{\rho_i}} - \sqrt{\frac{1-\rho_i}{\rho_i}}TN_{\varepsilon_i}{}^{-1}(x), \lambda_{Y,i}\right), \quad \text{(II.10)}$$

where $TN_Y(.)$ is the cumulative density function for the systemic risk factor Y with a Transmuted-Normal distribution and a weight associated with the quadratic term capturing skewness in the systemic risk factor Y given by $-1 \le \lambda_{Y,i} \le 1$; K_i: is the default barrier; and $TN_{\varepsilon_i}^{-1}(.)$ is the inverse of the cumulative density function for the idiosyncratic factor ε_i with a Transmuted-Normal distribution and a weight $-1 \le \lambda_{\varepsilon_i} \le 1$ associated with the quadratic term capturing skewness in the idiosyncratic risk factor. The risk-weight function (or the β-quantile) can be written as:

$$F_1\left(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}, \lambda_{Y,i}\right) = TN_{\varepsilon_i}\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}}TN_Y{}^{-1}(1-\beta), \lambda_{\varepsilon_i}\right). \quad \text{(II.11)}$$

Incorporating the definition of the cumulative density function of a Transmuted-Normal probability distribution yields the following expression:

$$F_1(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}, \lambda_{Y,i}) = (1+\lambda_{\varepsilon_i})\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i-\sqrt{\rho_i}{TN_Y}^{-1}(1-\beta)}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right) - \lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i-\sqrt{\rho_i}{TN_Y}^{-1}(1-\beta)}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right)\right]^2, \tag{II.12}$$

where the inverse of ${TN_Y}^{-1}(1-\beta)$ can be found by solving numerically the expression below for y given $\lambda_{Y,i}$

$$1-\beta = (1+\lambda_{Y,i})\Phi\left(\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}y-\frac{\lambda_{Y,i}}{\sqrt{\pi}}\right)-\lambda_{Y,i}\left[\Phi\left(\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}y-\frac{\lambda_{Y,i}}{\sqrt{\pi}}\right)\right]^2. \tag{II.13}$$

The barrier K_i for each borrower can be associated with a probability of default PD_i by taking expectation of the conditional probability $p_i(y)$:

$$E(p_i(y)) = E\left(TN_{\varepsilon_i}\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}},\lambda_{\varepsilon_i}\right)\right) = PD_i, \tag{II.14}$$

which is equivalent to:

$$E(p_i(y)) = \int_{-\infty}^{\infty} TN_{\varepsilon_i}\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}},\lambda_{\varepsilon_i}\right) f_1^{TN}(y,\lambda_{Y,i})\,dy = PD_i. \tag{II.15}$$

Substituting the definitions of the cumulative and probability density functions for a Transmuted-Normal random variable in (II.7) and (II.8), respectively, in (II.15) yields:

$$E(p_i(y)) = \int_{-\infty}^{\infty}\left\{(1+\lambda_{\varepsilon_i})\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}y\right) - \lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}y\right)\right]^2\right\}\frac{1}{\sqrt{2\pi}}\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}e^{\frac{-\left(\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}y-\frac{\lambda_i}{\sqrt{\pi}}\right)^2}{2}}\left[1+\lambda_{Y,i}-2\lambda_{Y,i}\Phi\left(\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}y-\frac{\lambda_{Y,i}}{\sqrt{\pi}}\right)\right]dy = PD_i. \tag{II.16}$$

If $u = \sqrt{1-\lambda_{Y,i}^2/\pi}\, y - \lambda_{Y,i}/\sqrt{\pi}$, a change in variables in the expression above results in:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty}\left\{(1+\lambda_{\varepsilon_i})\Phi\left(\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}u\right)\right.$$
$$-\lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}\right.\right.$$
$$\left.\left.\left.-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}u\right)\right]^2\right\}\left\{\frac{1}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}\left(1+\lambda_{Y,i}-2\lambda_{Y,i}\Phi(u)\right)\right\}du = PD_i.$$

(II.17)

This expression can be integrated by parts. The first one I_1 is given by:

$$I_1 = \int_{-\infty}^{\infty}\left[(1+\lambda_{\varepsilon_i})\Phi\left(\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}u\right)\right]$$
$$\frac{1}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}\left(1+\lambda_{Y,i}\right)du.$$

(II.18)

The integral for I_1 is based on formula 10,010.8 in Owen (1980):

$$I_1 = (1+\lambda_{\varepsilon_i})(1+\lambda_{Y,i})\Phi\left(\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}{1-\frac{\lambda_{Y,i}^2}{\pi}}}}\right),$$

(II.19)

which, after simplification, can be rewritten as follows:

$$I_1 = \left(1+\lambda_{\varepsilon_i}\right)\left(1+\lambda_{Y,i}\right)\Phi\left(\frac{\sqrt{\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}K_i - \sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}}\right), \tag{II.20}$$

The ratio can be interpreted as the default barrier adjusted by the weighted sum of the cross-products of the mean and the systemic and idiosyncratic risk factors and normalized by the weighted sum of their variances, where the weights are given by the correlation coefficient ρ_i and $1-\rho_i$.

The second part I_2 is given by:

$$I_2 = -2\int_{-\infty}^{\infty}\left\{\left(1+\lambda_{\varepsilon_i}\right)\Phi\left(\sqrt{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}{}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}{}^2}{\pi}}}u\right)\right\}\frac{1}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}\lambda_{Y,i}\Phi(u)du. \tag{II.21}$$

Based on formula 20,010.3 in Owen (1980), integration of the expression above yields the following:

$$I_2 = -2\lambda_{Y,i}\left(1+\lambda_{\varepsilon_i}\right)BvN\left(\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}{}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}{1-\frac{\lambda_{Y,i}{}^2}{\pi}}}}, 0;\ \rho_{12} = -\frac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}{}^2}{\pi}}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}}{1-\frac{\lambda_{Y,i}{}^2}{\pi}}}}\right), \tag{II.22}$$

where $BvN(.)$stands for the Bi-variate Normal distribution function. This expression can be simplified to:

$$I_2 = -2\lambda_{Y,i}\left(1+\lambda_{\varepsilon_i}\right)BvN\left(\frac{\sqrt{\left(1-\frac{\lambda_{Y,i}^{\;2}}{\pi}\right)\left(1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}\right)}K_i-\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^{\;2}}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^{\;2}}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}\right)}},0;\ \rho_{12}\right.$$
$$\left.=-\frac{\sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}\right)}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^{\;2}}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}\right)}}\right). \tag{II.23}$$

The third part I_3 is given by the following expression:

$$I_3 = \int_{-\infty}^{\infty} -\lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}u\right)\right]^2 \frac{1}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}\left(1+\lambda_{Y,i}\right)du. \tag{II.24}$$

Integration of the expression above based on formula 20,010.4 in Owen (1980) yields the following:

$$I_3 = -\lambda_{\varepsilon_i}\left(1+\lambda_{Y,i}\right)\left[\Phi\left(\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}}\right) -2T\left(\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}},\frac{1}{\sqrt{1+2\frac{\rho_i}{1-\rho_i}\frac{1-\frac{\lambda_{\varepsilon_i}^{\;2}}{\pi}}{1-\frac{\lambda_{Y,i}^{\;2}}{\pi}}}}\right)\right], \tag{II.25}$$

where $T(.,.)$ is the Owen's T function:

$$T(x,a)=\frac{1}{2\pi}\int_{0}^{a}\frac{e^{-(1+x^2)\frac{h^2}{2}}}{1+x^2}dx.$$

(II.26)

Simplifying the expression above yields:

$$I_3 = -\lambda_{\varepsilon_i}\left(1+\lambda_{Y,i}\right)\left[\Phi\left(\frac{\sqrt{\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}K_i-\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}}\right)-2T\left(\frac{\sqrt{\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}K_i-\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}},\frac{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}^2}{\pi}\right)+2\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^2}{\pi}\right)}}\right)\right].$$

(II.27)

Finally, the fourth part I_4 is as follows:

$$I_4 = 2\int_{-\infty}^{\infty}-\lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^2}{\pi}}}{\sqrt{1-\frac{\lambda_{Y,i}^2}{\pi}}}u\right)\right]^2\frac{1}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}\lambda_{Y,i}\Phi(u)du.$$

(II.28)

Based on formula 30.010,3 in Owen (1980), the integral of the expression above is as follows:

$$I_4 = -2\lambda_{X,i}\lambda_{Y,i} TvN\left(\begin{array}{c} \dfrac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}}{\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}}, \dfrac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}}{\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}}, 0; \\ \rho_{12} = \dfrac{\frac{\rho_i}{1-\rho_i}\frac{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}{1+\frac{\rho_i}{1-\rho_i}\frac{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}, \rho_{13} = -\dfrac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}}{\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}}{\sqrt{2+2\frac{\rho_i}{1-\rho_i}\frac{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}}, \rho_{23} = -\dfrac{\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}}{\sqrt{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}}{\sqrt{2+2\frac{\rho_i}{1-\rho_i}\frac{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}{1-\frac{{\lambda_{Y,i}}^2}{\pi}}}} \end{array}\right), \tag{II.29}$$

where $TvN(.)$ stands for a Tri-variate Normal distribution (cumulative density function). This expression can be simplified to:

$$I_4 = -2\lambda_{X,i}\lambda_{Y,i} TvN\left(\begin{array}{c} \dfrac{\sqrt{\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}K_i-\sqrt{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\sqrt{\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}, \\ \dfrac{\sqrt{\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}K_i-\sqrt{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}-\sqrt{\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}, 0; \\ \rho_{12} = \dfrac{\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}, \\ \rho_{13} = -\frac{1}{\sqrt{2}}\dfrac{\sqrt{\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}{\sqrt{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}, \rho_{23} = -\frac{1}{\sqrt{2}}\dfrac{\sqrt{\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}{\sqrt{(1-\rho_i)\left(1-\frac{{\lambda_{Y,i}}^2}{\pi}\right)+\rho_i\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}} \end{array}\right) \tag{II.30}$$

The sum of the four parts above can be rewritten as follows:

$$E\big(p_i(y)\big) = \big(1+\lambda_{Y,i}\big)\Phi(K_i^*) - 2\lambda_{Y,i}\big(1+\lambda_{\varepsilon_i}\big)BvN\big(K_i^*, 0; -\rho_{1,2}^*\big)$$
$$+ 2\lambda_{\varepsilon_i}\big(1+\lambda_{Y,i}\big)T\left(K_i^*, \frac{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)+2\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}}\right)$$
$$- 2\lambda_{\varepsilon_i}\lambda_{Y,i}TvN\left(\begin{matrix}K_i^*, K_i^*, 0;\\ \rho_{1,2}^{*2}, -\frac{1}{\sqrt{2}}\rho_{1,2}^*, -\frac{1}{\sqrt{2}}\rho_{1,2}^*\end{matrix}\right) = PD_i,$$

(II.31)

where K_i^* is the adjusted default barrier for borrower *i*:

$$K_i^* = \frac{\sqrt{\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}K_i - \sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}} - \sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}}$$

(II.32)

and $\rho_{1,2}^*$ is the adjusted correlation coefficient between the systemic and idiosyncratic risk factors:

$$\rho_{1,2}^* = \frac{\sqrt{\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}}{\sqrt{(1-\rho_i)\left(1-\frac{\lambda_{Y,i}{}^2}{\pi}\right)+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}{}^2}{\pi}\right)}}$$

(II.33)

Once the weight coefficient $\lambda_{Y,i}$ capturing skewness in the systemic risk factor is estimated by Maximum Likelihood and $SN_y^{-1}(1-\beta)$ is obtained, the total risk-weighted asset equation (1) with a Transmuted Normal risk-weight function equation (II.12) complemented by equation (II.13) for the systemic risk factor and the sum of the four parts above in equation (II.31) can be jointly numerically solved for the weight coefficients λ_{ε_i} and the default threshold K_i assuming that $E\big(p_i(y)\big) = PD_i$.

Idiosyncratic Risk Factor with a Transmuted-Normal Distribution

If the systemic and the idiosyncratic risk factors are, respectively, Normal ($\lambda_{Y,i} = 0$) and Transmuted-Normal ($\lambda_{\varepsilon_i} \neq 0$) random variables, the risk-weight function (or the β-quantile) can then be simplified to:

$$F_1\big(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}\big) = TN_{\varepsilon_i}\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}}N^{-1}(1-\beta), \lambda_{\varepsilon_i}\right).$$

(II.34)

Expanding the expression to include the definition of the cumulative density function for the idiosyncratic risk factor with a Transmuted Normal distribution yields:

$$F_1\left(\beta, K_i, \rho_i, \lambda_{\varepsilon_i}\right) = \left(1+\lambda_{\varepsilon_i}\right)\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i-\sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right) - \lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i-\sqrt{\rho_i}N^{-1}(1-\beta)}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right)\right]^2.$$

(II.35)

The barrier K for each borrower can be associated with its probability $p_i(y)$ by taking its expectation:

$$E\left(p_i(y)\right) = \int_{-\infty}^{\infty} TN_{\varepsilon_i}\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}, \lambda_{\varepsilon_i}\right) f_1^N(y)dy = PD_i,$$

(II.36)

which is equivalent to:

$$E\left(p_i(y)\right) = \int_{-\infty}^{\infty}\left\{\left(1+\lambda_{\varepsilon_i}\right)\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right)-\lambda_{\varepsilon_i}\left[\Phi\left(\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}\right)\right]^2\right\}\Phi(y)dy = PD_i.$$

(II.37)

Based on formulas 10,010.8 and 20,010.4 in Owen (1980), the integrals of the first and second parts are as follows:

$$\begin{aligned}
E\left(p_i(y)\right) &= \left(1+\lambda_{\varepsilon_i}\right)\Phi\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right) \\
&\quad - \lambda_{\varepsilon_i}\left[\Phi\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right)-2T\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}, \frac{1}{\sqrt{1+2\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right)\right] \\
&= \Phi\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right)+2\lambda_{\varepsilon_i}T\left(\frac{\sqrt{1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}}\frac{K_i}{\sqrt{1-\rho_i}}-\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1+\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}, \frac{1}{\sqrt{1+2\frac{\rho_i}{1-\rho_i}\left(1-\frac{{\lambda_{\varepsilon_i}}^2}{\pi}\right)}}\right) = PD_i.
\end{aligned}$$

(II.38)

Simplifying the above expression yields the following:

$$E\big(p_i(y)\big) = \Phi\left(\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\ 2}}{\pi}}K_i - \sqrt{1-\rho_i}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\ 2}}{\pi}\right)}}\right) + 2\lambda_{\varepsilon_i}T\left(\frac{\sqrt{1-\frac{\lambda_{\varepsilon_i}^{\ 2}}{\pi}}K_i - \sqrt{1-\rho_i}\frac{\lambda_{\varepsilon_i}}{\sqrt{\pi}}}{\sqrt{1-\rho_i+\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\ 2}}{\pi}\right)}}, \sqrt{\frac{1-\rho_i}{1-\rho_i+2\rho_i\left(1-\frac{\lambda_{\varepsilon_i}^{\ 2}}{\pi}\right)}}\right) = PD_i. \tag{II.39}$$

The first ratio above can also be interpreted as the default barrier adjusted by the mean of the idiosyncratic risk factor and normalized by the weighted sum of the variances of the systemic and idiosyncratic risk factors. The second ratio is the adjusted weight capturing skewness in the idiosyncratic risk factor proxied by the squared root of the ratio of the weighted variance of the systemic risk to the weighted sum of the variance of the systemic and idiosyncratic risk factors, with weights given by $1-\rho_i$ and ρ_i.

Finally, the nonlinear system of equations (1) complemented by equation (II.35) and equation (II.39) are jointly solved for K_i, and λ_{ε_i}given all the other parameters.

Systemic Risk Factor with a Transmuted-Normal Distribution

If the systemic and the idiosyncratic risk factors are, respectively, Transmuted-Normal ($\lambda_{Y,i} \neq 0$) and Normal ($\lambda_{\varepsilon_i} = 0$) random variables, the credit loss function is also simplified to:

$$F_1\big(\beta, K_i, \rho_i, \lambda_{Y,i}\big) = \Phi\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}}TN_Y^{-1}(1-\beta)\right), \tag{II.40}$$

where the inverse $TN_Y^{-1}(1-\beta)$ is given by:

$$1-\beta = \big(1+\lambda_{Y,i}\big)\Phi\left(\sqrt{1-\frac{\lambda_{Y,i}^{\ 2}}{\pi}}y - \frac{\lambda_{Y,i}}{\sqrt{\pi}}\right) - \lambda_{Y,i}\left[\Phi\left(\sqrt{1-\frac{\lambda_{Y,i}^{\ 2}}{\pi}}y - \frac{\lambda_{Y,i}}{\sqrt{\pi}}\right)\right]^2. \tag{II.41}$$

The barrier K_i is for each borrower can also be associated with its probability $p_i(y)$ by taking its expectation:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty}\Phi\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}\right) f_1^{TN}\big(y,\lambda_{Y,i}\big)\,dy = PD_i. \tag{II.42}$$

This expression is equivalent to:

$$E\big(p_i(y)\big) = \int_{-\infty}^{\infty}\Phi\left(\frac{K_i-\sqrt{\rho_i}y}{\sqrt{1-\rho_i}}\right)\frac{1}{\sqrt{2\pi}}\sqrt{1-\frac{\lambda_{Y,i}^{\ 2}}{\pi}}\,e^{\frac{-\left(\sqrt{1-\frac{\lambda_{Y,i}^{\ 2}}{\pi}}y-\frac{\lambda_i}{\sqrt{\pi}}\right)^2}{2}}\left[1+\lambda_{Y,i}-2\lambda_{Y,i}\Phi\left(\sqrt{1-\frac{\lambda_{Y,i}^{\ 2}}{\pi}}y-\frac{\lambda_{Y,i}}{\sqrt{\pi}}\right)\right]dy = PD_i. \tag{II.43}$$

If $u = \sqrt{1 - {\lambda_{Y,i}}^2/\pi}\, y - \lambda_{Y,i}/\sqrt{\pi}$, a change in variables in the expression above results in:

$$E(p_i(y)) = PD_i = \int_{-\infty}^{\infty} \Phi\left(\frac{K_i}{\sqrt{1-\rho_i}} - \sqrt{\frac{\rho_i}{1-\rho_i}}\frac{\left(u + \frac{\lambda_{Y,i}}{\sqrt{\pi}}\right)}{\sqrt{1 - \frac{{\lambda_{Y,i}}^2}{\pi}}}\right)\frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}}\left[1 + \lambda_{Y,i} - 2\lambda_{Y,i}\Phi(u)\right] du = PD_i,$$

(II.44)

Integration of the first and second parts of the expression above is based on formulas 10,010.8 and 20,010.3 in Owen (1980) and is given by:

$$E(p_i(y)) = (1 + \lambda_{Y,i})\Phi\left(\frac{\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\sqrt{\rho_i}}{\sqrt{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1 + \frac{\rho_i}{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}}}\right)$$

$$- 2\lambda_{Y,i} BvN\left(\frac{\frac{K_i}{\sqrt{1-\rho_i}} - \frac{\sqrt{\rho_i}}{\sqrt{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{1 + \frac{\rho_i}{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}}}, 0; \rho_{12} = -\frac{1}{\sqrt{2}}\frac{\sqrt{\frac{\rho_i}{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}}}{\sqrt{1 + \frac{\rho_i}{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}}}\right)$$

$$= PD_i$$

(II.45)

This expression can be simplified to:

$$E(p_i(y)) = (1 + \lambda_{Y,i})N\left(\frac{\sqrt{\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}K_i - \sqrt{\rho_i}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right) + \rho_i}}\right)$$

$$- 2\lambda_{Y,i} BvN\left(\frac{\sqrt{\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right)}K_i - \sqrt{\rho_i}\frac{\lambda_{Y,i}}{\sqrt{\pi}}}{\sqrt{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right) + \rho_i}}, 0; \rho_{12} = -\frac{1}{\sqrt{2}}\frac{\sqrt{\rho_i}}{\sqrt{(1-\rho_i)\left(1 - \frac{{\lambda_{Y,i}}^2}{\pi}\right) + \rho_i}}\right) = PD_i$$

(II.46)

The first ratio can be interpreted as the default barrier adjusted by the weight capturing skewness in the systemic risk factor and normalized by the weighted sum of the variances associated with the systemic and idiosyncratic risk factors, with weights given by $1 - \rho_i$and ρ_i. The second ratio is the adjusted variance of the idiosyncratic risk factor normalized by the weighted sum of the variances of the systemic and idiosyncratic risk factors, with the weights also given by $1 - \rho_i$and ρ_i, respectively.

Finally, the nonlinear system of equations (1) complemented by equation (II.40) and equations (II.41) and (II.46) are jointly solved for K_i, $TN_Y^{-1}(1-\beta)$, and $\lambda_{Y,i}$.

Annex III. The Dynamics of Probabilities of Default

The dynamics of nonperforming loans at time t is simply described by the sum of nonperforming loans at time t-1 and new defaulted loans at time t net out of the write-offs at time t:

$$NPL_t = NPL_{t-1} + DR_t(Loans_{t-1} - NPL_{t-1}) - NetWriteOffs_t, \tag{IV. 1}$$

where DR_t stands for the default rate at time t. If net write-offs are assumed to be equal to zero, the default rate at time t can be written as following:

$$DR_t = \frac{NPL_t - NPL_{t-1}}{Loans_{t-1} - NPL_{t-1}}. \tag{IV. 2}$$

The projection of nonperforming loans at time t based on macro variables and bank specific variables implies that the expected value of default rates is equivalent to its probability of default:

$$PD_t = \frac{NPL_t - NPL_{t-1}}{Loans_{t-1} - NPL_{t-1}}. \tag{IV. 3}$$

Changes in probabilities of default can be written as following:

$$PD_{t+1} - PD_t = \frac{NPL_{t+1} - NPL_t}{Loans_t - NPL_t} - \frac{NPL_t - NPL_{t-1}}{Loans_{t-1} - NPL_{t-1}}. \tag{IV. 4}$$

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