18. Economic Approach

A. Introduction

18.1 The economic approach differs from the fixed basket, axiomatic, and stochastic approaches outlined in Chapters 16 and 17 in an important respect: quantities are no longer assumed to be independent of prices. Consider a price index for the output produced by establishments. If, for example, it is assumed that the establishments behave as revenue maximizers, it follows that they would switch production to commodities with higher relative price changes. This behavioral assumption about the firm allows something to be said about what a "true" index number formula should be and the suitability of different index number formula as approximations to it. For example, the Laspeyres price index uses fixed period revenue shares to weight its price relatives and ignores the substitution of production toward products with higher relative price changes. The Laspeyres price index will thus understate aggregate price changes—be biased downward against its true index. The Paasche price index uses fixed current period weights and will thus overstate aggregate price changes—be biased upward against its true index.

18.2 Alternatively, producers may try to anticipate demand changes by producing less of products with above-average price changes. A Laspeyres price index will thus overstate aggregate price changes—be biased upwards against its true index and a Paasche price index will understate aggregate price changes—be biased downwards against its true index.

18.3 Consider a price index for the intermediate inputs to establishments. If, for example, it is assumed that firms behave as cost minimizers, it follows that they would purchase more of products with below-average price changes and again the behavioral assumption would have implications for the nature of substitution bias in Laspeyres and Paasche price index number formulas. The economic approach can be seen to be very powerful, since it has identified a type of bias in Laspeyres and Paasche indices not apparent from other approaches: *substitution bias*.

18.4 The approach from economic theory of production is thus first to develop theoretical index number formulas based on what are considered to be reasonable models of economic behavior by the producer. A mathematical representation of the production activity—whereby capital and labor conjoin to turn intermediate inputs into outputs—is required. An assumption of optimizing behavior (cost minimization or revenue maximization) is also required. A theoretical index is then derived that is "true" for both the form of the representation of the production activity and behavioral assumption. The economic approach then examines practical index number formulas such as Laspeyres, Paasche, Fisher, and Törnqvist, and considers how they compare with the "true" formulas defined under different assumptions. Important findings are that: (i) Laspeyres and Paasche price indices act as bounds on their true indices and, under certain conditions, also are bounds for more generally applicable theoretical true indices that adequately incorporates *substitution effects*; (ii) such generally applicable theoretical indices fall between the Laspeyres and Paasche price indices arguing for an index that is a symmetric mean of the two—the Fisher price index formula is the only symmetric average of Laspeyres and Paasche that satisfies the *time reversal test*; (iii) index number formula correspond exactly to specific functional forms of the mathematical representation of the revenue/cost functions—for example the Törnqvist index is exact for a revenue function represented by a translogarithmic functional form; (iv) a class of *superlative* index number formulas exist that are exact for flexible functional forms giving strong support to their use, since flexible functional forms incorporate substitution behavior. The Fisher, Törnqvist, and Walsh index formulas are all superlative.

18.5 In section B the stage will be set for the analysis. First, two approaches will be distinguished each serving different analytical purposes, the resident's and non-resident's approaches. The *resident's approach* identifies exports as outputs from domestic economic producers—the behavioral assumption would be one of revenue maximization and the Laspeyres price index would be expected to be biased downward against its true index and the Paasche price index biased upward against its true index. The *non-resident's approach* identifies exports from the domestic economy as imports to the rest of the world and the perspective taken is the importer's whose behavioral assumption is cost minimization: the biases in the Laspeyres and Paasche price index number formulas would be reversed. The resident's approach to *imports* as inputs to the domestic economic may take cost minimization as its behavioral assumption—Laspeyres would be expected to be biased upwards against its true index and the Paasche index biased downwards, with the position reversed from the non-resident's perspective.

18.6 The use of a symmetric means of Laspeyres and Paasche price index formulas would accord with both the resident's and non resident's perspectives since their mean is unaffected by the direction of the bounds. Thus for deliberations of the nature of the bias in Laspeyres and Paasche price indices it is necessary to consider the behavioral assumption of the economic agents which in turn require consideration of the perspective from which exports and imports are regarded.

B. Economic theory and the resident's and non-resident's approach

18.7 This chapter considers two perspectives on XMPIs:

- Nonresident perspective: exports of an economic territory are viewed from the *nonresident* establishment or household user's perspective as an input, and imports of an economic territory are viewed from the *nonresident* producer's or supplier's perspective as an output—see Dridi and Zieschang (2004) for details; and

• Resident perspective: exports of an economic territory are viewed from the *resident* producer's or supplier's perspective as an output, and imports of an economic territory viewed from the *resident* establishment or household user's perspective as an input.

18.8 The *1993 System of National Account*'s adopts a "non-residents' perspective" to the treatment of exports and imports in external account of goods and services. Exports and imports are treated as "uses" and "resources/supply" respectively. Exports are the non-residents' use of goods and services produced by residents, and imports are the non-residents' supply of goods and services to the residents of an economic territory. Thus the appropriate economic theory underlying imports and exports in the *System's* external account of goods and services should be based on this non-resident perspective and, as such, would carry over to be the appropriate economic theory for the price indices used to deflate these aggregates. The behavioral assumptions for the economic theory of such price indices would be of cost minimizing non-resident economic agents, including establishments, households and government, purchasing exports and revenue maximizing non-resident economic agents supplying imports.

18.9 However, a "resident's perspective" would have the exports of an economic territory viewed from the resident producer's or supplier's perspective as an output, and imports of an economic territory viewed from the resident establishment or household user's perspective as an input. The resident's perspective would be appropriate for import and export price and volume series used for the analysis of (the resident country's) productivity change, changes in the terms of trade, and transmission of inflation. The counterpart aggregates to such price and volume measures would be for imports as uses to the residents, and exports as supply. The *System's* production account include intermediate consumption from the resident producer's perspective and a component of this is served by imports. The *System's* production account also include output as a supply, some of which is a supply to domestic markets and some to non-domestic markets, that is exports, again from the resident producers perspective. Thus the appropriate economic theory underlying imports and exports for the analysis of (the resident country's) productivity change, changes in the terms of trade, and transmission of inflation would be based on this resident's perspective and, as such, would carry over to be the appropriate economic theory for the price indices used to deflate these aggregates. The behavioral assumptions for the economic theory of such price indices would be of cost minimizing resident economic agents purchasing imports and revenue maximizing resident economic agents supplying exports.

18.10 The perspective taken dictates the behavioral assumptions applied. In Section D1 of this Chapter it will be demonstrated how the behavior assumptions in turn dictate the direction of the substitution bias in terms of the relationship between Laspeyres and its theoretical "true" counterpart and the Paasche price index in relation to its theoretical "true" counterpart.

18.11 The analysis equates cost minimizing behavior with purchasers substituting away from commodities with above average price increases and revenue maximizing behavior with producers substituting output towards commodities with above average price increases. However, these general patterns need not hold for all commodities. It may well be that, for example, some resident exporters produce more of commodities with relatively low or falling price changes due to changes in preferences and technological change that allow producers to both cut prices and increase demand. The strength of the economic analysis is that it identifies a type of bias, and demonstrates how answers to questions as to its nature and extent depend on assumptions as to the behavior of exporting and importing economic agents.

Table 18.1, Behavioral assumptions for resident's and non-resident's approaches

	Exports	Imports
Resident's approach	Revenue maximizer	Cost minimizer
Non-resident's approach	Cost minimizer	Revenue maximizer

18.12 Section C sets the stage for the economic analysis by defining the economic agents involved and the some assumptions implicit in the analysis. *The analysis proceeds from the resident's perspective.* This is for two reasons: First, the treatment from the non-resident's perspective is well documented in Dridi and Zieschang (2004). Second, the distinguishing feature of the two approaches for the purpose of economic theory is the behavioral assumptions. There are essentially two sets of theory—those from cost minimizing behavioral assumptions and those from revenue maximizing assumptions. As is apparent from Table 18.1, the findings for exports from the resident's approach applies to those of imports from the non-resident's approach, and findings for imports from the resident's approach applies to those of exports from the non-resident's approach. There is simply no need to replicate the outline of the theory from one perspective given it has been undertaken from the other. As will be demonstrated in Sections D1 and F2 for exports and imports respectively, the nature of these assumptions affect the results for the direction of the bounds on the theoretical "true" indices. However, the behavioral assumptions, and thus distinction between resident's and non-resident's perspectives, do not affect the validity of superlative indices; as averages of these bounds it does not matter which direction they take.

18.13 Sections D2 and D2 respectively demonstrate how Fisher and Törnqvist price indices can be justified as appropriate export price index number formulas using economic theory. Section E outlines the justification for superlative export price index number formulas and Section F adapts considers the economic theory of import price index number formulas.

C Setting the Stage

C.1 The production accounts of *SNA 1993*

18.14 In the remainder of this chapter, the resident approach will be pursued, though as noted above in Table 18.1, there is an immediately apparent correspondence to the results from this perspective and those from the non-resident's perspective, though details are available in Dridi and Zieschang (2004). Establishments undertake the basics of international goods and services trade but, as Chapter 15 notes, households may undertake trade for final consumption in the form of cross-border shopping and of rentals of housing units and general government units also undertake international procurements and asset sales.[1] The economic approach to the XMPIs thus begins not at the industry or institutional sector level, but at the *establishment* and *household* level. Readers of the *Producer Price Index (PPI) Manual* will note in its Chapter 17 a parallel approach to the economic index number theory of input and output price indices for the part of trade flows establishments undertake. Similarly, readers of the *Consumer Price Index (CPI) Manual* will note in its Chapter 17 a parallelism with the economic theory of price indices for consumption for the part of trade flows households undertake. As outlined in Chapter 15, in order to provide a coherent theoretical framework for implementing the resident approach, it turns out that the main production accounts in *SNA 1993* require some elaboration.

18.15 There are a number of reasons why the main production accounts in the *SNA 1993* require some modifications so that the modified accounts can provide a theoretical framework for export and import price indices. The main reason is that exports and imports enter the main supply and use tables (Table 15.1) as additions (or subtractions) to total net supply or to total domestic final demand in the familiar C+I+G+X-M setup, where C and G are household and government final consumption expenditures, I gross capital formation, X exports, and M imports. This means that Table 15.1 in the main production accounts of the *SNA 1993* does not elaborate on which industries are actually

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[1] The establishments, of course, may be owned or controlled by units in any institutional sector: nonfinancial and financial corporations, general government, households, and nonprofit institutions serving households (NIPISHs). By definition, establishments, including those owned by general government, households, and NPISHs, combine nonfinancial assets and intermediate consumption to produce output, and they can engage in capital formation, but do not make final consumption expenditures. The "use of income accounts" of these non-corporation institutional units thus includes not only the capital formation expenditure these non-corporations have made, but also final consumption expenditure. This chapter distinguishes between the international trade non-corporation institutional units undertake for their own intermediate consumption and capital formation, and the international trade they undertake for final consumption.

using the imports or on which industries are actually doing the exporting by commodity.[2] Hence, the main additions to the *SNA 1993* Chapter 15 for XMPI Manual purposes are to add tables to the main production accounts that provide industry by commodity detail on exports and imports. With these additional tables on the industry by commodity allocation of exports and imports, the resident's approach to collecting export and import price indexes can be imbedded in the SNA framework.

18.16 A second main reason for expanding the existing SNA production accounts is that the present set of accounts does not allow the export and import price indexes to be related to the PPI for gross outputs and the PPI for intermediate imports by industry. Thus for the purposes of this chapter a modification of the SNA production accounts is considered so that export price indexes by industry become subindexes of the gross output PPI for that industry and import price indexes by industry become subindexes of the intermediate input PPI for that industry and thus the augmented accounts provide an integrated approach to all of the price indexes that affect producers. In principle, the way to achieve this reconciliation is conceptually simple: all that is required is a decomposition of the present Supply and Use matrices into two parts for each matrix; one part that lists transactions involving domestic goods and services and one part that lists transactions involving internationally traded goods and services. Of course, this is not going to be as easy in practice as it is conceptually.[3]

18.17 Besides the residency orientation of the accounts and their associated price and volume indices, the *1993 SNA* distinguishes between market and nonmarket goods and services. As noted in Chapter 15, market goods and services are transacted at "economically significant prices," largely covering their cost of production, while nonmarket goods and services are transacted at lower prices, including zero. Non-market goods and services are defined to include both "output produced for own final use" (*SNA* transaction code P.12) and "other non-market output" (*SNA* transaction code P.13). The former is output retained by households for own consumption or by establishments for capital formation, and is not traded internationally and thus of no concern to XMPIs. Other non-market goods and services include those produced by government or non-profit institutions serving households (NPISHs) that are supplied free or at a price of no economic significance. These, for the large part, will be aimed at residents, though government, for example, may have as its output some goods and services that benefit

[2] It should be noted that *SNA 1993* does have a recommended optional Table 15.5 which is exactly suited to our present needs; i.e., this table provides the detail for imports by commodity and by industry. However, *SNA 1993* does not provide a recommendation for a corresponding commodity by industry table for exports.

[3] One detail which is troublesome is that it is necessary to decide when an internationally traded commodity becomes a domestic commodity. In this chapter, the transportation industry is treated as a margin industry and it will be assumed that this industry does not transform an imported good into a domestic good; it is the industry to which the good was delivered that counts as the industry which transforms the imported good into a domestic good. A similar treatment could be extended to the retailing and wholesaling industries; i.e., an imported good that is held as a retail or wholesale inventory item could be regarded as an import into the purchasing industry or household. However, this treatment of retailing and wholesaling as margin industries is not recommended because of difficulties that arise if there are inventory accumulations; i.e., the retailer may be adding a certain amount of domestic value to the imported good by making it available to purchasers at a convenient time.

nonresidents and these will be exports. Because the share of non-market output in exports and imports is generally small, our theory focuses on market output. For any relevant non-market output the *SNA* values production by imputing the prices of similarly dated market transactions in comparable goods and services.

18.18 As foreshadowed at the beginning of this section, the XMPIs cover both household and establishment units in both their production and consumption activities. XMPIs thus comprise subindices of the output, intermediate consumption, final consumption, and capital formation price indices of units resident in the economic territory. Our theory of international trade price indices thus is a theory of international trade sub-indices of the CPI and PPI, as well as the other price indices for the supply and use of goods and services mentioned in Chapter 15.

18.19 *Production* is an activity that transforms or combines material inputs and services into other material outputs (for example, agricultural, mining, manufacturing, or construction activities) and services, including transportation of materials from one location to another. Production also includes storage activities, which in effect transport materials in the same location from one time period to another, as well as all types of other services.[4] Production occurs in *establishments*. An establishment is an economic entity that undertakes *production* or *productive activity* at a specific geographic location in the country and is capable of providing basic accounting information on the prices and quantities of the outputs it produces, and on the inputs it uses during an accounting period.

C.2 The price data

18.20 There are two major problems with making the definition of an establishment operational. The first is that many production units at specific geographic locations do not have the capability of providing basic accounting information on inputs used and outputs produced. These production units may be only a division or single plant of a large firm, and detailed accounting information on prices may be available only at the head office (or not at all). If this is the case, the definition of an establishment is modified to include production units at a number of specific geographic locations in the country instead of just one location. The important aspect of the definition of an establishment is that it be able to provide accounting information on prices and quantities.[5] A second problem is that while the establishment may be able to report accurate quantity information, its price information may be based on *transfer prices* set by a head office. These transfer prices are *imputed prices* and may not be very closely related to market prices.[6] Potentially large

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[4]See Hill (1999) for a taxonomy for services.

[5]In this modified definition of an establishment, it is generally a smaller collection of production units than a *firm* since a firm may be multinational. Thus, another way of defining an establishment for our practical purposes is as follows: an establishment is the smallest aggregate of national production units able to provide accounting information on its inputs and outputs for the time period under consideration.

[6]For many highly specialized intermediate inputs in a multistage production process using proprietary technologies, market prices may simply not exist. Furthermore, several alternative concepts could be used to define transfer prices; see Diewert (1985) and Eden (1998), and Chapter 18 of this *Manual*. The *SNA* 1993 (6.82) notes that for deliveries between establishments belonging to the same enterprise: "Goods and services that one establishment provides to a different establishment belonging to the same enterprise are

shares of international trade occur between related enterprises resident in different countries at such transfer prices. This problem is deferred until Chapter 19, which addresses the issue squarely.

18.21 Thus the problems involved in obtaining the correct commodity prices for establishments are generally more difficult than the corresponding problems associated with obtaining market prices for households. However, in this chapter, these problems will be ignored for the most part, and it will be assumed that representative market prices are available for each output produced by an establishment and for each intermediate input used by the same establishment for at least two accounting periods.[7] Price indices for the supply aggregates of goods and services (output price indices) follow valuation at basic prices, which is what the producer would receive for output excluding taxes on products and including subsidies on products.

18.22 For price indices of the aggregates for the establishment and household users of goods and services (input price indices), the economic approach to price indices requires that input prices follow valuation at purchasers' prices, adding taxes on products to, and subtracting subsidies on, products from the basic prices producers receive. The indirect taxes are included because users pay them, even though the producing establishments may collect them for government. The subsidies on products are excluded because the cost of goods and services purchased by establishments and households is lowered by these payments. Chapter 15 and section B in this chapter considers in more detail these national accounting and microeconomic conventions on the treatment of indirect taxes and subsidies on production.

18.23 In this chapter, an *export price index* and an *import price index* will be defined for a *single establishment* or *household* from the economic perspective of a producer in sections D and F. Household import and export price indices will be defined in section F.2.

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counted as part of the output of the producing establishment. Such goods and services may be used for intermediate consumption by the receiving establishment, but they also could be used for gross fixed capital formation. The goods and services should be valued by the producing establishment at current basic prices; the receiving establishment should value them at the same prices plus any additional transportation costs paid to third parties. The use of artificial transfer prices employed for internal accounting purposes within the enterprise should be avoided, if possible." The difficulties in ascertaining such prices are recognized however: "From an accounting point of view it can be difficult to partition a vertically integrated enterprise into establishments because values have to be imputed for the outputs from the earlier stages of production which are not actually sold on the market and which become intermediate inputs into later stages. Some of these enterprises may record the intra-enterprise deliveries at prices that reflect market values, but others may not. Even if adequate data are available on the costs incurred at each stage of production, it may be difficult to decide what is the appropriate way in which to allocate the operating surplus of the enterprise among the various stages. One possibility is that a uniform rate of profit could be applied to the costs incurred at each stage." (*SNA 1993*, 5.33).

[7]These pricing problems are pursued in Chapter 6, where the concept of a market price for each product produced by an establishment during the accounting period under consideration is the value of production for that product divided by the quantity produced during that period; that is, the price is the average price for that product. There are also practical difficulties in separating domestic transport costs out of the prices of imported goods and services.

18.24 Note it is assumed that the list of commodities produced by the establishment and the list of inputs used by the establishment *remains the same* over the two periods of a price comparison. In real life, the list of commodities used and produced by an establishment does not remain constant over time. New commodities appear and old commodities disappear. The reasons for this churning of commodities include the following:

(i) Producers substitute new technologies for older ones that may reduce the prices of exiting varieties, but may also enable some new varieties to be (technologically and/or economically) feasible and some old ones to be no longer so. Such "technologies" may involve new capital formation, a change in the way production is organized, and/or a change in the primary and intermediate inputs used to generate the outputs. The introduction of new technologies may be in response to changes in relative prices, households' tastes, or strategic marketing.

(ii) Existing processes are sufficiently flexible to produce newly differentiated varieties in addition to, or as a replacement for, existing varieties. The introduction of new varieties may be in response to changes in relative prices, households' tastes, or strategic marketing.

(iii) Seasonal fluctuations in the demand (or supply) of commodities cause some commodities to be unavailable in certain periods of the year.

The introduction of new commodities or different varieties of existing ones is dealt with in Chapters 8, 9 and 22 and the problems associated with seasonal commodities in Chapter 23. In the present chapter, these complications are ignored, and it is assumed that the list of commodities remains the *same* over the two periods under consideration. It also will be assumed that all establishments are present in both periods under consideration; that is, there are no new or disappearing establishments.[8]

18.25 When convenient, the notation will be simplified to match the notation used in Chapters 16 and 17.

[8]Rowe (1927, pp. 174–75) was one of the first economists to appreciate the difficulties statisticians faced when attempting to construct price or quantity indices of production: "In the construction of an index of production there are three inherent difficulties which, inasmuch as they are almost insurmountable, impose on the accuracy of the index, limitations, which under certain circumstances may be somewhat serious. The first is that many of the products of industry are not capable of quantitative measurement. This difficulty appears in its most serious form in the case of the engineering industry. ... The second inherent difficulty is that the output of an industry, even when quantitatively measurable, may over a series of years change qualitatively as well as quantitatively. Thus during the last twenty years there has almost certainly been a tendency towards an improvement in the average quality of the yarn and cloth produced by the cotton industry .... The third inherent difficulty lies in the inclusion of new industries which develop importance as the years go on." These three difficulties still exist today: think of the difficulties involved in measuring the outputs of the insurance and gambling industries; an increasing number of industries produce outputs that are one of a kind, and, hence, price and quantity comparisons are necessarily difficult if not impossible; and, finally, the huge increases in research and development expenditures by firms and governments have led to ever increasing numbers of new products and industries. Chapter 8 considers the issues for index compilation arising from new and disappearing goods and services, as well as establishments.

18.26 To most practitioners in the field, our basic framework, which assumes that detailed price and quantity data are available for each of the possibly millions of establishments in the economy, will seem to be utterly unrealistic. However, two answers can be directed at this very valid criticism:

- The spread of the computer and the ease of storing transaction data have made the assumption that the statistical agency has access to detailed price and quantity data less unrealistic. With the cooperation of businesses, it is now possible to calculate price and quantity indices of the type studied in Chapters 16 and 17 using very detailed data on prices and quantities.[9]

- Even if it is not realistic to expect to obtain detailed price and quantity data for every transaction made by every establishment in the economy on a monthly or quarterly basis, it is still necessary to accurately specify the *universe* of transactions in the economy. Once the target universe is known, sampling techniques can be applied in order to reduce data requirements. The principles and practice of sampling establishments for XMPIs were outlined in Chapter 6

C.3 An Overview of the Chapter

18.27 This subsection gives a brief overview of the contents of this chapter. In section Chapters 15 (and 20) the present system of production accounts in the *System of National Accounts 1993* was extended to accommodate exports and imports in the resident framework. With this expanded system of production accounts in hand, in section D, economic approaches to the *export price index* for a single establishment are developed. This theory is basically an adaptation of the theory of the *output price index* due to Fisher and Shell (1972) and Archibald (1977) and it follows closely the exposition of the export price index made by Alterman, Diewert and Feenstra (1999). Section E follows up on this material by Diewert's (1976) theory of *superlative indices*. A superlative index can be evaluated using observable price and quantity data, but under certain conditions it can give exactly the same answer as the theoretical output price index. Section F.1 presents an economic approach to an *import price index* for a single establishment. This theory is again due to Alterman, Diewert and Feenstra (1999). It can also be regarded as an adaptation of the theory of the *intermediate input price index* for an establishment that was developed in Chapter 17 of the *Producer Price Index Manual* and in fact, the establishment import price index can simply be regarded as a subindex of the entire intermediate input price index for an establishment, using the expanded system of production accounts that will be explained in section B below. Section F.2 concludes by developing an economic approach to the *household import price index* for imported goods and services that do not pass through the domestic production sector. This theory is an adaptation of the standard *cost of living index* theory that originated with Konüs (1924) and may be found in Chapter 17 of the *Consumer Price Index Manual*.[10] Thus the

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[9]An early study that computed Fisher ideal indices for a distribution firm in western Canada for seven quarters aggregating over 76,000 inventory items is found in Diewert and Smith (1994).
[10] See the International Labour Office (2004).

theories of the export and import price indexes that will be developed in sections C through E are substantially the same as corresponding theories developed in the *Producer Price Index Manual* and *Consumer Price Index Manual*.

18.28 In the previous two chapters, the Fisher (1922) ideal price index and the Törnqvist (1936) price index emerged as very good choices because they are supported by both the test and stochastic approaches to index number theory. These two indices also will emerge as very good choices from the economic perspective. However, a practical drawback to their use is that current-period information on quantities is required, information that the statistical agency will usually not have on a current period basis. An important recommendation of the manual will be that if responding establishments can provide current period quantity data in a timely manner, they should be used to enable the compilation of such indices`

D The Export Price Index for a Single Establishment

D.1 The Export Price Index and Observable Bounds

18.29 In this subsection, an outline of the theory of the export price index is presented for a single establishment. This theory was developed by Alterman, Diewert and Feenstra (1999; 10-16), which in turn was based on the theory of the output price index developed by Fisher and Shell (1972) and Archibald (1977). This theory is the producer theory counterpart to the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). These economic approaches to price indices rely on the assumption of (competitive) *optimizing behavior* on the part of economic agents (consumers or producers). Thus in the case of the export price index, given a vector of output or export prices that the agent faces in a given time period t, it is assumed that the corresponding period t quantity vector is the solution to a revenue maximization problem that involves the producer's production function f or production possibilities set. The export price index considered in this section ids defined using the theory of the producer and is referred to as an "export (output) price index to reinforce the fact that the approach takes a resident producer's perspective.

In contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two export quantity vectors pertaining to periods 0 and 1 are independent of the corresponding two export price vectors. In the economic approach, the period 0 export quantity vector is determined by the producer's period 0 production function and the period 0 vector of export prices that the producer faces and the period 1 export quantity vector is determined by the producer's period 1 production function and the period 1 vector of export prices.

18.30 Before the export price index is defined for an establishment, it is first necessary to describe the establishment's technology in period t. In the economics literature, it is traditional to describe the technology of a firm or industry in terms of a production function, which tells us what the maximum amount of output that can be produced using a given vector of inputs. However, since most establishments produce more than one output, it is more convenient to describe the establishment's technology in period t by means of a *production possibilities set*, S^t. The set S^t describes what output vectors [y,x] can be produced in period t if the establishment has at its disposal the vector of inputs [z,m,v] where y is a vector of domestic outputs produced by the establishment, x is a vector of exports produced by the establishment, z is a vector of domestic intermediate inputs utilized by the establishment, m is a vector of imported intermediate inputs utilized by the establishment and v is a vector of primary inputs utilized by the establishment. Thus if [y,x,z,m,v] belongs to S^t, then the nonnegative output vectors y and z can be produced by the establishment in period t if it can utilize the nonnegative vectors z, m and v of inputs. Note the relationship of this establishment production structure with the industrial structure that was explained in section B above; the only differences are that primary inputs are now introduced into the establishment production possibilities sets and establishments have replaced industries.

18.31 Let $p_x \equiv (p_{x1},\ldots p_{xN})$ denote a vector of positive export prices that the establishment might face in period t[11] and let y be a vector of domestic outputs that the establishment is asked to produce, z be a vector of domestic intermediate inputs that the establishment has available during the period, m be a vector of imports that the establishment can utilize during the period and v be a vector of primary inputs that are available to the establishment. Then the establishment's *conditional export revenue function* using period t technology is defined as the solution to the following revenue maximization problem:

$$(18.28)\quad R^t(p_x,y,z,m,v) \equiv \max_x \{\sum_{n=1}^{N} p_{xn}x_n : x \equiv (x_1,\ldots,x_N) \text{ and } (y,x,z,m,v) \text{ belongs to } S^t\}.$$

Thus $R^t(p_x,y,z,m,v)$ is the maximum value of exports, $p_x \cdot x \equiv \sum_{n=1}^{N} p_{xn}x_n$, that the establishment can produce, given that it faces the vector of export prices p and is given the vector y of domestic output targets to produce and given that the input vectors z,m and v are available for use, using the period t technology.[12] Note that the export revenue function is conditioned on domestic export targets being given. This has the merit of allowing the behavioral assumption of exports revenue maximization to be invoked, and economic export output indices to be defined, without confounding the theory with substitution effects between the domestic and foreign markets. The reader must, however, bear in mind that this is also a limitation of the theory.

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[11] Depending on the context, these export prices may be either the per unit amounts that foreign demanders pay to the establishment or these prices may be adjusted for commodity tax or subsidy payments as in section B.

[12] The function R^t is closely related to the *GDP function* or the *national product function* in the international trade literature; see Kohli (1978)(1991) or Woodland (1982). It was introduced into the economics literature by Samuelson (1953). Alternative terms for this function include: (i) the *gross profit function*; see Gorman (1968); (ii) the *restricted profit function*; see Lau (1976) and McFadden (1978); and (iii) the *variable profit function*; see Diewert (1973) (1974a). The mathematical properties of the conditional revenue function are laid out in these references.

18.32 The period t revenue function R^t can be used to define the establishment's *period t technology export output price index* P^t between any two periods, say period 0 and period 1, as follows:

$$(18.29)\quad P^t(p_x^0,p_x^1,y,z,m,v) = R^t(p_x^1,y,z,m,v)/R^t(p_x^0,y,z,m,v)$$

where p_x^0 and p_x^1 are the vectors of export prices that the establishment faces in periods 0 and 1 respectively and y, z, m and v are reference vectors of domestic outputs, domestic intermediate inputs, imports and primary inputs respectively.[13] If N = 1 so that there is only one output that the establishment produces, then it can be shown that the output price index collapses down to the single output price relative between periods 0 and 1, p_{x1}^1/p_{x1}^0. In the general case, note that the output price index defined by(18.29) is a ratio of hypothetical revenues that the establishment could realize, given that it has the period t technology, the set of domestic output targets y and the vectors of inputs z,m and v to work with. The numerator in(18.29) is the maximum export revenue that the establishment could attain if it faced the output prices of period 1, p_x^1, while the denominator in(18.29) is the maximum export revenue that the establishment could attain if it faced the export prices of period 0, p_x^0. Note that all of the variables in the numerator and denominator functions are exactly the same, except that the export price vectors differ. This is a defining characteristic of an economic price index: all environmental variables are held constant with the exception of the prices in the domain of definition of the price index.

18.33 Note that there are a wide variety of price indices of the form(18.29) depending on which reference technology t and reference input vector v that is chosen. Thus there is not a single economic price index of the type defined by(18.29): there is an entire *family* of indices.

18.34 In order to simplify the notation in what follows, define the composite vector of *reference quantities* u as follows:

$$(18.30)\quad u \equiv (y,z,m,v).$$

As an additional notational simplification, let p^t denote the vector of export prices, p_x^t, for periods t = 0,1.

18.35 Usually, interest lies in two special cases of the general definition of the export price index(18.29): (i) $P^0(p^0,p^1,u^0)$ which uses the period 0 technology set and the

[13] This concept of the export price index was defined in Alterman, Diewert and Feenstra (1999; 10-13) and it is closely related to output price indices defined by Fisher and Shell (1972; 56-58), Samuelson and Swamy (1974; 588-592), Archibald (1977; 60-61), Diewert (1980; 460-461) (1983; 1055) and Balk (1998; 83-89). Readers who are familiar with the theory of the true cost of living index will note that the output price index defined by(18.29) is analogous to the *true cost of living index* which is a ratio of cost functions, say $C(p^1,u)/C(p^0,u)$ where u is a reference utility level: R replaces C and the reference utility level u is replaced by the vector of reference variables (t,y,z,m,v). The optimizing behavior for the cost of living index is one of minimization while that for the export output price index is revenue maximization. For references to the theory of the true cost of living index, see Konüs (1924), Pollak (1983) or the CPI counterpart to this manual, ILO *et al.* (2004).

reference quantity vector $u^0 \equiv (y^0,z^0,m^0,v^0)$ that was actually produced and used by the establishment in period 0 and (ii) $P^1(p^0,p^1,u^1)$ which uses the period 1 technology set and the reference quantity vector $u^1 \equiv (y^0,z^0,m^0,v^0)$ that was actually produced and used by the establishment in period 1. Let x^0 and x^1 be the observed export vectors for the establishment in periods 0 and 1 respectively. If there is revenue maximizing behavior on the part of the establishment in periods 0 and 1, then observed revenue in periods 0 and 1 should be equal to $R^0(p^0,u^0)$ and $R^1(p^1,u^1)$ respectively; i.e., the following equalities should hold:

$$(18.31)\quad R^0(p^0,u^0) = \sum_{n=1}^{N} p_n^0 x_n^0 \qquad \text{and} \qquad R^1(p^1,u^1) = \sum_{n=1}^{N} p_n^1 x_n^1 .$$

18.36 Under these revenue maximizing assumptions, Alterman, Diewert and Feenstra (1999; 11), adapting the arguments of Fisher and Shell (1972; 57-58) and Archibald (1977; 66), have shown that the two theoretical indices, $P^0(p^0,p^1,u^0)$ and $P^1(p^0,p^1,u^1)$ described in (i) and (ii) above, satisfy the following inequalities(18.32) and(18.33):

$$\begin{aligned}
(18.32)\quad P^0(p^0,p^1,u^0) &\equiv R^0(p^1,u^0)/R^0(p^0,u^0) && \text{using definition(18.29)}\\
&= R^0(p^1,u^0)/\sum_{n=1}^{N} p_n^0 x_n^0 && \text{using(18.31)}\\
&\geq \sum_{n=1}^{N} p_n^1 x_n^0 / \sum_{n=1}^{N} p_n^0 x_n^0 && \text{since } x^0 \text{ is feasible for the maximization}\\
&\qquad \text{problem which defines } R^0(p^1,u^0) \text{ and so } R^0(p^1,u^0) \geq \sum_{n=1}^{N} p_n^1 x_n^0\\
&\equiv P_L(p^0,p^1,x^0,x^1)
\end{aligned}$$

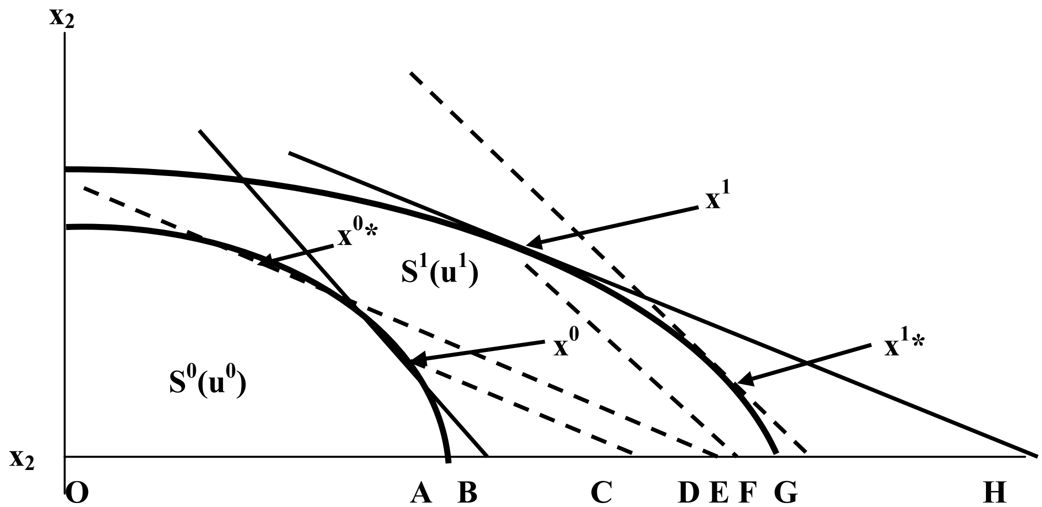
where P_L is the Laspeyres (1871) price index. Similarly,:

$$\begin{aligned}
(18.33)\quad P^1(p^0,p^1,u^1) &\equiv R^1(p^1,u^1)/R^1(p^0,u^1) && \text{using definition(18.29)}\\
&= \sum_{n=1}^{N} p_n^1 x_n^1 / R^1(p^0,u^1) && \text{using(18.31)}\\
&\leq \sum_{n=1}^{N} p_n^1 x_n^1 / \textstyle\sum_{n=1}^{N} p_n^0 x_n^1 && \text{since } x^1 \text{ is feasible for the maximization}\\
&\qquad \text{problem which defines } R^1(p^0,u^1) \text{ and so } R^1(p^0,u^1) \geq \sum_{n=1}^{N} p_n^0 x_n^1\\
&\equiv P_P(p^0,p^1,x^0,x^1)
\end{aligned}$$

where P_P is the Paasche (1874) price index. Thus the inequality(18.32) says that the observable Laspeyres index of output prices P_L is a *lower bound* to the theoretical export output price index $P^0(p^0,p^1,u^0)$ and the inequality(18.33) says that the observable Paasche index of export output prices P_P is an *upper bound* to the theoretical export output price index $P^1(p^0,p^1,u^1)$. Note that these inequalities are in the *opposite direction* compared to their counterparts in the theory of the true cost of living index.[14]

18.37 It is possible to illustrate the two inequalities(18.31) and(18.32) if there are only two commodities; see Figure 1 below, which is based on diagrams due to Hicks (1940; 120) and Fisher and Shell (1972; 57).

Figure 1: The Laspeyres and Paasche bounds to the output price index



18.38 First the inequality(18.32) is illustrated for the case of two exports that are both produced in both periods. The solution to the period 0 export revenue maximization problem is the period 0 export vector x^0 and the straight line through B represents the revenue line that is just tangent to the period 0 export production possibilities set, $S^0(u^0) \equiv \{(x_1,x_2,u^0)$ belongs to $S^0\}$. The curved line through x^0 and A is the frontier to the producer's period 0 export production possibilities set $S^0(u^0)$. The solution to the period 1 revenue maximization problem is the vector x^1 and the straight line through H represents the export revenue line that is just tangent to the period 1 export production possibilities set, $S^1(u^1) \equiv \{(x_1,x_2,u^1)$ belongs to $S^1\}$. The curved line through x^1 and F is the frontier to the producer's period 1 export production possibilities set $S^1(u^1)$. The point x^{0*} solves the hypothetical maximization problem of maximizing export revenues when facing the period 1 price vector $p^1 = (p_1^1,p_2^1)$ but using the period 0 technology and reference quantity vector u^0. This hypothetical export revenue is given by $R^0(p^1,u^0) = p_1^1x_1^{0*} + p_2^1x_2^{0*}$ and the dashed line through D is the corresponding isorevenue line $p_1^1x_1 + p_2^1x_2 = R^0(p^1,u^0)$. Note that the hypothetical export revenue line through D is parallel to the actual period 1 revenue line through H. From(18.32), the hypothetical export price index, $P^0(p^0,p^1,u^0)$, is $R^0(p^1,u^0)/[p_1^0x_1^0 + p_2^0x_2^0]$ while the ordinary Laspeyres export price index

[14] This is due to the fact that the optimization problem in the cost of living theory is a cost *minimization* problem as opposed to our present revenue *maximization* problem. The method of proof used to derive(18.32) and(18.33) dates back to Konüs (1924), Hicks (1940) and Samuelson (1950).

is $[p_1^1x_1^0 + p_2^1x_2^0]/[p_1^0x_1^0 + p_2^0x_2^0]$. Since the denominators for these two indices are the same, the difference between the indices is due to the differences in their numerators. In Figure 1, this difference in the numerators is expressed by the fact that the revenue line through C lies *below* the parallel revenue line through D. Now if the producer's period 0 export production possibilities set were block shaped with vertex at x^0, then the producer would not change his or her production pattern in response to a change in the relative export prices of the two commodities while using the period 0 technology and inputs. In this case, the hypothetical vector x^{0*} would coincide with x^0, the dashed line through D would coincide with the dashed line through C and the true export price index $P^0(p^0,p^1,u^0)$, would *coincide* with the ordinary Laspeyres export price index. However, block shaped production possibilities sets are not generally consistent with producer behavior; i.e., when the price of a commodity increases, producers generally supply more of it. Thus in the general case, there will be a gap between the points C and D. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *less* than the corresponding true export price index, $P^0(p^0,p^1,u^0)$.

18.39 Figure 1 can also be used to illustrate the inequality(18.33) for the two export case. Note that technical progress or increases in input availability have caused the period 1 export production possibilities set $S^1(u^1) \equiv \{(x_1,x_2) : (x_1,x_2,u^1) \text{ belongs to } S^1\}$ to be much bigger than the corresponding period 0 export production possibilities set $S^0(u^0) \equiv \{(x_1,x_2) : (x_1,x_2,u^0) \text{ belongs to } S^0\}$.[15] Secondly, note that the dashed lines through E and G are parallel to the period 0 isorevenue line through B. The point x^{1*} solves the hypothetical revenue maximization problem of maximizing export revenue using the period 1 technology and inputs when facing the period 0 export price vector $p^0 = (p_1^0,p_2^0)$. This is given by $R^1(p^0,u^1) = p_1^0x_1^{1*} + p_2^0x_2^{1*}$ and the dashed line through G is the corresponding isorevenue line $p_1^1x_1 + p_2^1x_2 = R^1(p^0,u^1)$. From(18.33), the theoretical export price index using the period 1 technology and inputs is $[p_1^1x_1^1 + p_2^1x_2^1]/R^1(p^0,u^1)$ while the ordinary Paasche export price index is $[p_1^1x_1^1 + p_2^1x_2^1]/[p_1^0x_1^1 + p_2^0x_2^1]$. Since the numerators for these two indices are the same, the difference between the indices is due to the differences in their denominators. In Figure 1, this difference in the denominators is expressed by the fact that the revenue line through E lies *below* the parallel cost line through G. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *greater* than the corresponding true export price index using current period technology and inputs, $P^1(p^0,p^1,u^1)$. Note that this inequality goes in the opposite direction to the previous inequality,(18.32). The reason for this change in direction is due to the fact that one set of differences between the two indices takes place in the numerators of the two indices (the Laspeyres inequalities) while the other set takes place in the denominators of the two indices (the Paasche inequalities).

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[15] However, validity of the inequality(18.33) does not depend on the relative position of the two output production possibilities sets. To obtain the strict inequality version of(18.33), it is necessary that two conditions be satisfied: (i) the frontier of the period 1 output production possibilities set needs to be "curved" and (ii) relative output prices must change going from period 0 to 1 so that the two price lines through G and H in Figure 1 are tangent to *different* points on the frontier of the period 1 output production possibilities set.

18.40 There are two problems with the inequalities(18.32) and(18.33):

- There are *two* equally valid economic price indices, $P^0(p^0,p^1,u^0)$ and $P^1(p^0,p^1,u^1)$, that could be used to describe the amount of price change that took place between periods 0 and 1 whereas the public will demand that the statistical agency produce a *single* estimate of price change between the two periods.
- Only *one sided* observable bounds to these two theoretical price indices[16] result from this analysis and what are required for most practical purposes are *two sided* bounds.

In the following subsection, it will be shown how a possible solution to these two problems can be found.

D.2 The Fisher Ideal Index as an Approximation to an Economic Export Output Price Index

18.41 It is possible to define a theoretical export price index that falls *between* the observable Paasche and Laspeyres export price indices. To do this, first define a hypothetical export revenue function, $R(p,\alpha)$, that corresponds to the use of an α weighted average of the technology sets S^0 and S^1 for periods 0 and 1 as the reference technology and that uses an α weighted average of the period 0 and period 1 reference input and export output vectors u^0 and u^1 as the reference quantity vector:

$$(18.34)\quad R(p,\alpha) \equiv \max_x \{\sum_{n=1}^{N} p_n x_n : (x,(1-\alpha)u^0 + \alpha u^1) \text{ belongs to } (1-\alpha)S^0 + \alpha S^1\}.$$

Thus the revenue maximization problem in(18.34) corresponds to the use of a weighted average of the period 0 and 1 reference quantity vectors u^0 and u^1 where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α and an "average" is used of the period 0 and period 1 technology sets where the period 0 set gets the weight $1-\alpha$ and the period 1 set gets the weight α , where α is a number between 0 and 1.[17] The meaning of the weighted average technology set in definition(18.34) can be explained in terms of Figure 1 as follows. As α changes continuously from 0 to 1, the export output production possibilities set changes in a continuous manner from the set $S^0(u^0)$ (whose frontier is the curve which ends in the point A) to the set $S^1(u^1)$ (whose frontier is the curve which ends in the point F). Thus for any α between 0 and 1, a hypothetical establishment export output production possibilities set is obtained which lies between the base period set $S^0(u^0)$ and the current period set $S^1(u^1)$. For each α, this hypothetical output production possibilities set can be used as the constraint set for a theoretical export output price index.

[16]The Laspeyres export price index is a lower bound to the theoretical index $P^0(p^0,p^1,u^0)$ while the Paasche output price index is an upper bound to the theoretical index $P^1(p^0,p^1,u^1)$.
[17] When α=0, $R(p,0) = R^0(p,u^0)$ and when $\alpha = 1$, $R(p,1) = R^1(p,u^1)$.

18.42 The new revenue function defined by(18.34) is now used in order to define the following family (indexed by α) of theoretical net export output price indices:

(18.35) $P(p^0,p^1,\alpha) \equiv R(p^1,\alpha)/R(p^0,\alpha)$.

The important advantage that theoretical export output price indices of the form defined by(18.29) or(18.35) have over the traditional Laspeyres and Paasche export output price indices P_L and P_P is that these theoretical indices deal adequately with *substitution effects*; i.e., when an export output price increases, the producer supply should increase, holding inputs and the technology constant.[18]

18.43 Diewert (1983; 1060-1061) showed that, under certain conditions[19], there exists an α between 0 and 1 such that the theoretical export output price index defined by(18.35) lies between the observable (in principle) Paasche and Laspeyres export output indices, P_P and P_L ; i.e., there exists an α such that

(18.36) $P_L \le P(p^0,p^1,\alpha) \le P_P$ or $P_P \le P(p^0,p^1,\alpha) \le P_L$.

18.44 The fact that the Paasche and Laspeyres export output price indices provide upper and lower bounds to a "true" export output price $P(p^0,p^1,\alpha)$ in(18.35) is a more useful and important result than the one sided bounds on the "true" indices that were derived in(18.32) and(18.33) above. If the observable (in principle) Paasche and Laspeyres indices are not too far apart, then taking a symmetric average of these indices should provide a good approximation to an economic export output price index where the reference technology is somewhere between the base and current period technologies. The precise symmetric average of the Paasche and Laspeyres indices was determined in section C.1 of Chapter 16 above on axiomatic grounds and led to the geometric mean, the Fisher price index, P_F:

(18.37) $P_F(p^0,p^1,x^0,x^1) \equiv [P_L(p^0,p^1,x^0,x^1)\ P_P(p^0,p^1,x^0,x^1)]^{1/2}$.

[18] This is a normal output substitution effect. However, empirically, it will often happen that observed period to period decreases in price are not accompanied by corresponding decreases in supply. However, these abnormal "substitution" effects can be rationalized as the effects of technological progress. For example, suppose the price of computer chips decreases substantially from period 0 to 1. If the technology were constant over these two periods, we would expect domestic producers to decrease their supply of chips going from period 0 to 1. In actual fact, the opposite happens. The fall in price is driven by technological progress arising from a reduction in the cost of producing chips which is passed on to demanders of chips. Thus the effects of technological progress should not be ignored in the theory of the output price index. The counterpart to technological change in the theory of the cost of living index is taste change, which is often ignored!

[19] Diewert adapted a method of proof due originally to Konüs (1924) in the consumer context. Sufficient conditions on the period 0 and 1 technology sets for the result to hold are given in Diewert (1983; 1105). Our exposition of the material in sections B.2, B.3 and C.1 also draws on Chapter 2 in Alterman, Diewert and Feenstra (1999).

Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical export output price index.[20]

18.45 The bounds given by(18.32),(18.33) and(18.36) are the best bounds that can be obtained on economic export output price indices without making further assumptions. In the next subsection, further assumptions are made on the two technology sets S^0 and S^1 or equivalently, on the two revenue functions, $R^0(p,u)$ and $R^1(p,u)$. With these extra assumptions, it is possible to determine the geometric average of the two theoretical export output price indices that are of primary interest, $P^0(p^0,p^1,u^0)$ and $P^1(p^0,p^1,u^1)$.

D.3 The Törnqvist Index as an Approximation to an Economic Export Output Price Index

18.46 An alternative to the Laspeyres and Paasche or the Fisher index defined by(18.37) above is to use the Törnqvist (1936)(1937) Theil (1967) price index P_T, whose natural logarithm is defined as follows:

$$(18.38)\quad \ln P_T(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) \ln (p_n^1/p_n^0)$$

where $s_n^t \equiv p_n^t x_n^t / \sum_{j=1}^{N} p_j^t x_j^t$ is the revenue share of commodity n in the total value of export sales in period t.

18.47 Recall the definition of the period t revenue function, $R^t(p,u)$, defined earlier by(18.28) above. Now assume that the period t revenue function has the following *translog functional form*[21] : for t = 0,1:[22]

$$(18.39)\ \ln R^t(p,u) = \alpha_0^t + \sum_{n=1}^{N} \alpha_n^t \ln p_n + \sum_{m=1}^{M} \beta_m^t \ln u_m + (1/2)\sum_{n=1}^{N} \sum_{j=1}^{N} \alpha_{nj}^t \ln p_n \ln p_j$$
$$+ \sum_{n=1}^{N} \sum_{m=1}^{M} \beta_{nm}^t \ln p_n \ln u_m + (1/2)\sum_{m=1}^{M} \sum_{k=1}^{M} \gamma_{mk}^t \ln u_m \ln u_k$$

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[20] It should be noted that Fisher (1922) constructed Laspeyres, Paasche and Fisher output price indices for his U.S. data set. Fisher also adopted the view that the product of the price and quantity index should equal the value ratio between the two periods under consideration, an idea that he already formulated in Fisher (1911; 403). He did not consider explicitly the problem of deflating value added but by 1930, his ideas on deflation and the measurement of quantity growth being essentially the same problem had spread to the problem of deflating nominal value added; see Burns (1930).

[21] This functional form was introduced and named by Christensen, Jorgenson and Lau (1971). It was adapted to the revenue function or profit function context by Diewert (1974a).

[22] Recall that the vector of reference quantities u was defined by(18.30) and is equal to (y,z,m,v). If the same commodity classification is used for domestically produced goods y, for domestic intermediate inputs z and for imports and if the number of primary inputs v is K, then the u vector will have dimension 3N + K, which we will denote by M.

where the α_n^t coefficients satisfy the restrictions:

(18.40) $\sum_{n=1}^{N} \alpha_n^t = 1 \quad \text{for } t = 0,1$

and the α_{nj}^t and the β_{nm}^t coefficients satisfy the following restrictions:[23]

(18.41) $\sum_{j=1}^{N} \alpha_{nj}^t = 0 \quad \text{for } t = 0,1 \text{ and } n = 1,2,\ldots,N;$

(18.42) $\sum_{n=1}^{N} \beta_{nm}^t = 0 \quad \text{for } t = 0,1 \text{ and } m = 1,2,\ldots,M.$

The restrictions(18.41)-(18.43) are necessary to ensure that $R^t(p,u)$ is linearly homogeneous in the components of the export price vector p, which is a property that a revenue function must satisfy[24]. Note that at this stage of the argument, the coefficients that characterize the technology in each period (the α's, β's and γ's) are allowed to be completely different in each period. It should also be noted that the translog functional form is an example of a *flexible* functional form[25]; i.e., it can approximate an arbitrary technology to the second order.

18.48 A result in Caves, Christensen and Diewert (1982; 1410) can now be adapted to the present context: if the quadratic price coefficients in(18.39) are equal across the two periods of the index number comparison (i.e., $\alpha_{nj}^0 = \alpha_{nj}^1$ for all n,j), then the geometric mean of the economic export price index that uses period 0 technology and the period 0 reference vector u^0, $P^0(p^0,p^1,u^0)$, and the economic export price index that uses period 1 technology and the period 1 reference quantity vector u^1, $P^1(p^0,p^1,u^1)$, is *exactly equal* to the Törnqvist export output price index P_T defined by(18.38) above; i.e.,

(18.43) $P_T(p^0,p^1,x^0,x^1) = [P^0(p^0,p^1,u^0)\, P^1(p^0,p^1,u^1)]^{1/2}.$

where $P^0(p^0,p^1,u^0)$ takes the form here as the period 0 export sales share weighted geometric mean of price relatives and $P^1(p^0,p^1,u^1)$ the period 1 export sales share weighted geometric mean of price relatives. The assumptions required for this result seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and our assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic export output price indices and it corresponds to a flexible functional form, the Törnqvist export output price index number formula is said to be *superlative,* following the terminology used by Diewert (1976).

[23] It is also assumed that the symmetry conditions $\alpha_{nj}^t = \alpha_{jn}^t$ for all n,j and for t = 0,1 and $\gamma_{mk}^t = \gamma_{km}^t$ for all m,k and for t = 0,1 are satisfied.
[24] See Diewert (1973) (1974a) for the regularity conditions that a revenue or profit function must satisfy.
[25] The concept of flexible functional form was introduced by Diewert (1974a; 113).

18.49 For the reader who has read Chapter 17 in the *PPI Manual*, the above economic theories of the export price index for an establishment will seem very similar to the economic approaches to the *gross output price index* that appeared in that Manual. In fact, the theories are exactly the same; only some of the terminology has changed. Also, another way of viewing the establishment export price index is as a *subindex* of a gross output price index that encompasses both domestically produced outputs as well as outputs which are exported. Thus once the commodity by industry production accounts for the SNA are expanded along the lines suggested in Chapter 15 and section B above, the establishment export output price index can be viewed as a subindex of a more complete system of industry by commodity output price indexes.

18.50 In the following section, additional superlative export output price formulae are derived. However, this section concludes with a few words of caution on the applicability of the economic approach to Export Price Indices. The above economic approaches to the theory of export price indices have been based on the assumption that producers take the prices of their exports as given fixed parameters that they cannot affect by their actions. However, a *monopolistic exporter* of a commodity will be well aware that the average price that can be obtained in the market for their commodity will depend on the number of units supplied during the period. Thus under noncompetitive conditions when outputs are monopolistically supplied (or when intermediate inputs are monopsonistically demanded), the economic approach to producer price indices breaks down. The problem of modeling noncompetitive behavior does not arise in the economic approach to consumer price indices because, usually, a single household does not have much control over the prices it faces in the marketplace. The economic approach to producer output price indices can be modified to deal with certain monopolistic situations. The basic idea is due to Frisch (1936; 14-15) and it involves linearizing the demand functions a producer faces in each period around the observed equilibrium points in each period and then calculating shadow prices which replace market prices. Alternatively, one can assume that the producer is a markup monopolist and simply adds a markup or premium to the relevant marginal cost of production.[26] However, in order to implement these techniques, econometric methods will usually have to be employed and hence, these methods are not really suitable for use by statistical agencies, except in very special circumstances when the problem of noncompetitive behavior is thought to be very significant and the agency has access to econometric resources.

18.51 The approach is a conditional one; it is assumed that the output of similar commodities to domestic and foreign markets are independent of changes in the relative prices of these similar commodities between the two markets. A revenue maximizing producer would, for example, shift output to the export market if the price in that market relative to the domestic market, increased. However, the expectation is that such a response may be "sticky", since changes in relative prices may be due to exchange rate changes which may be relatively volatile. Further, there will be costs attached to shifting output between markets, including the loss of customer loyalty.

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[26] See Diewert (1993; 584-590) for a more detailed description of these techniques for modeling monopolistic behavior and for additional references to the literature.

E Superlative Export Output Price Indexes

Section D.2 demonstrated that the Paasche and Laspeyres export output price indices provide upper and lower bounds to a "true" export output price, $P(p^0,p^1,\alpha)$ in(18.35). Given no preference for Laspeyres and Paasche, or their theoretical counterparts $P^0(p^0,p^1,u^0)$ and $P^1(p^0,p^1,u^1)$, a symmetric average of Laspeyres and Paasche was advocated as an approximation to a true index. More particularly, the Fisher price index, as a geometric mean of Laspeyres and Paasche price indices, was justified on the basis of its axiomatic properties which are superior to other symmetric averages. In this section economic theory is used to justify the Fisher index formula as one of a class of superlative index number formulas. An index number is said to be *exact* when it equals its theoretical true counterpart defined for a particular functional form of its reference quantity vector, $u \equiv (y,z,m,v)$. A *superlative* index is defined as a an index that is exact for a flexible functional form that can provide a second-order approximation to other twice-differentiable functions around the same point. Flexible functional forms allow different outputs to be realized in response to relative price changes and thus a realization of a more realistic representation of revenue maximizing behavior: producers substitute away from commodities with below average price increases. To develop an economic theory of superlative indices it is first necessary to outline in Section E.1 separability conditions that allow a aggregate export output price index to be defined. Two results are then required that enable specific functional forms for the aggregator function to be related to specific index number formulas; Wold's Identity and Hotelling's Lemma and these are outlined in Section E.2. Fisher as a superlative index number formula is derived in section E.3 and other superlative formulas in Section E.4, and their properties for two-stage aggregation considered in Section E.5,

E.1 Homogeneous Separability and the Export Output Price Index

18.52 Instead of representing the period t technology by a set S^t, the period t technology is now represented by a *factor requirements function* F^t; i.e., $v_1 = F^t(x,y,z,m,v_2,v_3, \ldots ,v_K)$ is set equal to the minimum amount of primary input 1 that is required in period t in order to produce the vector of exports x and domestic outputs y, given that the vector of imports m and the amounts $v_2, v_3, \ldots ,v_K$ of the remaining primary inputs are available for use. It is assumed that a linearly homogeneous aggregator function f exists for exports; i.e., assume that functions f and G^t exist such that[27]

$$(18.44)\ F^t(x,y,z,m,v_2,v_3, \ldots ,v_K) = G^t(f(x),y,z,m,v_2,v_3, \ldots ,v_K); \quad t = 0,1.$$

In technical terms, period t exports are said to be *homogeneously weakly separable* from the other commodities in the technology.[28] The intuitive meaning of the separability

[27] This method for justifying aggregation over commodities is due to Shephard (1953; 61-71). It is assumed that f(q) is an increasing ,positive and convex function of q for positive q. Samuelson and Swamy (1974) and Diewert (1980; 438-442) also develop this approach to index number theory.
[28] This terminology follows that used by Geary and Morishima (1973). The concept of weak separability dates back to Sono (1945). A survey of separability concepts can be found in Blackorby, Primont and Russell (1978).

assumption that is defined by(18.44) is that an export aggregate $Q \equiv f(x_1,...,x_N)$ *exists*; i.e., a measure of the aggregate contribution to production of the amounts x_1 of the first export, x_2 of the second export, ... , and x_N of the Nth export is the number $Q = f(x_1,x_2,...,x_N)$. Note that it is assumed that the linearly homogeneous output aggregator function f does not depend on t. These assumptions are quite restrictive from the viewpoint of empirical economics[29] but strong assumptions are required in order to obtain the existence of export aggregates from the viewpoint of this variant of economic approach.[30]

18.53 It turns out that the *export aggregator function* f has a corresponding *unit revenue function*, r, defined as follows:

$$\text{(17.45)}\quad r(p) \equiv \max_x \{\sum_{n=1}^{N} p_n x_n : f(x) = 1\}$$

where $p \equiv [p_1,...,p_N]$ and $x \equiv [x_1,...,x_N]$. Thus r(p) is the maximum export revenue that the establishment can make, given that it faces the vector of export prices p and is asked to produce a combination of exports $[x_1,...,x_N] = x$ that will produce a unit level of aggregate exports.[31]

18.54 Let $Q > 0$ be an aggregate level of exports. Then it is straightforward to show that[32]:

$$\text{(18.46)}\quad \max_x \{\sum_{n=1}^{N} p_n x_n : f(x) = Q\} = \max_x \{\sum_{n=1}^{N} p_n x_n : (1/Q)f(x) = 1\}$$

$$= \max_x \{\sum_{n=1}^{N} p_n x_n : f(x/Q) = 1\}$$

using the linear homogeneity of f

$$= Q \max_x \{\sum_{n=1}^{N} p_n x_n/Q : f(x/Q) = 1\}$$

[29] Suppose that in period 0, the vector of inputs v^0 produces the vector of outputs q^0. Our separability assumptions imply that the same vector of inputs v^0 could produce *any* vector of outputs q such that $f(q) = f(q^0)$. In real life, as q varied, we would expect that the corresponding input requirements would also vary instead of remaining fixed.

[30] The assumptions on the technology of the establishment that are made in section D of this Chapter are considerably stronger than the assumptions that were made in section C above, where we made no separability assumptions at all. However, in the previous section, the export aggregates were conditional on a reference vector of quantities u, whereas in the present section, unconditional export aggregates are obtained.

[31] It can be shown that r(p) has the following mathematical properties: r(p) is a nonnegative, nondecreasing, convex and positively linearly homogeneous function for strictly positive p vectors; see Diewert (1974b) or Samuelson and Swamy (1974). A function r(p) is *convex* if for every strictly positive p^1 and p^2 and number λ such that $0 \le \lambda \le 1$, $r(\lambda p^1+(1-\lambda)p^2) \le \lambda r(p^1)+(1-\lambda)r(p^2)$. A function r(p) is *positively linearly homogeneous* if for every positive vector p and positive number λ, we have $r(\lambda p) = \lambda r(p)$.

[32] For additional material on revenue and factor requirements functions, see Diewert (1974b).

$$= Q \max_q \{\sum_{n=1}^{N} p_n q_n/Q : f(q) = 1\} \qquad \text{letting } q \equiv x/Q$$
$$= Q\, r(p) \qquad \text{using definition(18.45).}$$

Thus r(p)Q is the maximum export revenue that the establishment can make, given that it faces the vector of output prices p and is asked to produce a combination of exports $[x_1,...,x_N] = x$ that will produce the level Q of aggregate exports.

18.55 Now recall the export revenue maximization problem defined by(18.28) above. Using the factor requirements function defined by(18.44) in place of the period t production possibilities set S^t, this revenue maximization problem can be rewritten as follows:

$$(18.47)\quad R^t(p,u) = \max_q \{\sum_{n=1}^{N} p_n x_n : v_1 = G^t(f(x),y,z,m,v_2,v_3, \ldots ,v_K)\}$$
$$= \max_{q,\,Q} \{\sum_{n=1}^{N} p_n x_n : v_1 = G^t(f(x),y,z,m,v_2,v_3, \ldots ,v_K);\ Q = f(x)\}$$
$$= \max_Q \ \{ r(p)Q : v_1 = G^t(Q,y,z,m,v_2,v_3, \ldots ,v_K)\}$$

where the last equality follows using(18.46). Now make assumptions(18.31); i.e., that the observed period t export vector q^t solves the period t export revenue maximization problems, which are given by(18.47) under our separability assumption(18.44), with (p,u) $= (p^t,u^t)$ for t = 0,1. Using(18.47), the following equalities result:

$$(18.48)\qquad Q^t = f(q^t) \quad ;\ t = 0,1;$$
$$(18.49)\quad R^t(p^t,v^t) = r(p^t)Q^t \ ;\ t = 0,1.$$

18.56 Consider the following export revenue maximization problem which uses the period 0 technology, the period 1 export price vector p^1 and conditions on the period 0 reference quantity vector u^0 :

$$(18.50)\quad R^0(p^1,u^0) = \max_{x,\,Q} \{\sum_{n=1}^{N} p_n^1 x_n :\ v_1^0 = G^0(Q,y^0,z^0,m^0, v_2^0,v_3^0,\ldots ,v_{M+K}^0) ;\ Q = f(q)\}$$
$$= \max_{x,\,Q} \{\sum_{n=1}^{N} p_n^1 x_n :\ v_1^0 = G^0(Q^0,y^0,z^0,m^0,v_2^0,v_3^0,\ldots ,v_{M+K}^0) ;\ Q^0 = f(q)\}$$

since Q^0 will be the only Q that satisfies the constraint $v_1^0 = G^0(Q, y^0,z^0,m^0,v_2^0,v_3^0,\ldots ,v_{M+K}^0)$

$$= \max_x \{\sum_{n=1}^{N} p_n^1 x_n : Q^0 = f(x)\}$$
$$= r(p^1)Q^0 \qquad \text{using(18.47) with } p = p^1 \text{ and } Q = Q^0.$$

18.57 Now using the first equality in(18.49) and the last equality in(18.50) in order to evaluate *the base period version of the theoretical export price index*, $P^0(p^0,p^1,u^0)$, defined above in(18.32):

$$\begin{aligned}(18.51)\quad P^0(p^0,p^1,u^0) &\equiv R^0(p^1,u^0)/R^0(p^0,u^0)\\ &= r(p^1)Q^0 / r(p^0)Q^0\\ &= r(p^1)/r(p^0).\end{aligned}$$

Note that the base period export price index $P^0(p^0,p^1,v^0)$ no longer depends on the base period reference quantity vector u^0; it is now simply a ratio of export unit revenue functions evaluated at the period 1 prices p^1 in the numerator and at the period 0 prices p^0 in the denominator. This is the simplification that the separability assumptions on the technologies for the two periods imply.

18.58 Using the same technique of proof that was used to establish(18.50), it can be shown that under the separability assumptions(18.44):

$$(18.52)\quad R^1(p^0,u^0) = r(p^0)Q^1.$$

18.59 Now the second equality in(18.49) and equation (18.52) can be used in order to evaluate *the current period version of the theoretical export price index* $P^1(p^0,p^1,u^1)$ defined above in(18.33):

$$\begin{aligned}(18.53)\quad P^1(p^0,p^1,u^1) &\equiv R^1(p^1,u^1)/R^1(p^0,u^1)\\ &= r(p^1)Q^1 / r(p^0)Q^1\\ &= r(p^1)/r(p^0).\end{aligned}$$

Again, the current period export price index $P^1(p^0,p^1,u^1)$ no longer depends on the current period reference quantity vector u^1; it is again the ratio of unit export revenue functions evaluated at the period 1 prices p^1 in the numerator and at the period 0 prices p^0 in the denominator.

18.60 Note that under the present homogeneous weak separability assumptions, both theoretical export price indices defined in(18.32) and(18.33) collapse down to the same thing, the ratio of unit export revenues pertaining to the two periods under consideration, $r(p^1)/r(p^0)$.[33]

18.61 Under the separability assumptions(18.44) on the establishment technologies for periods 0 and 1, the following decompositions for establishment export revenues in periods 0 and 1 can be obtained:

$$(18.54)\quad R^0(p^0,u^0) = \sum_{n=1}^{N} p_n{}^0 q_n{}^0 = r(p^0)f(q^0)\ ;$$

$$(18.55)\quad R^1(p^1,u^1) = \sum_{n=1}^{N} p_n{}^1 q_n{}^1 = r(p^1)f(q^1).$$

[33] The separability assumptions(18.44) play the same role in the economic theory of output price indices as the assumption of homothetic preferences does in the economic theory of cost of living indices.

The ratio of unit revenues, $r(p^1)/r(p^0)$, has already been identified as the economic output price index under our separability assumptions,(18.44), so if the ratio of establishment export revenues in period 1 to revenues in period 0, $\sum_{n=1}^{N} p_n^1 x_n^1/\sum_{n=1}^{N} p_n^0 x_n^0$, is divided by the export price index, the corresponding *implicit export quantity index*, $Q(p^0,p^1,x^0,x^1)$ is obtained:

$$(18.56)\quad Q(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^{N} p_n^1 x_n^1/\sum_{n=1}^{N} p_n^0 x_n^0]/[r(p^1)/r(p^0)] = f(x^1)/f(x^0).$$

Thus under the separability assumptions, the economic export quantity index is found to be equal to $f(x^1)/f(x^0)$.[34]

18.62 Now a position has been reached to apply the theory of exact index numbers. In the following subsections, some specific assumptions will be made about the functional form for the export unit revenue function $r(p)$ or the export aggregator function $f(x)$[35] and these specific assumptions will enable price index number formulae that are exactly equal to the theoretical output price index, $r(p^1)/r(p^0)$, to be determined. However, before this, it is necessary to develop the mathematics of the revenue maximization problems for periods 0 and 1 in a bit more detail. This is done in the next subsection.

E.2 The Mathematics of the Revenue Maximization Problem

18.63 In subsequent material, two additional results from economic theory will be needed: Wold's Identity and Hotelling's Lemma. These two results follow from the assumption that the establishment is maximizing export revenue during the two periods under consideration subject to the constraints of technology. Wold's Identity tells us that the partial derivative of an export aggregator function with respect to an export quantity is proportional to its export price while Hotelling's Lemma tells us that the partial derivative of an export unit revenue function with respect to an export price is proportional to the equilibrium export quantity. These two results enable specific functional forms for the aggregator function $f(q)$ or for the unit revenue function $r(p)$ to be related to bilateral price and quantity indices, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, that depend on the observable price and quantity vectors pertaining to the two periods under consideration. In particular, Wold's Identity,(18.58), is used to establish(18.68) in section E.3 and(18.80) in section E.4 while Hotelling's Lemma,(18.64), is used to establish(18.64) in section E.3 and(18.85) in section E.4. The less mathematically inclined reader can simply note these results and skip over to section E.3.

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[34] Note that under the separability assumptions(18.44), the family of export price indices defined by(18.29) simplifies to the unit export revenue function ratio $r(p^1)/r(p^0)$ which depends *only* on export prices (and not the reference quantity vector u) and the corresponding export quantity index is $f(x^1)/f(x^0)$ which depends *only* on quantities of exports produced during the two periods under consideration.

[35] In the following section, in order to make the notation more comparable with the notation used in previous chapters, the export quantity vector x will be replaced by the quantity vector q.

18.64 *Wold's* (1944; 69-71) (1953; 145) *Identity* is the following result[36]. Assume that the establishment technologies satisfy the separability assumptions(18.44) for periods 0 and 1. Assume in addition that the observed period t export vector q^t solves the period t export revenue maximization problems, which are defined by(18.47) under our separability assumptions, with $(p,u) = (p^t,u^t)$ for t = 0,1. Finally, assume that the export aggregator function f(q) is differentiable with respect to the components of q at the points q^0 and q^1. Then it can be shown[37] that the following equations hold:

$$(18.57)\quad p_n^t / \sum_{k=1}^{N} p_k^t q_k^t = [\partial f(q^t)/\partial q_n] / \sum_{k=1}^{N} p_k q_k^t\, \partial f(q^t)/\partial q_k\ ; \quad t = 0,1\ ; \quad n = 1,\ldots,N$$

where $\partial f(q^t)/\partial q_n$ denotes the partial derivative of the export revenue function f with respect to the nth export quantity q_n evaluated at the period t quantity vector q^t.

18.65 Since the export aggregator function f(q) has been assumed to be linearly homogeneous, Wold's Identity(18.57) simplifies into the following equations which will prove to be very useful:[38]

$$(18.58)\quad p_n^t / \sum_{k=1}^{N} p_k^t q_k^t = [\partial f(q^t)/\partial q_n]/f(q^t)\ ; \qquad n = 1,\ldots,N\ ;\ t = 0,1.$$

In words,(18.58) says that the vector of period t establishment export prices p^t divided by period t establishment export revenues $\sum_{k=1}^{N} p_k^t q_k^t$ is equal to the vector of first order partial derivatives of the establishment export aggregator function $\nabla f(p^t) \equiv [\partial f(q^t)/\partial q_1,\ldots,\partial f(q^t)/\partial q_N]$ divided by the period t export aggregator function $f(q^t)$.

18.66 Under assumptions(18.31) and our separability assumptions(18.44), q^t solves the following export revenue maximization problem:

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[36] Actually, Wold derived his result in the context of a consumer utility maximization problem but his result carries over to the present production context.

[37] To prove this, consider the first order necessary conditions for the strictly positive vector q^t to solve the period t export revenue maximization problem, $\max_q \{\sum_{n=1}^{N} p_n^t q_n : f(q_1,\ldots,q_N) = f(q_1^t,\ldots,q_N^t) \equiv Q^t\}$. The necessary conditions of Lagrange for q^t to solve this problem are: $p^t = \lambda^t \nabla f(q^t)$ where λ^t is the optimal Lagrange multiplier and $\nabla f(q^t)$ is the vector of first order partial derivatives of f evaluated at q^t. Now take the inner product of both sides of this equation with respect to the period t quantity vector q^t and solve the resulting equation for λ^t. Substitute this solution back into the vector equation $p^t = \lambda^t \nabla f(q^t)$ and we obtain(18.57).

[38] Differentiate both sides of the equation $f(\lambda q) = \lambda f(q)$ with respect to λ and then evaluate the resulting equation at $\lambda = 1$. The equation $\sum_{n=1}^{N} f_n(q)q_n = f(q)$ results where $f_n(q) \equiv \partial f(q)/\partial q_n$.

$$(18.59)\quad \max_q \{\sum_{n=1}^{N} p_n^t q_n : f(q_1,...,q_N) = f(q_1^t,...,q_N^t)\} = r(p^t)Q^t ; \qquad t = 0,1$$

where $Q^t \equiv f(q^t)$ and the last equality follows using(18.49). Consider the period t export revenue maximization problem defined by (5.30) above. *Hotelling's* (1932; 594) *Lemma* is the following result. If the unit export revenue function $r(p^t)$ is differentiable with respect to the components of the price vector p, then the period t export quantity vector q^t is equal to the period t export aggregate Q^t times the vector of first order partial derivatives of the unit export revenue function with respect to the components of p evaluated at the period t price vector p^t; i.e.,

$$(18.60)\quad q_n^t = Q^t \partial r(p^t)/\partial p_n ; \qquad n = 1,\ldots,N ; t = 0,1.$$

To explain why(18.60) holds, consider the following argument. Because it is being assumed that the observed period t export quantity vector q^t solves the export revenue maximization problem that corresponds to $r(p^t)Q^t$, then q^t must be feasible for this maximization problem so it is necessary that $f(q^t) = Q^t$. Thus q^t is a feasible solution for the following export revenue maximization problem where the general export price vector p has replaced the specific period t export price vector p^t:

$$\begin{aligned}(18.61)\quad r(p^t)Q^t &\equiv \max_q \{\sum_{n=1}^{N} p_n q_n : f(q_1,\ldots,q_N) = Q^t\} \\ &\geq \sum_{n=1}^{N} p_n q_n^t\end{aligned}$$

where the inequality follows from the fact that $q^t \equiv (q_1^t,\ldots,q_N^t)$ is a feasible (but usually not optimal) solution for the export revenue maximization problem in(18.61). Now for each strictly positive export price vector p, define the function g(p) as follows:

$$(18.62)\quad g(p) \equiv \sum_{n=1}^{N} p_n q_n^t - r(p)Q^t$$

where as usual, $p \equiv (p_1,\ldots,p_N)$. Using(18.59) and(18.61), it can be seen that g(p) is maximized (over all strictly positive price vectors p) at $p = p^t$. Thus the first order necessary conditions for maximizing a differentiable function of N variables hold, which simplify to equations(18.60).

18.67 Combining equations(18.48),(18.54) and(18.55), yields the following equations:

$$(18.63)\quad \sum_{n=1}^{N} p_n^t q_n^t = r(p^t)f(q^t) = r(p^t)Q^t \qquad \text{for } t = 0,1.$$

Combining equations(18.60) and(18.63), yields the following system of equations:

$$(18.64)\quad q_n^t / \sum_{k=1}^{N} p_k^t q_k^t = [\partial r(p^t)/\partial p_n]/r(p^t) ; \qquad n = 1,\ldots,N ; t = 0,1.$$

In words,(18.64) says that the vector of period t establishment exports q^t divided by period t establishment export revenues $\sum_{k=1}^N p_k^t q_k^t$ is equal to the vector of first order partial derivatives of the establishment unit export revenue function $\nabla r(p^t) \equiv [\partial r(p^t)/\partial p_1,...,\partial r(p^t)/\partial p_N]$ divided by the period t unit export revenue function $r(p^t)$.

Note the symmetry of equations(18.64) with equations(18.58). It is these two sets of equations that shall be used in subsequent material.

E.3 Superlative indices: the Fisher ideal index

18.68 Suppose the producer's export aggregator function has the following functional form:

$$(18.65)\quad f(q_1,\ldots,q_N) \equiv [\sum_{i=1}^{N} \sum_{k=1}^{N} a_{ik} q_i q_k]^{1/2} ; \quad a_{ik} = a_{ki} \quad \text{for all i and k.}$$

Differentiating the f(q) defined by(18.65) with respect to q_i yields the following equations:

$$(18.66)\quad f_i(q) = (1/2) [\sum_{j=1}^{N} \sum_{k=1}^{N} a_{jk} q_j q_k]^{-1/2} 2\sum_{k=1}^{n} a_{ik} q_k ; \qquad i = 1,\ldots,N$$

$$= \sum_{k=1}^{N} a_{ik} q_k / f(q) \qquad \text{using(18.65)}$$

where $f_i(q) \equiv \partial f(q^t)/\partial q_i$. In order to obtain the first equation in(18.66), the symmetry conditions, $a_{ik} = a_{ki}$ are needed. Now evaluate the second equation in(18.66) at the observed period t quantity vector $q^t \equiv (q_1^t,\ldots,q_N^t)$ and divide both sides of the resulting equation by $f(q^t)$. We obtain the following equation:

$$(18.67)\quad f_i(q^t)/f(q^t) = \sum_{k=1}^{N} a_{ik} q_k^t / [f(q^t)]^2 \qquad t = 0,1 ; i = 1,\ldots,N.$$

Assume export revenue maximizing behavior for the producer in periods 0 and 1. Since the aggregator function f defined by(18.65) is linearly homogeneous and differentiable, equations(18.58) will hold (Wold's Identity). Now recall the definition of the Fisher ideal price index, P_F defined by(18.37) above. If the period 1 export revenues are divided by the period 0 export revenues and then this value ratio is divided by P_F, then the Fisher ideal quantity index, $Q_{F,}$ results:

$$(18.68)\quad Q_F(p^0,p^1,q^0,q^1) \equiv [\sum_{i=1}^{N} p_i^1 q_i^1 / \sum_{i=1}^{N} p_i^0 q_i^0]/P_F(p^0,p^1,q^0,q^1)$$

$$= [\sum_{i=1}^{N} p_i^0 q_i^1 / \sum_{k=1}^{N} p_k^0 q_k^0]^{1/2} [\sum_{i=1}^{N} p_i^1 q_i^1 / \sum_{k=1}^{N} p_k^1 q_k^0]^{1/2}$$

$$= [\sum_{i=1}^{N} f_i(q^0) q_i^1 / f(q^0)]^{1/2} [\sum_{i=1}^{N} p_i^1 q_i^1 / \sum_{k=1}^{N} p_k^1 q_k^0]^{1/2} \qquad \text{using(18.58) for } t = 0$$

$$= [\sum_{i=1}^{N} f_i(q^0) q_i^1 / f(q^0)]^{1/2} / [\sum_{k=1}^{N} p_k^1 q_k^0 / \sum_{i=1}^{N} p_i^1 q_i^1]^{1/2}$$

$$= [\sum_{i=1}^{N} f_i(q^0) q_i^1 / f(q^0)]^{1/2} / [\sum_{i=1}^{N} f_i(q^1) q_i^0 / f(q^1)]^{1/2} \qquad \text{using(18.58) for } t = 1$$

$$= [\sum_{i=1}^{N} \sum_{k=1}^{N} a_{ik} q_k^0 q_i^1 / [f(q^0)]^2]^{1/2} / [\sum_{i=1}^{N} \sum_{k=1}^{N} a_{ik} q_k^1 q_i^0 / [f(q^1)]^2]^{1/2} \qquad \text{using(18.67)}$$

$$= [1/[f(q^0)]^2]^{1/2} / [1/[f(q^1)]^2]^{1/2} \qquad \text{using(18.65) and canceling terms}$$

$$= f(q^1)/f(q^0).$$

Thus under the assumption that the producer engages in export revenue maximizing behavior during periods 0 and 1 and has technologies in periods 0 and 1 that satisfy the separability assumptions(18.44), then the Fisher ideal quantity index Q_F is *exactly* equal to the true quantity index, $f(q^1)/f(q^0)$.[39]

18.69 As was noted in earlier chapters, the price index that corresponds to the Fisher quantity index Q_F using the product test is the Fisher price index P_F defined by(18.37). Let r(p) be the export unit revenue function that corresponds to the homogeneous quadratic export aggregator function f defined by(18.65). Then using(18.54),(18.55) and(18.68), it can be seen that

$$(18.69)\quad P_F(p^0,p^1,q^0,q^1) = r(p^1)/r(p^0).$$

Thus under the assumption that the producer engages in export revenue maximizing behavior during periods 0 and 1 and has production technologies that satisfy the separability assumptions(18.44) during periods 0 and 1,then the Fisher ideal export price index P_F is exactly equal to the true price index, $r(p^1)/r(p^0)$.

18.70 A twice continuously differentiable function f(q) of N variables $q \equiv (q_1,\ldots,q_N)$ can provide a *second order approximation* to another such function $f^*(q)$ around the point q^* if the level and all of the first and second order partial derivatives of the two functions coincide at q^*. It can be shown[40] that the homogeneous quadratic function f defined by(18.65) can provide a second order approximation to an arbitrary f^* around any (strictly positive) point q^* in the class of linearly homogeneous functions. Thus the homogeneous

[39] For the early history of this result in the consumer context, see Diewert (1976; 184).
[40] See Diewert (1976; 130) and let the parameter r equal 2.

quadratic functional form defined by(18.65) is a *flexible functional form*.[41] Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[42] Equation (18.68) and the fact that the homogeneous quadratic function f defined by(18.65) is a flexible functional form shows that the Fisher ideal quantity index Q_F is a superlative index number formula. Since the Fisher ideal price index P_F also satisfies(18.69) where r(p) is the unit export revenue function that is generated by the homogeneous export quadratic aggregator function, P_F is also a superlative index number formula.

18.71 It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the producer's export aggregator function is the homogeneous quadratic function defined by(18.65), start with the assumption that the producer's unit export revenue function is a homogeneous quadratic.[43] Thus suppose that the producer has the following unit export revenue function:

$$(18.70)\quad r(p_1,\ldots,p_N) \equiv [\sum_{i=1}^{N} \sum_{k=1}^{N} b_{ik}\, p_i p_k]^{1/2}$$

where the parameters b_{ik} satisfy the following symmetry conditions:

$$(18.71)\quad b_{ik} = b_{ki} \quad \text{for all i and k.}$$

Differentiating r(p) defined by(18.70) with respect to p_i yields the following equations:

$$(18.72)\quad r_i(p) = (1/2)[\sum_{j=1}^{N} \sum_{k=1}^{N} b_{jk}\, p_j p_k]^{-1/2}\, 2\sum_{k=1}^{N} b_{ik}\, p_k\,; \qquad i = 1,\ldots,N$$

$$= \sum_{k=1}^{N} b_{ik}\, p_k / r(p) \qquad \text{using(18.70)}$$

where $r_i(p) \equiv \partial r(p^t)/\partial p_i$. In order to obtain the first equation in(18.72), it is necessary to use the symmetry conditions,(18.71). Now evaluate the second equation in(18.72) at the observed period t price vector $p^t \equiv (p_1^t,\ldots,p_N^t)$ and divide both sides of the resulting equation by $r(p^t)$. The following equations result:

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[41] Diewert (1974a; 133) introduced this term to the economics literature.

[42] Fisher (1922; 247) used the term *superlative* to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

[43] Given the producer's unit export revenue function r(p), it is possible to modify a technique in Diewert (1974a; 112) and show that the corresponding export aggregator function f(q) can be defined as follows: for a strictly positive quantity vector q, $f(q) \equiv \max_p \{\sum_{i=1}^{N} p_i q_i : r(p) = 1\}$.

$$(18.73)\quad r_i(p^t)/r(p^t) = \sum_{k=1}^{N} b_{ik}\, p_k^{\,t} / [r(p^t)]^2 \qquad t = 0,1\ ;\ i = 1,\ldots,N.$$

As export revenue maximizing behavior is assumed for the producer in periods 0 and 1 and since the unit export revenue function r defined by(18.70) is differentiable, equations(18.64) will hold (Hotelling's Lemma). Now recall the definition of the Fisher ideal price index, P_F defined by(18.37) above:

$$\begin{aligned}
(18.74)\quad P_F(p^0,p^1,q^0,q^1) &= [\sum_{i=1}^{N} p_i^{\,1} q_i^{\,0} / \sum_{k=1}^{N} p_k^{\,0} q_k^{\,0}]^{1/2}\, [\sum_{i=1}^{N} p_i^{\,1} q_i^{\,1} / \sum_{k=1}^{N} p_k^{\,0} q_k^{\,1}]^{1/2} \\
&= [\sum_{i=1}^{N} p_i^{\,1} r_i(p^0)/r(p^0)]^{1/2}\, [\sum_{i=1}^{N} p_i^{\,1} q_i^{\,1} / \sum_{k=1}^{N} p_k^{\,0} q_k^{\,1}]^{1/2} && \text{using(18.64) for } t = 0 \\
&= [\sum_{i=1}^{N} p_i^{\,1} r_i(p^0)/r(p^0)]^{1/2} / [\sum_{k=1}^{N} p_k^{\,0} q_k^{\,1} / \sum_{i=1}^{N} p_i^{\,1} q_i^{\,1}]^{1/2} \\
&= [\sum_{i=1}^{N} p_i^{\,1} r_i(p^0)/r(p^0)]^{1/2} / [\sum_{i=1}^{N} p_i^{\,0} r_i(p^1)/r(p^1)]^{1/2} && \text{using(18.65) for } t = 1 \\
&= [\sum_{i=1}^{N} \sum_{k=1}^{N} b_{ik}\, p_k^{\,0} p_i^{\,1} / [r(p^0)]^2]^{1/2} / [\sum_{i=1}^{N} \sum_{k=1}^{N} b_{ik}\, p_k^{\,1} p_i^{\,0} / [r(p^1)]^2]^{1/2} && \text{using(18.73)} \\
&= [1/[r(p^0)]^2]^{1/2} / [1/[r(p^1)]^2]^{1/2} && \text{using(18.71) and canceling terms} \\
&= r(p^1)/r(p^0).
\end{aligned}$$

Thus under the assumption that the producer engages in revenue maximizing behavior during periods 0 and 1 and has technologies that satisfy the separability assumptions(18.44) and the functional form for the unit revenue function that corresponds to the output aggregator function f(q) is given by(18.70), then the Fisher ideal price index P_F is *exactly* equal to the true price index, $r(p^1)/r(p^0)$.[44]

18.72 Since the homogeneous quadratic unit revenue function r(p) defined by(18.70) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true export price index $r(p^1)/r(p^0)$ means that P_F is a *superlative index number formula*.[45]

18.73 Suppose that the b_{ik} coefficients in(18.70) satisfy the following restrictions:

$$(18.75)\quad b_{ik} = b_i b_k \qquad \text{for } i,k = i,\ldots,N$$

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[44] This result was obtained by Diewert (1976; 133-134) in the consumer context.

[45] Note that we have shown that the Fisher index P_F is exact for the output aggregator function defined by(18.65) as well as the output aggregator function that corresponds to the unit revenue function defined by(18.70). These two output aggregator functions do not coincide in general. However, if the N by N symmetric matrix A of the a_{ik} has an inverse, then it can readily be shown that the N by N matrix B of the b_{ik} will equal A^{-1}.

where the N numbers b_i are nonnegative. In this special case of(18.70), it can be seen that the unit export revenue function simplifies as follows:

$$(18.76)\ \ r(p_1,\ldots,p_N) \equiv [\sum_{i=1}^{N}\ \sum_{k=1}^{N} b_i\, b_k\, p_i\, p_k]^{1/2}$$
$$= [\sum_{i=1}^{N} b_i\, p_i\ \sum_{k=1}^{N} b_k\, p_k]^{1/2}$$
$$= \sum_{i=1}^{N} b_i\, p_i\ .$$

Substituting(18.76) into Hotelling's Lemma(18.60) yields the following expressions for the period t quantity vectors, q^t:

$$(18.77)\ \ q_n{}^t = Q^t\, \partial r(p^t)/\partial p_n = b_n\, Q^t \qquad\qquad i = 1,\ldots,N\ ;\ t = 0,1.$$

Thus if the producer has the export aggregator function that corresponds to the unit export revenue function defined by(18.70) where the b_{ik} satisfy the restrictions(18.75), then the period 0 and 1 quantity vectors are equal to a multiple of the vector $b \equiv (b_1,\ldots,b_N)$; i.e., $q^0 = bQ^0$ and $q^1 = bQ^1$. Under these assumptions, the Fisher, Paasche and Laspeyres indices, P_F, P_P and P_L, *all coincide.* However, the export aggregator function f(q) which corresponds to this unit export revenue function is not consistent with normal producer behavior since the output production possibilities set in this case is block shaped and hence the producer will not substitute towards producing more expensive commodities from cheaper commodities if relative prices change going from period 0 to 1.

E.4 Quadratic Mean of Order r Superlative Indices

18.74 It turns out that there are many other superlative index number formulae; i.e., there exist many export quantity indices $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many export price indices $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $r(p^1)/r(p^0)$ where the export aggregator function f or the export unit revenue function r is a flexible functional form. Two families of superlative indices are defined below.

18.75 17.120 Suppose that the producer's output aggregator function is the *following quadratic mean of order r aggregator function*:[46]

$$(18.78)\ \ f^r(q_1,\ldots,q_N) \equiv [\sum_{i=1}^{N}\ \sum_{k=1}^{N} a_{ik}\, q_i{}^{r/2}\, q_k{}^{r/2}\,]^{1/r}$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the aggregator

[46] This terminology is due to Diewert (1976; 129).

function f^r defined by(18.78) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when r = 2, f^r equals the homogeneous quadratic function defined by(18.65) above.

18.76 Define the quadratic mean of order r export quantity index Q^r by:

$$(18.79)\quad Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^{\,0}\,(q_i^{\,1}/q_i^{\,0})^{r/2}\}^{1/r}\ \{\sum_{i=1}^N s_i^{\,1}\,(q_i^{\,1}/q_i^{\,0})^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^N p_k^t q_k$ is the period t export revenue share for export output i as usual. It can be verified that when r = 2, Q^r simplifies into Q_F, the Fisher ideal quantity index.

18.77 Using exactly the same techniques as were used in section E.3 above, it can be shown that Q^r is exact for the aggregator function f^r defined by(18.78); i.e.,

$$(18.80)\quad Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Thus under the assumption that the producer engages in export revenue maximizing behavior during periods 0 and 1 and has technologies that satisfy assumptions(18.44) where the output aggregator function f(q) is defined by(18.78), then the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[47] Since Q^r is exact for f^r and f^r is a flexible functional form, the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative quantity indices.

18.78 For each quantity index Q^r, the product test can be used in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

$$(18.81)\quad P^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^N p_i^{\,1} q_i^{\,1}/\{\sum_{i=1}^N p_i^{\,0} q_i^{\,0}\, Q^r(p^0,p^1,q^0,q^1)\}$$
$$= r^{r*}(p^1)/r^{r*}(p^0)$$

where r^{r*} is the unit revenue function that corresponds to the aggregator function f^r defined by(18.78) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

18.79 When r = 2, Q^r defined by(18.79) simplifies to Q_F, the Fisher ideal quantity index and P^{r*} defined by(18.81) simplifies to P_F, the Fisher ideal price index. When r = 1, Q^r defined by(18.79) simplifies to:

[47] See Diewert (1976; 130).

$$(18.82)\ \ Q^1(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^{N} s_i^{\,0}\ (q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^{N} s_i^{\,1}\ (q_i^1/q_i^0)^{-1/2}\}$$

$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]\{\ \sum_{i=1}^{N} p_i^0 q_i^0 (q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^{N} p_i^1 q_i^1 (q_i^1/q_i^0)^{-1/2}\}$$

$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]\{\ \sum_{i=1}^{N} p_i^0 (q_i^0 q_i^1)^{1/2}\}/\{\sum_{i=1}^{N} p_i^1 (q_i^0 q_i^1)^{1/2}\}$$

$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]/\{\ \sum_{i=1}^{N} p_i^1 (q_i^0 q_i^1)^{1/2}/\sum_{i=1}^{N} p_i^0 (q_i^0 q_i^1)^{1/2}\}$$

$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]/\ P_W(p^0,p^1,q^0,q^1)$$

where P_W is the *Walsh price index* defined previously by (16.19) in Chapter 16. Thus P^{1*} is equal to P_W, the *Walsh price index*, and hence it is also a superlative price index.

18.80 **17.125** ic mean of order r unit revenue function:[48]

$$(18.83)\ \ r^r(p_1,\ldots,p_n) \equiv [\sum_{i=1}^{N}\ \ \sum_{k=1}^{N} b_{ik}\, p_i^{r/2}\, p_k^{r/2}\,]^{1/r}$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the unit revenue function r^r defined by(18.83) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when r = 2, r^r equals the homogeneous quadratic function defined by(18.70) above.

18.81 Define the quadratic mean of order r price index P^r by:

$$(18.84)\ \ P^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^{N} s_i^{\,0}\ (p_i^1/p_i^0)^{r/2}\}^{1/r}\ \{\sum_{i=1}^{N} s_i^{\,1}\ (p_i^1/p_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^{N} p_k^t q_k^t$ is the period t revenue share for output i as usual. It can be verified that when r = 2, P^r simplifies into P_F, the Fisher ideal price index.

18.82 Using exactly the same techniques as were used in section D.3 above, it can be shown that P^r is exact for the unit revenue function r^r defined by(18.83); i.e.,

$$(18.85)\ \ P^r(p^0,p^1,q^0,q^1) = r^r(p^1)/r^r(p^0).$$

[48] This terminology is due to Diewert (1976; 130). This functional form was first defined by Denny (1974) as a unit cost function.

Thus under the assumption that the producer engages in export revenue maximizing behavior during periods 0 and 1 and has technologies that satisfy assumptions(18.44) where the output aggregator function f(q) corresponds to the unit revenue function $r^r(p)$ defined by(18.83), then the quadratic mean of order r price index P^r is *exactly* equal to the true export price index, $r^r(p^1)/r^r(p^0)$.[49] Since P^r is exact for r^r and r^r is a flexible functional form, that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative price indices.

18.83 For each price index P^r, the product test (16.3) can be used in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

$$(18.86)\quad Q^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{N} p_i^1 q_i^1 / \{\sum_{i=1}^{N} p_i^0 q_i^0\, P^r(p^0,p^1,q^0,q^1)\}$$
$$= f^{r*}(p^1)/f^{r*}(p^0)$$

where f^r* is the aggregator function that corresponds to the unit revenue function r^r defined by(18.83) above.[50] For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

18.84 When r = 2, P^r defined by(18.84) simplifies to P_F, the Fisher ideal price index and Q^{r*} defined by(18.86) simplifies to Q_F, the Fisher ideal quantity index. When r = 1, P^r defined by(18.84) simplifies to:

$$(18.87)(5.58)\quad P^1(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^{N} s_i^0\,(p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^{N} s_i^1\,(p_i^1/p_i^0)^{-1/2}\}$$
$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]\{\sum_{i=1}^{N} p_i^0 q_i^0 (p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^{N} p_i^1 q_i^1 (p_i^1/p_i^0)^{-1/2}\}$$
$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]\{\sum_{i=1}^{N} q_i^0 (p_i^0 p_i^1)^{1/2}\}/\{\sum_{i=1}^{N} q_i^1 (p_i^0 p_i^1)^{1/2}\}$$
$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]/\{\sum_{i=1}^{N} q_i^1 (p_i^0 p_i^1)^{1/2}/\sum_{i=1}^{N} q_i^0 (p_i^0 p_i^1)^{1/2}\}$$
$$= [\sum_{i=1}^{N} p_i^1 q_i^1/\sum_{i=1}^{N} p_i^0 q_i^0]/Q_W(p^0,p^1,q^0,q^1)$$

where Q_W is the *Walsh quantity index* defined previously by (16.34) in Chapter 16. Thus Q^{1*} is equal to Q_W, the Walsh (1901) (1921) quantity index, and hence it is also a superlative quantity index.

[49] See Diewert (1976; 133-134).

[50] The function f^r* can be defined by using r^r as follows: $f^r*(q) \equiv \max_p \{\sum_{i=1}^N p_i q_i : r^r(p) = 1\}$.

18.85 Essentially, the economic approach to index number theory provides reasonably strong justifications for the use of the Fisher price index P_F, the Törnqvist-Theil price index P_T defined by (16.48) or(18.38), the implicit quadratic mean of order r price indices P^{r*} defined by(18.81) (when r = 1, this index is the Walsh price index defined by (16.19) in Chapter 16) and the quadratic mean of order r price indices P^r defined by(18.84). It is now necessary to ask if it matters which one of these formula is chosen as "best".

E.5 The Approximation Properties of Superlative Indices

18.86 The results in sections C.2, C.3, D.3 and D.4 provide a large number of superlative index number formulae which appear to have good properties from the viewpoint of the economic approach to index number theory.[51] Two questions arise as a consequence of these results:

- Does it matter which of these formulae is chosen?
- If it does matter, which formula should be chosen?

18.87 With respect to the first question, Diewert (1978; 888) showed that all of the superlative index number formulae listed above in sections E.3 and E.4 approximate each other to the second order around any point where the two price vectors, p^0 and p^1,are equal and where the two quantity vectors, q^0 and q^1, are equal. In particular, this means that the following equalities exist for all r and s not equal to 0 provided that $p^0 = p^1$ and $q^0 = q^1$:[52]

(18.88) $P_T(p^0,p^1,q^0,q^1) = P^r(p^0,p^1,q^0,q^1) = P^{s*}(p^0,p^1,q^0,q^1)$;

(18.89) $\partial P_T(p^0,p^1,q^0,q^1)/\partial p_i^t = \partial P^r(p^0,p^1,q^0,q^1)/\partial p_i^t = \partial P^{s*}(p^0,p^1,q^0,q^1)/\partial p_i^t$; $i = 1,\ldots,N$; $t = 0,1$;

(18.90) $\partial P_T(p^0,p^1,q^0,q^1)/\partial q_i^t = \partial P^r(p^0,p^1,q^0,q^1)/\partial q_i^t = \partial P^{s*}(p^0,p^1,q^0,q^1)/\partial q_i^t$; $i = 1,\ldots,N$; $t = 0,1$;

(18.91) $\partial^2 P_T(p^0,p^1,q^0,q^1)/\partial p_i^t\partial p_k^t = \partial^2 P^r(p^0,p^1,q^0,q^1)/\partial p_i^t\partial p_k^t = \partial^2 P^{s*}(p^0,p^1,q^0,q^1)/\partial p_i^t\partial p_k^t$; $i,k = 1,\ldots,N$; $t = 0,1$;

(18.92) $\partial^2 P_T(p^0,p^1,q^0,q^1)/\partial p_i^t\partial q_k^t = \partial^2 P^r(p^0,p^1,q^0,q^1)/\partial p_i^t\partial q_k^t = \partial^2 P^{s*}(p^0,p^1,q^0,q^1)/\partial p_i^t\partial q_k^t$; $i,k = 1,\ldots,N$; $t = 0,1$;

(18.93) $\partial^2 P_T(p^0,p^1,q^0,q^1)/\partial q_i^t\partial q_k^t = \partial^2 P^r(p^0,p^1,q^0,q^1)/\partial q_i^t\partial q_k^t = \partial^2 P^{s*}(p^0,p^1,q^0,q^1)/\partial q_i^t\partial q_k^t$; $i,k = 1,\ldots,N$; $t = 0,1$;

where the Törnqvist-Theil price index P_T is defined by(18.38), the implicit quadratic mean of order r price indices P^{s*} are defined by(18.61) and the quadratic mean of order r

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[51] The justifications for the Fisher and Törnqvist indices presented in sections C.2 and C.3 are stronger than the justifications for the other superlative indices presented in sections D.3 and D.4 because the arguments in C.2 and C.3 did not rely on restrictive separability assumptions.

[52] To prove the equalities in(18.89)-(18.93), simply differentiate the various index number formulae and evaluate the derivatives at $p^0 = p^1$ and $q^0 = q^1$. Actually, equations(18.88)-(18.93) are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$; i.e., provided that the period 1 price vector is proportional to the period 0 price vector and that the period 1 quantity vector is proportional to the period 0 quantity vector.

price indices P^r are defined by(18.84). Using the above results, Diewert (1978; 884) concluded that "all superlative indices closely approximate each other".

18.88 However, the above conclusion is not true even though the equations(18.88)-(18.93) are true. The problem is that the quadratic mean of order r price indices P^r and the implicit quadratic mean of order s price indices P^{s*} are (continuous) functions of the parameters r and s respectively. Hence as r and s become very large in magnitude, the indices P^r and P^{s*} can differ substantially from say $P^2 = P_F$, the Fisher ideal index . In fact, using definition(18.84) and the limiting properties of means of order r[53], Robert Hill (2006) showed that P^r has the following limit as r approaches plus or minus infinity:

$$(18.94)\quad \lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = [\min_i\{p_i^1/p_i^0\}\max_i\{p_i^1/p_i^0\}]^{1/2} .$$

Using Hill's method of analysis, it can be shown that the implicit quadratic mean of order r price index has the following limit as r approaches plus or minus infinity:

$$(18.95)\quad \lim_{r\to+\infty} P^{r*}(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^{r*}(p^0,p^1,q^0,q^1)$$
$$= \sum_{i=1}^{N} p_i^1 q_i^1 / \sum_{i=1}^{N} p_i^0 q_i^0\, [\min_i\{q_i^1/q_i^0\}\max_i\{q_i^1/q_i^0\}]^{1/2} .$$

Thus for r large in magnitude, P^r and P^{r*} can differ substantially from P_T, P^1, $P^{1*} = P_W$ (the Walsh price index) and $P^2 = P^{2*} = P_F$ (the Fisher ideal index).[54]

18.89 Although Robert Hill's theoretical and empirical results demonstrate conclusively that all superlative indices will not necessarily closely approximate each other, there is still the question of how well the more commonly used superlative indices will approximate each other. All of the commonly used superlative indices, P^r and P^{r*}, fall into the interval $0 \le r \le 2$.[55] Robert Hill (2006) summarized how far apart the Törnqvist and Fisher indices were making all possible bilateral comparisons between any two data points for his time series data set as follows:

"The superlative spread S(0,2) is also of interest since, in practice, Törnqvist (r = 0)and Fisher (r = 2) are by far the two most widely used superlative indices. In all 153 bilateral comparisons, S(0,2) is less than the Paasche-Laspeyres spread and on average, the superlative spread is only 0.1 percent. It is because attention, until now, has focused almost exclusively on superlative indices in the range $0 \le r \le 2$ that a general misperception has persisted in the index number literature that all superlative indices approximate each other closely."

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[53] See Hardy, Littlewood and Polyá (1934). Actually, Allen and Diewert (1981; 434) obtained the result (5.65) but they did not appreciate the significance of the result.

[54] Robert Hill (2000) documents this for two data sets. His time series data consists of annual expenditure and quantity data for 64 components of U.S. GDP from 1977 to 1994. For this data set, Hill (2000; 16) found that "superlative indices can differ by more than a factor of two (i.e., by more than 100 percent), even though Fisher and Törnqvist never differ by more than 0.6 percent."

[55] Diewert (1980; 451) showed that the Törnqvist index P_T is a limiting case of P^r as r tends to 0.

Thus for Hill's time series data set covering 64 components of U.S. GDP from 1977 to 1994 and making all possible bilateral comparisons between any two years, the Fisher and Törnqvist price indices differed by only 0.1 percent on average. This close correspondence is consistent with the results of other empirical studies using annual time series data.[56] Additional evidence on this topic may be found in Chapter 20 below.

18.90 A reasonably strong justification has been provided by the economic approach for a small group of index numbers: the *Fisher ideal index* $P_F = P^2 = P^{2*}$ defined by(18.37), the *Törnqvist-Theil index* P_T defined by(18.38), and the *Walsh index* P_W defined by (16.19) (which is equal to the implicit quadratic mean of order r price indices P^{r*} defined by(18.81) when r = 1). They share the property of being *superlative* and approximate each other to the second order around any point. This indicates that for "normal" time series data, these three indices will give virtually the same answer. The economic approach gave particular support to the Fisher and Törnqvist-Theil indices, albeit on different grounds. The Fisher index was advocated as the only symmetrically weighted average of Laspeyres and Paasche bounds that satisfied the time reversal test. Economic theory argued for the existence of Laspeyres and Paasche bounds on a suitable 'true' theoretical index. The support for the Törnqvist-Theil index arose from it requiring less restrictive assumptions to show it was superlative than the Fisher and Walsh indices. The Törnqvist-Theil index seemed to be best from the stochastic viewpoint, and the Fisher ideal index was supported from the axiomatic viewpoint, in that it best satisfied the quite reasonable tests presented. The Walsh index seemed to be best from the viewpoint of the "pure" price index. To determine precisely which one of these three alternative indices to use as a theoretical target or actual index, the statistical agency will have to decide which approach to bilateral index number theory is most consistent with its goals. It is reassuring that, as illustrated in Chapter 20, for "normal" time series data, these three indices gave virtually the same answer.

E.6 Superlative Indices and Two Stage Aggregation

18.91 Most statistical agencies use the Laspeyres formula to aggregate prices in two stages. At the first stage of aggregation, the Laspeyres formula is used to aggregate components of the overall index (e.g., agricultural output prices, other primary industry output prices, manufacturing prices, service output prices, etc.) and then at the second stage of aggregation, these component subindices are further combined into the overall index. The following question then naturally arises: does the index computed in two stages coincide with the index computed in a single stage? This question is initially addressed in the context of the Laspeyres formula.[57]

18.92 Now suppose that the price and quantity data for period t, p^t and q^t, can be written in terms of J subvectors as follows:

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[56] See for example Diewert (1978; 894) or Fisher (1922), which is reproduced in Diewert (1976; 135).

[57] Much of the initial material in this section is adapted from Diewert (1978) and Alterman, Diewert and Feenstra (1999). See also Vartia (1976a) (1976b) and Balk (1996) for a discussion of alternative definitions for the two stage aggregation concept and references to the literature on this topic.

$$(18.96)\quad p^t = (p^{t1}, p^{t2}, \ldots ,p^{tJ}) \;;\; q^t = (q^{t1}, q^{t2}, \ldots ,q^{tJ}) \;;\; t = 0,1$$

where the dimensionality of the subvectors p^{tj} and q^{tj} is N_j for j = 1,2,…,J with the sum of the dimensions N_j equal to N. These subvectors correspond to the price and quantity data for subcomponents of the export output price index for period t. Construct subindices for each of these components going from period 0 to 1. For the base period, the price for each of these subcomponents, say P_j^0 for j = 1,2,…J, is set equal to 1 and the corresponding base period subcomponent quantities, say Q_j^0 for j = 1,2,…,J, is set equal to the base period value of production for that subcomponent. For j = 1,2,…,J; i.e.,:

$$(18.97)\quad P_j^0 \equiv 1 \;;\; Q_j^0 \equiv \sum_{i=1}^{Nj} p_i^{0j} q_i^{0j} \qquad \text{for } j = 1,2,\ldots,J.$$

Now use the Laspeyres formula in order to construct a period 1 price for each subcomponent, say P_j^1 for j = 1,2,…,J, of the export price index. Since the dimensionality of the subcomponent vectors, p^{tj} and q^{tj} , differ from the dimensionality of the complete period t vectors of prices and quantities, p^t and q^t , different symbols for these subcomponent Laspeyres indices will be used, say P_L^j for j = 1,2,…J. Thus the period 1 subcomponent Laspeyres price indices are defined as follows:

$$(18.98)\quad P_j^1 \equiv P_L^j(p^{0j},p^{1j},q^{0j},q^{1j}) \equiv \sum_{i=1}^{Nj} p_i^{1j} q_i^{0j} / \sum_{i=1}^{Nj} p_i^{0j} q_i^{0j} \quad \text{for } j = 1,2,\ldots J.$$

Once the period 1 J price subindices have been defined by(18.98), then corresponding J subcomponent period 1 quantity indices Q_j^1 for j = 1,2,…,J can be defined by deflating the period 1 subcomponent values $\sum_{i=1}^{Nj} p_i^{1j} q_i^{1j}$ by the price indices P_j^1 defined by(18.98); i.e.,:

$$(18.99)\quad Q_j^1 \equiv \sum_{i=1}^{Nj} p_i^{1j} q_i^{1j} / P_j^1 \qquad \text{for } j = 1,2,\ldots,J.$$

Subcomponent price and quantity vectors for each J in each period t = 0,1 can now be defined using equations(18.97) to(18.99) above. Define the period 0 and 1 subcomponent price vectors P^0 and P^1 as follows:

$$(18.100)\quad P^0 = (P_1^0, P_2^0,\ldots,P_J^0) \equiv 1_J \;;\; P^1 = (P_1^1, P_2^1,\ldots,P_J^1)$$

where 1_J denotes a vector of ones of dimension J and the components of P^1 are defined by(18.98). The period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

$$(18.101)\quad Q^0 = (Q_1^0, Q_2^0,\ldots,Q_J^0) \;;\; Q^1 = (Q_1^1, Q_2^1,\ldots,Q_J^1)$$

where the components of Q^0 are defined in(18.97) and the components of Q^1 are defined by(18.99). The price and quantity vectors in(18.100) and(18.101) represent the results of the first stage aggregation. These vectors can now be used as inputs into the second stage aggregation problem; i.e., the Laspeyres price index formula can be applied using the information in(18.100) and(18.101) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second stage aggregation problem have dimension J instead of the first stage formula each of which utilized vectors of dimension N_j, a different symbol is needed for our new Laspeyres price index which is chosen to be P_L^*. Thus the Laspeyres price index computed in two stages can be denoted as $P_L^*(P^0,P^1,Q^0,Q^1)$. It is now appropriate to ask whether this two stage Laspeyres price index equals the corresponding single stage price index P_L that was studied in the previous sections of this chapter; i.e., whether

$$(18.102)\quad P_L^*(P^0,P^1,Q^0,Q^1) = P_L(p^0,p^1,q^0,q^1).$$

If the Laspeyres formula is used at each stage of each aggregation, the answer to the above question is yes: straightforward calculations show that the Laspeyres index calculated in two stages equals the Laspeyres index calculated in one stage. However, the answer is yes if the Paasche formula is used at each stage of aggregation; i.e., the Paasche formula is consistent in aggregation just like the Laspeyres formula.

18.93 Now suppose the Fisher or Törnqvist formula is used at each stage of the aggregation; i.e., in equations(18.98), suppose the Laspeyres formula $P_L^j(p^{0j},p^{1j},q^{0j},q^{1j})$ is replaced by the Fisher formula $P_F^j(p^{0j},p^{1j},q^{0j},q^{1j})$ (or by the Törnqvist formula $P_T^j(p^{0j},p^{1j},q^{0j},q^{1j})$ and in equation (18.102), $P_L^*(P^0,P^1,Q^0,Q^1)$ is replaced by P_F^* (or by P_T^*) and $P_L(p^0,p^1,q^0,q^1)$ replaced by P_F (or by P_T). Then do counterparts to the two stage aggregation result for the Laspeyres formula,(18.102) hold? The answer is no; it can be shown that, in general,

$$(18.103)\quad P_F^*(P^0,P^1,Q^0,Q^1) \neq P_F(p^0,p^1,q^0,q^1) \text{ and } P_T^*(P^0,P^1,Q^0,Q^1) \neq P_T(p^0,p^1,q^0,q^1).$$

Similarly, it can be shown that the quadratic mean of order r index number formula P^r defined by(18.84) and the implicit quadratic mean of order r index number formula P^{r*} defined by(18.81) are also not consistent in aggregation.

18.94 However, even though the Fisher and Törnqvist formulae are not *exactly* consistent in aggregation, it can be shown that these formulae are *approximately* consistent in aggregation. More specifically, it can be shown that the two stage Fisher formula P_F^* and the single stage Fisher formula P_F in(18.103), both regarded as functions of the 4N variables in the vectors p^0,p^1,q^0,q^1, approximate each other to the second order around a point where the two price vectors are equal (so that $p^0 = p^1$) and where the two quantity vectors are equal (so that $q^0 = q^1$) and a similar result holds for the two stage and single stage Törnqvist indices in(18.103).[58] As it was shown in the previous section, the

[58] See Diewert (1978; 889), who utilized some results due to Vartia (1976a) (1976b). In other words, a string of equalities similar to(18.88)-(18.93) hold between the two stage indices and their single stage

single stage Fisher and Törnqvist indices have a similar approximation property so all four indices in(18.103) approximate each other to the second order around an equal (or proportional) price and quantity point. Thus for normal time series data, single stage and two stage Fisher and Törnqvist indices will usually be numerically very close.[59] This result is illustrated in Chapter 20 below for an artificial data set.

18.95 A similar approximate consistency in aggregation results (to the results for the Fisher and Törnqvist formulae explained in the previous paragraph) can be derived for the *quadratic mean of order r indices*, P^r, and for the implicit quadratic mean of order r indices, P^{r*}; see Diewert (1978; 889). However, the results of Hill (2006) again imply that *the second order approximation property of the single stage quadratic mean of order r index P^r to its two stage counterpart will break down as r approaches either plus or minus infinity*. To see this, consider a simple example where there are only four commodities in total. Let the first price relative p_1^1/p_1^0 be equal to the positive number a, let the second two price relatives p_i^1/p_i^0 equal b and let the last price relative p_4^1/p_4^1 equal c where it is assumed that a < c and a ≤ b ≤ c. Using a result in Hill's result (2006), the limiting value of the single stage index is:

$$(18.104)\quad \lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1)$$
$$= [\min_i\{p_i^1/p_i^0\}\max_i\{p_i^1/p_i^0\}]^{1/2}$$
$$= [ac]^{1/2}.$$

Now if commodities 1 and 2 are aggregated into a subaggregate and commodities 3 and 4 into another subaggregate. Using Hill's result again, it is found that the limiting price index for the first subaggregate is $[ab]^{1/2}$ and the limiting price index for the second subaggregate is $[bc]^{1/2}$. Now apply the second stage of aggregation and use Hill's result once again to conclude that the limiting value of the two stage aggregation using P^r as our index number formula is $[ab^2c]^{1/4}$. Thus the limiting value as r tends to plus or minus infinity of the single stage aggregate over the two stage aggregate is $[ac]^{1/2}/[ab^2c]^{1/4} = [ac/b^2]^{1/4}$. Now b can take on any value between a and c and so the ratio of the single stage limiting P^r to its two stage counterpart can take on any value between $[c/a]^{1/4}$ and $[a/c]^{1/4}$. Since c/a is less than 1 and a/c is greater than 1, it can be seen that the ratio of the single stage to the two stage index can be arbitrarily far from 1 as r becomes large in magnitude with an appropriate choice of the numbers a, b and c.

18.96 The results in the previous paragraph show that caution is required in assuming that *all* superlative indices will be approximately consistent in aggregation. However, for the three most commonly used superlative indices (the Fisher ideal P_F, the Törnqvist-Theil P_T and the Walsh P_W), the available empirical evidence indicates that these indices

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counterparts. In fact, these equalities are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.

[59] For an empirical comparison of the four indices, see Diewert (1978; 894-895). For the Canadian consumer data considered there, the chained two stage Fisher in 1971 was 2.3228 and the corresponding chained two stage Törnqvist was 2.3230, the same values as for the corresponding single stage indices.

satisfy the consistency in aggregation property to a sufficiently high enough degree of approximation that users will not be unduly troubled by any inconsistencies.[60]

F Import Price Indices

F.1 The Economic Import Price Index for an Establishment

18.97 Attention is now turned to the economic theory of the *import input price index for an establishment*. Note the nomenclature: it is an import price index that treats imports as inputs to a resident producing unit. This theory is analogous to the economic theory of the export output price index explained in sections D and E above but now uses the *joint cost function* or the *conditional cost function* C in place of the revenue function R that was used in section D and the behavioral assumption of minimizing costs as opposed to maximizing revenue. Our approach in this section turns out to analogous to the Konüs (1924) theory for the true cost of living index in consumer theory.

18.98 Recall that in section D above, the set S^t described the technology of the establishment. Thus if (y,x,z,m,v) belongs to S^t, then the nonnegative output vectors y of domestic outputs and x of exports can be produced by the establishment in period t if it can utilize the nonnegative vectors of z of domestic intermediate inputs, m of imports and v of primary inputs.

18.99 Let $p_m \equiv (p_{m1},\ldots p_{mN})$ denote a positive vector of import prices that the establishment might face in period t,[61] let y be a nonnegative vector of domestic output targets, x be a vector of export targets and let z and v be a nonnegative vectors of domestic intermediate inputs and primary inputs respectively that the establishment might have available for use during period t. Then the establishment's *conditional import cost function* using period t technology is defined as the solution to the following import cost minimization problem:

$$(18.105)\quad C^t(p_x,y,x,z,v) \equiv \min{}_x \{\sum_{n=1}^{N} p_{xn}m_n : (y,x,z,m,v) \text{ belongs to } S^t\}.$$

Thus $C^t(p_x,y,x,z,v)$ is the minimum import cost, $\Sigma_n\, p_{xn}m_m$, that the establishment must pay in order to produce the vectors of outputs y and x, given that it faces the vector of

[60] See Chapter 19 below for some additional evidence on this topic.

[61] From the viewpoint of economic theory, these prices should include all taxes and transportation margins, since when the establishment chooses its cost minimizing import quantities, what is relevant is the total cost of delivering these inputs to the establishment door. However, as was seen in section B above, it often does no harm if these total import cost prices are decomposed into two or more separate terms, with the foreign price shown as one term and the tax and transportation terms shown as additional terms. However, these tax and transportation margin terms will affect establishment behavior according to the economic approach to price indices and so these terms cannot be ignored.

intermediate input prices p_x and given that it has the input vectors z and v available for use, using the period t technology.[62]

18.100 In order to make the notation for the import price index comparable to the notation used in previous chapters for price and quantity indices, in the remainder of this subsection, the import price vector p_m is replaced by the vector p and the vector of import quantities m is replaced by the vector q. Thus $C^t(p_m,y,x,z,v)$ is rewritten as $C^t(p,y,x,z,v)$. In order to further simplify the notation, the entire vector of reference quantities, [y,x,z,v], will be written as the composite quantity reference vector u. Thus $C^t(p,y,x,z,v)$ is rewritten as $C^t(p,u)$.

18.101 The period t conditional import input cost function C^t can be used to define the economy's *period t technology import price index* P^t between any two periods, say period 0 and period 1, as follows:

$$(18.106)\quad P^t(p^0,p^1,u) = C^t(p^1,u)/C^t(p^0,u)$$

where p^0 and p^1 are the vectors of import prices that the establishment faces in periods 0 and 1 respectively and u is the reference vector of establishment quantities defined in the previous paragraph.[63] If N = 1 so that there is only one imported commodity that the establishment uses, then it can be shown that the import price index collapses down to the single import price relative between periods 0 and 1, $p_1^{\,1}/p_1^{\,0}$. In the general case, note that the import price index defined by(18.106) is a ratio of hypothetical import costs that the establishment must pay in order to produce the vector of domestic outputs y and the vector of exports x, given that it has the period t technology, the vector of domestic intermediate inputs z and the vector of primary inputs v to work with. The numerator in(18.106) is the minimum import cost that the establishment could attain if it faced the import prices of period 1, p^1, while the denominator in(18.106) is the minimum import cost that the establishment could attain if it faced the import prices of period 0, p^0. Note that all variables in the numerator and denominator of(18.106) are held constant except the vectors of intermediate import input prices.

18.102 As was the case with the theory of the export price index, there are a wide variety of price indices of the form(18.106) depending on which (t,y,x,z,v) reference quantity vector is chosen; (the reference technology is indexed by t, the reference domestic output vector is indexed by y, the reference export vector is indexed by x, the reference domestic intermediate input vector is indexed by z and the reference primary input vector is indexed by v). As in the theory of the export price index, two special cases of the general definition of the import price index(18.106) are of interest: (i) $P^0(p^0,p^1,u^0)$ which uses the period 0 technology set, the output vector y^0 that was actually produced in period 0, the export vector x^0 that was produced in period 0 by the establishment, the domestic

[62] See McFadden (1978) for the mathematical properties of a conditional cost function. Alternatively, we note that $-C^t(p_m,y,x,z,v)$ has the same mathematical properties as the revenue function R^t defined earlier in this chapter.
[63] This concept of the import price index is the same as the concept defined in Alterman, Diewert and Feenstra (1999). This concept is related to the physical production cost index defined by Court and Lewis (1942-43; 30).

intermediate vector z^0 that was used in period 0 and the primary input vector v^0 that was used in period 0 and (ii) $P^1(p^0,p^1,u^1)$ which uses the period 1 technology set and reference quantities u^1. Let q^0 and q^1 be the observed import quantity vectors for the establishment in periods 0 and 1 respectively. If there is import cost minimizing behavior on the part of the producer in periods 0 and 1, then the observed import cost in periods 0 and 1 should be equal to $C^0(p^0,u^0)$ and $C^1(p^1,u^1)$ respectively; i.e., the following equalities should hold:

$$(18.107)\quad C^0(p^0,u^0) = \sum_{m=1}^{M} p_m{}^0 q_m{}^0 \quad \text{and} \quad C^1(p^1,u^1) = \sum_{m=1}^{M} p_m{}^1 q_m{}^1 .$$

18.103 Under these cost minimizing assumptions, the arguments of Fisher and Shell (1972; 57-58) and Archibald (1977; 66) can again be adapted to show that the two theoretical indices, $P^0(p^0,p^1,u^0)$ and $P^1(p^0,p^1,u^1)$ described in (i) and (ii) above, satisfy the following inequalities(18.108) and(18.109):

$$(18.108)\quad P^0(p^0,p^1,y^0,z^0) \equiv C^0(p^1,y^0,z^0)/C^0(p^0,y^0,z^0) \quad \text{using definition(18.106)}$$

$$= C^0(p^1,y^0,z^0)/ \sum_{m=1}^{M} p_m{}^0 q_m{}^0 \quad \text{using(18.107)}$$

$$\leq \sum_{m=1}^{M} p_m{}^1 q_m{}^0 \Big/ \sum_{m=1}^{M} p_m{}^0 q_m{}^0 \quad \text{since } q^0 \text{ is feasible for the minimization}$$

problem which defines $C^0(p^1,u^0)$ and so $C^0(p^1,u^0) \leq \sum_{m=1}^{M} p_m{}^1 q_m{}^0$

$$\equiv P_L(p^0,p^1,q^0,q^1)$$

where P_L is the Laspeyres import input price index. Similarly,:

$$(18.109)\quad P^1(p^0,p^1,y^1,z^1) \equiv C^1(p^1,u^1)/C^1(p^0,u^1) \quad \text{using definition(18.106)}$$

$$= \sum_{m=1}^{M} p_m{}^1 q_m{}^1 / C^1(p^0,u^1) \quad \text{using(18.107)}$$

$$\geq \sum_{m=1}^{M} p_m{}^1 q_m{}^1 \Big/ \sum_{m=1}^{M} p_m{}^0 q_m{}^1 \quad \text{since } q^1 \text{ is feasible for the minimization}$$

problem which defines $C^1(p^0,u^1)$ and so $C^1(p^0,u^1) \leq \sum_{m=1}^{M} p_m{}^0 q_m{}^1$

$$\equiv P_P(p^0,p^1,q^0,q^1)$$

where P_P is the Paasche import price index. Thus the inequality(18.108) says that the observable Laspeyres index of import prices P_L is an *upper bound* to the theoretical import index $P^0(p^0,p^1,u^0)$ and the inequality(18.109) says that the observable Paasche index of import prices P_P is a *lower bound* to the theoretical import price index $P^1(p^0,p^1,u^1)$. Note that these inequalities are the reverse of our earlier inequalities(18.32) and(18.33) that was found for the export price index but our new inequalities are analogous to their counterparts in the theory of the true cost of living index.

18.104 As was the case in section D.2 above, it is possible to define a theoretical import price index that falls *between* the observable Paasche and Laspeyres intermediate input price indices. To do this, first define a *hypothetical import cost function*, $C(p,\alpha)$, that corresponds to the use of an α weighted average of the technology sets S^0 and S^1 for periods 0 and 1 as the reference technology and that uses an α weighted average of the period 0 and period 1 reference quantity vectors, u^0 and u^1:

$$(18.110)\ C(p,\alpha) \equiv \min_q \{ \sum_{m=1}^{M} p_m q_m : [q, (1-\alpha)u^0 + \alpha u^1] \text{ belongs to } (1-\alpha)S^0 + \alpha S^1 \}.$$

Thus the intermediate import input cost minimization problem in(18.110) corresponds to the α and $(1-\alpha)$ weighted average of the reference quantity target vectors, $(1-\alpha)u^0 + \alpha u^1$ where the period 0 reference quantity vector u^0 gets the weight $1-\alpha$ and the period 1 reference quantity vector u^1 gets the weight α, where α is a number between 0 and 1. The new import cost function defined by(18.110) can now be used to define the following *family of theoretical intermediate import input price indices*:

$$(18.111)\ P(p^0,p^1,\alpha) \equiv C(p^1,\alpha)/C(p^0,\alpha).$$

18.105 Adapting the proof of Diewert (1983; 1060-1061) shows that there exists an α between 0 and 1 such that the theoretical import price index defined by(18.111) lies between the observable (in principle) Paasche and Laspeyres import price indices, P_P and P_L ; i.e., there exists an α such that

$$(18.112)\ P_L \le P(p^0,p^1,\alpha) \le P_P \quad \text{or} \quad P_P \le P(p^0,p^1,\alpha) \le P_L .$$

18.106 If the Paasche and Laspeyres indices are numerically close to each other, then (18.112) tells us that a "true" economic import price index is fairly well determined and a reasonably close approximation to the "true" index can be found by taking a symmetric average of P_L and P_P such as the geometric average which again leads to Irving Fisher's (1922) ideal price index, P_F defined earlier by(18.37).

18.107 It is worth noting that the above theory of an economic import price index was very general; in particular, no restrictive functional form or separability assumptions were made on the technology.

18.108 The arguments used in section D.3 above to justify the use of the Törnqvist Theil export price index as an approximation to a theoretical export price index can be adapted to yield a justification for the use of the Törnqvist Theil import price index as an approximation to a theoretical import price index. Recall the definition of the period t conditional import cost function, $C^t(p_x,y,x,z,v) \equiv C^t(p,u)$, defined by(18.105) above. Now assume that the period t conditional import cost function has the following *translog functional form* : for t = 0,1 :

$$(17.113)\ \ln C^t(p,u) = \alpha_0^t + \sum_{n=1}^{N} \alpha_n^t \ln p_n + \sum_{k=1}^{3N+K} \beta_k^t \ln u_k + (1/2)\sum_{n=1}^{N}\sum_{j=1}^{N} \alpha_{nj}^t \ln p_n \ln p_j + \sum_{n=1}^{N}\sum_{k=1}^{3N+K} \beta_{nk}^t \ln p_n \ln u_k + (1/2)\sum_{k=1}^{3N+K}\sum_{j=1}^{3N+K} \gamma_{kj}^t \ln u_k \ln u_j$$

where the coefficients satisfy the following restrictions:

$$(18.114)\ \alpha_{nj}^t = \alpha_{jn}^t \quad \text{for all n,j and for t = 0,1;}$$
$$(18.115)\ \gamma_{kj}^t = \gamma_{jk}^t \quad \text{for all k,j and for t = 0,1;}$$
$$(18.116)\ \sum_{n=1}^{N} \alpha_n^t = 1 \quad \text{for t = 0,1;}$$
$$(18.117)\ \sum_{j=1}^{N} \alpha_{nj}^t = 0 \quad \text{for t = 0,1 and n = 1,2,…,N;}$$
$$(18.118)\ \sum_{n=1}^{N} \beta_{nk}^t = 0 \quad \text{for t = 0,1 and k = 1,2,...,3N+K.}$$

The restrictions(18.116),(18.117) and(18.118) are necessary to ensure that $C^t(p,u)$ is linearly homogeneous in the components of the import price vector p (which is a property that a conditional cost function must satisfy). Note that at this stage of the argument the coefficients that characterize the technology in each period (the α's, β's and γ's) are allowed to be completely different in each period.

18.109 Adapting again the result in Caves, Christensen and Diewert (1982; 1410) to the present context[64]: if the quadratic price coefficients in(18.113) are equal across the two periods where an index number comparison (i.e., $\alpha_{nj}^0 = \alpha_{nj}^1$ for all n,j) is being made, then the geometric mean of the economic import price index that uses period 0 technology and period 0 reference quantities, $P^0(p^0,p^1,u^0)$, and the economic import price index that uses period 1 technology and period 1 reference quantities, $P^1(p^0,p^1,u^1)$, is *exactly* equal to the Törnqvist import price index P_T defined by(18.38) above[65]; i.e.,

$$(18.119)\ P_T(p^0,p^1,q^0,q^1) = [P^0(p^0,p^1,u^0)\, P^1(p^0,p^1,u^1)]^{1/2}\,.$$

18.110 As was the case with the previous result(18.112), the assumptions required for the result(18.119) seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and the assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic import price index and this corresponds to a flexible functional form, the Törnqvist import index number formula is said to be *superlative*.

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[64] The Caves, Christensen and Diewert translog exactness result is slightly more general than a similar translog exactness result that was obtained earlier by Diewert and Morrison (1986; 668); Diewert and Morrison assumed that all of the quadratic terms in(18.113) were equal to each other during the two periods under consideration whereas Caves, Christensen and Diewert assumed only that $\alpha_{nj}^0 = \alpha_{nj}^1$ for all n,j. See Kohli (1990) for closely related results.

[65] Of course, in the present context, export prices are replaced by import prices.

18.111 It is possible to adapt the analysis of the output price index that was developed in sections E.3 and E.4 above to the import price index and show that the two families of superlative output price indices, P^{r*} defined by(18.81) and P^r defined by(18.84), are also superlative import price indices. However, the details are omitted here since in order to derive these results, rather restrictive separability restrictions are required on the technology of the establishment.[66]

18.112 For the reader who has read Chapter 17 in the *PPI Manual*, the above economic theories for the import price index for an establishment will seem very similar to the economic approaches to the *intermediate input price index* that appeared in that Manual. In fact, the theories are exactly the same; only some of the terminology has changed. Also, another way of viewing the establishment import price index is as a *subindex* of a comprehensive intermediate input price index that encompasses both domestically and foreign sourced intermediate inputs that are used by the establishment.

In the following section, the analysis presented in this section is modified to provide an economic approach to determining a household import price index.

F.2 The Economic Import Price Index for a Household

18.113 The theory of the cost of living index for a single consumer (or household) was first developed by the Russian economist, A. A. Konüs (1924). This theory relies on the assumption of *optimizing behavior* on the part of households. Thus given a vector of commodity prices p^t that the household faces in a given time period t, this approach assumes that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves the consumer's preference or utility function F.[67] Thus in contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors q^0 and q^1 are independent of the two price vectors p^0 and p^1. In the economic approach, the period 0 quantity vector q^0 is determined by the consumer's preference function F and the period 0 vector of prices p^0 that the consumer faces and the period 1 quantity vector q^1 is determined by the consumer's preference function f and the period 1 vector of prices p^1.

18.114 This household cost of living approach to an import price index is necessary in the present context because a small proportion of household consumption does not pass through the domestic production sector of the economy. The main expenditures of this type are tourist expenditures made abroad by domestic residents. In some countries expenditure on cross-border shopping may be a significant proportion of aggregate household consumption expenditure.

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[66] The counterpart to our earlier separability assumption(18.44) is now: $v_1 = F^t(y,x,z,m,v_2,...,v_K) = G^t(y,x,z,f(m),v_2,...,v_K)$ for t = 0,1 where the import aggregator function f is linearly homogeneous and independent of t.

[67] For a description of the economic theory of the input and output price indexes, see Balk (1998). In the economic theory of the output price index, q^t is assumed to be the solution to a revenue maximization problem involving the output price vector p^t.

18.115 It is assumed that a household has preferences over combinations of imported goods and services, $m \equiv (m_1,...,m_N)$, and domestically supplied goods and services, $y \equiv (y_1,...,y_N)$ and these preferences can be represented by the utility function, $u = F(m,y)$, where u is the utility the household receives if it consumes the services of the import vector m and the domestically supplied commodities y.

18.116 Given a target utility level u and a vector of domestic commodity availabilities, y, and given that the household faces the import price vector p_m, the *consumer's conditional import cost function* is defined as follows:

$$(18.120)\ C(p_m,y,u) \equiv \min{}_m \{\sum_{n=1}^{N} p_{mn}m_n : F(m,y) = u\}.$$

18.117 As usual, in order to make the notation in this chapter more comparable to the notation used in previous chapters, the import vector m will be replaced by the quantity vector q and the import price vector p_m will be replaced by the vector p.

18.118 Suppose the household faces the import price vector p^0 in period 0 and p^1 in period 1. Suppose also that the household has available the domestic quantity vector y for use in both periods. Finally suppose that the household wants to achieve the same standard of living in each period; i.e., the household wants to achieve the utility level u in each period at minimum import cost. Under these conditions, the household's conditional import cost function defined above can be used in order to define the following family of *household import price indices*:

$$(18.121)\ P(p^0,p^1,y,u) \equiv C(p^1,y,u)/C(p^0,y,u).$$

18.119 There is a family of household import price indices; i.e., as the standard of living indexed by the utility level u changes and as the reference vector of domestic quantity availabilities y changes, the import price index defined by(18.121) will change.

18.120 It is natural to choose two specific reference quantity vectors y and reference utility levels in definition(18.121): the observed base period domestic quantity vector y^0 that the household had available in period 0 along with the period 0 level of utility that was achieved by the household, u^0, and the period 1 counterparts, y^1 and u^1. It is also reasonable to assume that the household period 0 observed import vector $m^0 = q^0$, solves the following period 0 conditional cost minimization problem:

(18.122) $C(p^0,y^0,u^0) \equiv \min_q \{\sum_{n=1}^{N} p_n^0 q_n : F(q,y^0) = u^0\} = \sum_{n=1}^{N} p_n^0 q_n^0$.

18.121 Similarly, it is reasonable to assume that the household period 1 observed import vector $m^1 = q^1$, solves the following period 1 conditional cost minimization problem:

(18.123) $C(p^1,y^1,u^1) \equiv \min_q \{\sum_{n=1}^{N} p_n^1 q_n : F(q,y^1) = u^1\} = \sum_{n=1}^{N} p_n^1 q_n^1$.

Using assumptions(18.122) and(18.123), it is easy to establish the following bounds on two special cases of the family of import price indices defined by(18.121).

18.122 Consider the import price index that results when u is set equal to u^0 and y is set equal to y^0:

$P(p^0,p^1,y^0,u^0) \equiv C(p^1,y^0,u^0)/C(p^0,y^0,u^0)$

$= C(p^1,y^0,u^0)/\sum_{n=1}^{N} p_n^0 q_n^0$ using(18.122)

$= \min_q \{\sum_{n=1}^{N} p_n^1 q_n : F(q,y^0) = u^0\}/\sum_{n=1}^{N} p_n^0 q_n^0$ using definition(18.120)

$\leq \sum_{n=1}^{N} p_n^1 q_n^0 / \sum_{n=1}^{N} p_n^0 q_n^0$

since $q^0 \equiv (q_1^0,\ldots,q_N^0)$ is feasible for the minimization problem

$= P_L(p^0,p^1,q^0,q^1)$

where P_L is the Laspeyres price index defined in earlier chapters.[68]

18.123 The second of the two natural choices for a reference domestic quantity vector y and utility level u in definition(18.121) is y^1 and u^1. In this case the household import price index becomes:

[68] This type of inequality was first obtained by Konüs (1924) (1939; 17). See also Pollak (1983).

$P(p^0,p^1,y^1,u^1) \equiv C(p^1,y^1,u^1)/C(p^0,y^1,u^1)$

$= \sum_{n=1}^{N} p_n^1 q_n^1 / C(p^0,y^1,u^1)$ using(18.123)

$= \sum_{n=1}^{N} p_n^1 q_n^1 / \min_q \{\sum_{n=1}^{N} p_n^0 q_n : F(q,y^1) = u^1\}$ using definition(18.120)

$\geq \sum_{n=1}^{N} p_n^1 q_n^1 / \sum_{n=1}^{N} p_n^0 q_n^1$

since $q^1 \equiv (q_1^1,\ldots,q_N^1)$ is feasible for the minimization problem

$= P_P(p^0,p^1,q^0,q^1)$

where P_P is the Paasche price index defined in earlier.[69]

18.124 At this stage, the reader will realize that the household theory of the import price index is more or less isomorphic to the establishment theory of the import price index that was developed in the previous section: the household conditional cost function replaces the establishment conditional cost function and the household price index concept defined by(18.121) replaces the establishment price index concept defined by(18.106). The same type of results that were established in the previous section can be established in the household context. Again, the Fisher and Törnqvist import price indices can be given strong justifications. The quadratic mean of order r price indices can also be justified in the present context with an appropriate separability assumption.[70]

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[69] This type of inequality is also due to Konüs (1924) (1939; 19). See also Pollak (1983).
[70] The counterpart to the earlier separability assumption(18.44) is now: $u = F(y,m) = G(y,f(m))$ where the import aggregator function f is linearly homogeneous.

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