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Abstract

We present a model that describes how different types of bank regulation can interact to affect the likelihood of fire sales in a crisis. In our model, risk shifting motives drive how banks recapitalize following a negative shock, leading banks to concentrate their portfolios. Regulation affects the likelihood of fire sales by giving banks the incentive to sell certain assets and retain others. Ex-post incentives from high risk weights and the interaction of capital and liquidity requirements can make fire sales more likely. Time-varying risk weights may be an effective tool to prevent fire sales.

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How is the likelihood of fire sales in a crisis affected by the interaction of various bank regulations?

Divya Kirti and Vijay Narasiman*

Abstract

We present a model that describes how different types of bank regulation can interact to affect the likelihood of fire sales in a crisis. We focus on a situation in which banks experience a shock to asset values, face binding capital requirements, and therefore must recapitalize. We assume that banks act in the interest of their shareholders, which leads to the optimal recapitalization being influenced by risk-shifting motives. This causes banks to choose to concentrate their portfolios into one asset while discarding others in the process of recapitalizing. There are three main results. First, the design of the capital requirement strongly affects whether fire sales of risky assets can occur in the recapitalization process. For example, with a simple leverage requirement, banks choose to sell relatively safe assets in response to a shock. With a risk-weighted capital requirement, high risk weights can actually push banks toward fire sales of risky assets. Second, the interaction between capital and liquidity requirements causes banks to become large in scale and can also make fire sales more likely. Third, mandatory equity issuance can be a useful policy for limiting fire sales of risky assets, but only if binding. Collectively, our findings suggest that high risk weights, and the interaction between capital and liquidity requirements, might create undesirable ex-post incentives from a macro-prudential perspective. In addition, time-varying risk weights may be a more effective macro-prudential tool than time-varying capital requirements.

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1 Introduction

It is commonly believed that the financial crisis of 2008 was made significantly worse by banks engaging in fire sales of risky assets (Brunnermeier 2009). These fire sales are generally attributed to two sources. First, short-term creditors of banks refused to roll over their loans, forcing banks to sell assets in order to repay these loans (Shleifer & Vishny 2011). Second, after an initial shock led banks to suffer losses on their holdings, banks sold assets in order to recapitalize (Hanson, Kashyap & Stein 2011). Though these two explanations for fire sales are well-accepted, there is little theoretical work on the optimal course of action for banks facing creditor withdrawals or with insufficient capital. One could argue that in the former situation, expediency might take precedence over optimality. Following a shock, however, banks generally have a longer horizon over which to recapitalize, making this situation suitable for a theoretical model.

In this paper, we model the recapitalization process in a setting where banks face a (potentially risk weighted) regulatory capital requirement. After experiencing an initial shock that causes them to fall short of the requirement, banks choose the optimal combination of asset sales and equity issuance that restores their capital ratio.[1] We then analyze how bank behavior is impacted when additional regulations, such as liquidity requirements and mandatory equity issuance, are in place. It is important to think about how regulations put in place to solve different problems interact with each other.[2] Can high risk weights be counterproductive? Can liquidity requirements interact with capital requirements in a harmful manner?

We assume that banks are risk-neutral, act in the interests of their existing shareholders, and that all assets are priced in a risk-neutral manner. Under these assumptions, we show that the optimal bank choice is shaped solely by risk-shifting motives. Intuitively, as banks act on behalf of shareholders with limited liability, actions that allow them to retain more risk, while still satisfying regulatory requirements, are desirable because value can be transferred from creditors to sharehold-

[1]We assume that the calculation of a bank's capital ratio involves marking-to-market the value of its assets. According to the new Basel III regulations, this is the case for assets designated as "trading" and "available for sale" but not "held to maturity". The first two buckets make up over 50% of total bank assets (source).

[2]The post crisis regulatory agenda includes many separate regulatory thrusts. Andy Haldane listed ten separate areas at a speech given at a 2015 conference (slides here). These different areas have largely been treated as independent or complementary of each other.

ers. Importantly, we assume that in the absence of shocks, risk-shifting motives play a very limited role in bank decision-making. In particular, the composition of the asset side of bank balance sheets is taken to be exogenous in our model before the shock takes place.

While these assumptions are admittedly strong, the purpose of our model is simply to show how risk-shifting can influence banks' recapitalization decisions in the midst of a crisis. We argue that such analysis is useful to the extent that shareholder value maximization, which is the root of risk-shifting, is an important consideration for banks. Consistent with the risk shifting view, weaker banks in the Euro area concentrated their balance sheets into domestic sovereign debt following the Euro area crisis (Acharya & Steffen 2015, Crosignani 2015).[3] Prior work on the crisis, including Hanson et al. (2011) and French, Baily, Campbell, Cochrane, Diamond, Duffie, Kashyap, Mishkin, Rajan, Scharfstein, Shiller, Shin, Slaughter, Stein & Stulz (2010), uses debt overhang as the overarching framework, which implicitly makes the same assumption. There is also a literature on risk shifting by banks in more 'normal' circumstances (Stiglitz & Weiss 1981, Hellmann, Murdock & Stiglitz 2000, Acharya & Viswanathan 2011, Dell'Ariccia, Laeven & Suarez 2016).[4] In addition, it is precisely in the midst of crises, when the probability of insolvency is nontrivial, that shareholders have the greatest ability to shift risk on to creditors. Similarly, in the absence of a crisis, shareholders are almost fully exposed to any risks that they take because the probability of insolvency is so low. Therefore, it seems appropriate for banks to incorporate risk-shifting into their post-shock recapitalization plans while focusing on other factors before the shock occurs.

Having established that the bank recapitalization process is shaped by risk-shifting motives, we then solve the model. When the banks experience a shock and face risk-weighted capital requirements, banks choose to recapitalize by concentrating their holdings into one particular asset and shedding the others. The "desired" asset is selected based on two criteria: the underlying risk of the asset's return and the asset's risk weight. This is a manifestation of risk-shifting and resembles regulatory arbitrage: banks choose to concentrate their portfolios in the asset that provides the

[3]While risk shifting is one explanation for this pattern, high expected returns following fire sales (Shleifer & Vishny 2011, Diamond & Rajan 2011) and financial repression (Becker & Ivashina 2014) might also be relevant.

[4]Bahaj & Malherbe (2016) provide empirical evidence on banks' responses to changes in capital requirements consistent with a model based on maximizing value for existing shareholders.

greatest risk relative to the amount of capital that must be held against it.

This result leads to the first main result of our paper: when banks recapitalize after a shock, the choice of risk weights in the capital requirement strongly affects whether fire sales[5] can occur in equilibrium. If all risk weights are identical, as in a simple leverage ratio requirement, then the optimal choice for banks is to sell relatively safe assets. However, if risky assets are given sufficiently high risk weights, banks will find it optimal to sell these assets in a fire sale. This result is driven by banks' risk-shifting motives. If a risky asset has a high risk weight, retaining this asset requires banks to hold relatively more capital for a given value of total assets, which represents a transfer of value from shareholders to creditors. For a high enough risk weight, banks would rather sell the risky asset and retain a lower risk-weight asset, even though it is safer, because it does not require as much capital to be held.

In contrast, when risk weights are identical, all assets have the same capital charge regardless of riskiness. In this case, risk-shifting motives lead to a simple choice: retain the risky assets and sell the relatively safer ones. These results are pertinent in light of the recently adopted Basel III capital regulations. One component of the regulations is a simple leverage ratio requirement, which lowers the risk of fire sales according to our model. However, the regulations also modify the existing regime by assigning higher risk weights to a variety of risky assets, which arguably raises the risk of fire sales.

Our second result is that if, in addition to a capital requirement, banks face a liquidity requirement that requires them to retain a minimum amount of "safe" assets, banks have an incentive to become large in scale and could be pushed toward a fire sale of risky assets in response to a shock. The rationale for the scale is that liquidity requirements force banks to hold assets they would otherwise want to sell. Banks make up for this by building up holdings of desired assets to the maximum extent possible, to dilute the holdings of the undesired assets.

[5] The term fire sale is generally used for assets that have substantial illiquidity such that if a large quantity was sold, the price would drop substantially. An example would be subprime MBS. On the other hand, if a large quantity of a relatively safe, liquid asset like GSE debt was sold, there would likely be little price impact. Therefore, we do not think of the latter situation as a fire sale.

Suppose that in the absence of the liquidity requirement, banks prefer to sell the relatively safe assets and retain risky assets. The liquidity requirement can be interpreted as diminishing the appeal of this action. Banks want to hold a concentrated portfolio of risky assets, but the liquidity requirement forces them to hold a diversified portfolio of risky and safe assets, which is costly when there is a risk-shifting motive. Alternatively, the bank can recapitalize by selling risky assets, which are likely not subject to liquidity requirements. Though this action involves holding a concentrated portfolio of safer assets, this portfolio could still be more appealing than a diversified portfolio of the risky and safe assets, especially if the liquidity requirement is strict (more diversification required) and risky assets have high risk weights (making them less desirable). This result is particularly important given that Basel III introduces enhanced liquidity requirements to be implemented alongside stricter risk-weighted capital requirements.[6]

Our first two results suggest that undercapitalized banks may respond to a shock by engaging in fire sales of risky assets. One common view is that these fire sales can be averted by forcing banks to issue equity (Hanson et al. 2011), the idea being that equity issuance recapitalizes banks while rendering it unnecessary to sell assets. Our third result is that there is theoretical justification for mandatory equity issuance. In the absence of the mandate, banks wish to concentrate their portfolios in one asset and dispose of the others to the maximum extent possible given liquidity requirements. Mandatory equity issuance forces banks to hold assets they would otherwise sell, which could mitigate fire sales of illiquid assets. However, if mandatory equity issuance levels are not large enough, banks may still have an incentive to engage in fire sales as part of the recapitalization process.

Collectively, our results suggest that the assignment of risk-weights can be an important determinant of how banks choose to recapitalize in a crisis. While assigning high risk weights to risky, illiquid assets may have favorable ex-ante incentives, doing so might generate unintended ex-post

[6] Our model does not include incompleteness in the market for insurance against aggregate risks or other features that could make liquidity requirements optimal (Allen 2014). We simply highlight potential negative side effects of combining capital and liquidity requirements.

incentives for banks to engage in fire sales, particular if liquidity requirements are in place.[7] Moreover, system-wide cyclical risk weights (CRWs) could be a powerful tool in the midst of a crisis, potentially more powerful than cyclical capital requirements (CCRs). In addition to implicitly lowering the capital requirement, lowering risk weights on certain risky, illiquid assets makes it more worthwhile for banks to retain these assets and sell safe ones, for the same risk-shifting related reasons described earlier. Therefore, CRWs might be a more effective macro-prudential tool than CCRs.

Our approach contributes to the literature by studying the joint impact of multiple types of bank regulation in a setting where banks adjust their portfolio allocation to multiple assets. Much of the literature of interactions between capital requirements and other policy tools focuses on the interaction of capital or macro-prudential regulation with monetary policy. See IMF (2013) and BOE (2015) for a discussion of the literature in this area. Closer to our work, Walther (2016) and Goodhart, Kashyap, Tsomocos & Vardoulakisd (2013) study the effects and design of capital and other types of bank regulation, including liquidity requirements. Both of these papers abstract away from multiple asset classes. Similarly, work on the role of cyclical capital requirements typically focuses on the time dimension rather than the cross section of assets (Kashyap & Stein 2004, Repullo & Suarez 2013). We emphasize the impact of the interaction of various types of bank regulation on the tradeoffs between multiple assets.

The paper proceeds as follows. In Section 2, we describe a basic version of the model in order to build intuition about the recapitalization problem that banks face in a crisis. In Section 3, we lay out a more generalized and detailed version of the model. In Section 4, we use the generalized model in Section 3 to prove various propositions related to our main results. Section 5 concludes.

2 Basic Model

In this section, we describe a very basic version of the model in order to show why banks are driven by risk-shifting motives when deciding how to recapitalize in a crisis. The analysis of the banks'

[7] Similarly, exposure limits with respect to risky assets, or risk weights that increase with concentration on the balance sheet, might have good ex-ante properties but generate bad ex-post incentives.

actual decisions will be left for the more generalized model in the Section 3.

2.1 Setup and Assumptions

There are three periods: 0, 1, and 2. In period 0, a representative bank holds assets A_0, debt with face value D_0 that matures in period 2, and adequate regulatory capital. In period 1, asset fundamental value is shocked to A_1, after which some combination of asset sales and equity issuance is done in order to restore the capital ratio. Negative asset sales (i.e. asset purchases) and negative equity issuance (equity repurchases) are permitted. After these actions, bank assets are $A_{1,post}$ (implying asset sales were $A_1 - A_{1,post}$) and the face value of debt that remains until period 2 is D_2. In period 2, the value of assets evolves from $A_{1,post}$ to A_2, where the latter is uncertain. Assets are then liquidated and debt/equity holders are paid accordingly.

We assume that all assets (including debt and equity) are priced fairly for a risk neutral investor with a discount rate of zero. This assumption implies that $E(A_2) = A_{1,post}$. In addition, the proceeds of asset sales and equity issuance in period 1 are used to buy back debt. The price at which $1 of debt is bought back in period 1 (β) must equal the value of the debt per unit face value that remains after the action is taken:

$$\beta = \frac{E\left(\min\left[A_2, D_2\right]\right)}{D_2} \tag{1}$$

Finally, in period 1 banks choose the combination of equity issuance and asset sales that maximizes the expected payoff to the existing shareholders, the only restriction on asset sales being that the bank cannot short any asset.

2.2 Objective function

In period 1, the bank chooses equity issuance (e) and asset sales ($A_1 - A_{1,post}$) to maximize the expected payoff to existing shareholders in period 2, subject to restoring its capital ratio. Therefore, the objective function is

$$
\begin{aligned}
&\left(1-\frac{e}{E\left(\max\left[A_2-D_2,0\right]\right)}\right)E\left(\max\left[A_2-D_2,0\right]\right)\\
=&E\left(\max\left[A_2-D_2,0\right]\right)-e \qquad (2)\\
=&E\left(A_2-\min\left[A_2,D_2\right]\right)-e\\
=&A_{1,post}-\beta D_2-e \qquad (3)
\end{aligned}
$$

where the last equality uses the definition of β in (1). We will show that this objective function is equivalent to minimizing the price of debt β.

Since debt is bought back at a price of β using the proceeds of asset sales and equity issuance, the following is true:

$$
D_0-D_2=\frac{1}{\beta}\left(\left(A_1-A_{1,post}\right)+e\right) \qquad (4)
$$

Substituting for D_2 into the objective (3), we get

$$
\begin{aligned}
&A_{1,post}-\beta\left(D_0-\frac{1}{\beta}\left(\left(A_1-A_{1,post}\right)+e\right)\right)-e\\
=&A_1-\beta D_0 \qquad (5)
\end{aligned}
$$

Note that A_1, the fundamental value of the assets after the shock, and D_0, the face value of the debt in period 0, do not depend on the specific action taken. As a result, the objective function is equivalent to minimizing β, the price of the debt outstanding after the action (also the price at which the debt is repaid in period 1).

This version of the objective in (5) characterizes the risk shifting problem. Since all assets are priced at fair value, there is no NPV to be gained by taking any specific action. However, different actions can result in different transfers from shareholders to creditors. Shareholders want to choose the action that minimizes this transfer, which is equivalent to minimizing the price of the remaining debt, β. There is a risk-shifting motive: shareholders want to take as much risk as possible because the creditors absorb the downside risk. This result extends to the more general version of the model described in the next section but is perhaps more easily seen in this simple setting. Banks want to pursue a recapitalization strategy that minimizes the price of its remaining debt per unit face value.

3 Generalized model

In this section, we lay out a more generalized version of the model described in the previous section. This model will allow for analysis of banks' optimal recapitalization plans and will be used to establish the main results of the paper.

3.1 Setup

There are three periods: 0, 1, and 2. In period 0, there is a continuum of identical banks, each with total assets A and debt dA. The banks hold n different types of assets, the weight and risk-weight on asset i being w_i and r_i, respectively. Regulation requires that banks maintain a risk-weighted capital ratio of at least $\theta \in (0,1)$ and we assume that this requirement is binding for banks in period 0.[8] The binding capital requirement in period 0 implies:

$$\theta = \frac{A(1-d)}{\sum_{i=1}^{n} A w_i r_i} = \frac{1-d}{\bar{r}}$$

$$\iff 1 - \theta\bar{r} = d \tag{6}$$

where $\bar{r} = \sum_{i=1}^{n} w_i r_i$. In period 1, there is an unanticipated shock to asset fundamental values. Specifically, every asset j experiences a percentage decline of $1-\lambda_j$, where $\lambda_j \in (0,1)$. The shock causes banks' total assets to fall to $\sum_{i=1}^{n} A w_i \lambda_i = \bar{\lambda} A$, causing the capital ratio to fall under the

[8] This assumption can be justified in the data by observing that banks generally maintain their capital ratios close to the requirement and tend to resist proposed increases in the required ratio.

regulatory minimum. This can be shown as follows:

$$\lambda_i < 1,\ \theta r_i < 1\ \forall i \implies \sum_{i=1}^{n} w_i \lambda_i (1-\theta r_i) < \sum_{i=1}^{n} w_i (1-\theta r_i) \tag{7}$$

$$\implies \bar{\lambda} - \theta \sum_{i=1}^{n} w_i \lambda_i r_i < 1 - \theta \bar{r} = d \tag{8}$$

$$\implies \frac{A(\bar{\lambda} - d)}{\sum_{i=1}^{n} A w_i \lambda_i r_i} < \theta \tag{9}$$

where (8) employs (6) and the left side of (9) represents the risk-weighted capital ratio of the bank after the unanticipated shock in period 1. Note the inclusion of $\theta r_i < 1\ \forall i$ in (7). We assume this is true and believe it is reasonable given the level of capital requirements and range of risk weights in the new Basel III regulations.[9]

In period 2, every asset j experiences a net return of $\frac{\eta_j}{\lambda_j} - 1$, where $\{\eta_i\}_{i=1}^{n}$ are jointly distributed according to $f(\eta_1, \eta_2, \ldots \eta_n)$, marginally distributed according to $f_i(\eta_i)$, and the support of η_i is $[0, \eta_{iH}]$ where $\eta_{iH} > 0$. We assume that in period 1, all assets are priced in a risk-neutral manner based on fundamental values with a risk-free rate of zero, which implies $E[\eta_i] = \lambda_i$. At the end of period 2, banks liquidate their assets at fundamental values, repay debt to the extent possible, and give the residual to shareholders.

In order to recapitalize in period 1, banks undertake some combination of asset sales/purchases and equity issuance/repurchase. By assumption, they choose whatever combination maximizes the expected value of existing shareholders' equity. The dollar amount of asset i sold is given by $s_i \bar{\lambda} A$, where s_i can be positive (asset sale) or negative (asset purchase). There is a no shorting constraint, meaning a bank cannot sell more of an asset than it has. There is also a maximum amount of each asset that can be purchased (the market supply of the asset). The dollar amount of equity issued is given by eA, where e can also be positive or negative. The amount of equity repurchased cannot exceed the equity value of the entire firm.

[9]http://usbasel3.com/tool/

Any cash excess (deficit) from transactions in assets and equity is offset by debt repurchase (issuance). After all transactions, the outstanding debt of the bank must be weakly positive. Like the assets the banks hold, bank debt and equity are priced in a risk-neutral manner based on fundamental values. Moreover, there is a no-arbitrage condition that the price at which debt and equity are issued or repurchased must be the same as the price of outstanding debt and equity.

3.2 The bank's problem

In this section, we formally present the bank's problem. We then show how this problem collapses into the bank simply wanting to minimize the price of its debt per unit face value, just as in basic model presented in Section 2. Finally, we eliminate some redundant constraints to produce the most parsimonious version of the problem.

3.2.1 Statement of the problem

In period 1, the bank maximizes the expected equity value of existing shareholders in period 2. This is equivalent to solving the following problem, a general version of (2):

$$\max_{\{s_i\}_{i=1}^{n},\, e} \; E\left(max\left[\sum_{i=1}^{n}\frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}) - \left(d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)\right),\; 0\right]\right) - e \tag{10}$$

subject to

$$\theta = \frac{\bar{\lambda}\left(1 - \sum_{i=1}^{n} s_i\right) - \left(d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)\right)}{\sum_{i=1}^{n} r_i(w_i\lambda_i - s_i\bar{\lambda})} \tag{11}$$

$$\beta = \frac{E\left(min\left[\sum_{i=1}^{n}\frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}),\; d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)\right]\right)}{d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)} \tag{12}$$

$$\beta d - \bar{\lambda} \leq e \leq \beta d - \bar{\lambda}\sum_{i=1}^{n} s_i \tag{13}$$

$$-\kappa_i \leq s_i\bar{\lambda} \leq w_i\lambda_i,\; \forall i \in \{1, 2, \ldots n\} \tag{14}$$

3.2.2 Objective function

The objective function (10) can be explained as follows. The term $w_i\lambda_i - s_i\bar{\lambda}$ represents the amount of asset i that the bank holds after selling or purchasing some of the asset.[10] Multiplying this quantity by $\frac{\eta_i}{\lambda_i}$ gives the value of the bank's holdings of asset i in period 2 and summing across all i gives the total value of the bank's assets.

The term $e + \bar{\lambda}\sum_{i=1}^{n} s_i$ represents the net amount the bank raises in equity issuance and asset sales. By assumption, this excess or deficit goes toward debt repurchase or issuance at the price of β. Therefore, $d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)$ is the face value of debt that remains after all transactions. The total value of the bank's equity is the greater of zero and the difference between total assets and remaining debt. The expected value of this quantity is taken (since η_i is uncertain) and in order to isolate the value to existing shareholders, the amount of equity issued is subtracted.

3.2.3 Constraints

Constraint (11) is the capital requirement that must be satisfied in period 1 after the unanticipated shock. On the right hand side, $\bar{\lambda}\left(1 - \sum_{i=1}^{n} s_i\right)$ is the value of assets that remain after all sales or purchases. As discussed before, the second term in the numerator is the face value of debt that remains after all transactions, making the numerator the book value of equity in period 1. The denominator of (11) is the bank's cumulative risk-weighted assets after all purchases and sales: the weighted sum of the amount of each asset held, where the weight on asset i is its regulatory risk weight r_i. Therefore, constraint (11) says that after all transactions in period 1, the ratio of book equity to risk-weighted assets must exceed the regulatory requirement θ. The equality condition in (11) is by assumption, though it can be shown that if constraint was an inequality, it would be binding.[11]

Constraint (12) is the definition of β, the price of the bank's outstanding debt per unit of face value after all transactions. The numerator is the expectation of what debtholders will receive in period 2: the smaller of total asset value and the face value of remaining debt. This quantity is scaled

[10] The constant A (the value of the bank's initial assets) is technically a multiplier on all of the above equations but can be dropped or canceled. The problem is invariant to the original scale of the banks.

[11] The proof involves showing that if the capital ratio exceeds θ, the bank can raise shareholder value by simply issuing debt and repurchasing equity.

by the face value of remaining debt to put β in the correct units. By assumption, β is also the price at which debt is purchased or issued, as seen in (10) and (11). Note that constraint (12) is undefined for $d-\frac{1}{\beta}\left(e+\bar{\lambda}\sum_{i=1}^{n}s_i\right)=0$. In the event this condition is true, β is simply defined by this condition.

Constraint (13) defines the bounds on the choice of e. The right inequality states that the proceeds of equity issuance (e) cannot exceed the market value of debt that was originally outstanding (βd), net of the proceeds of asset sales ($\bar{\lambda}\sum_{i=1}^{n}s_i$). This constraint is equivalent to condition that the amount of debt that remains after the proceeds of equity issuance and asset sales, $d-\frac{1}{\beta}\left(e+\bar{\lambda}\sum_{i=1}^{n}s_i\right)$, must exceed zero.

The left inequality of (13) puts a limit on how much equity can be repurchased. This inequality can be restated as $-e \leq \bar{\lambda}-\beta d$, where $-e$ is the amount of equity repurchased. The right hand side is the market value of equity that is available to be repurchased after asset sales are used to repurchase debt at the price of β:

$$\bar{\lambda}\left(1-\sum_{i=1}^{n}s_i\right)-\beta\left(d-\frac{1}{\beta}\bar{\lambda}\sum_{i=1}^{n}s_i\right)$$

which simplifies to $\bar{\lambda}-\beta d$.

Finally, constraint (14) puts limits on asset sales and purchases. The right inequality says that the amount of asset i that is sold ($s_i\bar{\lambda}$) cannot exceed the holdings of asset i after the unanticipated shock in period 1 ($w_i\lambda_i$). The left inequality says that the amount of asset i that is purchased ($-s_i\bar{\lambda}$) cannot exceed market supply κ_i. One can imagine that κ_i is large for liquid assets and small for illiquid assets.

Note that constraints (13) and (14) ensure that the numerator and denominator of β as defined in (12) are weakly positive, meaning $\beta \geq 0$. In addition, the nature of the definition of β in (12) implies $\beta \leq 1$. Therefore, we have $\beta \in [0,1]$.

3.2.4 Equity issuance feasibility

In order for equity issuance to be feasible after the unanticipated shock in period 1, the amount of equity issued must be less than or equal to the total equity value to new and old shareholders after all asset sales and debt repurchases have taken place:

$$\begin{aligned} & e \leq \bar{\lambda}\left(1-\sum_{i=1}^{n} s_i\right) - \beta\left(d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)\right) \\ \iff\ & \bar{\lambda} \geq \beta d \end{aligned}$$

In other words, the bank's assets must exceed the market value of its debt. Since (12) implies that $\beta \leq 1$, a sufficient condition for equity issuance being feasible in period 1 is $\bar{\lambda} \geq d$: after the unanticipated shock in period 1, the bank's assets must exceed the face value of its debt.

3.3 Simplifying the problem

In this section, we show how in the general model, just as in the basic model presented in section 2, the bank's objective is equivalent to minimizing β (the price of the bank's outstanding debt per unit face value). We start with the fact that for any X and Y, $max(X-Y,0) = X - min(X,Y)$. This means that the bank's objective in (10) can be rewritten as

$$E\left(\sum_{i=1}^{n} \frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}) - min\left[\sum_{i=1}^{n} \frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}), d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)\right]\right) - e$$

Using (12) and $E[\eta_i] = \lambda_i$, this becomes

$$\bar{\lambda}\left(1-\sum_{i=1}^{n} s_i\right) - \beta\left(d - \frac{1}{\beta}\left(e + \bar{\lambda}\sum_{i=1}^{n} s_i\right)\right) - e \tag{15}$$

$$= \bar{\lambda} - \beta d \tag{16}$$

Since $\bar{\lambda}$ and d are parameters that do not depend on the bank's decision, the bank's problem can be restated as

$$\min_{\{s_i\}_{i=1}^{n},\, e} \quad \beta$$

subject to (11)-(14). Intuitively, the shareholders want to take the action that minimizes the market value of the debt (per unit of face value) because that maximizes the market value of equity, given that total asset values ($\bar{\lambda}$) are unaffected by the action taken. This is the risk-shifting motive: shareholders want to take on as much risk as possible, offloading the downside on to the creditors or equivalently, lowering the price of outstanding debt. The preceding argument proves that this is in fact the only consideration. From (15), asset sales reduce the market value of assets and debt by the same amount. In addition, equity issuance adds to the total equity value of the firm but is subtracted out for the existing shareholders. Therefore, asset sales/purchases and equity issuance/repurchases only affect the objective through their effects on β.

Since the definition of β in (12) is self-referencing, it is useful to express it differently. Using (11) and noting that

$$\bar{\lambda}\left(1-\sum_{i=1}^{n} s_i\right)=\sum_{i=1}^{n}(w_i\lambda_i - s_i\bar{\lambda}) \tag{17}$$

we find

$$d-\frac{1}{\beta}\left(e+\bar{\lambda}\sum_{i=1}^{n} s_i\right)=\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda}) \tag{18}$$

Substituting (18) into (12), the objective function becomes

$$g(s_1, s_2, \ldots s_n)=\frac{E\left(min\left[\sum_{i=1}^{n}\frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}),\ \sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})\right]\right)}{\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})} \tag{19}$$

The problem can therefore be restated as

$$\min_{\{s_i\}_{i=1}^{n}} \quad g(s_1, s_2, ..., s_n)$$

subject to (11), (13), and (14), with $g(s_1, s_2, ..., s_n)$ being substituted for β. To be clear, $g(\cdot)$ is the value of β (the price per unit face value of the debt outstanding) that results from a particular choice of asset sales $\{s_1, s_2, ..., s_n\}$, assuming that the capital ratio requirement in (11) is met. Note that the choice of e is no longer part of the bank's problem. This is because e is pinned down by the choice of asset sales and (11), with $g(\cdot)$ in place of β.

3.4 Eliminating redundant constraints

In this section, we show that constraint (13) is redundant. Let $\{\{s_i\}_{i=1}^{n}, e\}$ be any feasible solution that satisfies (11) and (14), with $g(\cdot)$ in place of β. Using (17) and rearranging (11), we have

$$e = \left(d - \sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})\right) g(s_1, s_2, \ldots s_n) - \bar{\lambda}\sum_{i=1}^{n} s_i \tag{20}$$

$$= g(\cdot)d - \bar{\lambda}\sum_{i=1}^{n} s_i - g(\cdot)\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda}) \tag{21}$$

$$= g(\cdot)d - \bar{\lambda} + \left[\bar{\lambda}\left(1 - \sum_{i=1}^{n} s_i\right) - g(\cdot)\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})\right]$$

$$= g(\cdot)d - \bar{\lambda} + \left[\bar{\lambda}\left(1 - \sum_{i=1}^{n} s_i\right)(1-g(\cdot)) + g(\cdot)\sum_{i=1}^{n}\theta r_i(w_i\lambda_i - s_i\bar{\lambda})\right] \tag{22}$$

Using $g(\cdot) \geq 0$, $\theta r_i \leq 1\ \forall i$, and (14), we have

$$g(\cdot)\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda}) \geq 0$$

Combining this with (21) implies that $e \leq g(\cdot)d - \bar{\lambda}\sum_{i=1}^{n} s_i$, which is the right inequality of (13).

Using $g(\cdot) \in [0,1]$ and (14), the bracketed term in (22) is weakly positive, which implies that $e \geq g(\cdot)d - \bar{\lambda}$, which is the left inequality of (13). To summarize, any choice of $\{s_1, s_2, ..., s_n\}$ that satisfies (14) is feasible as long as e is set according to (20).

3.5 Final statement of problem

The problem can be stated as

$$\min_{\{s_i\}_{i=1}^{n}} \quad g(s_1, s_2, \ldots s_n) = \frac{E\left(min\left[\sum_{i=1}^{n}\frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}),\ \sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})\right]\right)}{\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})} \tag{23}$$

subject to

$$\begin{gathered}
-\kappa_i \leq s_i\bar{\lambda} \leq w_i\lambda_i \ \forall i \in \{1,2,...,n\} \\
s_j\bar{\lambda} < w_j\lambda_j \text{ for at least one } j \in \{1,2,...,n\}
\end{gathered} \tag{24}$$

Note that the second constraint in (24) has been introduced to eliminate the possibility of selling the entire balance sheet, which would lead to zero capital and risk-weighted assets.

In summary, the bank's problem is to choose asset sales $\{s_1, s_2, ..., s_n\}$ that minimize the price per unit face value of the outstanding debt, or $g(\cdot)$. As stated earlier, this is the action that maximizes equity value, because it minimizes the transfer made to the creditors. Importantly, apart from (24), the problem is otherwise unconstrained. This is because for any choice of $\{s_1, s_2, ..., s_n\}$, a value of e can be obtained from (20) that satisfies the constraints (11) and (13).

4 Results

In this section, we solve the model introduced in the previous section to establish the main results of the paper. To build intuition, we begin by analyzing the case in which banks hold just one asset. As in (Admati, DeMarzo, Hellwig & Pfleiderer 2013), we find that banks are indifferent over all combinations of asset sales and equity issuance that restore the capital ratio. We then move to the n-asset case and show that the solution involves banks concentrating their portfolio in the asset that permits the most risk-taking. Finally, we use the n-asset case to prove the main results of the paper regarding how risk weights, liquidity requirements, and mandatory equity issuance can affect whether banks engage in fire sales when recapitalizing in a crisis.

4.1 One asset case

Using (23) with $n = 1$, we have

$$\begin{aligned}
g(s_1) &= \frac{E\left(min\left[\frac{\eta_1}{\lambda_1}\lambda_1(1-s_1),\ (1-\theta r_1)\lambda_1(1-s_1)\right]\right)}{(1-\theta r_1)\lambda_1(1-s_1)} \\
&= \frac{E\left(min\left[\frac{\eta_1}{\lambda_1},\ 1-\theta r_1\right]\right)}{1-\theta r_1}
\end{aligned} \tag{25}$$

Note that $g(s_1)$ does not depend on s_1. Therefore, any $\{s_1, e\}$ that satisfies (20) and (24) is a valid solution. As long as the capital ratio is restored, the particular combination of asset sales and equity issuance does not affect the price of debt per unit face value and therefore does not matter.

This result is better understood by looking at the capital requirement (11) when $n = 1$:

$$\begin{aligned}
\theta &= \frac{\lambda_1(1-s_1) - \left(d - \frac{1}{\beta}(e + \lambda s_1)\right)}{r_1 \lambda_1 (1-s_1)} \\
\implies 1 - \theta r_1 &= \frac{d - \frac{1}{\beta}(e + \lambda s_1)}{\lambda_1(1-s_1)}
\end{aligned}$$

The last line shows that regardless of the choice of $\{s_1, e\}$, the ratio of the face value of debt outstanding (numerator) to assets (denominator) after any asset sales and equity issuance in period 1 equals $1 - \theta r_1$. The choice of $\{s_1, e\}$ only changes the scale of the bank's balance sheet. This is the nature of the capital requirement with one asset.

Let p be the probability of default in period 2. With probability $1-p$, the value of assets exceeds the face value of debt and creditors receive \$1 for every dollar of face value. With probability p, creditors only recover some fraction μ of the face value of debt, where μ equals the ratio of asset value to the face value of debt. With this terminology, we can express the period 1 price of debt outstanding per unit face value as

$$g(s_1) = (1-p)(1) + pE(\mu \mid \text{default})$$

Earlier, we showed that the ratio of debt outstanding to assets at the end of period 1 is fixed at $1-\theta r_1$. This means that p and $E(\mu \mid \text{default})$ will depend only on this ratio and return distribution of the asset between periods 1 and 2. They will not depend on the choice of $\{s_1, e\}$. These dependencies are clearly observed in (25). Different actions simply scale up or down how large the bank is but leave the fundamental riskiness of the debt (and therefore its price per unit of face value) unchanged.

4.2 Multiple asset case

In the one asset case, we showed that the price of the bank's debt per unit face value depends on the ratio of debt to assets at the end of period 1 and the return distribution of the asset between between periods 1 and 2, neither of which depend on the choice of $\{s_1, e\}$. Therefore, banks are indifferent between any choice of $\{s_1, e\}$ that satisfies the capital requirement. When there are two or more assets, this indifference result no longer holds.

The first reason is that if the assets have different risk weights, the ratio of debt to assets at the end of period 1 depends on the choice of $\{\{s_i\}_{i=1}^n, e\}$. If the bank chooses to sell assets with low risk weights (retain assets with high risk weights), the capital requirement forces the bank to hold less debt for the same level of assets. If the bank chooses to sell assets with high risk weights (retain assets with low risk weights), the bank is allowed to hold more debt for the same level of assets. The second reason is that if the assets have different return distributions, the distribution of the return of the bank's assets between periods 1 and 2 depends on the composition of the bank's portfolio at the end of period 1. This composition in turn depends on which assets the bank chooses to sell and retain in period 1.

4.2.1 Characterizing the solution

We have shown that the recapitalization decision is non-trivial for banks when there is more than one asset. To find the optimal action, we formally solve the problem described by (23) and (24): choosing the combination of asset sales that minimizes the price of the bank's outstanding debt per unit face value subject to no-shorting constraints, where equity issuance is pinned down by the capital requirement according to (20). The bank's decision is characterized by the following proposition.

Proposition 1. *The solution to the recapitalization problem described by (23) and (24) involves concentrating the portfolio into one asset and selling all other assets. The asset that is retained is the solution to*

$$\arg\min_i \frac{E\left(min\left[\frac{\eta_i}{\lambda_i}, 1-\theta r_i\right]\right)}{1-\theta r_i}$$

If multiple assets solve this problem, banks still choose to retain a single asset as long as there is *imperfect correlation across asset returns. However, the bank is indifferent between retaining any of the assets that solve the problem.*

Proof. See Appendix A. □

The intuition for this result is fairly straightforward. Since the objective is to minimize the price of debt, the shareholders want to take on as much risk as possible. It is therefore undesirable to hold multiple assets at once, as this could only provide unwanted diversification. Even though all assets are priced fairly, the one that allows for the most risk-taking is best for shareholders because it minimizes the transfer to creditors. As a result, the optimal choice is to sell out of the other assets entirely. A similar argument holds if two assets provide equal amounts of risk when held on their own. While the solution is not unique (the bank is indifferent between holding either asset by itself), the bank does not want to hold these assets simultaneously due to the unwanted diversification effect.

In Appendix A, one can see in (29) that if only one asset (k) is retained, the objective function takes the form

$$\frac{E\left(min\left[\frac{\eta_k}{\lambda_k},\ 1-\theta r_k\right]\right)}{1-\theta r_k} \tag{26}$$

Note that (26) does not depend s_k, which means (26) does not depend on how much of asset k the bank holds after recapitalizing. While this may seem puzzling, it is because a bank that completely discards all but one asset is equivalent to the one-asset bank analyzed in the previous section. We showed that in this case, the bank is indifferent between all actions that satisfy the capital requirement.

The decision of which asset to sell involves comparing the value of (26) across all assets. The comparison is equivalent to the following. In period 1, suppose there are n banks. Each bank holds only one asset, holds a different asset than all other banks, and is adequately capitalized. Note that these hypothetical banks are what would result from choosing to sell all assets other than one in the course of recapitalizing. The relevant question is: which bank's debt has the lowest price per unit face value? The optimal decision is to retain only the asset that this bank holds and sell all other assets. This decision can be interpreted as banks engaging in "regulatory arbitrage": concentrating the portfolio in assets that allow the most risk-taking while satisfying regulatory requirements.

It is worth noting here that Proposition 1 is quite extreme. For several reasons, it is unreasonable for banks to close out their entire holdings of all assets except for one in the process of recapitalizing (we will revisit this assumption in a later section). However, Proposition 1 is just meant to underscore the risk-shifting motives that banks face when choosing how to recapitalize in a crisis. Holding constant all of the other factors that drive banks' recapitalization decision, banks have an incentive to concentrate their portfolios in assets that allow them to take the most risk. This basic intuition leads to the main results of the paper, which we turn to next.

4.2.2 The effect of risk weights

The effect of risk weights on the recapitalization decision is summarized by the following two propositions.

Proposition 2. *For each asset k, there exists $\bar{r}_k < \frac{1}{\theta}$ such that for all $r_k > \bar{r}_k$, asset k is sold in the process of recapitalizing, holding fixed all other assets' risk weights.*

Proof. See Appendix B. □

The intuition for this result is as follows. Proposition 1 established that when deciding how to recapitalize in a crisis, banks choose to retain the single asset that minimizes the bank's price of debt per unit face value. Suppose the bank is evaluating the option of retaining asset k. As r_k rises, the probability of solvency in period 2 goes up because the bank is forced to hold less debt when risk weights on asset k are higher. However, this effect is ultimately second-order for creditors because the benefit of newly solvent states of the world in which creditors are made whole is offset by the fact that prior to the increase in r_k, creditors were just barely insolvent in these states. Creditors are benefiting only slightly in these newly solvent states in terms of the market value of debt per unit of face value.

A higher r_k also does not affect the attractiveness of retaining any other asset by itself. The only first-order effect of a higher r_k is that when banks holding asset k are forced to hold less debt, creditors recover a greater percentage of their face value in insolvent states of period 2. This is bad for shareholders, as it raises the price of debt per unit face value and constitutes a transfer from to creditors. If r_k is high enough, shareholders will eventually prefer to sell asset k and retain a different asset since retaining the former forces the bank to be underleveraged.

In risk-weighted capital requirements, "risky" assets tend to be assigned higher risk weights. This fact, along with Proposition 2, leads to the first main result of the paper: the design of capital requirements, specifically the assignment of risk weights, can affect whether banks engage in fire sales of risky assets when recapitalizing in a crisis. Holdings other things constant, the higher the risk weights are on "risky" assets, the more likely it is that banks choose to sell these risky assets in a recapitalization. If these risky assets are illiquid, fire sales could occur. In other words, while high risk weights on risky assets may have ex-ante benefits, they are not necessarily a panacea because of the adverse ex-post incentives they create.

The next proposition establishes what happens when risk weights are uniform across assets.

Proposition 3. *Suppose the risk weights of all assets are identically equal to one. If asset j's net return $(\frac{\eta_j}{\lambda_j} - 1)$ is a mean-preserving spread of the return of asset k (i.e. asset j is riskier), then the bank will not sell asset j in the process of recapitalizing.*

Proof. See Appendix C. □

Proposition 3 can be interpreted as follows: a system of uniform risk weights across assets can push banks toward the more desirable outcome of selling "safer" assets when recapitalizing in a crisis. In this paper, we denote an asset being "safer" than another if the latter's net return in period 2 is a mean-preserving spread of the former's. In other words, while a safer asset has the same expected net return as a riskier asset, it also has less volatility.

The intuition behind Proposition 3 is straightforward. When risk weights are identical, the choice of which asset to retain has no influence on the amount of leverage banks can take. Therefore, the decision for an equity-maximizing bank is simple: retain the asset whose return profile offers the most risk. Since all assets provide the same expected return of zero, an asset whose return is a mean-preserving spread of another's is attractive to shareholders because it must perform relatively well in good states of the world. In contrast, retaining a safe asset provides less upside, raises the price of the bank's debt per unit face value, and constitutes a transfers from shareholders to creditors.

4.2.3 The effect of liquidity requirements

Proposition 1 says that the solution to the recapitalization problem involves retaining just one asset and selling all of the others completely. One reason such an action would not be feasible in practice is that banks are subject to internal or regulatory liquidity requirements, whereby certain assets must be retained for liquidity purposes. In this section, we introduce a simplified version of liquidity requirements in order to understand how it changes the bank's decision from the base case in Proposition 1.

We model liquidity requirements by modifying constraint (24) as follows.

$$-\kappa_i \leq s_i \bar{\lambda} \leq \gamma_i w_i \lambda_i \ \forall i \in \{1, 2, \dots n\}\,,\ \gamma_i \in [0, 1) \tag{27}$$

The modified constraint (27) says that asset i can be sold only up to a fraction γ_i of current holdings. One can imagine that for illiquid assets $\gamma_i \approx 1$ and for liquid assets $\gamma_i << 1$, the idea being that banks must retain a certain amount of liquid assets while there is no such requirement on illiquid assets.[12] While this setup may not be how liquidity requirements work in practice, it offers a simple way of building intuition about the problem.[13]

The next proposition examines how banks recapitalize in the presence of both capital and liquidity requirements.

Proposition 4. *With constraint (27) in place of (24) to reflect liquidity requirements, the solution to the bank's recapitalization problem is an extreme point of the feasible set: sell all assets but one and expand holdings of the remaining asset, both to the maximum extent allowable under (27).*

[12] For transaction costs reasons, it may be the case that the opposite is true: banks can sell a large amount of liquid assets and only a small amount of illiquid assets. In this paper, we assume that since banks do not have to recapitalize all at once, transaction costs have less influence on banks' decision than liquidity requirements.

[13] In practice, Basel III requires banks to hold liquid assets to match short-term liabilities. This corresponds with effective limits on selling certain assets after facing a negative shock.

Proof. See Appendix D. □

The intuition underlying Proposition 4 is similar to that of Proposition 1: banks wants to take as much risk as possible. In the presence of liquidity constraints, this means expanding the holdings of one asset as much as possible while simultaneously holding as little as possible of every other asset. This produces the least diversified (riskiest) portfolio. Indeed, we can show that if there is no restriction on how much asset holdings can be built up ($\kappa = \infty$ in (27)), banks can achieve the same minimized objective as they do without liquidity requirements.[14]

To prove this, we take the limit of the objective (23) as s_k converges to $-\infty$ (expanding holdings of asset k to infinity).

$$
\begin{aligned}
&\lim_{s_k \to -\infty} \frac{E\left(min\left[\sum_{i=1}^{n} \frac{\eta_i}{\lambda_i}(w_i\lambda_i - s_i\bar{\lambda}),\ \sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})\right]\right)}{\sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})} \\
=&\lim_{s_k \to -\infty} \frac{E\left(min\left[\sum_{i\neq k} \frac{\eta_i}{\lambda_i}\frac{w_i\lambda_i - s_i\bar{\lambda}}{w_k\lambda_k - s_k\bar{\lambda}} + \frac{\eta_k}{\lambda_k},\ \sum_{i\neq k}(1-\theta r_i)\frac{w_i\lambda_i - s_i\bar{\lambda}}{w_k\lambda_k - s_k\bar{\lambda}} + 1 - \theta r_k\right]\right)}{\sum_{i\neq k}(1-\theta r_i)\frac{w_i\lambda_i - s_i\bar{\lambda}}{w_k\lambda_k - s_k\bar{\lambda}} + 1 - \theta r_k} \\
=&\frac{E\left(min\left[\frac{\eta_k}{\lambda_k},\ 1-\theta r_k\right]\right)}{1-\theta r_k}
\end{aligned}
$$

The last line is equivalent to (29), the objective when retaining only asset k in the absence of liquidity requirements. The intuition here is that building up holdings of one asset to infinity offers the same risk profile as holding a finite amount of that asset and selling all other assets entirely. While the liquidity requirements affect the structure of the bank's portfolio, they do not affect the minimized value of the objective function.

Therefore, one of the main effects of liquidity requirements is that banks are no longer invariant to their scale, as they were in Proposition 1. Instead, banks want to expand holdings of one asset

[14] This thought experiment corresponds to the case in which the bank grows by issuing long-term debt (and equity to meet the capital requirement), so there is no increase in the level of the liquidity requirement.

as much as they can in order to "dilute" holdings of other assets that cannot be sold because of liquidity requirements.

The limit analysis above shows that when asset holdings can be built up indefinitely ($\kappa = \infty$ in (27)), banks will choose to build up the same asset that is retained in the absence of liquidity requirements (see Proposition 1). However, when asset holdings cannot be built up indefinitely ($\kappa < \infty$ in (27)), the asset that banks choose to build up is not necessarily the asset that is retained in the absence of liquidity requirements.

Proposition 1 shows that if there are no liquidity requirements, asset j is retained based on two factors only: the inherent riskiness of its gross return $\frac{\eta_j}{\lambda_j}$ and the leverage it permits through its risk weight r_j. With liquidity requirements, there are two additional factors at play: the extent to which an asset's holdings can be built up (κ_j) and how small the liquidity requirements of the remaining assets are ($\gamma_k \; \forall k \neq j$).

As an example, suppose there are two assets: j and k. Asset j is risky and illiquid, meaning it has a low liquidity requirement and minimal market supply. In contrast, asset k is safer and more liquid, with a higher liquidity requirement and plenty of market supply. Suppose also that the assets' risk weights are the same. By Proposition 3, asset k should be is sold and asset j retained in the process of recapitalizing when there are no liquidity requirements. However, it is possible that this decision flips with liquidity requirements in place. The appeal of "concentrating" the portfolio in j is limited by the fact that a certain amount of diversification is unavoidable: a minimum amount of asset k must be held and holdings of asset j cannot be built up tremendously due to limited market supply. Meanwhile, it is possible to concentrate the portfolio in asset k, since its holdings can be built up substantially and asset j does not need to be held for liquidity purposes.

In summary, liquidity requirements have two main effects on banks' recapitalization decision. First, banks have an incentive to become very large in the process of accumulating one asset and diluting the holdings of others. Second, the desire to retain risky assets for risk-shifting purposes may be undone if there is not an abundant supply of these assets or the liquidity requirements on other asset are high. If either of these is true, banks may choose to sell risky assets en masse (a fire sale) in the process of recapitalizing in a crisis.

4.2.4 Mandatory equity issuance

In this section, we explore how banks' recapitalization decisions are affected by mandatory equity issuance when both capital and liquidity requirements are in place. The principal issue is whether mandatory equity issuance can prevent fire sales of illiquid assets that banks would otherwise engage in as part of their recapitalization decisions.

Recall that when equity issuance is not mandatory, banks can choose any combination of asset sales/purchases $\{s_i\}_{i=1}^n$ that complies with (27). Based on this choice of $\{s_i\}_{i=1}^n$, equity issuance is pinned down by (20). When there is mandatory equity issuance of $\bar{e}$, the bank's choice of $\{s_i\}_{i=1}^n$ is subject to the following additional constraint.[15]

$$\left(d - \sum_{i=1}^{n}(1-\theta r_i)(w_i\lambda_i - s_i\bar{\lambda})\right) g(s_1, s_2, \ldots s_n) - \bar{\lambda}\sum_{i=1}^{n} s_i \geq \bar{e} \tag{28}$$

The left hand side of (28) is the same as the right hand side of (20). The first term is the market value of the debt that is repurchased and the second term is the net proceeds of asset sales. Any difference between these two values must be made up with equity issuance, which the constraint requires to be greater than $\bar{e}$.

It is difficult to articulate a general solution to the bank's problem with mandatory equity issuance due to the dependence on multiple parameter values (the issuance requirement $\bar{e}$, the liquidity requirement γ_i for asset i, and the market supply κ_i of asset i). Instead, we can describe the nature of the deviations that mandatory equity issuance causes when it is binding, compared to what banks choose to do when they are not subject to (28) (see Proposition 4). We can also describe whether there are situations in which mandatory equity issuance is ineffective (i.e. non-binding) and if so, whether banks engage in fire sales of illiquid assets in such situations.

[15] Note that $\bar{e}$ is expressed as a fraction of the bank's total assets.

First, we analyze the binding case. Suppose that a bank has implemented its optimal recapitalization plan according to Proposition 4, meaning the bank has built up holdings of one asset and sold the remaining, both to the maximum extent possible under (27). Since mandatory equity issuance is binding, the bank's choice of e in (20) is less than $\bar{e}$ and it must issue more equity. The main impact of additional issuance is that the bank is forced to use the proceeds to purchase assets. But because the bank has already expanded holdings of its most desired asset to the maximum extent possible, the proceeds of equity issuance can only be used to "buy back" assets the bank originally wanted to sell.

While this is bad for the bank's shareholders, who are forced to have a more diversified asset portfolio, mandatory equity issuance also causes banks to engage in less selling of the assets that would otherwise be sold to the maximum extent possible. If these "undesired" assets happen to be illiquid, mandatory equity issuance can reduce the severity of fire sales that might otherwise occur in the process of bank recapitalization.

It is also possible that the constraint imposed by mandatory equity issuance is not binding, i.e. banks choose to issue sufficient equity on their own. Since equity issuance equals the difference between debt repurchases and asset sales, mandatory equity issuance would be non-binding if either debt repurchases are large, the bank is a large net buyer of assets, or both. None of these circumstances rule out fire sales of illiquid assets though, since even if a bank is net buyer of assets it could be selling certain assets in large quantities.

The conclusion from this section is that mandatory equity issuance can potentially mitigate the severity of fire sales when it is binding by forcing banks to do less net selling of potentially illiquid assets. However, if the constraint is not binding, fire sales may still occur. This suggests that mandatory equity issuance amounts should be set aggressively by policymakers.

5 Conclusion

In this paper, we model how banks that are subject to capital requirements choose to recapitalize after being hit by a shock to their asset values. We then use the model to assess how policy levers such as risk weights, liquidity requirements, and mandatory equity issuance can affect whether banks engage in fire sales of illiquid assets in the process of recapitalizing.

We first show that if banks act in the interest of shareholders, they will be influenced by risk-shifting motives and choose the combination of asset sales and equity issuance that minimizes the value of their debt per unit face value. In line with this objective, banks choose to concentrate their portfolios in one asset and discard the others, as this allows for minimal diversification and maximal risk-taking. The asset that is retained is the one that provides the best tradeoff between risk and allowable leverage, a function of the asset's regulatory risk weight.

Based on this result, we show that the assignment of risk weights can affect whether banks engage in fire sales when recapitalizing. If the risk weight on a particular asset is sufficiently high, banks will choose to sell it because retaining it forces them to be underleveraged, limiting the amount of risk that can be shifted on to creditors. Since high risk-weight assets are likely to be illiquid, fire sales can result. However, fire sales may be prevented if the risk weights on all assets are the same because banks will retain the asset that has the greatest underlying risk and sell assets that are safer and probably more liquid.

When liquidity requirements are introduced, banks again choose to concentrate their portfolios into one asset but instead of selling the others completely, they are sold to the maximum extent that the liquidity requirements allow. This has two main effects on the bank recapitalization process. First, banks have an incentive to build up holdings of the desired asset as much as possible in order to dilute the holdings of the less desired assets that must be held because of liquidity requirements. Second, the appeal of retaining a risky, illiquid asset is diminished because banks are forced to hold a certain amount of safe, liquid assets, creating unwanted diversification. To avoid this, banks might instead choose to build up holdings of a different asset and sell the illiquid asset in a fire sale.

Finally, mandatory equity issuance can potentially mitigate the severity of fire sales when it is binding because it forces banks to do less net selling of potentially illiquid assets. However, if the circumstances are such that banks are issuing sufficient amounts of equity on their own, fire sales may still occur. This issue can potentially be addressed by setting mandatory equity issuance amounts aggressively.

Overall, our model suggests that regulations can impact how banks choose to recapitalize in a crisis. While assigning high risk weights to risky, illiquid assets may have favorable ex-ante incentives, doing so might generate unintended ex-post incentives for banks to engage in fire sales. In contrast, a system of uniform risk weights probably reduces the risk of fire sales. These findings are interesting in light of the fact that the new Basel III accords both raise the risk weights on a variety of risky, illiquid assets and introduce a simple leverage requirement (which is equivalent to uniform risk weights). According to the results of this paper, these two policies may actually have different effects on the tendency of banks to engage in fire sales when recapitalizing.

Our model also suggests that the interaction between different types of regulations can affect bank decision making. For example, it is possible that banks choose not to engage in fire sales when just capital requirements are in place but do engage in fire sales when liquidity requirements are introduced.

Cyclical capital requirements (CCRs) are a well established component of the macro-prudential toolkit – see, for example, IMF-FSB-BIS (2016). A policy prescription of our model is cyclical risk weights (CRWs) for illiquid assets. In addition to implicitly lowering the capital requirement after a shock occurs (as with CCRs), CRWs would also lower risk weights on certain risky, illiquid assets, making it more worthwhile for banks to retain these assets in the process of recapitalizing. In principle, if some risk weights are increased while others are reduced, CRWs could mitigate destabilizing fire sales of risky assets while leaving the overall level of capital requirements unchanged. CRWs therefore offer additional ammunition against the risk of fire sales compared to CCRs. As with CCRs, CRWs should be implemented in a consistent manner across all banks, ideally by building ex-ante buffers that can be released (IMF-FSB-BIS 2016). All cyclical adjustments to capital requirements call for particular regulatory attention to actions that could jeopardize future financial stability.

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A Proof of Proposition 1

Proposition 1. *The solution to the recapitalization problem described by (23) and (24) involves concentrating the portfolio into one asset and selling all other assets. The asset that is retained is the solution to*

$$\arg\min_i \frac{E\left(min\left[\frac{\eta_i}{\lambda_i}, 1-\theta r_i\right]\right)}{1-\theta r_i}$$

If multiple assets solve this problem, banks still choose to retain a single asset as long as there is imperfect correlation across asset returns. However, the bank is indifferent between retaining any of the assets that solve the problem.

Proof. The first step of the proof is establishing the following lemma.

Lemma 1. *Let $x, y \in \mathbb{R}_n^{++}$ and let α be in the n-dimensional unit simplex. Then*

$$\frac{x_j}{y_j} = \min\left\{\frac{x_i}{y_i}\right\} \implies \frac{x_j}{y_j} \leq \frac{\sum_{i=1}^{n} \alpha_i x_i}{\sum_{i=1}^{n} \alpha_i y_i}$$

with strict inequality if $\alpha_j \neq 1$ and $\frac{x_j}{y_j}$ is a unique minimum.

Proof. The proof is by induction. First, we prove the result for $n = 2$. Without loss of generality, suppose $\frac{x_1}{y_1} \leq \frac{x_2}{y_2}$. Based on the conditions of the Lemma, $\alpha_1 \in [0, 1]$. If $\alpha_1 = 1$, then the result is trivially true. Suppose $\alpha_1 \in (0, 1)$. This implies

$$\frac{\alpha_1 x_1}{\alpha_1 y_1} \leq \frac{\alpha_2 x_2}{\alpha_2 y_2}$$

$$\implies \alpha_1 x_1 \alpha_2 y_2 \leq \alpha_1 y_1 \alpha_2 x_2$$

$$\implies \alpha_1 x_1(\alpha_1 y_1 + \alpha_2 y_2) \leq \alpha_1 y_1(\alpha_1 x_1 + \alpha_2 x_2)$$

$$\implies \frac{x_1}{y_1} \leq \frac{\alpha_1 x_1 + \alpha_2 x_2}{\alpha_1 y_1 + \alpha_2 y_2}$$

and the result holds. Note that if $\frac{x_1}{y_1}$ is the unique minimum ($\frac{x_1}{y_1} < \frac{x_2}{y_2}$), the result would hold with strict inequality.

Now suppose $\alpha_1 = 0$ (which implies $\alpha_2 = 1$). Then we have

$$\frac{x_1}{y_1} \leq \frac{x_2}{y_2} = \frac{\alpha_1 x_1 + \alpha_2 x_2}{\alpha_1 y_1 + \alpha_2 y_2}$$

and the result still holds, noting again that if $\frac{x_1}{y_1}$ is the unique minimum, the result would hold with strict inequality.

Now suppose that the result is true for n. With $x, y \in \mathbb{R}^{++}_{n+1}$, $\frac{x_j}{y_j} = \min\{\frac{x_i}{y_i}\}$, and α in the $(n+1)$-dimensional unit simplex, we must show that the result is true for $n+1$:

$$\frac{x_j}{y_j} \leq \frac{\sum_{i=1}^{n+1} \alpha_i x_i}{\sum_{i=1}^{n+1} \alpha_i y_i}$$

Again, the result is trivially true for $\alpha_j = 1$. Suppose that $\alpha_j \in [0, 1)$. We start with

$$\frac{\sum_{i=1}^{n+1} \alpha_i x_i}{\sum_{i=1}^{n+1} \alpha_i y_i} = \frac{\sum_{i \neq j} \alpha_i x_i + \alpha_j x_j}{\sum_{i \neq j} \alpha_i y_i + \alpha_j y_j}$$

Given the assumptions that $\frac{x_j}{y_j} = \min\left\{\frac{x_i}{y_i}\right\}$ and the result is true for n, we have

$$\frac{x_j}{y_j} = \min\left\{\frac{x_i}{y_i}\right\} \leq \min_{i \neq j}\left\{\frac{x_i}{y_i}\right\} \leq \frac{\sum_{i \neq j} \frac{\alpha_i}{1-\alpha_j} x_i}{\sum_{i \neq j} \frac{\alpha_i}{1-\alpha_j} y_i}$$

with the first inequality being strict if $\frac{x_j}{y_j}$ is the unique minimum. Since we established that the result is true for $n = 2$, we can use the outer terms of the above to show

$$\frac{x_j}{y_j} \leq \frac{(1-\alpha_j)\sum_{i \neq j} \frac{\alpha_i}{1-\alpha_j} x_i + \alpha_j x_j}{(1-\alpha_j)\sum_{i \neq j} \frac{\alpha_i}{1-\alpha_j} y_i + \alpha_j y_j} = \frac{\sum_{i=1}^{n+1} \alpha_i x_i}{\sum_{i=1}^{n+1} \alpha_i y_i}$$

with the inequality being strict if $\frac{x_j}{y_j}$ is the unique minimum. This completes the proof of the lemma. $\square$

The proof continues as follows. When only asset j is retained, $s_i\bar{\lambda} = w_i\lambda_i$ for all $i \neq j$. Using this, the objective function (23) when only asset j is retained simplifies to

$$\frac{E\left(min\left[\frac{\eta_j}{\lambda_j}, 1-\theta r_j\right]\right)}{1-\theta r_j} \tag{29}$$

Therefore, if the bank wants to minimize the objective while retaining only one asset, that asset must be contained in the set S defined below:

$$S = \left\{ j \in \{1,2,...,n\} : j = \arg\min_i \frac{E\left(min\left[\frac{\eta_i}{\lambda_i}, 1-\theta r_i\right]\right)}{1-\theta r_i} \right\} \tag{30}$$

The set S is non-empty and may contain multiple elements. Let k be any element in S. We will now show that the objective function when retaining only asset k is strictly less than the objective when the bank holds any portfolio containing more than just one asset. Clearly, if the bank holds a portfolio with just one asset, the objective will either be the same (if this asset is in S) or higher (if this asset is not in S). In other words, the solution to the recapitalization problem involves retaining only asset k, where k is any element of S.

Let $\{s_1, s_2, ..., s_n\}$ be a decision of the bank in which more than one asset is retained. Denote the bank's portfolio weight on asset j after all sales and purchases by α_j, defined below:

$$\alpha_j = \frac{w_j\lambda_j - s_j\bar{\lambda}}{\sum_{i=1}^{n}(w_i\lambda_i - s_i\bar{\lambda})} \tag{31}$$

where α is in the n-dimensional unit simplex because of (24). With this terminology, the condition that more than one asset is retained in the portfolio is equivalent to $\alpha_i \in (0,1)$ for at least two assets.

Now, we divide the numerator and denominator of bank's objective (23) by $\sum_{i=1}^{n}(w_i\lambda_i - s_i\bar{\lambda})$ to get

$$g(s_1, s_2, ..., s_n) = \frac{E\left(min\left[\sum_{i=1}^{n} \alpha_i \frac{\eta_i}{\lambda_i},\ \sum_{i=1}^{n} \alpha_i (1-\theta r_i)\right]\right)}{\sum_{i=1}^{n} \alpha_i (1-\theta r_i)} \tag{32}$$

Our goal is to show that if $\alpha_i \in (0,1)$ for at least two assets, then (29) for asset k, where $k \in S$ defined in (30), is strictly less than (32) . Since the function $min(x,y)$ is concave and α is in the n-dimensional unit simplex, Jensen's inequality implies

$$min\left[\sum_{i=1}^{n} \alpha_i \frac{\eta_i}{\lambda_i},\ \sum_{i=1}^{n} \alpha_i (1-\theta r_i)\right] \geq \sum_{i=1}^{n} \alpha_i min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i\right] \tag{33}$$

$$\implies \frac{min\left[\sum_{i=1}^{n} \alpha_i \frac{\eta_i}{\lambda_i},\ \sum_{i=1}^{n} \alpha_i (1-\theta r_i)\right]}{\sum_{i=1}^{n} \alpha_i (1-\theta r_i)} \geq \frac{\sum_{i=1}^{n} \alpha_i min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i\right]}{\sum_{i=1}^{n} \alpha_i (1-\theta r_i)} \tag{34}$$

$$\implies g(s_1, s_2, ..., s_n) \geq \frac{\sum_{i=1}^{n} \alpha_i E\left(min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i\right]\right)}{\sum_{i=1}^{n} \alpha_i (1-\theta r_i)} \tag{35}$$

Note that (34) is true because $\theta r_i < 1$ $\forall i$ and $\alpha_i > 0$ for at least two assets. The expression in (35) takes expectations of both sides of (34) and uses (32).

In (33), the inequality is generally weak but it will be useful to understand when it would be strong. Define the sets $D_1(\eta, \alpha)$ and $D_2(\eta, \alpha)$ as follows:

$$\begin{aligned} D_1(\eta, \alpha) &= \left\{i \in \{1, , ..., n\} : \frac{\eta_i}{\lambda_i} < 1-\theta r_i \text{ and } \alpha_i \in (0,1)\right\} \\ D_2(\eta, \alpha) &= \left\{i \in \{1, , ..., n\} : \frac{\eta_i}{\lambda_i} > 1-\theta r_i \text{ and } \alpha_i \in (0,1)\right\} \end{aligned} \tag{36}$$

The set D_1 is the set of assets that have positive weights in portfolio α and for a particular realization of η, such that a well-capitalized bank in period 1 holding only that asset defaults in period 2. This is because as can be seen from (29), the ratio of assets to debt in period 2 for a well-capitalized bank (capital ratio equals θ) in period 1 holding only asset x is $\frac{1}{1-\theta r_x}\frac{\eta_x}{\lambda_x}$. Similarly, the set D_2 is the set of assets that have positive weights in portfolio α and for a particular realization of η, such that a well-capitalized bank in period 1 holding only that asset does not default in period 2.

Based on the nature of the min function in (33), it is clear that the inequality in (33) would be strong if both D_1 and D_2 were non-empty for all possible α's. This would be true if, for every possible portfolio α and for a particular realization of η, there exists two particular assets with positive weight. For one asset, a well-capitalized bank holding just that asset in period 1 defaults in period 2. For the other asset, a well-capitalized bank holding just that asset does not default in period 2.

This may not be true for a particular realization of η. However, in (35), expectations of (34) are taken over all possible realizations of η. Therefore, the inequality in (35) would be strict if, for all possible portfolios α, there exists a realization of η such that the sets D_1 and D_2 are both non-empty. This condition can be stated more concisely as $D_3 \neq \emptyset$, where D_3 is defined below in (37).

$$D_3 = \{\{\eta\} : D_1(\eta, \alpha) \neq \emptyset \text{ and } D_2(\eta, \alpha) \neq \emptyset \ \forall \alpha\} \tag{37}$$

The condition $D_3 \neq \emptyset$ can be thought of as imposing imperfect correlation across asset returns. The condition is fairly weak and will be useful in establishing the uniqueness of the solution in certain corner cases (see the end of the proof).

Now, define the following:

$$x_i = E\left(min\left[\frac{\eta_i}{\lambda_i},\ 1 - \theta r_i\right]\right)$$

$$y_i = 1 - \theta r_i$$

which imply the following:

$$\frac{x_i}{y_i} = \frac{E\left(min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i\right]\right)}{1-\theta r_i}$$
$$\frac{\sum_{i=1}^{n}\alpha_i x_i}{\sum_{i=1}^{n}\alpha_i y_i} = \frac{\sum_{i=1}^{n}\alpha_i E\left(min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i\right]\right)}{\sum_{i=1}^{n}\alpha_i(1-\theta r_i)} \tag{38}$$

Note that $\frac{x_i}{y_i}$ takes the same form as (29), the objective function when retaining only one asset. In addition, $x_i, y_i \in \mathbb{R}_n^{++}$ and α defined in (31) is in the n-dimensional unit simplex. Furthermore, according to (30), we assumed that among the options of retaining each asset by itself, retaining asset $k \in S$ minimizes the bank's objective function: $\frac{x_k}{y_k} = \min\left\{\frac{x_i}{y_i}\right\}$. With all of these conditions, we can demonstrate the following.

$$\frac{E\left(min\left[\frac{\eta_k}{\lambda_k},\ 1-\theta r_k\right]\right)}{1-\theta r_k} = \frac{x_k}{y_k} \leq \frac{\sum_{i=1}^{n}\alpha_i x_i}{\sum_{i=1}^{n}\alpha_i y_i} = \frac{\sum_{i=1}^{n}\alpha_i E\left(min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i\right]\right)}{\sum_{i=1}^{n}\alpha_i(1-\theta r_i)} \leq g(s_1, s_2, ..., s_n)$$

where the equalities are from (38), the first inequality is from Lemma 1, and the second inequality is from (35).

Note that $\alpha_k \neq 1$ because at least two assets in the portfolio have positive weight. Therefore, if S contains only one element or equivalently, if the solution to (30) is unique, the first inequality would be strict based on Lemma 1 and the proof is complete. The intuition here is that if retaining asset k by itself is strictly preferred to retaining any other asset by itself, retaining asset k must be superior to any portfolio of two or more assets, since the portfolio can be thought of as a weighted average of retaining the constituent assets by themselves. This logic holds even if the assets are perfectly correlated.

If the set S has multiple elements, the first inequality is not strict. In this case, the bank is indifferent between retaining different individual assets by themselves. However, if the set D_3

as defined in (37) is non-empty, the second inequality is strict and the proof is complete. This condition can be thought of as imposing imperfect correlation across the returns of the assets contained in S. The intuition is that imperfect correlation creates diversification and lowers risk. As a result, it is undesirable for the bank to hold two assets at once, even if they are both contained in S.

The only case in which the proof breaks down is if S contains multiple elements and the assets in S are almost perfectly correlated (i.e. D_3 as defined in (37) is empty), a situation that would only occur if two assets were close to identical. Even in this case, the solution prescribed in Proposition 1 is valid, just not necessarily unique.

□

B Proof of Proposition 2

Proposition 2. *For each asset k, there exists $\bar{r}_k < \frac{1}{\theta}$ such that for all $r_k > \bar{r}_k$, asset k is sold in the process of recapitalizing, holding fixed all other assets' risk weights.*

Proof. Expression (26) is the value of the bank's objective if it chooses to retain asset k. Note that (26) only depends on r_k, not any of the other assets' risk weights. Also, note that (26) converges to 1 as r_k converges to $\frac{1}{\theta}$ from below:

$$\lim_{r_k \to \frac{1}{\theta}^-} \frac{E\left(min\left[\frac{\eta_k}{\lambda_k},\, 1-\theta r_k\right]\right)}{1-\theta r_k} = 1$$

Note also that since the objective is the price of debt per unit face value, its value must be less than one. Therefore, if we can show that (26) is strictly increasing in r_k, there exists some $\bar{r}_k < \frac{1}{\theta}$ above which (26) will exceed the value of the objective when retaining any other asset. For any $r_k \in (\bar{r}_k, \frac{1}{\theta})$, asset k will be sold and a different asset will be retained.

We begin by rewriting (26) as follows

$$g(\cdot) = \frac{1}{1-\theta r_k}\int_{-\infty}^{\infty} min\left[\frac{\eta_k}{\lambda_k},\ 1-\theta r_k\right] f_k(\eta_k)d\eta_k \tag{39}$$
$$= 1 - F_k(\lambda_k(1-\theta r_k)) + \frac{1}{\lambda_k(1-\theta r_k)}\left(\int\limits_{-\infty}^{\lambda_k(1-\theta r_k)} \eta_k f_k(\eta_k)d\eta_k\right)$$

then differentiating with respect to r_k

$$\frac{\partial g(\cdot)}{\partial r_k} = f_k(\lambda_k(1-\theta r_k))\theta\lambda_k + \frac{1}{(\lambda_k(1-\theta r_k))^2}\Bigg(\lambda_k(1-\theta r_k)(-\theta\lambda_k)\lambda_k(1-\theta r_k)f_k(\lambda_k(1-\theta r_k))$$
$$+\theta\lambda_k \int\limits_{-\infty}^{\lambda_k(1-\theta r_k)} \eta_k f_k(\eta_k)d\eta_k\Bigg)$$
$$= \frac{\theta\lambda_k}{(\lambda_k(1-\theta r_k))^2}\left(\int\limits_{0}^{\lambda_k(1-\theta r_k)} \eta_k f_k(\eta_k)d\eta_k\right) > 0$$

because θ, λ_k, and $1-\theta r_k$ are all strictly greater than zero. This completes the proof.

□

C Proof of Proposition 3

Proposition 3. *Suppose the risk weights of all assets are identically equal to one. If asset j's net return ($\frac{\eta_j}{\lambda_j} - 1$) is a mean-preserving spread of the return of asset k (i.e. asset j is riskier), then the bank will not sell asset j in the process of recapitalizing.*

Proof. Suppose that the return of asset j ($R_j = \frac{\eta_j}{\lambda_j} - 1$) is a mean-preserving spread of the return of asset k ($R_k = \frac{\eta_k}{\lambda_k} - 1$). According to (26) with $r_j = r_k = 1$, asset k is sold in the process of recapitalizing if

$$
\begin{aligned}
\frac{E\left(min\left[\frac{\eta_k}{\lambda_k},\ 1-\theta\right]\right)}{1-\theta} &\geq \frac{E\left(min\left[\frac{\eta_j}{\lambda_j},\ 1-\theta\right]\right)}{1-\theta} \\
\iff E\left(min\left[R_k,\ -\theta\right]\right) &\geq E\left(min\left[R_j,\ -\theta\right]\right) \\
\iff E\left(h(R_k)\right) &\geq E\left(h(R_j)\right) \qquad (40)
\end{aligned}
$$

where the function $h(\cdot)$ is defined as $h(x) = \min\left[x, -\theta\right]$.

Since R_j is a mean-preserving spread of R_k, it must be the case that $E\left(u(R_k)\right) \geq E\left(u(R_j)\right)$ for any concave, non-decreasing function $u(\cdot)$. Since $h(\cdot)$ in (40) is concave and non-decreasing, the proof is complete. □

D Proof of Proposition 4

Proposition 4. *With constraint (27) in place of (24) to reflect liquidity requirements, the solution to the bank's recapitalization problem is an extreme point of the feasible set: sell all assets but one and expand holdings of the remaining asset, both to the maximum extent allowable under (27).*

Proof. The proof has three steps. The first step is showing that any feasible solution can be represented as a convex combination of the extreme points of the feasible set.[16] The second step is showing that the objective function is quasiconcave.[17] Combining the first two steps, the third step is showing that the value of the objective at any non-extreme feasible solution must exceed the value of the objective at some extreme point of the feasible set. Therefore, the objective must achieve its minimum at an extreme point.

Let the feasible set for $s = \{s_1, s_2, ..., s_n\}$ be given by Ω_s. The set Ω_s is defined by (27) and is clearly closed, bounded, and therefore compact. In addition, Ω_s is an n-dimensional box and is therefore convex.

[16] An extreme point of a set S is any point in S which does not lie in any open line segment joining two points of S.

[17] A function $f(x)$ is quasiconcave if and only if for every $t \in [0, 1]$ and any x_1, x_2 in the domain of $f(\cdot)$, $f(tx_1 + (1-t)x_2) \geq \min\{f(x_1), f(x_2)\}$.

The objective function can be restated in terms of the vector α in the n-dimensional unit simplex, as in (32), with α defined in (31). Furthermore, based on (31), the function that maps s to α is a linear fractional function of the form

$$\alpha = \frac{As + b}{c's + d}$$

where $A = -\bar{\lambda} I_n$, $b_i = w_i \lambda_i$, $c_i = -\bar{\lambda}$, and $d = \bar{\lambda}$. Since linear fractional functions preserve convexity of sets, it follows that the feasible set for α, Ω_α, is also convex. Moreover, the function that maps s to α is also continuous, which implies that Ω_α is also compact.

The Krein-Millman theorem states that if Ω_α is convex and compact, Ω_α is the convex hull of its extreme points, given by the set $e(\Omega_\alpha)$. By the definition of a convex hull, this implies that for every non-extreme feasible solution α^f to the problem in Ω_α, there exists $t \in (0,1)$ and $\alpha^1, \alpha^2 \in e(\Omega_\alpha)$ such that $\alpha^f = t\alpha^1 + (1-t)\alpha^2$. Note that the definition of extreme points implies that the set $e(\Omega_\alpha)$ contains all the portfolios that result from accumulating one asset while selling all other assets, both to the maximum extent possible under (27).

Below, we show that the objective function (32) evaluated at any non-extreme feasible solution α^f is weakly greater than the minimum of the objective evaluated at all of the extreme points of Ω_α.

$$
\begin{aligned}
g(\alpha^f) &= g(t\alpha^1 + (1-t)\alpha^2) \\
&= \frac{E\left(min\left[\sum_{i=1}^{n} (t\alpha_i^1 + (1-t)\alpha_i^2)\frac{\eta_i}{\lambda_i},\ \sum_{i=1}^{n} (t\alpha_i^1 + (1-t)\alpha_i^2)(1-\theta r_i) \right] \right)}{\sum_{i=1}^{n} (t\alpha_i^1 + (1-t)\alpha_i^2)(1-\theta r_i)} \\
&\geq \frac{E\left(\sum_{i=1}^{n} (t\alpha_i^1 + (1-t)\alpha_i^2) min\left[\frac{\eta_i}{\lambda_i},\ 1-\theta r_i \right] \right)}{t\sum_{i=1}^{n} \alpha_i^1(1-\theta r_i) + (1-t)\sum_{i=1}^{n} \alpha_i^2(1-\theta r_i)} \qquad (41) \\
&= \frac{tE\left(\sum_{i=1}^{n} min\left[\alpha_i^1\frac{\eta_i}{\lambda_i},\ \alpha_i^1(1-\theta r_i) \right] \right) + (1-t)E\left(\sum_{i=1}^{n} min\left[\alpha_i^2\frac{\eta_i}{\lambda_i},\ \alpha_i^2(1-\theta r_i) \right] \right)}{t\sum_{i=1}^{n} \alpha_i^1(1-\theta r_i) + (1-t)\sum_{i=1}^{n} \alpha_i^2(1-\theta r_i)} \qquad (42)
\end{aligned}
$$

where (41) uses the fact that the function $min(x, y)$ is concave and $t\alpha_i^1 + (1-t)\alpha_i^2$ is in the n-dimensional unit simplex to apply Jensen's inequality. Note that for the same reason discussed at length in Appendix A (see the discussion of equations (36) and (37)), the inequality in (41) would be strict under fairly weak conditions that amount to imperfect correlation across assets.

Let $g_n(\cdot)$ and $g_d(\cdot)$ represent the numerator and denominator, respectively, of the objective (32). Then, continuing from (42) we have

$$
g(a^f) \geq \frac{tg_n(\alpha^1) + (1-t)g_n(\alpha^2)}{tg_d(\alpha^1) + (1-t)g_d(\alpha^2)}
$$

Since $t \in (0, 1)$, we can use Lemma 1 to show

$$
g(a^f) \geq \min\{g(\alpha^1), g(\alpha^2)\} \geq \min_{\alpha \in e(\Omega_\alpha)} \{g(\alpha)\}
$$

The first inequality is equivalent to the objective function $g(\cdot)$ being quasiconcave. The second inequality is strict if one of the extreme points uniquely minimizes the objective among all extreme points.

This completes the proof. To summarize, we have shown that the objective function evaluated at any non-extreme, feasible solution is weakly greater than the minimum objective across all extreme points of the feasible set. This means that at least one of the extreme points minimizes the objective. Moreover, if assets are imperfectly correlated or one of the extreme points uniquely minimizes the objective among all extreme points, no non-extreme point minimizes the objective. Finally, if multiple extreme points minimize the objective, they all represent valid solutions.

$\square$